Numerical Solution of one-dimensional Burgers' equation using finite difference method

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Declaration

I undersigned declare that this thesis entitled 'Numerical solution of one-dimensional Burgers' equation using finite difference method' is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this thesis, a finite difference method resulting from cubic spline is presented for solving the one-dimensional Burgers' equation. First, the nonlinear Burgers' equation is transformed to one-dimensional heat equation via Cole-Hopf transformation. The stability of the method is well established and it is found to be unconditionally stable. The numerical solutions presented in tables confirm that the present method approximates the exact solution very well.

Key words: *Burgers' equation, Cubic spline functions, Cole-Hopf transformation, Reynolds number.*

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CHAPTER ONE

1. Introduction

1.1. Background of the Study

Numerical analysis is the area of mathematics and computer science that creates, analyzes, and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real world applications of algebra, geometry and calculus, and they involve variables which vary continuously; these problems occur throughout the natural sciences, social sciences, engineering, medicine and business.

During the past half century the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science and engineering, and numerical analysis of increasing sophistication has been needed to solve these more detailed mathematical models of the world. The formal academic area of numerical analysis varies from quite theoretical mathematical studies to computer science issues (Atkinson and Han, 2001).With the growth in importance of using computers to carry out numerical procedures in solving mathematical models of the world an area known as scientific computing or computational science has taken shape during the 1980s and 1990s. This area looks at the use of numerical analysis from a computer science perspective. It is concerned with using the most powerful tools of numerical analysis, computer graphics, symbolic mathematical computations, and graphical user interfaces to make it easier for a user to set up, solve and interpret complicated mathematical models of the real world.

Numerical analysis plays a significant role and helps us to find an approximate solution for problems which are difficult to solve analytically. One of these methods is finite difference method. Finite difference methods (FDMs) are numerical methods for solving differential equations by approximating them with difference equations in which finite differences approximate the derivatives(Christianetal,2007). FDMs are thus discretization methods which convert a linear (non-linear) ODEs or PDEs into a system of linear (non-linear) equations and then be solved by matrix algebra techniques. The reduction of the differential equation to a system of algebraic equations makes the problem of finding the solution to a given ODE or PDE ideally suited to modern computers, hence the widespread use of FDMs in modern numerical analysis.

Burgers' equation

Burgers' equation arises in the theory of shock waves, in turbulence problems and in continuous stochastic processes. It has a large variety of applications in modeling of water in unsaturated soil, gas dynamics, heat conduction, elasticity, statics of flow problems, mixing and turbulent diffusion, cosmology, seismology, are the popular ones (Burger,1948).

Burgers' equation is an important and simple model in understanding the physical flows. It describes various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave travelling in a viscous fluid. This equation provides the simplest nonlinear models of turbulence in the phenomena process. The study of the general properties of the Burgers' equation has motivated considerable attention due to its applications in the various fields. These equations are decisive in determining the appearance of shock waves in the supersonic motion of gas. A study of Burgers' equation is important since it appears in the approximate theory of flow through a shock wave propagating in a viscous fluid and in the modeling of turbulence. Burgers' equation and the Navier-Stokes equation are similar in the form of their nonlinear terms and in the occurrence of higher order derivatives with small coefficients. Numerical techniques based on Variation iterative methods (Caldwell, 1985), Spectral and finite difference method (Basdevant, 1986), finite difference method (Kutlay et al, 1999), finite element method (Mittal and Jain, 2012), Homotopy analysis method (Inc, 2008), a numerical method based on Crank-Nicolson scheme for Burgers' equation (Kadalbajoo and Awashti,2006) and boundary element method (Bahadır and Saˇglam, 2005) have been developed in the efforts to solve Burgers' equation numerically. The modified Burgers' equation (MBE) is called the nonlinear advection-diffusion equation. The MBE has been solved by several researchers by analytically and/or numerically (Oruc, 2015).

Even if numerical solution of the Burgers' equation is well documented in the literature, a detailed literature survey indicates that still there exist a need for improving its solution for comparative discussion regarding the physical and mathematical significance of this equation.

The one-dimensional burgers' equation is described as follows:

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}=v\frac{\partial^2 u(x,t)}{\partial x^2}, 0<x<1, 0<t\le T. \qquad (1.1)$$

Subject to initial condition

$$u(x,0)=g(x), 0\le x\le 1$$

and boundary conditions

$$u(0,t)=h_1(t) \text{ and } u(1,t)=h_2(t), 0\le t\le T.$$

where $v=\frac{1}{\text{Re}}$ (Re is Reynolds number) is the positive coefficient of kinematic Viscosity and g, h_1 and h_2 are the sufficiently smooth given functions.

Linearization of one-dimensional Burgers' equation by Cole-Hopf transformation

Equation in Eq. (1.1) can be reduced to linear heat equation by the nonlinear transformation .The transformation $u=-2v\frac{\Phi_x}{\Phi}$ relates $u(x,t)$ and Φ, Where Φ is a solution of the linear heat equation $\frac{\partial\Phi}{\partial t}=v\frac{\partial^2\Phi}{\partial x^2}$ and u is the solution of the Burgers' equation Eq. (1.1).

