NUMERICAL SOLUTION OF SINGULARLY PERTURBED SELF-ADJOINT BOUNDARY VALUE PROBLEM USING GALERKIN METHOD

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

(Numerical Analysis)

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Declaration

I, undersigned declare that this thesis entitled "NUMERICAL SOLUTION OF SINGULARLY PERTURBED SELF-ADJOINT BOUNDARY VALUE PROBLEM USING GALERKIN METHOD "is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Acronyms

1D -One dimensional

BVP -Boundary value problem.

DE -Differential equation

ODE -Ordinary deferential equation.

PDE -Partial deferential equation

SP -Singular perturbation.

SPBVP -Singularly perturbed boundary value problem.

SPP -Singularly perturbed problem

WIM -Weighted integral method

WRM-Weighted residual method

Abstract

In this thesis, numerical solution of self-adjoint Boundary value problems has been presented for solving second order singularly perturbed problem using Galerkin method. First, for the given problem, the residue was computed using appropriate approximated basis function which satisfies all the boundary conditions. Then, using the chosen weighting function integrating the weighted residue over the domain and the given differential equation is transformed to linear systems of algebraic equations. Further, these algebraic equations were to solved by using Galerkin method. To validate the applicability of the proposed method, two model examples have been considered and solved for different values of perturbation parameter and with different order of basis function. Additionally convergence of error bounds has been established for the method. As it can be observed from the numerical results presented in Tables and graphs, the present method approximates the exact solution very well. Moreover, the present method gives better results when the order of basis function is increased than some existing numerical method reported in the literature.

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CHAPTER ONE

INTRODUCTION

1.1. Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundation of numerical algorithms for the solution of problems arising in scientific applications (wasow, 1992). Further, numerical analysis plays a significant role when difficulties encountered in finding the exact solution of an equation using a direct method and when it becomes very difficult or impossible to apply analytical methods to find the exact solution. Due to the advancement in the field of computational mathematics, numerical methods are most widely utilized to solve the equation arising in the field of applied medical science, engineering and technology. Science and technology develop many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations and almost all physical phenomena in nature are modeled using differential equations, and singularly perturbed problems are vital class of these kinds of problems.

In real life, we often encounter many problems which are described by parameter dependent differential equations. The problems in which the highest order derivative term is multiplied by small positive parameters are known to be perturbed problems and the parameter is known as the perturbation parameter (Vasil'eva, 1976). The behaviors of the solutions of these types of differential equations depend on the magnitude of the parameters. The property that its solution is an integrating factor of the other is known as adjoint differential equation. A differential equation that has the same solution as its adjoint differential equation is called self-adjoint differential equation and if it's highest order derivative is multiplied by a small positive parameter, $\varepsilon\,(0 < \varepsilon << 1)$, which has the form:

$$-\varepsilon \frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x) = f(x)$$

is called self-adjoint singular perturbation problem (Byme and Mishra, 2009).

Depending on the behavior of solution of the problem in the limiting case when perturbation parameter goes to zero, such types of problems can be categorized into two. They are regularly perturbed and singularly perturbed problems. If the solution of the original problem tends to the solution of the reduced problem (*i.e.*, the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed otherwise, it is known as singularly perturbed. Singularly perturbed problems arise in various branches of applied mathematics and physics such as fluid mechanics, quantum mechanics, elasticity, plasticity, semiconductor device physics, geophysics, optimal control theory, aerodynamics, oceanography, and mathematical models of chemical reactions (Firdous *et al*, 2016) and also in engineering, biology and lubrication theory (Kumar, 2012).

Classical computational approaches to singularly perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory solutions (Roos *et al*., 1996; Farell *et al*., 2000). Detailed discussions on the theory of asymptotical and numerical solutions of singular perturbation problems have been published (Boglave, 1981), (Kadalbajoo and Kumar, 2008), (Mishra et al, 2009), (Gupta and Pankaj, 2011). So, the treatment of singularly perturbed problems presents several difficulties that have to be addressed to ensure accurate numerical solutions (Roos *et al.*, 1996; Kadalbajoo and Kumar, 2010).

