NUMERICAL SOLUTION OF SECOND ORDER ONE DIMENSIONAL LINEAR HYPERBOLIC TELEGRAPH EQUATION

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS.

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DECLARATION

I, the undersigned declare that, this research entitled "**Numerical solution of second order one dimensional linear hyperbolic telegraph equation**" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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ABSTRACT

In this study, the numerical solution of second order one dimensional linear hyperbolic telegraph equations using crank Nicholson and stable finite difference method have been presented. First, the given domain or region is discritized and the derivatives of the differential equation were replaced by finite difference approximations and then, transformed to system of equations which can be solved by matrix inverse method. The stability and consistency of the method are established which shows convergence of the method. To validate the applicability of the method, model examples have been considered and solved at different mesh sizes. As it can be observed from the numerical results presented in Tables and graphs, the present method approximates the exact solution very well.

CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

An equation involving one or more dependent variable and its derivatives with respect to one or more independent variables is called Differential Equation (DE). Differential equations occur in connection with numerous problems that are encountered in the various branches of science and engineering. Some of these are: the problem of determining the motion of projectile, rocket, satellite, or planet, the problem of determining the charge or current in an electric circuit, the problem of the conduction of heat in a rod or in a slab, the problem of determining the vibrations of a wire or a membrane, the study of the rate of decomposition of a radio-active substances or the rate of growth of population, the study of the reaction of chemicals, the problem of the determination of the curves that have certain geometric properties (Shepley, 1980)**.** Partial differential equation (PDE) is an equation involving one or more derivatives of an unknown function, say u, that depend on two or more variables, often time t and one or several variables in space. PDEs have an enormous applications compared to Ordinary Differential Equations (ODEs), to mention some of these: dynamics, electricity, heat transfer, electromagnetic theory, quantum mechanics and so on (Erwin, 2006).

Telegraph equations are pairs of coupled, linear differential equations that describe the voltage and current on an electrical transmission line with distance and time. The telegraph equation is one of the important equations of mathematical physics with applications in many different fields such as transmission and propagation of electrical signals (Kajiwara *et al*., 2010), vibration systems, random walk theory and mechanical systems (Chakraverty and Behera, 2013), etc. The heat diffusion and wave propagation equations are particular cases of the telegraph equation. The telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion (Dosti and Nazemi, 2012).

Biologists encounter these equations in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge (Eftimie, 2012). Also the propagation of acoustic waves in Darcy-type porous media (Heider *et al*, 2012), and parallel floes of viscous Maxwell fluids (Liu *et a*l., 2011) are just some of the phenomena modeled by the telegraph equation.

Hence, the one dimensional hyperbolic telegraph equation is the significant class of the partial differential equations because of its wide range of applications in several fields. For instance, the hyperbolic type partial differential equations model atomic physics, aerospace, industry, biology and engineering problems such as vibrations of structures, beams and buildings (Mirzaee and Bimesl, 2013). One-dimensional hyperbolic telegraph equation with constant coefficients, models mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation. Also, hyperbolic telegraph equation is commonly used in signal analysis for transmission and propagation of electrical signals and also has applications in other fields (Stutzman and Thiele, 2012).

Furthermore with the appropriate coefficient and forcing terms, the one-dimensional telegraph equation describes a diverse array of physical system: for instance, the propagation of voltage and current signals in coaxial transmission lines of negligible leakage conductance and/or resistance.

In recent years, different methods have been applied to find the numerical solution of the hyperbolic one dimensional telegraph equation. To mention some: Radial basis function approximation (Saadamandi and Dehghan, 2010), Chebyshev Tau method, He’s variational iteration method (Dehghan *et al*., 2011), Laguerre-Legendre spectral collocation method (Tatari and Haghighi, 2014), differential quadrature method (Jiwari *et al*., 2012), differential transform method (Srivastava and Mukesh, 2014), method of weighted residuals (Odejide and Binuyo, 2014), Fibonacci polynomials (Kurt and Yalçınbaş, 2016) and meshless local radial point interpolation (Elyas and Hamid, 2015).

However, it is necessary to present a more accurate and convergent numerical method for the one dimensional linear hyperbolic telegraph equation. The fourth order stable central difference method to find the numerical solution for second order self-adjoint singularly perturbed ordinary differential equation subject to certain types of boundary conditions is presented by (Terefe *et al*., 2016). Hence, our work is an extension of the method described by (Terefe *et al*., 2016) together with the Crank Nicholson to find the accurate numerical solution of the one dimensional linear hyperbolic telegraph equation.

1.2 Statement of the problem

Due to the wide range of the application of the one dimensional linear hyperbolic telegraph equation, several numerical methods have been developed. Even though many numerical methods were applied to solve these types of equations, still it is possible to find a more accurate numerical method than that presented by different scholars. Therefore, it is important to describe the more accurate and convergent method for the second order one-dimensional linear hyperbolic telegraph equation.

Owing to this, the present study attempt to answer the following questions:

1. How does the present method be described for solving one-dimensional linear hyperbolic telegraph equation?
2. How to establish the stability of the present method?
3. To what extent the present method approximate the solution?
4. What is the advantage of the proposed method over the others?

1.3. Objectives of the study

1.3.1. General objective

The general objective of this study is to find the numerical solution of the second order one dimensional linear hyperbolic telegraph equation.

1.3.2 Specific Objectives

The specific objectives of the present study are to:

- Describe the method using Crank Nicholson and stable finite difference methods for solving second order one dimensional linear hyperbolic telegraph equation.
- Establish the stability of the present method.
- Investigate the accuracy of the present method.
- Describe the advantage of the present method over the others.

1.4. Significance of the Study

This study may help to find the numerical solution of second order one dimensional linear hyperbolic telegraph equations.

1.5. Delimitation of the Study

The present study delimited to the second order one dimensional linear hyperbolic telegraph equation, of the form:

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \; 0 \le x \le b, \; 0 \le t \le T \qquad (1.1)$$

subject to the initial conditions:

$$\begin{aligned} u(x,0) &= f_0(x) \\ \frac{\partial u}{\partial t}(x,0) &= f_1(x) \end{aligned} \qquad (1.2)$$

and with the boundary conditions:

$$\begin{aligned} u(0,t) &= g_0(t) \\ u(b,t) &= g_1(t) \end{aligned} \qquad (1.3)$$

where *b*, *T*, α and β are given positive real constants, $f_0(x)$, $f_1(x)$ and their derivatives are continuous functions of *x*, and $g_0(t), g_1(t)$ and their derivatives are continuous functions of *t*. Further, to find the accurate numerical solution and convergent scheme for solving Eq. (1.1) with respect to the given conditions in Eqs. (1.2) and (1.3), we applied the Crank Nicholson and stable finite difference method.

CHAPTER TWO
LITRATURE REVIEW

2.1. Partial Differential Equations

Equations involving one or more partial derivatives of a function of two or more independent variables are called Partial differential Equation (PDE). Historically, partial differential equations originated from the study of surfaces in geometry and a wide variety of problems in mechanics. During the second half of the nineteenth century, a large number of famous mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations (Debnath, 2011). The primary reason for this research was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering.