Theorem 1 (Kadalbajoo and Awashti, 2006).In the context with initial and boundary conditions of Eq.(1.1), if $\Phi(x,t)$ is any solution to the heat equation $\frac{\partial\Phi}{\partial t}=v\frac{\partial^2\Phi}{\partial x^2}$, then the nonlinear transformation (Hopf and Cole, 1951)

$$u=-2v\frac{\Phi_x}{\Phi}, \qquad (1.2)$$

is a solution to Eq. (1.1)

From this, the Burgers' equation in Eq. (1.1) is transformed to linear heat equation of the form:-

$$\frac{\partial \Phi}{\partial t}(x,t) = v\frac{\partial^2 \Phi}{\partial x^2}(x,t), 0 < x < 1, 0 < t \le T \tag{1.3}$$

with initial condition

$$\Phi(x,0) = \exp\left(\frac{-1}{2v}\int_0^x u_0(\zeta)d\zeta\right), 0 \le x \le 1$$

and boundary conditions

$$\Phi_x(0,t) = 0 = \Phi_x(1,t),\ 0 \le t \le T$$

1.2 Statement of the Problem

The Burgers' equation given in Eq. (1.1) is nonlinear and parabolic and one expects to find phenomena similar to turbulence. It is a fundamental partial differential equation (PDE) that occurs in various fields of applied mathematics such as fluid mechanics, nonlinear acoustic, gas dynamics and traffic flow(Burger, 1948).It is a simplified form of Navier stokes equation that very well represents their non-linear features. In this equation continuity and pressure components of the Navier stoke is omitted. This equation is widely used in many physical phenomena. Because of it is widely used, these equations have been studied by many researchers and are appropriate context for research activities.

In literature, many numerical methods have been proposed and implemented for approximating solution of the Burgers' equation. Numerically it has been treated by many researchers. Evans and Abdullah, 1948 introduced the Group explicit method in order to solve this equation. These methods are semi-explicit and shown to be unconditionally stable and accurate of order $O\left(\Delta t + (\Delta x)^2 + \frac{\Delta t}{\Delta x}\right)$.It is observed that this methods having severed consistency condition, i.e., methods are consistent if and only if $\frac{\Delta t}{\Delta x} \to 0$ when $\Delta t \to 0, \Delta x \to 0$.

Others have used numerical techniques based on the explicit exponential finite difference method was originally developed by (Bhattacharya, 1985), exponential finite difference method for solving this equation (Handschuh and Keith, 1992), finite difference (Zhang and Wang, 2012), finite element (Mittal and Jain, 2012) and boundary element methods (Bahadır and Saˇglam, 2005) in attempting to solve the equation. Kadalbajoo used a parameter uniform implicit difference scheme for solving time-dependent Burgers' equation *(*Kadalbajoo etal, 2005).

Burgers' equation is solved on the real line with finite element method (Konzena etal, 2016) numerical simulations of this method is considered by a sequence of auxiliary spatially dimensionless Dirichlet's problems parameterized by the domain's semi diameter L. This equation is solved by explicit finite difference method and implicit finite difference method (Ramlee and Rusli, 2017).In this solution process, the explicit exponential finite difference method is used a directly to solve the Burgers' equation while the implicit exponential finite difference method leads to a system of nonlinear equation. A numerical method based on Crank-Nicolson scheme for solving one-dimensional Burgers' equation (Kadalbajoo and Awasthi, 2006) is developed. The method has shown to be unconditionally stable and is second order accurate in space and time. This method gives accurate solution for small step sizes ($h = 0.025$ and $h = 0.0125$).This leads to large system of linear equation which requires large CPU time and memory storage. Inan and Ahmet in 2013 also developed implicit and fully implicit exponential finite difference-method for solving one-dimensional Burgers' equation. In this method there is restriction in choosing the mesh sizes. One-dimensional nonlinear Burgers' equation is solved using the Adomian decomposition Method (Haghighi and Shojaeifard, 2015). This method includes the unknown function $U(x)$ in which each equation is defined and solved by an infinite series of unbounded functions. Velocity parameters u in the direction of the x-axis are examined at different times with different Reynolds numbers over a fixed time step. Even though a considerable number of numerical techniques have been applied to solve one-dimensional Burgers' equation; still there is an attempt to improve the accuracy and efficiency of one-dimensional Burgers' equation. Hence the aim of this thesis is to construct accurate and efficient numerical method for one-dimensional Burgers' equation.

1.3 Objective of the Study

1.3.1 General Objective

The aim of this study is to formulate finite difference method for solving one-dimensional Burgers' equation.

1.3.2 Specific Objectives

The specific objectives of the study are:

- To apply the finite difference method resulting from the cubic spline to solve one-dimensional Burgers' equation;
- To establish the stability of the present method;
- To compare the advantage of the present method over other existing methods in the literature.

1.4 Significance of the study

This study provides back ground information for other researchers who want to work on similar and/or related areas.

1.5. Delimitation of the Study

This thesis is delimited to the numerical solution of the one-dimensional Burgers' equation:

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}=v\frac{\partial^2 u(x,t)}{\partial x^2}, 0<x<1, 0<t\le T .$$

Subject to initial condition

$$u(x,0)=g(x), 0\le x\le 1$$

and boundary conditions

$$u(0,t)=h_1(t) \text{ and } u(1,t)=h_2(t), 0\le t\le T$$

using the finite difference method resulting from cubic spline.