Classical numerical methods which have been known to be effective for solving most problems that arises in application have failed when applied to singular perturbation problems, so most of these methods are not effective for solving singular perturbation problems because, as ε goes to zero, the error in numerical solutions increases and often becomes not comparable in magnitude to the exact solutions (Farell *et al*, 2000). There are so many authors who have worked in the field of Self-adjoint singularly perturbed problems, Such as (Boglav, 1981); (Miller, 1979); (Mishra, 2009); (Kumar and Kadalbajoo, 2008); (Gupta and Pankaj, 2011)and (Gupta, 2011). A finite element method for solving self-adjoint singularly perturbed boundary value problem was presented by (Vukoslavcevic and Surla, 1996). Recently, scholars like Terefe *et al*., (2016)and Kalid et al., 2018 present's fourth order stable central difference method for solving self-adjoint singularly perturbed problems, study of self-adjoint singularly perturbed two-point boundary value problems using collocation method respectively. Even if those authors that a considerable amount of work has been done for the development of numerical methods to solve the singularly

perturbed self-adjoint boundary values problem. Solving singularly perturbed problem is not simple as the solution of singularly perturbed problem depends on perturbation parameter ε and mesh size h (Doolan *et al*., 1980).

Due to this, numerical treatment of singularly perturbed boundary values problems needs improvement. Therefore, we present a numerical method which is independent of h or mesh size and relatively more accurate than other methods presented in the literature for solving self-adjoint singularly perturbed boundary values problems for small values of perturbation parameter ε. Convergence of the Galerkin method was presented.

1.2. Objectives of the study

1.2.1. General Objective

The general objective of this study is to solve singularly perturbed self-adjoint boundary value problems using Galerkin method.

1.2.2. Specific Objectives

The specific objectives of the present study are:

- To apply Galerkin method for solving singularly perturbed self-adjoint boundary value problems.
- To investigate the accuracy of the present method.
- To establish convergence of the proposed method.

1.3. Significance of the Study

The outcomes of this study may help to introduce the application of numerical methods in solving problems arising in different field of studies and serve as reference material for scholars' who works on this area.

1.4. Delimitation of the Study

Singularly perturbed problems arise in verity of mathematical models in engineering and physical system. However, this study is delimited to solve singularly perturbed self-adjoint boundary value problems of the form:

$$-\varepsilon \frac{d}{dx}\left(p(x)\frac{d}{dx}y(x) \right)+r(x)y(x)=z(x) \quad \text{for a} < x < b$$

subject to the boundary conditions:

$$y(a)=\alpha, \qquad y(b)=\beta$$

where $p(x)\geq p>0, \varepsilon \ \ (0<\varepsilon <<1)$, is a small positive perturbation parameter α and β are given constant and $p(x)$, $r(x)$ and $z\left(x\right)$ are assume to be sufficiently smooth functions on domain $\left(a,b\right)$.

1.5. Important definition of terms

For any differential equations of the form:

$$Ay=z, \quad \text{in } \Omega.$$

Linearity in A: Let S be a space containing functions that are admissible over the domain $\Omega=(a,b)$. Then, $\forall \phi,\psi \in S$ and $\forall \alpha,\beta \in R$, we must show that

$$A(\alpha\phi+\beta\psi)=\alpha A(\phi)+\beta A(\psi).$$

Adjoint of an operator: Let A be a linear differential operator and let Ω be its domain of definition. If

$$(A\phi,\psi)=(\phi,A^{*}\psi)+(A\phi,\psi)_{\Gamma}, \qquad \forall \phi,\psi \in S$$

holds, then A^{*} is called the adjoint of A. A^{*} is an adjoint operator.

Symmetric operator: Let A be a linear operator with its domain of definition Ω, then A is symmetric if

$$(A\phi,\psi)=(\phi,A\psi),\qquad \forall\phi,\psi\in S$$

If we use the definition of the scalar product, the Symmetry of A implies:

$$\int_{\Omega}(A\phi)\psi\,d\Omega=\int_{\bar{\Omega}}\phi(A\phi)d\Omega$$

Self-adjoint operator: If an operator A is linear and symmetric, then it is self-adjoint. Thus, for self-adjoint operators we have:

$$\begin{cases}A(\alpha\phi+\beta\psi)=\alpha A(\phi)+\beta A(\psi),\\(A\phi,\psi)=(\phi,A\psi),\qquad \forall\phi,\psi\in S \text{ and } \forall\alpha,\beta\in\mathrm{R}\end{cases}$$

CHAPTER TWO

REVIEW RELATED LITERATURE

2.1. Boundary Value Problem

A boundary value problem is a problem, typically an ordinary or partial differential equation that has values assigned on physical boundary of the domain in which the problem is specified. A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions**.** Finding the numerical solution of a boundary value problem (BVP) is more difficult than that of the corresponding initial values problem. Difficulties in applying the asymptotic expansion outer region, which are not routine exercises but it require skill, insight and experimentation.