The next phase of the development of linear partial differential equations was characterized by efforts to develop the general theory and various methods of solution of linear equations (Myint and Debnath, 2007).

Almost all physical phenomena obey mathematical laws that can be formulated by differential equations. This striking fact was first discovered by Isaac Newton (1642–1727) when he formulated the laws of mechanics and applied them to describe the motion of the planets. During the three centuries since Newton's fundamental discoveries, many partial differential equations that govern physical, chemical, and biological phenomena have been found and successfully solved by numerous methods. Partial Differential equations include Euler's equations for the dynamics of rigid bodies and for the motion of an ideal fluid, Lagrange's equations of motion, Hamilton's equations of motion in analytical mechanics, Fourier's equation for the diffusion of heat, Cauchy's equation of motion and Navier's equation of motion in elasticity, the Navier–Stokes equations for the motion of viscous fluids, the Cauchy–Riemann equations in complex function theory, the Cauchy–Green equations for the static and dynamic behavior of elastic solids, Kirchhoff's equations for electrical circuits, Maxwell's equations for electromagnetic fields, and the Schr¨odinger equation and the Dirac equation in quantum mechanics.

In its early stages of development, the theory of second-order linear partial differential equations was concentrated on applications to mechanics and physics. All such equations can be classified into three basic categories: the wave equation, the heat equation, and the Laplace equation or potential equation. Thus, a study of these three different kinds of equations yields much information about more general second-order linear partial differential equations.

Jean d'Alembert (1717–1783) first derived the one dimensional wave equation for vibration of an elastic string and solved this equation in 1746. His solution is now known as the d'Alembert solution. The wave equation is one of the oldest equations in mathematical physics. Some form of this equation, or its various generalizations, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. In fact, the studies of water waves, acoustic waves, elastic waves in solids, and electromagnetic waves are all based on this equation. A technique known as the method of separation of variables is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation (Aubert and Kornprobst, 2006).The wave equation and its methods of solution attracted the attention of many famous mathematicians including Leonhard Euler (1707–1783), James Bernoulli (1667–1748), Daniel Bernoulli (1700–1782), J.L. Lagrange (1736–1813), and Jacques Hadamard (1865–1963).

Hence, hyperbolic telegraph equation is a significant class of the partial differential equation due to its wide range of applications in many areas of science and engineering as mentioned in the introduction part.

2.2 The telegraph equation

The telegrapher's equations (or just telegraph equations) are a pair of coupled, linear differential equations that describe the voltage and current on an electrical transmission line with distance and time. The equations come from Oliver Heaviside who in the 1880s developed the transmission line model. The model demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line. The theory applies to transmission lines of all frequencies including high-frequency transmission lines (such as telegraph wires and radio frequency conductors), audio frequency (such as telephone lines), low frequency (such as power lines) and direct current.

To deal with such equation, various mathematical methods have been proposed for obtaining exact and approximate analytic solutions. For instance, (Dehghan and Shokri, 2008) proposed a numerical scheme to solve the one-dimensional hyperbolic telegraph equation using collocation points and approximating the solution using thin plate splines radial basis function. (Mohebbi and Dehghan, 2008) combined a high-order compact finite difference scheme to approximate the spatial derivative and the collocation technique for the time component to numerically solve the one-dimensional linear hyperbolic equation. In solving the second-order linear hyperbolic equation,(Dehghan and Lakestani, 2009) used a numerical technique consisting of expanding the approximate solution as the elements of Chebyshev cardinal functions. (Biazar *et al*., 2009) applied the variational iteration method to obtain an approximate solution of the telegraph equation. (Saadatmandi and Dehghan, 2010) used the Chebyshev Tau method in numerically solving the telegraph equation. Solutions of the telegraph equation are still an attractive and interesting topic. Due to this, we are interested in finding the numerical solution of hyperbolic telegraph equation using finite difference method.

2.3 Routh-Hurwitz stability criterion

Important criteria that give necessary and sufficient conditions for all of the roots of the characteristic polynomial, with real coefficients to lie in the left hand side of the complex plane are known as the Routh-Hurwtiz criteria. The name refers to E.J. Routh and A.Hurwtiz, who contribute to the formulation of this criterion. In 1875, Routh, a British mathematician, developed an algorithm to determine the number of roots that lie in the right half of the complex plane (Gantmacher, 1964). In 1985 Hurwtiz, a German mathematician verified the determinant criteria for roots to lie in the left half of the complex plane.

This criterion states that if the roots of the characteristic polynomial lie in the left half of the complex plane, then any solution to the linear, homogenous differential equation converges to zero.

CHAPTER THREE
METHODOLOGY

3.1. Study site and Time

This study would be conducted in Jimma University department of Mathematics under the numerical Analysis stream from September 2016 G.C to June 2017 G.C. conceptually, the study focus on one dimensional linear hyperbolic telegraph equations

3.2. Study Design

This study was employed by mixed-design (documentary review design and experimental design) on one dimensional linear hyperbolic telegraph equation type.

3.3. Source of Information

The relevant sources of information for this study are Journals, books, published articles & related studies from internet and the experimental result was obtained by writing MATLAB code.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the study follows the following procedures:

1. Defining the problem,
2. Discretizing the domain for the defined problem,
3. Replace the derivatives in the partial differential equation by finite difference approximations to obtain the scheme,
4. The obtained scheme form system of equations that can be solved by matrix inverse method,
5. Writing MATLAB code for the systems obtained and to use matrix inverse method.
6. Demonstrate the validity of the scheme using model examples.

CHAPTER FOUR
DESCRIPTION OF THE METHODS AND RESULTS

4.1 Description of the method

Consider the second order one dimensional linear hyperbolic telegraph equation of the form:

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}+\beta u=\frac{\partial^2 u}{\partial x^2}+f(x,t),\quad 0\le x\le b,\ 0\le t\le T \tag{4.1}$$

subject to the initial conditions:

$$\begin{aligned} u(x,0)&=f_0(x)\\ \frac{\partial u}{\partial t}(x,0)&=f_1(x)\end{aligned} \tag{4.2}$$

and boundary conditions:

$$\begin{aligned} u(0,t)&=g_0(t)\\ u(b,t)&=g_1(t)\end{aligned} \tag{4.3}$$

where α and β are given positive constants and we assume that $f_0(x)$, $f_1(x)$, $g_0(t)$ and $g_1(t)$ are real continuous functions.

To describe the scheme, we divide the interval $[0,b]$ and $[0,T]$ into N and M equal subintervals of mesh length h and k respectively. Let $0=x_0<x_1<x_2<\ldots<x_N=b,$ and $0=t_0<t_1<t_2<\ldots<t_N=T$ be the mesh points with $x_i=x_0+ih$ and $t_j=t_0+jk,$ for $i=1,2,\ldots,N$ and $j=0,1,\ldots,M$. For the sake of simplicity, we use $u(x_i,t_j)=u_i^j$, $\frac{\partial^n u}{\partial x^n}(x_i,t_j)=\frac{\partial^n u_i^j}{\partial x^n}$, $\frac{\partial^n u}{\partial t^n}(x_i,t_j)=\frac{\partial^n u_i^j}{\partial t^n}$ and $f(x_i,t_j)=f_i^j$ ($n\ge 1$ called n^{th} order derivatives).