CHAPTER TWO

2. Review Literature

2.1. One-dimensional Burgers’ Equation

The history of Burgers' equation dates back to 1915 when (Bateman, 1915) derived it in a physical context. One of the most interesting solutions of Burgers' equations in a series form is due to (Fay, 1923) when it was derived in the acoustic framework. In 1940, Burgers gave special solutions to it and emphasized its importance and (Burgers, 1948) concluded its form as a model in the theory of turbulence. In connection with Burgers' equation had discovered that the Burgers' equation can be transformed to the linear heat equation which was published by Cole. This transformation at about the same time was discovered independently by (Blackstock, 1950) and from which it is known as Cole -Hopf transformation (Cole and Hopf, 1951). The Fay series was rediscovered by Hopf a quasi-linear parabolic equation occurring in aerodynamics Quart (Hopf, 1950) as an approximate solution of the Burgers' equation for a sinusoidal initial condition. Then independently (Lighthil, 1956) and Black stock employed Burgers' equations in studying the propagation of the one-dimensional acoustic signals of finite amplitude. (Lagerstrom, 1969) used the equation in the discussions of shock structure in Navier. Burgers’ equations can be treated as a qualitatively correct approximation of the Navier-Stocks equations. That means it can be considered as a simplified form of the Navier-Stock equation. The characteristics feature of Burgers’ equation is the combination of two terms $v\frac{\partial^2 u}{\partial x^2}$ (diffusion term) and $u\frac{\partial u}{\partial x}$ (convection term), which gives rise to the appearance of dissipation layers. The one-dimensional Burgers’ equation has received an enormous amount of attention since the studies by J.M. Burgers’ in the 1940’s, principally as a model problem of the interaction between nonlinear and dissipative phenomena. The Burgers’ equation is nonlinear and one expects to find Phenomena similar to turbulence.

However, as it has been shown by Cole the homogeneous Burgers’ equation lacks the most important property attributed to turbulence: The solutions do not exhibit chaotic features like sensitivity with respect to initial conditions. This can explicitly be shown using the Cole-Hopf transformation which transforms Burgers’ equation into a linear parabolic equation. From the numerical point of view, this is of importance since it allows one to compare numerically obtained solutions of the nonlinear equation with the exact one. This comparison is important to investigate the quality of the applied numerical scheme.

The method to solve one-dimensional Burgers' equation represented by the boundary and initial conditions (Malek and Mansi, 2000).Then the solution to solve the analytic is must close form by specific choice of boundary condition.

According to (Whitham, 1927) one-dimensional Burgers' equation is the simple combining with nonlinear and diffusive effect. Burgers' equation is originally proposed by simplifying model of turbulence which is exhibit by Navier-Stokes equations. The behavior of turbulence the Burgers' equation sometimes dubbed turbulence. Kaneda and Ishihara (2006) said that it is phenomena exhibited by solutions to particular partial differential equations that are Navier-Stokes equations.

Cole-Hopf transformation is one of the applications that can be used to generalize the nonlinear equation (Humi, 2013). Moreover, dimensional convection of steady state represents to generalize the Burgers'equation with convective terms.

According to (Kuo and Lee, 2015) to generalize one-dimensional Burgers' equation with nonlinear had been used by the researchers. There are two identical solutions that independently by a direct integration method and Bernoulli equation which is the simplest method to do. The nonlinear advance toward zero when the exact solution discontinuity and reduce to linearity.

The derivation of the exact solution will get coefficient of the nonlinear term so that same as Burgers' equation.

Cole-Hopf method alters to get linear heat equation by non-linear partial differential with some domains of initial values. Exact solution for initial and boundary conditions is taken by Adomian with putrefaction method. Use to prove the point about the reliability and applied directly for all types of differential equation with constant coefficient or variable coefficients.

Furthermore (Malek and Mansi,2000) estimate solution of the one-dimensional Burgers' equation which is get by Cole. Lighthill (1956) and Blackstock (1964) use one-dimensional Burgers' equation to study the dimensional acoustic signals of propagation final amplitude.

The equation shock structure in Navier-Stokes fluids on 1958 used by Hyes. Benton found an exact solution for one-dimensional Burgers’ equation on 1966.
On 1970, Riccati solution without using auxiliary conditions use for one-dimensional Burger's equation. Painleve property of the partial differential equation from integrability on 1983, while for the Backlund transforms and Lux pairs of the Burger's equation as well as KDV equation and modified of it by (Malek and Mansi, 2000).

CHAPTER THREE

3. Methodology

3.1 Study Area and Period

The study is conducted in Jimma University in 2018 /19Academic year.

3.2 Study Design

The Study employed mixed design. That is review of documents and experimental results from coding.

3.3 Source of Information

Related articles and books were used as a source of information.

3.4 Mathematical Procedures

In order to achieve the above mentioned objectives, the study follows

1. Describing the problem;

2. Apply the finite difference method resulting from cubic spline to the transformed Burgers' equation;

3. Prove the stability of the method for solving the transformed Burgers'equation;

4. Validate the present method via some numerical examples.

CHAPTER FOUR

4. Description of the Method, Results and Discussion

4.1. Description of the Method

Finite difference scheme resulting from cubic spline

Cubic spline

Let $S_i(x)$ be the cubic spline defined in the interval $\left[x_{i-1}, x_i\right]$ and

$S_i(x) = y_i$, $i = 0,1,2,...,n$, $S_i(x)$, $S_i^{'}(x)$ and $S_i^{''}(x)$ are continuous in $\left[x_0, x_n\right]$.