2.2. Singularly Perturbed Boundary Values Problems

The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed boundary value problems that have a multi scale characters. However, most of the problem cannot be completely solved by analytic techniques consequently, numerical simulation of fundamental importance in gaining some useful in sights on the solutions of the singularly perturbed differential equations (Kadalbajoo and Gupta, 2010) These singularly perturbed problems arise in the modeling of varies modern. Complicated processes, such as fluid flow at high Reynolds numbers, water quality problems in rivers networks, and convective heat transport problem with large peclet number drift diffusion equation of semiconductor device modeling, electromagnetic field problem in moving media.

Spline approximation method for numerical solutions of singulary perturbed two-point boundary value problems have been studied by various researchers. Uzelac and Surla (2003) constructed a uniformly accurate scheme using collocation with classical quadratic polynomial splines on Shishkin meshes. Stojanovic (2006) introduced the spline collocation method for singular perturbation problem using piecewise quadratic interpolating polynomials as an approximate. Sakai and Usmani (1986) gave a concept of B-spline in terms of hyperbolic and trigonometric splines which are different from earlier ones.

It is proved that the hyperbolic and trigonometric B-splines are characterized by a convolution of some special exponential functions and a characteristic function on the interval [0,1]. Again Sakai and Usmani (1989), considered an application of simple exponential splines to the numerical solution of singular perturbation problem. They found that computational effort involved in their collocation method was less than that required for other exponential type splines. Kadalbajoo and Patidar (2002) derived uniformly convergent schemes of order two for these problems using splines in tension and splines in compression. (Kadalbajoo and Aggarwal ,2005) gave the B-spline collocation method of order two with Shishkin mesh for self-adjoint singularly perturbed two-point boundary value problem.

2.3. Finite Element Method

The fundamental idea of the finite element method is to discretize the domain into several sub domains, or finite elements. These elements can be irregular and possess different properties so that they form a basis to discretize complex structures, or structures with mixed material properties. Further, they can accurately model the domain boundary regardless of its shape.

The modern use of finite elements started in the field of structural engineering. The advent of jet engine in the 1940 and the resulting changes in aircraft speeds had led to the change from upswept to swept wind designs. The first attempt was by (Hrennikoff, 1941) who developed analogy between actual discrete elements and the corresponding portions of a continuous solid, and it was adapted to aircraft structural design. Based on displacement assumptions, (Turner et al., 1956) introduced the element stiffness matrix for a triangular element, and together with the direct stiffness method, described the method for assembling the elements. Finite element method is mostly computed by different method such as, variational method, weighted residual method etc.

2.4. Method of Weighted Residuals

Prior to development of the Finite Element Method, there existed an approximation technique for solving differential equations called the Method of Weighted Residuals (MWR). The weighted-residual method is the generalization of the Ritz method in that the weight function can be chosen from an independent set of functions, and it requires only the weighted integral form

to determine the parameters (Reddy.J.N., 2006). In applied mathematics, methods of weighted residuals are methods for solving differential equations. The solutions of these differential equations are assumed to be well approximated by a finite sum of test functions ϕ_i. In such cases, the selected method of weighted residuals is used to find the coefficient value of each corresponding test function. The resulting coefficients are made to minimize the error between the linear combination of test functions, and actual solution, in a chosen norm. The Weighted residual method is illustrated on a simple one-dimensional problem. First the problem is given a general mathematical form that is relevant for any differential equation. It is assumed that a problem is governed by the differential equation. This method will be presented as an introduction, before using a particular subclass of MWR, the Galerkin Method of Weighted Residuals, to derive the element equations for the finite element method. Galerkin methods provide a fairly general framework for the numerical solution of differential equations within the context of the weighted-residual formalization (Carey and Oden, 1983).