Eq. (4.1) can be re-written at discretized points as:

$$\frac{\partial^2 u_i^j}{\partial t^2}=-\alpha\frac{\partial u_i^j}{\partial t}-\beta u_i^j+\frac{\partial^2 u_i^j}{\partial x^2}+f(x_i,t_j) \tag{4.4}$$

Assume that $u(x,t)$ has continuous higher order partial derivatives on the region $[0,b]\times[0,T]$. Using Taylor's series expansion for any point $u(x_i,t_j)$ with uniform step mesh sizes *h* and *k* in the direction of *x* and for fixed *t*, we have:

$$u_{i+1}^{j}=u_i^j+h\frac{\partial u_i^j}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_i^j}{\partial x^2}+\frac{h^3}{3!}\frac{\partial^3 u_i^j}{\partial x^3}+\frac{h^4}{4!}\frac{\partial^4 u_i^j}{\partial x^4}+\frac{h^5}{5!}\frac{\partial^5 u_i^j}{\partial x^5}+\ldots \quad (4.5)$$

$$u_{i-1}^{j}=u_i^j-h\frac{\partial u_i^j}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_i^j}{\partial x^2}-\frac{h^3}{3!}\frac{\partial^3 u_i^j}{\partial x^3}+\frac{h^4}{4!}\frac{\partial^4 u_i^j}{\partial x^4}-\frac{h^5}{5!}\frac{\partial^5 u_i^j}{\partial x^5}+\ldots \quad (4.6)$$

In the same way, using Taylor's series expansion in the direction of *t*, for a fixed *x*, we have:

$$u_{i}^{j+1}=u_i^j+k\frac{\partial u_i^j}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 u_i^j}{\partial t^2}+\frac{k^3}{3!}\frac{\partial^3 u_i^j}{\partial t^3}+\frac{k^4}{4!}\frac{\partial^4 u_i^j}{\partial t^4}+\frac{k^5}{5!}\frac{\partial^5 u_i^j}{\partial t^5}+\ldots \quad (4.7)$$

$$u_{i}^{j-1}=u_i^j-k\frac{\partial u_i^j}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 u_i^j}{\partial t^2}-\frac{k^3}{3!}\frac{\partial^3 u_i^j}{\partial t^3}+\frac{k^4}{4!}\frac{\partial^4 u_i^j}{\partial t^4}-\frac{k^5}{5!}\frac{\partial^5 u_i^j}{\partial t^5}+\ldots \quad (4.8)$$

Adding Eqs. (4.5) with (4.6), Subtracting Eq.(4.8) from Eq.(4.7),and adding (4.7) with (4.8) respectively, gives:

$$\frac{\partial^2 u_i^j}{\partial x^2}=\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{h^2}+\tau_1 \quad (4.9)$$

$$\frac{\partial u_i^j}{\partial t}=\frac{u_i^{j+1}-u_i^{j-1}}{2k}-\frac{k^2}{6}\frac{\partial^3 u_i^j}{\partial t^3}+\tau_2 \quad (4.10)$$

$$\frac{\partial^2 u_i^j}{\partial t^2}=\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{k^2}-\frac{k^2}{12}\frac{\partial^4 u_i^j}{\partial t^4}+\tau_3 \quad (4.11)$$

where: $\tau_1=\frac{-h^2}{12}\frac{\partial^4 u_i^j}{\partial x^4}$, $\tau_2=\frac{-k^4}{120}\frac{\partial^5 u_i^j}{\partial t^5}$ and $\tau_3=\frac{-k^4}{360}\frac{\partial^6 u_i^j}{\partial t^6}$

From Crank Nicholson finite difference method, we have the following average values for u_i^j and $\frac{\partial^2 u_i^j}{\partial x^2}$ as:

$$u_i^j=\frac{1}{3}\left(u_i^{j+1}+u_i^j+u_i^{j-1}\right) \quad (4.12)$$

$$\frac{\partial^2 u_i^j}{\partial x^2}=\frac{1}{3}\left(\frac{\partial^2 u_i^{j+1}}{\partial x^2}+\frac{\partial^2 u_i^j}{\partial x^2}+\frac{\partial^2 u_i^{j-1}}{\partial x^2}\right) \quad (4.13)$$

Now, substituting Eqs. (4.10 - 4.13) into Eq. (4.4) yields:

$$\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{k^2}+\alpha(\frac{u_i^{j+1}-u_i^{j-1}}{2k})-\frac{\alpha k^2}{6}\frac{\partial^3 u_i^j}{\partial t^3}-\frac{k^2}{12}\frac{\partial^4 u_i^j}{\partial t^4}$$
$$+\frac{\beta}{3}\left(u_i^{j+1}+u_i^j+u_i^{j-1}\right)=\frac{1}{3}(\frac{\partial^2 u_i^{j+1}}{\partial x^2}+\frac{\partial^2 u_i^j}{\partial x^2}+\frac{\partial^2 u_i^{j-1}}{\partial x^2})+f_i^j+\tau_4 \quad (4.14)$$

where: $\tau_4=-\tau_3-\alpha\tau_2$

Differentiating Eq. (4.1) successively with respect to *t*, and evaluated at (x_i,t_j) we obtain:

$$\frac{\partial^3 u_i^j}{\partial t^3}=-\alpha\frac{\partial^2 u_i^j}{\partial t^2}-\beta\frac{\partial u_i^j}{\partial t}+\frac{\partial}{\partial t}(\frac{\partial^2 u_i^j}{\partial x^2})+\frac{\partial}{\partial t}f_i^j \quad (4.15)$$

$$\frac{\partial^4 u_i^j}{\partial t^4} = (\alpha^2 - \beta)\frac{\partial^2 u_i^j}{\partial t^2} + \alpha\beta\frac{\partial u_i^j}{\partial t} - \alpha\frac{\partial}{\partial t}(\frac{\partial^2 u_i^j}{\partial x^2}) + \frac{\partial^2}{\partial t^2}(\frac{\partial^2 u_i^j}{\partial x^2}) - \alpha\frac{\partial}{\partial t} f_i^j + \frac{\partial^2}{\partial t^2} f_i^j \qquad (4.16)$$