The governing equation of the cubic spline is determined from the spline second derivatives. Hence, we have in $\left[x_{i-1}, x_i\right]$

$$S_i^{''}(x) = \frac{1}{h_i}\left[(x_i - x)M_{i-1} + (x - x_{i-1})M_i\right] \tag{4.1}$$

where $h_i = x_i - x_{i-1}$ and $S_i^{''}(x_i) = M_i$, for all i.

Obviously the spline second derivatives are continuous. Integrating Eq. (4.1) twice with respect to x, we get the cubic spline S(x) interpolate the function $y(x)$ at the knots $x_i = x_0 + ih$ $(i = 0,1,2,...,n)$ on the interval $x_{i-1} \le x \le x_i$

by the equation (Sastry, 2012).

$$S(x) = M_{i-1}\frac{\left(x_i - x\right)^3}{6h} + M_i\frac{\left(x - x_{i-1}\right)^3}{6h} + \left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)\left(\frac{x_i - x}{h}\right) + \left(y_i - \frac{h^2}{6}M_i\right)\left(\frac{x - x_{i-1}}{h}\right) \tag{4.2}$$

Where $M_i = S^{''}(x_i)$ and $y_i = y(x_i)$.

In Eq. (4.2), the spline second derivatives, M_i are still not known. To determine them we use the condition of continuity of $S^{'}(x)$. Differentiating Eq. (4.2), we get

$$S^{'}(x) = \frac{1}{h_i}\left[\frac{-3\left(x_i - x\right)^2}{6}M_{i-1} + \frac{3\left(x - x_{i-1}\right)^2}{6}M_i - \left(y_{i-1} - \frac{h_i^2}{6}M_{i-1}\right) + \left(y_i - \frac{h_i^2}{6}M_i\right)\right] \tag{4.3}$$

Setting $x = x_i$ in the above, we obtain the left hand derivative

$$S_i^{'}(x_i^-) = \frac{h_i}{2} M_i - \frac{1}{h_i}\left(y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right) + \frac{1}{h_i}\left(y_i - \frac{h_i^{\,2}}{6} M_i \right) = \frac{1}{h_i}\left(y_i - y_{i-1} \right) + \frac{h_i}{6} M_{i-1} + \frac{h_i}{3} M_i . \qquad (4.4)$$

$(i = 1, 2, ..., n)$.

To obtain the right hand derivative, we need frist to write down the equation of the cubic spline in the sub interval $\left(x_i, x_{i+1} \right)$. We do this by setting $i = i + 1$ in Eq. (4.2).

$$S_{i+1}(x) = \frac{1}{h_{i+1}}\left[\frac{\left(x_{i+1} - x \right)^3}{6} M_i + \frac{\left(x - x_i \right)^3}{6} M_{i+1} + \left(y_i - \frac{h_{i+1}^2}{6} M_i \right)\left(x_{i+1} - x \right) + \left(y_{i+1} - \frac{h_{i+1}^2}{6} M_{i+1} \right)\left(x - x_i \right) \right] \quad (4.5)$$

where $h_{i+1} = x_{i+1} - x_i$.

Differentiating Eq. (4.5) and setting $x = x_i$, we obtain the right hand derivative at $x = x_i$,

$$S_{i+1}^{'}(x_i^+) = \frac{1}{h_{i+1}}\left(y_{i+1} - y_i \right) - \frac{h_{i+1}}{3} M_i - \frac{h_{i+1}}{6} M_{i+1} , \ \left(i = 0, 1, ..., n-1 \right). \qquad (4.6)$$

Equality of Eq. (4.4) and Eq. (4.6) produces the recurrence relation

$$\frac{h_i}{6} M_{i-1} + \frac{1}{3}\left(h_i + h_{i+1} \right) M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} , \ \ (i = 1, 2, ..., n-1) \qquad (4.7)$$

For equal intervals, we have $h_i = h_{i+1} = h$ and Eq. (4.7) simplified to

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1} \right), (i = 1, 2, ..., n-1) \qquad (4.8)$$

Now, consider the one-dimensional Burgers' equation in Eq. (1.1) which is transformed to the heat equation in Eq. (1.3)

$$\frac{\partial \Phi}{\partial t} = v \frac{\partial^2 \Phi}{\partial x^2} \tag{4.9a}$$

with initial condition

$$\Phi(x,0) = \exp\left(-\frac{1}{2v}\int_0^x u_0(\xi)\, d\xi\right), 0 \le x \le 1 \text{and} \tag{4.9b}$$

and boundary conditions

$$\Phi_x(0,t) = 0 = \Phi_x(1,t), t \ge 0 \tag{4.9c}$$

Let the time derivative $\frac{\partial \phi}{\partial t}$ is replaced by forward finite difference approximation $\frac{\phi_{i,j+1} - \phi_{i,j}}{k}$ and the space derivative $\frac{\partial^2 \phi}{\partial x^2}$ by a cubic spline $S(x)$, then a suitable approximation to Eq. (4.9) due to (Sastry, 2012) is given by:

$$\frac{1}{k}\left[\phi_{i,j+1} - \phi_{i,j}\right] = (1-\theta) M_{i,j} + \theta M_{i,j+1} \tag{4.10}$$

$i = 0,1,2,...,n$; $j = 1,2,...$, where $M_i^j = S_i^{''}(x_i)$.