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

This study was conducted at Jimma University department of Mathematics from September 2017 to September 2018 G.C. Conceptually, the study focus on numerical solution of singularly perturbed self-adjoint boundary values problems.

3.2. Study Design

The study employed mixed-design (document review design and experimental design).

3.3. Source of Information

The relevant sources of information for this study were books, published articles and related studies from internets and experimental results obtained by writing MATLAB code.

3.4. Mathematical Procedures

In one's study, it is crucial to know where to go, how to go and what procedures to follow to achieve the target. Hence, in order to achieve the stated objectives, the following procedures were followed:

1. Define the problem.
2. Choose the approximation basis function which satisfies all the boundary conditions.
3. Compute the residual.
4. Choose the appropriate weighted function and integrate the weighted residual over domain and get the system of equations.
5. Establish the convergence of the method.
6. Writing MATLAB code for the scheme.
7. Validate the scheme by using numerical examples.
8. Present the numerical result in different forms.

CHAPTER FOUR

DESCRIPTION OF THE METHOD, RESULTS AND DISCUSSION

4.1. Description of the method

Consider the following self-adjoint singularly perturbed equation of the form:

$$-\varepsilon \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx} \right) + r(x)y(x) = z(x) \text{ for } a < x < b \tag{4.1}$$

subject to the boundary conditions: for

$$y(a) = \alpha, \qquad y(b) = \beta \tag{4.2}$$

where ε $(0 < \varepsilon << 1)$, is a small positive perturbation parameter, $p(x), r(x)$ and $z(x)$ are assumed to be sufficiently smooth known functions on (a, b), α and β are known parameters.

We find an approximate solution over the entire domain $\Omega = (a, b)$ in the form

$$y_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \tag{4.3}$$

where c_j are constant unknwon parameter to be determined and $\phi_j(x)$ and $\phi_0(x)$ are basis functions chosen such that the specified boundary conditions of the problem are satisfied by the N parameter approximate solution y_N. The particular form in Eq.(4.3) has two parts, one containing the unknowns $\sum_{j=1}^{N} c_j \phi_j$ that is termed the homogeneous part and the other is the non-homogeneous part ϕ_0 that has the main purpose of satisfying the specified boundary conditions of the problem.

To apply the method, we choose the approximation basis function which satisfies the Eq. (4.2). For a choice of algebraic polynomials:

We assume $\phi_0(x) = c + dx$ and the two conditions on ϕ_0 to determine the constant c and d we obtain

$$\phi_0(a) = \alpha \Rightarrow c + d(a) = \alpha \Rightarrow c = \alpha - ad$$

$$\phi_0(b) = \beta \Rightarrow c + d(b) = \beta \Rightarrow d = \frac{\beta - \alpha}{b - a}$$

$$\phi_0(x) = c + dx = \alpha - ad + dx = \alpha + \left(\frac{\beta - \alpha}{b - a}\right)(x - a)$$

Therefore,

$$\phi_0(x) = \alpha + \left(\frac{\beta - \alpha}{b - a}\right)(x - a) \tag{4.4}$$

Similarly, choose ϕ_j which satisfies the boundary conditions in homogeneous form. Since there are two homogeneous conditions, we must assume at least a three parameter polynomial to obtain a nonzero function.

Let us assume $\phi_1(x) = m + nx + ex^2$

Using the conditions on $\phi_1(x)$ we obtain:

$$\phi_1(a) = 0 \Rightarrow m + n(a) + e(a)^2 = 0 \Rightarrow m = -an - ea^2$$

$$\phi_1(b) = 0 \Rightarrow m + n(b) + e(b)^2 = 0 \Rightarrow n = -e(b + a)$$

$$\phi_1(x) = m + nx + ex^2 = -an - ea^2 + nx + ex^2 = -e[(x - a)[(b + a) - (x + a)]]$$

The constant e can be set equalto unity because it will be absorbed in to the parameter notation c_1.

$$\phi_1(x) = -[(x - a)[(b + a) - (x + a)]] \tag{4.5}$$

For $\phi_2(x)$ we can assume one of the forms:

$$\phi_2(x) = m + nx + tx^3 \text{ or } \phi_2(x) = m + ex^2 + tx^3$$

with $t \neq 0$; $\phi_2(x)$ does not contain all-order terms in either case, but the approximate solution is complete because $\{\phi_1, \phi_2\}$ contains all terms up to degree three.