Substituting Eqs. (4.15) and (4.16) into Eq. (4.14), gives:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \frac{\alpha}{2k}(u_i^{j+1} - u_i^{j-1}) + \frac{\beta}{3}\left(u_i^{j+1} + u_i^j + u_i^{j-1}\right) + \frac{k^2}{12}(\alpha^2 + \beta)\frac{\partial^2 u_i^j}{\partial t^2}$$
$$+ \frac{\alpha\beta k^2}{12}\frac{\partial u_i^j}{\partial t} - \frac{\alpha k^2}{12}\frac{\partial}{\partial t}(\frac{\partial^2 u_i^j}{\partial x^2}) - \frac{k^2}{12}\frac{\partial^2}{\partial t^2}(\frac{\partial^2 u_i^j}{\partial x^2}) = \frac{1}{3}(\frac{\partial^2 u_i^{j+1}}{\partial x^2} + \frac{\partial^2 u_i^j}{\partial x^2} + \frac{\partial^2 u_i^{j-1}}{\partial x^2}) \qquad (4.17)$$
$$+ f_i^j + \frac{\alpha k^2}{12}\frac{\partial}{\partial t} f_i^j + \frac{k^2}{12}\frac{\partial^2}{\partial t^2} f_i^j + \tau_4$$

where: $\tau_4 = -\tau_3 - \alpha\tau_2$

Using the finite difference approximation of Eqs. (4.9 – 4.11), we have:

$$\frac{\partial}{\partial t}(\frac{\partial^2 u_i^j}{\partial x^2}) = \frac{1}{2kh^2}(u_{i+1}^{j+1} - u_{i+1}^{j-1} - 2u_i^{j+1} + 2u_i^{j-1} + u_{i-1}^{j+1} - u_{i-1}^{j-1}) + \tau_5 \qquad (4.18)$$

$$\frac{\partial^2}{\partial t^2}(\frac{\partial^2 u_i^j}{\partial x^2}) = \frac{1}{k^2h^2}(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} - 2u_{i+1}^j + 4u_i^j$$
$$- 2u_{i-1}^j + u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}) + \tau_6 \qquad (4.19)$$

Where: $\tau_5 = \frac{-h^2}{12}\frac{\partial}{\partial t}(\frac{\partial^4 u_i^j}{\partial x^4})$ and $\tau_6 = \frac{-h^2}{12}\frac{\partial^2}{\partial t^2}(\frac{\partial^4 u_i^j}{\partial x^4})$

Putting Eqs. (4.18) and (4.19) into Eq. (4.17) and using the central finite difference approximation for $\frac{\partial u_i^j}{\partial t}$, $\frac{\partial^2 u_i^j}{\partial t^2}$ and $\frac{\partial^2 u_i^j}{\partial x^2}$, we obtain:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \frac{\alpha}{2k}(u_i^{j+1} - u_i^{j-1}) + \frac{\beta}{3}(u_i^{j+1} + u_i^j + u_i^{j-1}) - \frac{1}{3h^2}(u_{i+1}^{j+1} - 2u_i^{j+1}$$
$$+ u_{i-1}^{j+1} + u_{i+1}^j - 2u_i^j + u_{i-1}^j + u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}) + \frac{(\alpha^2 + \beta)}{12}(u_i^{j+1} - 2u_i^j + u_i^{j-1})$$
$$+ \frac{\alpha\beta k}{24}(u_i^{j+1} - u_i^{j-1}) - \frac{\alpha k}{24h^2}(u_{i+1}^{j+1} - u_{i+1}^{j-1} - 2u_i^{j+1} + 2u_i^{j-1} + u_{i-1}^{j+1} - u_{i-1}^{j-1}) \qquad (4.20)$$
$$- \frac{1}{12h^2}(u_{i+1}^{j+1} - 2u_{i+1}^j + u_{i+1}^{j-1} - 2u_i^{j+1} + 4u_i^j - 2u_i^{j-1} + u_{i-1}^{j+1} - 2u_{i-1}^j + u_{i-1}^{j-1})$$
$$= f_i^j + \frac{\alpha k^2}{12}\frac{\partial}{\partial t} f_i^j + \frac{k^2}{12}\frac{\partial^2}{\partial t^2} f_i^j + \tau_7$$

where:
$$\tau_7 = k^4(\frac{1}{360}\frac{\partial^6 u_i^j}{\partial t^6} + \frac{\alpha}{120}\frac{\partial^5 u_i^j}{\partial t^5} + \frac{(\alpha^2 + \beta)}{144}\frac{\partial^4 u_i^j}{\partial t^4} + \frac{\alpha\beta}{72}\frac{\partial^3 u_i^j}{\partial t^3})$$
$$- \frac{h^2k^2}{144}(\frac{\partial}{\partial t}(\frac{\partial^4 u_i^j}{\partial x^4}) + \frac{\partial^2}{\partial t^2}(\frac{\partial^4 u_i^j}{\partial x^4})) - h^2(\frac{1}{36}\frac{\partial^4 u_i^j}{\partial x^4})$$

Rearranging Eq. (4.20), we get the recurrence relation:

$$
\begin{aligned}
&-\frac{10+\alpha k}{24h^2}u_{i-1}^{j+1}+\left(\frac{1}{k^2}+\frac{\alpha}{2k}+\frac{5\beta}{12}+\frac{5}{6h^2}+\frac{\alpha^2}{12}+\frac{\alpha\beta k}{24}+\frac{\alpha k}{12h^2}\right)u_i^{j+1}-\frac{10+\alpha k}{24h^2}u_{i+1}^{j+1}\\
&=\frac{1}{6h^2}u_{i+1}^{j}+\left(\frac{2}{k^2}-\frac{\beta}{6}-\frac{1}{3h^2}+\frac{\alpha^2}{6}\right)u_i^{j}+\frac{1}{6h^2}u_{i-1}^{j}+\frac{10-\alpha k}{24h^2}u_{i+1}^{j-1}+\left(\frac{-1}{k^2}+\frac{\alpha}{2k}-\frac{5\beta}{12}\right.\\
&\left.-\frac{5}{6h^2}-\frac{\alpha^2}{12}+\frac{\alpha\beta k}{24}+\frac{\alpha k}{12h^2}\right)u_i^{j-1}+\frac{10-\alpha k}{24h^2}u_{i-1}^{j-1}+f_i^{\,j}+\frac{\alpha k^2}{12}\frac{\partial}{\partial t}f_i^{\,j}+\frac{k^2}{12}\frac{\partial^2}{\partial t^2}f_i^{\,j}+T_i^{\,j}
\end{aligned}
\tag{4.21}
$$

where: $T_i^j=\tau_7$

This can be re-written as:

$$Au_{i-1}^{j+1}+Bu_i^{j+1}+Au_{i+1}^{j+1}=Cu_{i-1}^{j}+Du_i^{j}+Cu_{i+1}^{j}+Eu_{i+1}^{j-1}+Fu_i^{j-1}+Eu_{i-1}^{j-1}+H_i^j+T_i^j \tag{4.22}$$

for $i=1,2,3,\ldots,N-1$ and $j=0,1,2,\ldots,M-1$

where: $A=-\dfrac{10+\alpha k}{24h^2}$, $B=\dfrac{1}{k^2}+\dfrac{\alpha}{2k}+\dfrac{5\beta}{12}+\dfrac{5}{6h^2}+\dfrac{\alpha^2}{12}+\dfrac{\alpha\beta k}{24}+\dfrac{\alpha k}{12h^2}$, $C=\dfrac{1}{6h^2}$,