Analogous with equation Eq. (4.8) the following relation holds:

$$M_{i-1,j} + 4M_{i,j} + M_{i+1,j} = \frac{6}{h^2}\left[\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}\right] \text{, and} \tag{4.11}$$

$$M_{i-1,j+1} + 4M_{i,j+1} + M_{i+1,j+1} = \frac{6}{h^2}\left[\phi_{i-1,j+1} - 2\phi_{i,j+1} + \phi_{i+1,j+1}\right] \tag{4.12}$$

Substituting Eq. (4.10) inEq. (4.12) yields the following finite difference scheme (Sastry, 2012).

$$\begin{aligned}(1-6r\theta)\Phi_{i-1}^{j+1}+(4+12r\theta)\Phi_{i}^{j+1}+(1-6r\theta)\Phi_{i+1}^{j+1}\\ =\left[1+6r(1-\theta)\right]\Phi_{i-1}^{j}+\left[4-12r(1-\theta)\right]\Phi_{i}^{j}+\left[1+6r(1-\theta)\right]\Phi_{i+1}^{j}\end{aligned} \tag{4.13}$$

In general, the finite difference scheme with cubic spline discretization to the linearized heat equation in Eq. (4.9a) with Neumann boundary conditions inEq. (4.9c) is given by:

$$(4+12r\theta)\Phi_{i}^{j+1}+2(1-6r\theta)\Phi_{i+1}^{j+1}=\left[4-12r(1-\theta)\right]\Phi_{i}^{j}+2\left[1+6r(1-\theta)\right]\Phi_{i+1}^{j},\ i=0 \tag{4.14a}$$

$$\begin{aligned}(1-6r\theta)\Phi_{i-1}^{j+1}+(4+12r\theta)\Phi_{i}^{j+1}+(1-6r\theta)\Phi_{i+1}^{j+1}\\ =\left[1+6r(1-\theta)\right]\Phi_{i-1}^{j}+\left[4-12r(1-\theta)\right]\Phi_{i}^{j}+\left[1+6r(1-\theta)\right]\Phi_{i+1}^{j}\end{aligned}\ ,\ i=1(1)N-1 \tag{4.14b}$$

$$2(1-6r\theta)\Phi_{i-1}^{j+1}+(4+12r\theta)\Phi_{i}^{j+1}=2\left[1+6r(1-\theta)\right]\Phi_{i-1}^{j}+\left[4-12r(1-\theta)\right]\Phi_{i}^{j}\ ,\ i=N \tag{4.14c}$$

Where $r=\dfrac{kv}{h^2}$, $j=0(1)M$, θ is a free parameter greater than zero and Φ_i^j is the discrete approximation to $\Phi(x_i,t_j)$ at the point (x_i,t_j).

Finally, the approximate solution to the Burgers' equation in Eq. (1.1) in terms of the approximate solution of heat equation in Eq. (4.9) by using Cole-Hopf transformation in Eq. (1.2) is given by:-

$$u_i^j=(-2v)\left\{\frac{\Phi_{i+1}^{j}-\Phi_{i-1}^{j}}{2h\Phi_i^j}\right\}=-v\left\{\frac{\Phi_{i+1}^{j}-\Phi_{i-1}^{j}}{h\Phi_i^j}\right\} \tag{4.15}$$

4.2 Stability Analysis

To investigate the stability of the approximate solution obtained by the present algorithm, we use the matrix form. Here we analyze the stability of the method by choosing the value of θ and Writing Eq. (4.13) in matrix form. The matrix form of Eq. (4.13) is given as:

$$\begin{pmatrix} 4+12r\theta & 2(1-6r\theta) & & & \\ 1-6r\theta & 4+12r\theta & 1-6r\theta & & \\ & \ddots & \ddots & \ddots & \\ & & 1-6r\theta & 4+12r\theta & 1-6r\theta \\ & & & 2(1-6r\theta) & 4+12r\theta \end{pmatrix} \begin{pmatrix} \Phi_0^{j+1} \\ \Phi_1^{j+1} \\ \vdots \\ \Phi_{N-1}^{j+1} \\ \Phi_N^{j+1} \end{pmatrix}$$

$$= \begin{pmatrix} (4-12r(1-\theta)) & 2\left[1+6r(1-\theta)\right] & & & \\ (1+6r(1-\theta)) & 4-12r(1-\theta) & 1+6r(1-\theta) & & \\ & \ddots & \ddots & \ddots & \\ & & 1+6r(1-\theta) & 4-12r(1-\theta) & 1+6r(1-\theta) \\ & & & 2\left[1+6r(1-\theta)\right] & 4-12r(1-\theta) \end{pmatrix} \begin{pmatrix} \Phi_0^{j} \\ \Phi_1^{j} \\ \ddots \\ \Phi_{N-1}^{j} \\ \Phi_N^{j} \end{pmatrix} \quad (4.16)$$