Using conditions like ϕ_1 for $\phi_2(x) = m + ex^2 + tx^3$ we obtain:

$$\phi_2(a)=0\Rightarrow m+e(a)^2+t(a)^3=0\Rightarrow m=-ea^2-ta^3$$

$$\phi_2(b)=0\Rightarrow m+e(b)^2+t(b)^3=0\Rightarrow e=-t\left(\frac{b^3-a^3}{b^2-a^2}\right)$$

Now, $\phi_2(x)=m+ex^2+tx^3=-t\left[\dfrac{a^2[a(b^2-a^2)-(b^3-a^3)]+x^2[(b^3-a^3)-x(b^2-a^2)]}{b^2-a^2}\right]$

The constant t can be set equalto unity because it will be absorbed in to the parameter notation c_2.

Therefore,

$$\phi_2(x)=-\left[\frac{a^2[a(b^2-a^2)-(b^3-a^3)]+x^2[(b^3-a^3)-x(b^2-a^2)]}{b^2-a^2}\right] \qquad (4.6)$$

In the same way, choose $\phi_3(x)$ which satisfies all the boundary conditions in homogeneous form

For $\phi_3(x)$ we can assume one of the forms:

$\phi_3(x)=m+nx+gx^4$ or $\phi_3(x)=m+ex^2+gx^4$ or $\phi_3(x)=m+tx^3+gx^4$.

Using the conditions on $\phi_1(x)$ and $\phi_2(x)$ for $\phi_3(x)=m+tx^3+gx^4$, we obtain;

$$\phi_3(a)=0\Rightarrow m+ta^3+ga^4=0\Rightarrow m=-ta^3-ga^4$$

$$\phi_3(b)=0\Rightarrow m+tb^3+gb^4=0\Rightarrow t=-g\left(\frac{b^4-a^4}{b^3-a^3}\right)$$

Now, $\phi_3(x)=m+tx^3+gx^4=a^3\left(g\left(\dfrac{b^4-a^4}{b^3-a^3}\right)\right)-ga^4-gx^3\left(\dfrac{b^4-a^4}{b^3-a^3}\right)+gx^4$

$$\phi_3(x)=-g\left[\frac{a^3[a(b^3-a^3)-(b^4-a^4)]+x^3[(b^4-a^4)-x(b^3-a^3)]}{b^3-a^3}\right]$$

The constant g can be set equalto unity because it will be absorbed in to the parameter notation c_3

Therefore, for the third choice $\phi_3(x)$ we obtain;

$$\phi_3(x)=-\left[\frac{a^3[a(b^3-a^3)-(b^4-a^4)]+x^3[(b^4-a^4)-x(b^3-a^3)]}{b^3-a^3}\right] \qquad (4.7)$$

In generally, follow as a chosen above ϕ_1, ϕ_2 *and* ϕ_3 for, $(j = 1, 2..., N)$ ϕ_j is given as follows

$$\phi_1(x) = -\left[(x-a)[(b+a)-(x+a)]\right]$$

$$\phi_2(x) = -\left[\frac{a^2[a(b^2-a^2)-(b^3-a^3)]+x^2[(b^3-a^3)-x(b^2-a^2)]}{b^2-a^2}\right]$$

$$\phi_3(x) = -\left[\frac{a^3[a(b^3-a^3)-(b^4-a^4)]+x^3[(b^4-a^4)-x(b^3-a^3)]}{b^3-a^3}\right]$$

$$\vdots$$

$$\phi_N(x) = -\left[\frac{a^N[a(b^N-a^N)-(b^{N+1}-a^{N+1})]+x^N[(b^{N+1}-a^{N+1})-x(b^N-a^N)]}{b^N-a^N}\right]$$

for N is a positive integer.

We choose a set of basis functions $\{\phi_j,\ j = 1, 2, ..., N\}$, and make an approximation of the form

of Eq. (4.3) $y_N = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)$

The basis functions can be polynomials functions, trigonometric functions or other functions. But in our case the basis functions are polynomial because polynomial functions are continuous, easily differentiable,integrable and suitable for programming.