$D=\dfrac{2}{k^2}-\dfrac{\beta}{6}-\dfrac{1}{3h^2}+\dfrac{\alpha^2}{6}$, $E=\dfrac{10-\alpha k}{24h^2}$, $F=\dfrac{-1}{k^2}+\dfrac{\alpha}{2k}-\dfrac{5\beta}{12}-\dfrac{5}{6h^2}-\dfrac{\alpha^2}{12}+\dfrac{\alpha\beta k}{24}+\dfrac{\alpha k}{12h^2}$

and $H_i^j=f_i^{\,j}+\dfrac{\alpha k^2}{12}\dfrac{\partial}{\partial t}f_i^{\,j}+\dfrac{k^2}{12}\dfrac{\partial^2}{\partial t^2}f_i^{\,j}$

But, for $j=0$, from Eq. (4.22), we get:

$$Au_{i-1}^{1}+Bu_i^{1}+Au_{i+1}^{1}=Cu_{i+1}^{0}+Du_i^{0}+Cu_{i-1}^{0}+Eu_{i+1}^{-1}+Fu_i^{-1}+Eu_{i-1}^{-1}+H_i^0 \tag{4.23}$$

Using the initial condition given in Eq. (4.2) and the relations with Eq. (4.10) at $j=0$ we have:

$$\frac{\partial u}{\partial t}(x,0)=\frac{u_i^1-u_i^{-1}}{2k} \tag{4.24}$$

From Eq. (4.24), we get the value for u_{i-1}^{-1}, u_i^{-1} and u_{i+1}^{-1}, and then putting these values into Eq. (4.23) and then, rearranging, yields:

$$
\begin{aligned}
&(A-E)u_{i-1}^{1}+(B-F)u_i^{1}+(A-E)u_{i+1}^{1}\\
&\quad=Cu_{i+1}^{0}+Du_i^{0}+Cu_{i-1}^{0}-2k\left(E\frac{\partial u_{i-1}^{0}}{\partial t}+F\frac{\partial u_i^{0}}{\partial t}+E\frac{\partial u_{i+1}^{0}}{\partial t}\right)+H_i^0
\end{aligned}
\tag{4.25}
$$

for $i=1,2,3,\ldots,N-1$

Hence, Eqs. (4.22) and (4.25) gives system of equations which can be solved by matrix inverse method.

To solve the obtained system of equations using matrix inverse method, we considered the schemes given in Eqs. (4.22) and (4.25) which can be written as a matrix vector form:

$$M\,x^{j+1} = r^{j} \tag{4.26}$$

where: $M = [m_{ij}]$ a square matrix of order $(N-1)\times(N-1)$, with x^{j+1} and r^{j} are column matrices and it can be expressed for both cases as:

Case-I: using Eq. (4.25), for $j=0$, $i=1, 2, \ldots, N-1$, we have:

$$M = \begin{bmatrix} B-F & A-E & 0 & 0 & \cdots & 0 \\ A-E & B-F & A-E & 0 & \cdots & 0 \\ 0 & A-E & B-F & A-E & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A-E & B-F & A-E \\ 0 & \ddots & \ddots & 0 & A-E & B-F \end{bmatrix},\ x^{1} = \begin{bmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ \vdots \\ u_{N-2}^1 \\ u_{N-1}^1 \end{bmatrix} \text{ and } r^{0} = \begin{bmatrix} \eta_1^0 \\ \eta_2^0 \\ \vdots \\ \vdots \\ \eta_{N-2}^0 \\ \eta_{N-1}^0 \end{bmatrix}$$

where: For $i=1$,

$$\eta_1^0 = Cu_2^0 + Du_1^0 + Cu_0^0 - 2k(E\frac{\partial u_0^0}{\partial t} + F\frac{\partial u_1^0}{\partial t} + E\frac{\partial u_2^0}{\partial t}) + H_i^0 - (A-E)u_0^1,$$

For $i = 2, 3, \ldots, N-2$, $\quad \eta_i^0 = Cu_{i+1}^0 + Du_i{}^0 + Cu_{i-1}{}^0 - 2k(E\frac{\partial u_{i-1}^0}{\partial t} + F\frac{\partial u_i{}^0}{\partial t} + E\frac{\partial u_{i+1}{}^0}{\partial t}) + H_i$

and for $i = N-1$,

$$\eta_{N-1}^0 = Cu_N^0 + Du_{N-1}^0 + Cu_{N-2}^0 - 2k(E\frac{\partial u_{N-2}^0}{\partial t} + F\frac{\partial u_{N-1}^0}{\partial t} + E\frac{\partial u_N^0}{\partial t}) + H_{N-1}^0 - (A-E)u_N^1$$

Case-II: using Eq. (4.22), for $j = 1, 2, \ldots\ M-1$ and $i = 1, 2, \ldots, N-1$, we have:

$$M = \begin{bmatrix} B & A & 0 & 0 & \cdots & 0 \\ A & B & A & 0 & \cdots & 0 \\ 0 & A & B & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A & B & A \\ 0 & \ddots & \ddots & 0 & A & B \end{bmatrix},\ x^{j+1} = \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ \vdots \\ u_{N-2}^{j+1} \\ u_{N-1}^{j+1} \end{bmatrix} \text{ and } r^{j} = \begin{bmatrix} \eta_1^j \\ \eta_2^j \\ \vdots \\ \vdots \\ \eta_{N-2}^j \\ \eta_{N-1}^j \end{bmatrix}$$

where: $\eta_1^j = Cu_0^j + Du_1^j + Cu_2^j + Eu_2^{j-1} + Fu_1^{j-1} + Eu_0^{j-1} + H_1^j - Au_0^{j+1}$, for $i=1$

$$\eta_i^j = Cu_{i-1}^j + Du_i^j + Cu_{i+1}^j + Eu_{i+1}^{j-1} + Fu_i^{j-1} + Eu_{i-1}^{j-1} + H_i^j, \text{ for } i = 2, 3, \ldots, N-2$$

$$\eta_{N-1}^j = Cu_{N-2}^j + Du_{N-1}^j + Cu_N^j + Eu_N^{j-1} + Fu_{N-1}^{j-1} + Eu_{N-2}^{j-1} + H_{N-1}^j - Au_N^{j+1}, \text{ for } i = N-1$$

Definition 4.1: As Farid, (1995), A square matrix $M=[m_{ij}]$ is said to be strictly diagonally dominant if for every row, the magnitude of the diagonal entry in a row is larger than the sum of the magnitude of all the other (non-diagonal) entries in that row, that is:

$$|m_{ii}|>\sum_{j\neq i}|m_{ij}| \text{ for all } 1\leq i\leq N-1$$

where m_{ij} denotes the entry in the i^{th} row and j^{th} column.

Theorem 4.1:(Levy Desplanques Theorm). If the matrix $M=[m_{ij}]$ is strictly diagonally dominant matrix, then M is invertible. (Horn & Johnson, 1985).