Factorization of LHS of (4.16) gives,

$$6\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} + \left(1-6r\theta\right) \begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix} \begin{bmatrix} \Phi_0^{j+1} \\ \Phi_1^{j+1} \\ \vdots \\ \Phi_{N-1}^{j+1} \\ \Phi_N^{j+1} \end{bmatrix}$$

This factorization can be written as;

$$\left(6I_{N+1} + \left(1-6r\theta\right)T_{N+1}\right)\Phi^{j+1}$$

Again factorization of RHS of (4.16) gives,

$$6\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}+\left(1+6r(1-\theta)\right)\begin{bmatrix}-2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1\\ & & & 2 & -2\end{bmatrix}\begin{bmatrix}\Phi_0^j\\ \Phi_1^j\\ \vdots\\ \Phi_{N-1}^j\\ \Phi_N^j\end{bmatrix}$$

This factorization also can be written as;

$$\left(6I_{N+1}+\left(1+6r(1-\theta)\right)T_{N+1}\right)\Phi^j$$

From both factorizations, we obtained the same tri-diagonal matrix of the form:

$$T_{N+1}=A=\begin{pmatrix}-2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1\\ & & & 2 & -2\end{pmatrix}$$

Hence, the above system can be written as:

$$\left[6I+\left(1-6r\theta\right)A\right]\phi^{j+1}=\left[6I+\left(1+6r(1-\theta)\right)A\right]\phi^j \qquad (4.17)$$

where $A=T_{N+1}$ is a matrix of order $N+1$ and

$\phi_m=(m=j,\ j+1)$ is component vector given by $\phi_m=\begin{pmatrix}\phi_{0,m}\\ \phi_{1,m}\\ \vdots\\ \phi_{N-1,m}\\ \phi_{N,m}\end{pmatrix}$

with **I** is identity matrix of order N+1 and $r=\dfrac{kv}{h^2}$ is the mesh ratio.

We can write Eq. (4.16) in the form of, $\phi^{j+1}=\left[6I+(1-6r\theta)A\right]^{-1}\left[6I+(1+6r(1-\theta))A\right]\phi^j$

We introduce a diagonal matrix $D = \begin{pmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \sqrt{2} \end{pmatrix}$.

In such a way that A is similar to a symmetric matrix $\widetilde{A} = D^{-1}AD$. (Kadalbajoo and Awashti, 2006)

Let $C = \left[6I + (1-6r\theta)A\right]^{-1}\left[6I + (1+6r(1-\theta))A\right]$

$$
\begin{aligned}
\widetilde{C} &= D^{-1}CD = D^{-1}\left[6I + (1-6r\theta)A\right]^{-1}\left[6I + (1+6r(1-\theta))A\right]D \\
&= \left[D^{-1}\left(6I + (1-6r\theta)A\right)^{-1} D\right]\left[D^{-1}\left(6I + (1+6r(1-\theta))A\right)D\right] \\
&= \left[D^{-1}\left(6I + (1-6r\theta)A\right)D\right]^{-1}\left[D^{-1}\left(6I + (1+6r(1-\theta))\right)A)D\right] \\
&= \left[6I + (1-6r\theta)\widetilde{A}\right]^{-1}\left[6I + (1+6r(1-\theta))\widetilde{A}\right]
\end{aligned}
$$

But $\left[6I + (1-6r\theta)\widetilde{A})\right]^{-1}$ *and* $\left[6I + (1+6r(1-\theta))\widetilde{A}\right]$ are symmetric and commute and so C is symmetric. Therefore, C is similar to a symmetric matrix $\widetilde{C}$.

Since the Eigen values of the real and symmetric matrix are equal to its 2-norm which is the same as its spectral radius. So we use this to show the stability of the method by showing the spectral radius of the matrix C is less than or equal to one. i.e.

$$\rho(C) \le 1. \tag{4.18}$$

The spectral radius of the matrix C is given by $\rho(C) = \max|\alpha_i|$, where $\alpha_i (i = 0,1,...,N)$ are the Eigen values of the matrix $\left[6I + (1-6r\theta)A\right]^{-1}\left[6I + (1+6r(1-\theta))A\right]$.

The values of α_i's are given by $\alpha_i = \dfrac{6 + \left[(1+6r(1-\theta))\lambda_i\right]}{6 + \left[(1-6r\theta)\lambda_i\right]}$ (4.19)

where $\lambda_i (i = 0,1,...,N)$ are the Eigen values of the matrix A.

The Eigen values λ_i of the matrix A are given by:

$$Det(A-\lambda I)=0\,. \tag{4.20}$$

Simplifying Eq. (4.20), we get $\left((\lambda+2)^2-4\right)\delta_{N-1}(\lambda)=0$

$$\left((\lambda+2)^2-4\right)\delta_{N-1}(\lambda)=0 \tag{4.21}$$

Where $\delta_N(\lambda)$ is the determinant of the matrix $\begin{pmatrix} -(2+\lambda) & 1 & & & \\ 1 & -(2+\lambda) & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -(2+\lambda) & 1 \\ & & & 1 & -(2+\lambda) \end{pmatrix}$

which is $\left(-1\right)^N \prod_{i=1}^{N}\left(4\cos^2\dfrac{\pi i}{2(N+1)}+\lambda\right)$ (4.22)