Weighted integral method (WIM) is a class of method used to obtain the approximate solution to the differential equations of the form

$$Ay = z, \quad \text{in } \Omega. \tag{4.8}$$

To apply the WRM, we can approximate $y(x)$ of the differential equation in Eq. (4.8) in to $y_N(x)$. When $y_N(x)$ is substituted into Eq. (4.8), it is unlikely that the equation is satisfied.

$$\Rightarrow Ay_N(x) - z \neq 0$$

$$\Rightarrow R \equiv Ay_N(x) - z \tag{4.9}$$

where R is residual.

Multiply Eq. (4.9) by an arbitrary weighted function $w(x)$ and integrating over the domain Ω to force this integral to vanish over the given domain to obtain the unknown parameters:

$$\int_{\Omega} w(x)R(x)d\Omega = 0 \tag{4.10}$$

In the weighted residual method, the unknown parameters c_j's are determine by minimize the residual R in some special cases. Different methods of minimizing the residual yield different approximate solutions.

When the weight functions are chosen to be the basis functions themselves, then it is known as Galerkin method. We set

$$\phi_i(x) = w_i(x), \quad \mathrm{i} = 1,2,...,N \tag{4.11}$$

The unknown coefficients in the approximate solution are determined by setting the integral over Ω of the weighted residual to zero.

For one-dimensional problem in the interval $(a,b),$ this procedure will result:

$$\int_a^b \phi_i(x)R(x)dx = \int_a^b w_i(x)R(x)dx = 0, \ (\mathrm{i} = 1,2,...,N)$$

The present method considers result in a system of equations of the form:

$$\int_{\Omega} (\phi_i[A\phi_0 + \sum_{j=1}^{N} c_j A\phi_j - z])dx = 0$$

$$c_j \int_{\Omega} \phi_i A\phi_j dx = \int_{\Omega} \phi_i[z - A\phi_0]d\Omega$$

$$DC = B \tag{4.12}$$

where $D_{ij} = \int_{\Omega} \phi_i A\phi_j dx, \ B_i = \int_{\Omega} \phi_i [z - A\phi_0]dx, \ D$ is matrices and a column vector B.

4.2. Convergence of the method

Consider the following self-adjoint singularly perturbed equation of the form:

$$-\varepsilon \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx} \right)+r(x)y(x)=z(x) \qquad \text{for } a<x<b$$

subject to the boundary conditions:

$$y(a)=\alpha, \qquad y(b)=\beta$$

The weak form of the above problem is:

$$\int_a^b w\left(-\varepsilon \frac{d}{dx}(p(x)\frac{dy(x)}{dx}) \right)dx+\int_a^b wr(x)y(x)dx-\int_a^b wz(x)dx \tag{4.13}$$

By using integration by part

$$\int_a^b w\left(-\varepsilon \frac{d}{dx}(p(x)\frac{dy(x)}{dx}) \right)dx=\int_a^b \varepsilon p\frac{dy}{dx}\frac{dw}{dx}dx-\left(\varepsilon pw\frac{dy}{dx} \right)\Big|_a^b \tag{4.14}$$

Now substituting Eq. (4.14) in to Eq. (4.13) we get

$$\int_a^b w\left(-\varepsilon \frac{d}{dx}(p(x)\frac{dy(x)}{dx}) \right)dx+\int_a^b wr(x)y(x)dx-\int_a^b wz(x)dx \;=\int_a^b \left(\varepsilon p\frac{dy}{dx}\frac{dw}{dx}+wr(x)y(x) \right)dx$$

$$-\int_a^b wz(x)dx \;-\left(\varepsilon pw\frac{dy}{dx} \right)\Big|_a^b$$

Since the give boundary condition Dirichlet boundary condition assume that $y(a)=y(b)=0$.