Proof:

Suppose $\det(M)=0$, then for some non-zero vector $u=(u_1,u_2,u_3,...,u_n)^t$, $Mu=\vec{0}$ Now, let k be the index where $u_k\geq u_i$ for all $i=1,2,3,..., N$ form the k^{th} row of $Mu=\vec{0}$, we obtain:

$$m_{k,1}u_1+m_{k,2}u_2+...+m_{k,k-1}u_{k-1}+m_{k,k}u_k+...+m_{k,k+1}u_{k+1}+m_{k,k+2}u_{k+2}+...+m_{k,n}u_n=0$$

Hence,

$$m_{k,1}u_1+m_{k,2}u_2+...+m_{k,k-1}u_{k-1}+...+m_{k,k+1}u_{k+1}+m_{k,k+2}u_{k+2}+...+m_{k,n}u_n=0-m_{k,k}u_k$$

$$|m_{k,k}||u_k|=\left|\sum_{i\neq k}m_{k,i}u_i\right|\leq\sum_{i\neq k}|m_{k,i}||u_i|\leq\sum_{i\neq k}|m_{k,i}||u_k|\leq|u_k|r_k(M)$$

where $r_k(M)=\sum_{i\neq k}|m_{k,i}|$ which is the contradiction with the fact that $|m_{k,k}|>r_k(M)$.

Hence $\det(M)\neq 0$ implying that the matrix A is nonsingular (invertible).

Now, from the system of equation given in the form of Eq. (4.26), we can show that the following conditions for both cases:

For the case $j=0$,

$$|B-F|=\left|\frac{2}{k^2}+\frac{5\beta}{6}+\frac{5}{3h^2}+\frac{\alpha^2}{6}\right| > |A-E|=\left|-\frac{5}{6h^2}\right|, \text{ which implies } |B-F|>|A-E|$$

for $i=1$ and $i=N-1$.

$$|B-F|=\left|\frac{2}{k^2}+\frac{5\beta}{6}+\frac{5}{3h^2}+\frac{\alpha^2}{6}\right| > 2|A-E|=\left|-\frac{5}{3h^2}\right|. \text{ Thus, } |B-F|>2|A-E|$$

for $i=2, 3, . . . , N-2$.

For the case $j = 1, 2, \ldots$

$$|B| = \left|\frac{1}{k^2} + \frac{\alpha}{2k} + \frac{5\beta}{12} + \frac{\alpha^2}{12} + \frac{\alpha\beta k}{24} + \frac{10+\alpha k}{12h^2}\right| > |A| = \left|-\frac{10+\alpha k}{24h^2}\right| \text{ which implies } |B| > |A|$$

for $i = 1$ and $i = N-1$.

$$|B| = \left|\frac{1}{k^2} + \frac{\alpha}{2k} + \frac{5\beta}{12} + \frac{\alpha^2}{12} + \frac{\alpha\beta k}{24} + \frac{10+\alpha k}{12h^2}\right| > |2A| = \left|-\frac{10+\alpha k}{12h^2}\right|,$$

which shows that $|B| > |2A|$, for $i = 2, 3, \ldots, N-2$

Thus, by definition 4.1 M is strictly diagonally dominant matrix. Therefore, by theorem 4.1, matrix M is invertible.

4.2 Stability Analysis

Definition 4.2: A Finite Difference Approximation is said to be stable if the errors (truncation, round off, etc) decay as the computation proceeds from one marching step to the next.

In this section, the Von Neumann stability technique is applied to investigate the stability of the proposed method. Such an approach has been used by many researchers like (Rashidinia *et.al*, 2013) and (Shokofeh and Rashidinia, 2016).

We assume that the solution of Eq. (4.22) at the grid point (ih, jk) is given by:

$$u_i^j = \zeta^j e^{ip\theta} \tag{4.27}$$

where $p = \sqrt{-1}$, θ is the real number and ζ is the complex number.

Now, putting Eq. (4.27) into the homogenous part of Eq. (4.22), gives:

$$A\zeta^{j+1}e^{(i-1)p\theta} + B\zeta^{j+1}e^{ip\theta} + A\zeta^{j+1}e^{(i+1)p\theta}$$
$$= C\zeta^{j}e^{(i-1)p\theta} + D\zeta^{j}e^{ip\theta} + C\zeta^{j}e^{ip\theta} + E\zeta^{j-1}e^{(i+1)p\theta} + F\zeta^{j-1}e^{ip\theta} + E\zeta^{j-1}e^{(i-1)p\theta}$$

This implies:

$$\zeta^{j+1}e^{ip\theta}(Ae^{-p\theta} + B + Ae^{p\theta}) + \zeta^{j}e^{ip\theta}(-Ce^{-p\theta} - D - Ce^{p\theta}) + \zeta^{j-1}e^{ip\theta}(-Ee^{p\theta} - F - Ee^{-p\theta}) = 0$$

Since, the value of $p = \sqrt{-1}$, $e^{-p\theta} = \cos\theta - p\sin\theta$ and $e^{p\theta} = \cos\theta + p\sin\theta$, the above equation can be written as:

$$\zeta^{j+1}e^{ip\theta}(2A\cos\theta + B) + \zeta^{j}e^{ip\theta}(-2C\cos\theta - D) + \zeta^{j-1}e^{ip\theta}(-2E\cos\theta - F) = 0 \tag{4.28}$$

Dividing both sides Eq. (4.28) by $\zeta^{j-1}e^{ip\theta}$, we obtain:

$$\zeta^2(2A\cos\theta + B) + \zeta(-2C\cos\theta - D) + (-2E\cos\theta - F) = 0 \tag{4.29}$$

Since, $\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$, Eq. (4.29) written in the form of:

$$P\zeta^2 + Q\zeta + R = 0 \tag{4.30}$$

where: $P = 2A + B - 4A\sin^2(\frac{\theta}{2})$

$Q = -2C - D + 4C\sin^2(\frac{\theta}{2})$ and $R = -2E - F + 4E\sin^2(\frac{\theta}{2})$

Using Routh-Hurwitz criterion and the transformation $\zeta = \frac{1+z}{1-z}$ into Eq. (4.30), we have:

$P\left(\frac{1+z}{1-z}\right)^2 + Q\left(\frac{1+z}{1-z}\right) + R = 0$, which can be reduced to:

$$(P - Q + R)z^2 + 2(P - R)z + (P + Q + R) = 0 \tag{4.31}$$

The necessary and sufficient condition for $|\zeta| < 1$, from Eq. (4.31) is:

$$P - Q + R > 0, \quad P - R > 0 \text{ and } P + Q + R > 0. \tag{4.32}$$

From Eq. (4.30), we have:

$$\begin{aligned} &P - Q + R = \frac{8}{3h^2}\sin^2\left(\frac{\theta}{2}\right) + \frac{\alpha^2 + 2\beta}{3} + \frac{4}{k^2} \\ &P - R = \alpha\left(\frac{k}{3h^2}\sin^2\left(\frac{\theta}{2}\right) + \frac{1}{k} + \frac{\beta k}{12}\right) \quad \text{and} \quad P + Q + R = \frac{4}{h^2}\sin^2\left(\frac{\theta}{2}\right) + \beta \end{aligned} \tag{4.33}$$

Since, α and β are positive real constants and from Eq. (4.33), it is clearly observed that the inequality of Eq. (4.32) are satisfied for any values of θ. Thus, our method is stable for the one dimensional hyperbolic telegraph equation.