Using Eq. (4.22) in Eq. (4.21) gives; $\left(-1\right)^{N-1}((\lambda+2)^2-4)\prod_{i}^{N-1}(4\cos^2\dfrac{\pi i}{2N}+\lambda)=0$

$$\lambda(\lambda+4)\prod_{i=1}^{N-1}(4\cos^2\frac{\pi i}{2N}+\lambda)=0$$

This gives $\lambda=0$ *or* $\lambda=-4$ *or* $\lambda=-4\cos^2\dfrac{\pi i}{2N}$ $,i=0,1,2,...,N.$

Now, Substituting $\lambda=0,\lambda=-4$ and $\lambda=-4\cos^2\dfrac{\pi i}{2N}$ $,i=0,1,2,...,N$ in equation Eq. (4.19) we get:

1. when $\lambda=0$

$$\alpha_i=\frac{6+\left[(1+6r(1-\theta))0\right]}{6+\left[(1-6r\theta)0\right]}=\frac{6+0}{6+0}=1$$

For this case it is true that $\rho(C)\leq 1$

2. When $\lambda = -4$

$$\alpha_i = \frac{6+\left[(1+6r(1-\theta))(-4)\right]}{6+\left[(1-6r\theta)(-4)\right]} = \frac{6+\left[-4-24r(1-\theta)\right]}{6+\left[-4+24r\theta\right]} = \frac{\left[2-24r+24r\theta\right]}{2+24r\theta} = \frac{\left[2+24r\theta-24r\right]}{2+24r\theta} \le 1$$

Because, $(2+24r\theta) > (2+24r\theta-24r)$ holds

3. When $\lambda_i = -4\cos^2\dfrac{\pi i}{2N}$, $i = 0,1,2,...,N.$

Again since $0 \le \cos^2\dfrac{\pi i}{2N} \le 1$, the Eigen values of the matrix A lies between 0 and -4

i.e. $-4 \le \lambda \le 0$, also for this case it is true that $\rho(C) = \max\left|\alpha_i\right| \le 1.$

Therefore, the method is unconditionally stable.

4.3. Numerical Results

Here, two model examples have been considered to confirm the applicability of the present method for solving one-dimensional Burgers' equation.

Example 1 (Kadalbajoo, 2006)

Consider Burgers' equation in Eq. (1.1)

with initial condition

$$u(x,0) = \sin(\pi x),\ 0 \le x \le 1$$

and homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, 0 \le t \le T$$

By Hopf-Cole transformation, $u(x,t) = -2v\dfrac{\Phi_x}{\phi}$.

Burgers' equation is transformed to the heat equation of the form

$$\frac{\partial \phi}{\partial t} = v\frac{\partial^2 \phi}{\partial x^2}\ \ , 0 < x < 1, t > 0$$

with initial condition

$$\phi(x,0) = \exp\left\{\frac{-1}{2v\pi}\left[1 - \cos(\pi x)\right]\right\},\ \ 0 \le x \le 1$$

and boundary conditions

$$\phi_x(0,t) = \phi_x(1,t) = 0,\ \ 0 \le t \le T$$

Table1. Comparison of the Numerical solution with the exact solution of Example 1at space points for T = 0.01,v= 10 and $\Delta t = 0.0001$

x	Computed solution for different values of N								Exact Solution
	Our Method N=10 $\theta = 2$	Kadal Bajoo N=10	Our Method N=20 $\theta = 0.225$	Kadal bajoo N=20	Our Method N=40 $\theta = 0.376$	Kadal bajoo N=40	Our Method N=80 $\theta = 0.5$	Kadal bajoo N=80	
0.1	0.11458	0.11365	0.11466	0.11437	0.12481	0.11455	0.11570	0.11460	0.11461
0.2	0.21811	0.21634	0.21825	0.21771	0.23742	0.21805	0.22024	0.21813	0.21816
0.3	0.30054	0.29810	0.30074	0.29999	0.31648	0.30046	0.30347	0.30057	0.30062
0.4	0.35382	0.35093	0.35405	0.35316	0.35941	0.35371	0.35726	0.35385	0.35390
0.5	0.37262	0.36957	0.37286	0.37192	0.37097	0.37250	0.37625	0.37264	0.37270
0.6	0.35494	0.35204	0.35518	0.35427	0.34503	0.35483	0.35840	0.35497	0.35502
0.7	0.30237	0.29989	0.30258	0.30179	0.28728	0.30227	0.30531	0.30238	0.30243
0.8	0.21993	0.21813	0.22009	0.21951	0.21139	0.21986	0.22208	0.21994	0.21997
0.9	0.11571	0.11476	0.11579	0.11549	0.11582	0.11567	0.11684	0.11572	0.11573

Table2. Comparison of the Numerical solution with the exact solution of Example 1 at space points for T = 0.1, v= 1 and $\Delta t = 0.001$