Therefore $\left(\varepsilon pw\frac{dy}{dx} \right)\Big|_a^b=0$

$$=\int_a^b \left(\varepsilon p\frac{dy}{dx}\frac{dw}{dx}+wr(x)y(x) \right)dx-\int_a^b wz(x)dx$$

$$=B(y,w)-l(w) \tag{4.15}$$

The quadratic functional of the Eq. (4.15) is:

$$I(y)=\frac{1}{2}B(y,y)-l(y)\ ,\ \sin ce\ \ y=w\ \ then$$

$$=\int_a^b\frac{1}{2}\left(\varepsilon p\frac{dy}{dx}\frac{dy}{dx}+yry\right)dx-\int_a^b yzdx$$

$$=\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy}{dx}\right)^2+y^2r\right)dx-\int_a^b yzdx$$

$$I(y)=\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy}{dx}\right)^2+y^2r-2yz\right)dx \qquad (4.16)$$

Similarly,

$$I(y_N)=\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy_N}{dx}\right)^2+{y_N}^2r-2y_Nz\right)dx \qquad (4.17)$$

Now $I(y_N)-I(y)=\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy_N}{dx}\right)^2+{y_N}^2r-2y_Nz\right)dx-\left(\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy}{dx}\right)^2+y^2r-2yz\right)dx\right)$

$$=\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy_N}{dx}\right)^2+{y_N}^2r-2y_Nz-\varepsilon p\left(\frac{dy}{dx}\right)^2-y^2r+2yz\right)dx$$

$$=\int_a^b\frac{1}{2}\left(\varepsilon p\left(\frac{dy_N}{dx}\right)^2-\varepsilon p\left(\frac{dy}{dx}\right)^2+{y_N}^2r-y^2r\right)dx+\int_a^b z\left(y-y_N\right)dx \qquad (4.18)$$

since $z(x)=-\varepsilon\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+r(x)y(x)$

$$\int_a^b z\left(y-y_N\right)dx=\int_a^b\left(-\varepsilon\frac{d}{dx}(p\frac{dy}{dx})+ry\right)(y-y_N)dx$$

$$= \int_a^b \left(\varepsilon p \frac{dy}{dx} \frac{d}{dx}(y - y_N) + ry(y - y_N) \right) dx - \left(\varepsilon p (y - y_N) \frac{dy}{dx} \right) \Big|_a^b \tag{4.19}$$

Substitute Eq. (4.19) in to Eq. (4.18) we get:

$$= \int_a^b \frac{1}{2} \left(\varepsilon p \left(\frac{dy_N}{dx} \right)^2 - \varepsilon p \left(\frac{dy}{dx} \right)^2 + {y_N}^2 r - y^2 r \right) dx$$

$$+ \int_a^b \left(\varepsilon p \frac{dy}{dx} \frac{d}{dx}(y - y_N) + ry(y - y_N) \right) dx - \left(\varepsilon p (y - y_N) \frac{dy}{dx} \right) \Big|_a^b$$

$$= \int_a^b \left(\frac{1}{2} \varepsilon p \left(\frac{dy_N}{dx} \right)^2 - \frac{1}{2} \varepsilon p \left(\frac{dy}{dx} \right)^2 + \frac{1}{2} {y_N}^2 r - \frac{1}{2} y^2 r + \varepsilon p \left(\frac{dy}{dx} \right)^2 - \varepsilon p \frac{dy}{dx} \frac{dy_N}{dx} + ry^2 - ryy_N \right) dx$$

$$= \int_a^b \left(\frac{1}{2} \varepsilon p \left(\frac{dy_N}{dx} \right)^2 + \frac{1}{2} \varepsilon p \left(\frac{dy}{dx} \right)^2 + \frac{1}{2} {y_N}^2 r + \frac{1}{2} y^2 r - \varepsilon p \frac{dy}{dx} \frac{dy_N}{dx} - ryy_N \right) dx$$

$$= \int_a^b \frac{1}{2} \left(\varepsilon p \left(\frac{dy_N}{dx} \right)^2 - 2\varepsilon p \frac{dy}{dx} \frac{dy_N}{dx} + \varepsilon p \left(\frac{dy}{dx} \right)^2 + r\left({y_N}^2 - 2yy_N + y^2 \right) \right) dx$$

$$= \int_a^b \left(\frac{1}{2} \varepsilon p \left(\frac{dy_N}{dx} - \frac{dy}{dx} \right)^2 + \frac{1}{2} r \left(y_N - y \right)^2 \right) dx \geq 0$$

$$\Rightarrow I(y_N) - I(y) \geq 0 \ ,$$

$$\Rightarrow I(y_N) \geq I(y) \tag{4.20}$$

Therefore, convergence of energy of the approximate solution to the exact solution is from the above and the exact solution minimizes the energy.The data of the problem is sefficiently contineues,diffireantiable and integrable.This guarantees convergence of the method.