4.3 Consistency

Definition4.2: (Vladimir, 2011). Given a partial differential equation $p\phi = f$ and its finite difference approximation $p_{\Delta x,\Delta t}\phi = f$, is said to be consistent with the partial differential equation if for sufficiently differentiable $\phi(x,t)$

$$p\phi - p_{\Delta x,\Delta t}\phi \to 0 \quad \text{as} \quad |\Delta x| \text{ and } |\Delta t| \to 0$$

Remark: As stated and proved in (Lax and Richtmyer, 1956), for a linear partial differential equation, consistency and stability of its finite difference approximation is equivalent to convergence,

consistency + stability $\Leftrightarrow$ convergence

Now, expand Eq. (4.4) in Taylor series and replace the derivatives involving *x* and *t* for the relation:

$$\frac{\partial^2 u_i^j}{\partial t^2} + \alpha\frac{\partial u_i^j}{\partial t} + \beta u_i^j = \frac{\partial^2 u_i^j}{\partial x^2} + f_i^j$$

and then we drive a local truncation error. The principal part of the local truncation error of the proposed method using Eqs. (4.9 - 4.13) for the one dimensional linear hyperbolic telegraph equation is:

$$T_i^j = k^4\left(\frac{1}{360}\frac{\partial^6 u_i^j}{\partial t^6} + \frac{\alpha}{120}\frac{\partial^5 u_i^j}{\partial t^5} + \frac{(\alpha^2+\beta)}{144}\frac{\partial^4 u_i^j}{\partial t^4} + \frac{\alpha\beta}{72}\frac{\partial^3 u_i^j}{\partial t^3}\right) - \frac{h^2k^2}{144}\left(\frac{\partial}{\partial t}\left(\frac{\partial^4 u_i^j}{\partial x^4}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 u_i^j}{\partial x^4}\right)\right) - h^2\left(\frac{1}{36}\frac{\partial^4 u_i^j}{\partial x^4}\right) \quad (4.34)$$

Thus, the right hand side of eq.(4.34) vanishes as $h \to 0$ and $k \to 0$ and implies $T \to 0$. Hence, the scheme is consistent with the order of $O(k^4 + h^2k^2 + h^2)$. Therefore, the scheme is convergent.

4.4 Numerical Examples and Results

To demonstrate the applicability of the method, two model examples of the one - dimensional linear hyperbolic telegraph equations have been considered and solved. For each positive integers *N* and *M*, the pointwise absolute errors (E_{er}) and Maximum absolute errors (mE_{abs}) are obtained by the formula, $|E_{er}(i,j)| = |u_e(i,j) - u_N(i,j)|$ and $mE_{abs} = \max_{0\le i,j\le N,M} E_{er}(i,j)$ for $i = 1,2,3,...,N$ and $j = 0,1,2,...,M$, where $u_E(i,j)$ and $u_N(i,j)$ are the exact and computed approximate solution of the given problem respectively, at the nodal point (i, j).

Also the Root Mean Square (RMS) errors is approximated by the formula, $RMS_{error} = \sqrt{\frac{\sum (u_E - u_N)^2}{n}}$ where n is the number of partition of the interval.

Example 1: Consider the telegraphic equation of the form:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2e^{-t}$$

subject to the initial conditions: $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$ for $0 \le x \le 1$

and the boundary conditions: $u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0$

The exact solution is given by $u(x,t) = (x - x^2)t^2e^{-t}$

Table 1: Pointwise absolute and root mean square errors for Example 1 at $t = 0.01$

x	Absolute errors	
	Present method	**Odejide and Binuyo, (2014)**
0.00	0.0000	0.0000
0.25	1.2454e-09	3.6735633e-07
0.50	1.6758e-09	4.8980846e-07
0.75	1.2454e-09	3.6735635e-07
1.00	0.0000	0.0000
RMS	2.6832e-09	3.193160467e-07

Table 2: Pointwise and Maximum absolute errors for example 1, in the region $(x,t) \in [0,1] \times [0,1]$

x_i	t_i	$h = k = 0.25$	$h = k = 0.125$	$h = k = 0.0625$
0.25	0.25	5.0503e-04	9.8387e-05	2.0727e-05
	0.5	8.9854e-04	3.1433e-04	7.9327e-05
	0.75	1.8188e-03	4.8147e-04	1.2420e-04
0.5	0.25	9.5221e-04	2.5029e-04	6.5151e-05
	0.5	7.7531e-04	3.2036e-04	9.5877e-05
	0.75	2.4344e-03	8.1746e-04	2.1688e-04
Max. Abs. errors		2.4344e-03	8.1746e-04	2.1688e-04

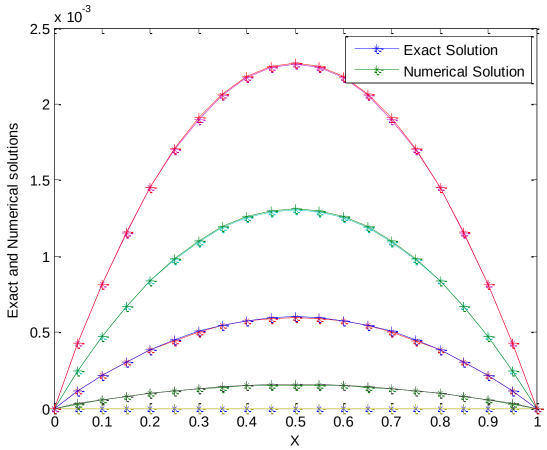
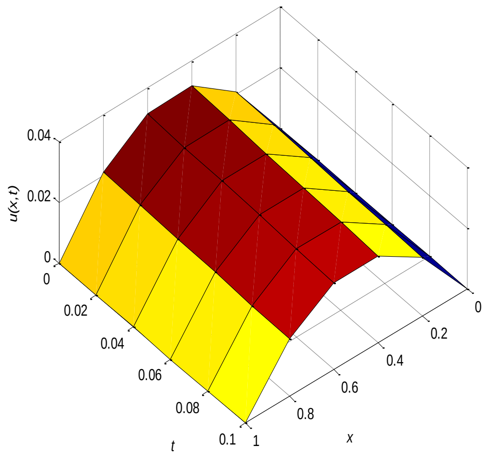


Figure 1: Comparison of exact and numerical solutions for Example 1 at $(x,t) \in [0,1] \times [0, 0.1]$ with $h = 0.05$ and $k = 0.025$



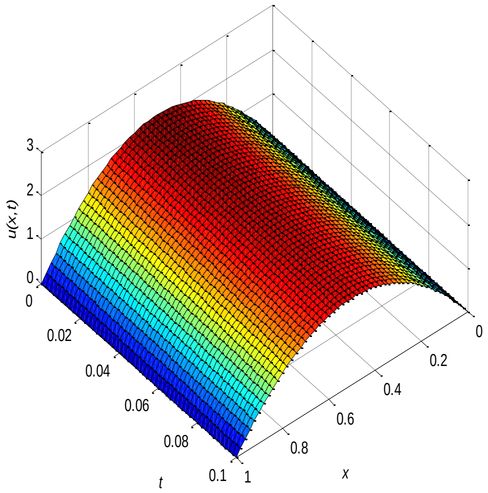


Figure 2: The physical behavior of Example 1 at different mesh sizes.