X	Computed solution for different values of N								Exact Solution
	Our Method N=10 $\theta = 2$	Kadal bajoo N=10	Our Method N=20 $\theta = 0.225$	Kadal bajoo N=20	Our Method N=40 $\theta = 0.39$	Kadal bajoo N=40	Our Method N=80 $\theta = 0.42$	Kadal bajoo N=80	
0.1	0.10948	0.10863	0.10958	0.10931	0.11020	0.10948	0.10989	0.10952	0.10954
0.2	0.20970	0.20805	0.20990	0.20936	0.21110	0.20968	0.21316	0.20976	0.20979
0.3	0.29106	0.28947	0.29209	0.29129	0.29382	0.29174	0.29493	0.29186	0.29190
0.4	0.34697	0.34502	0.34822	0.34720	0.35032	0.34774	0.35219	0.34788	0.34792
0.5	0.37062	0.36847	0.37198	0.37080	0.37420	0.37138	0.37369	0.37153	0.37158
0.6	0.35817	0.35603	0.35951	0.35829	0.36175	0.35886	0.36170	0.35900	0.35905
0.7	0.30919	0.30729	0.31038	0.30925	0.31243	0.30974	0.31355	0.30986	0.30991
0.8	0.22732	0.22589	0.22820	0.22734	0.22969	0.22734	0.22834	0.22779	0.22782
0.9	0.12043	0.11966	0.12090	0.12043	0.12161	0.12043	0.12302	0.12067	0.12069

Example 2(Kadalbajoo, 2006)

Consider Burgers' equation inEq. (1.1)

with initial condition

$$u(x,0) = 4x(1-x) \ , 0 \le x \le 1$$

and homogeneous boundary condition

$$u(0,t) = 0 = u(1,t) \, , 0 < t \le T$$

By Hopf-Cole transformation $u(x,t) = -2v \dfrac{\Phi_x}{\phi}$

Burgers' equation is transformed to the heat equation of the form

$$\frac{\partial \phi}{\partial t} = v \frac{\partial^2 \phi}{\partial x^2} \ , 0 < x < 1, t > 0$$

with initial condition:

$$\phi(x,0) = \exp\left\{ \frac{-1}{2v} \left[2x^2 - \frac{4}{3} x^3 \right] \right\}, 0 \le x \le 1$$

and boundary conditions

$$\phi_x(0,t) = \phi_x(1,t) = 0 \, , 0 \le t \le T$$

Table 3.Comparison of the Numerical solution with the exact solution of Example 2 at space points for T = 0.1, v= 1 and $\Delta t = 0.001$

	Computed solution for different values of N								
X	Our Method N=10 $\theta = 2$	Kadal bajoo N=10	Our Method N=20 $\theta = 0.30$	Kadal bajoo N=20	Our Method N=40 $\theta = 0.439$	Kadal bajoo N=40	Our Method N=80 $\theta = 0.489$	Kadal bajoo N=80	Exact Solution
0.1	0.11284	0.11196	0.11302	0.11266	0.11378	0.11283	0.11392	0.11288	0.11289
0.2	0.21616	0.21447	0.21651	0.21581	0.21791	0.21614	0.21822	0.21622	0.21625
0.3	0.30086	0.29847	0.30138	0.30034	0.30325	0.30081	0.30373	0.30092	0.30097
0.4	0.35876	0.35588	0.35943	0.35812	0.36161	0.35867	0.36220	0.35881	0.35886
0.5	0.38335	0.38022	0.38411	0.38262	0.38640	0.38322	0.38703	0.38337	0.38342
0.6	0.37063	0.36754	0.37141	0.36988	0.37359	0.37046	0.37419	0.37061	0.37066
0.7	0.32007	0.31737	0.32078	0.31939	0.32266	0.31990	0.32315	0.32002	0.32007
0.8	0.23540	0.23338	0.23594	0.23487	0.23733	0.23525	0.23766	0.23534	0.23537
0.9	0.12474	0.12366	0.12503	0.12445	0.12579	0.12465	0.12594	0.12470	0.12472

4.4. Discussion

In this section, two model examples are considered and the present method is applied to the transformed Burgers' equation. The solution of the Burgers' equation was recovered via the transformation $u = -2v\frac{\phi_x}{\phi}$.

As it can be seen from tables 1, 2 and 3, the solution of the Burgers' equation better converges to the exact solution for the step lengths $h = 0.1$ and $h = 0.05$ than when $h = 0.025$, and $h = 0.0125$. however, the solution obtained by (Kadalbajoo, 2006) better converges to the exact solution if the mesh size is refined further. Thus, the present method is more efficient and convergent than the method considered by Kadalbajoo (Kadalbajoo, 2006) in that it requires less memory size and CPU time to converge to the exact solution. Therefore, the present method is competitive enough for solving one-dimensional Burgers' equation.

CHAPTER FIVE

5. Conclusion and Future work

5.1. Conclusion

In this work, finite difference method resulting from cubic spline has been presented to solve one-dimensional Burgers' equation. The stability of the method is well established and is shown to be unconditionally stable.

As the numerical results presented in tables show, the solutions converge to the exact solutions for coarser step lengths than finer step lengths. But the solution obtained by Kadalbajoo (Kadalbajoo, 2006) converges to the exact solution for finer mesh lengths and requires more CPU time and memory size to converge. Hence, the present method is more efficient and accurate than the method considered by Kadalbajoo, 2006.

5.2. Future work

Here, the finite difference method resulting from cubic spline has been presented to solve one-dimensional Burgers' equation. The values of θ were chosen by trial and error for each step lengths restricting the convergence of the method for finer step lengths. Thus, one can calculate the optimum value of θ so that the convergence is enhanced further.

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