Example 2: Consider the telegraphic equation of the form:

$$\frac{\partial^2 u}{\partial t^2}+20\frac{\partial u}{\partial t}+25u=\frac{\partial^2 u}{\partial x^2}+(6t+60t^2)(x^2(1-x)^2)-t^3(12x^2-12x+2)$$

subject to the conditions: $\begin{cases} u(x,0)=0 \\ u_t(x,0)=0 \\ u(0,t)=0=u(1,t) \end{cases}$; $0\le x\le 1$, and $t>0$

The exact solution is given by $u(x,t)=t^3x^2(1-x)^2$

Table 3: Pointwise, maximum absolute and root mean square errors for Example 2

Points	**Pointwise errors**		
(x_i,t_j)	$h=k=0.2$	$h=k=0.1$	$h=k=0.05$
(0.2, 0.2)	5.4350e-05	1.2885e-05	3.9343e-06
(0.4, 0.4)	4.2612e-04	1.1950e-04	3.2055e-05
(0.6, 0.6)	8.4846e-04	2.3539e-04	6.1170e-05
(0.8, 0.8)	3.0432e-04	7.8122e-05	1.9923e-05
Max. Absolute errors	1.6476e-03	4.9860e-04	1.2690e-04
RMS	1.5581e-03	6.7579e-04	2.5208e-04

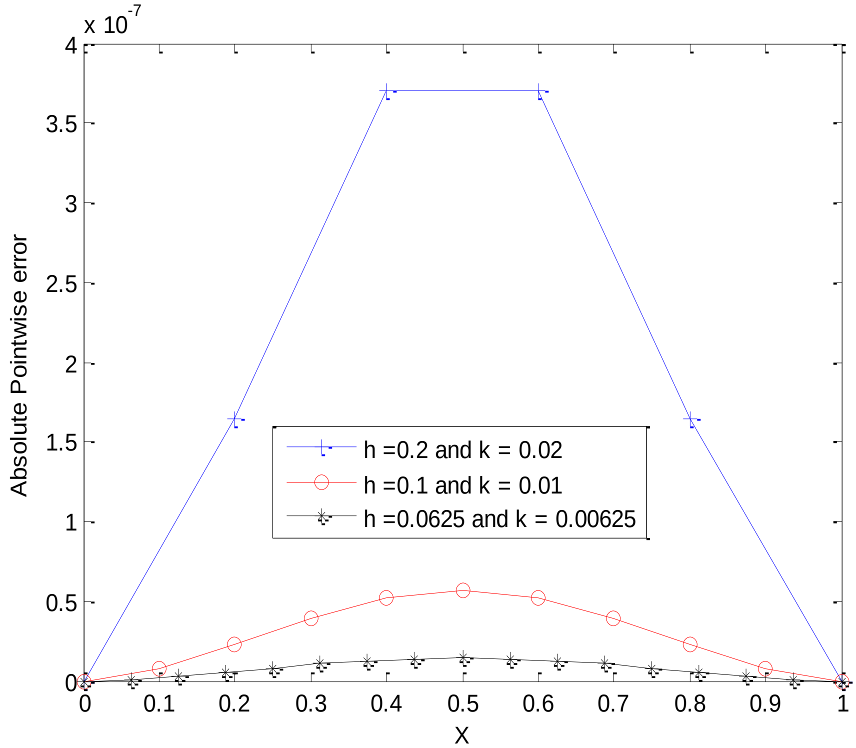


Figure 3: Absolute pointwise errors decreases as the number of mesh sizes decreases.

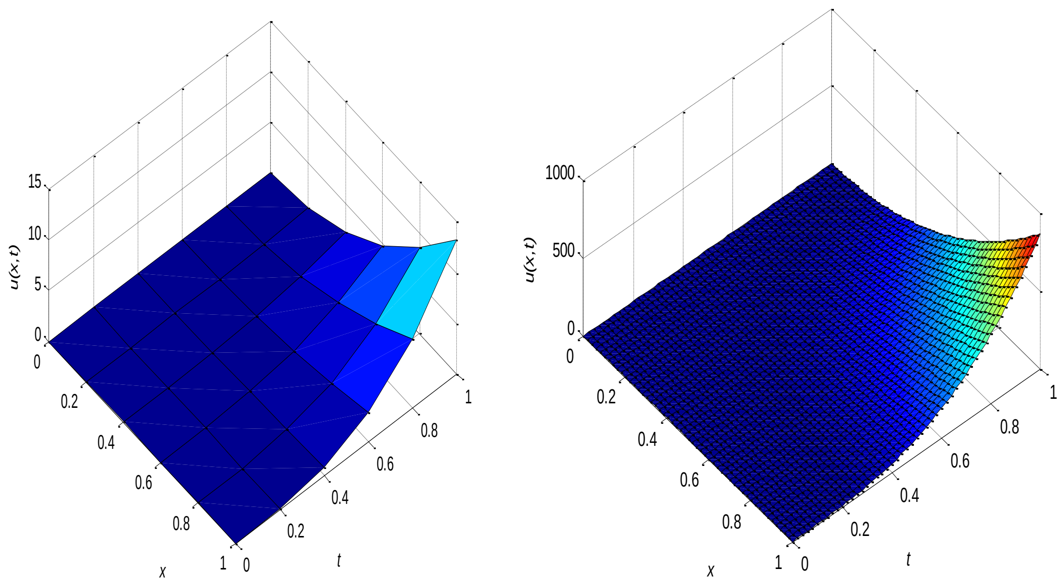


Figure 4: The physical behavior of Example 2 at different mesh sizes.

CHAPTER FIVE
DISCUSSION, CONCLUSION AND SCOPE FOR THE FUTURE WORK

5.1 Discussion and Conclusion

In this study, Crank Nicholson and fourth order stable finite difference method to obtain the scheme for solving one-dimensional linear hyperbolic telegraph equation were used. First, the given domain or region is discritized and the derivatives of the partial differential equation are replaced by finite difference approximations and then, transformed to system of equations which can be solved by matrix inverse method. The stability and consistency of the method is well established. To validate the applicability of the method, model examples have been considered and solved at different mesh sizes of h and k.

As it can be observed from the numerical results presented in Tables (1), (2) and (4) and graph (Figs. 1), the present method approximates the exact solution very well. Tables (2) and (3) show that as the values of h and k decreases, the accuracy of the method increases. Fig. 3 shows as the values of mesh sizes decrease, the pointwise absolute error also decreases. Also, results obtained by the presented method have been compared and shows betterment from the numerical result obtained by Odejide and Binuyo, (2014).

Therefore, the present scheme that obtained from the finite difference methods is more accurate and convergent method for solving one-dimensional linear hyperbolic telegraph equation.

5.2 **Scope for the Future Work**

In this study, the numerical solution for one dimensional linear hyperbolic telegraph equation based on Crank Nicholson and fourth order stable finite difference method, which give the more accurate numerical solution and convergent scheme. Hence, using those methods (Crank Nicholson and six order stable finite difference methods) can be obtaining another scheme.

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