4.3. Numerical Examples

To demonstrate the applicability of the methods, two model self-adjoint singularly perturbed problems have been considered. These examples have been chosen because they have been widely discussed in the literature and their exact solutions were available for comparison.

Example 1: Consider the singularly perturbed problem:

$$-\varepsilon((1+x^2)y')y'+(1+x-x^2)y=f(x),\quad 0<x<1\quad \text{with}\quad y(0)=y(1)=0$$

where $z(x)=1+x(1-x)-e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}[x(2x^2-3x+1)-(2\sqrt{\varepsilon}(2x^2-x(1+\sqrt{\varepsilon})+1)]$
$+e^{-\left(\frac{1-x}{\sqrt{\varepsilon}}\right)}[x^2(2x-1)+(2\sqrt{\varepsilon}(2x^2+x\sqrt{\varepsilon}+1)]$

The exact solution is given by:

$$y(x,\varepsilon)=1+(x-1)e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}-xe^{-\left(\frac{1-x}{\sqrt{\varepsilon}}\right)}$$

The numerical solutions in terms of maximum absolute errors are given in Tables 4.1-4.2 and figures 4.1– 4.2.

Example 2: Consider the singularly perturbed problem:

$$-\varepsilon y''+(1-x-x^2)y=f(x),\quad 0<x<1\quad \text{with}\quad y(0)=y(1)=0$$

where $z(x)=1+x-x^2+(2\sqrt{\varepsilon}-x^2+x^3)e^{-\left(\frac{1-x}{\sqrt{\varepsilon}}\right)}+(2\sqrt{\varepsilon}-x(1-x)^2)e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}$

The exact solution is given by:

$$y(x,\varepsilon)=1+(x-1)e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}-xe^{-\left(\frac{1-x}{\sqrt{\varepsilon}}\right)}$$

The numerical solutions in terms of maximum absolute errors are given in Tables 4.3-4.5 and figures 4.3– 4.4.

Table 4.1: Maximum absolute errors of example 1 for different order of base function

ε	Order 3	Order 5	Order 7	Order 9
2^{0}	$8.1767\text{x}10^{-6}$	$1.0615\text{x}10^{-8}$	$2.864\text{x}10^{-7}$	$8.6945\text{x}10^{-4}$
2^{-3}	$7.0081\text{x}10^{-4}$	$7.8164\text{x}10^{-6}$	$5.2035\text{x}10^{-8}$	$5.8425\text{x}10^{-8}$
2^{-5}	$7.5765\text{x}10^{-3}$	$3.1996\text{x}10^{-4}$	$8.1669\text{x}10^{-6}$	$5.4286\text{x}10^{-7}$
2^{-7}	$4.8755\text{x}10^{-2}$	$5.6437\text{x}10^{-3}$	$4.8615\text{x}10^{-4}$	$3.0480\text{x}10^{-5}$
2^{-9}	$1.8217\text{x}10^{-1}$	$4.9736\text{x}10^{-2}$	$8.9656\text{x}10^{-3}$	$1.6241\text{x}10^{-3}$
2^{-11}	$3.9651\text{x}10^{-1}$	$1.8364\text{x}10^{-1}$	$4.8905\text{x}10^{-2}$	$2.1081\text{x}10^{-2}$

Table 4.2: Maximum absolute errors for example 1for $n=16$ (using ninth order base function)

ε	Present method	Terefe et al.,2016
2^{-4}	$5.9456\text{x}10^{-8}$	$5.85\text{x}10^{-4}$
2^{-5}	$5.4286\text{x}10^{-7}$	$8.57\text{x}10^{-4}$
2^{-6}	$2.5381\text{x}10^{-6}$	$9.58\text{x}10^{-4}$
2^{-7}	$3.0480\text{x}10^{-5}$	$1.29\text{x}10^{-3}$
2^{-8}	$2.6089\text{x}10^{-4}$	$1.65\text{x}10^{-3}$
2^{-12}	$4.3854\text{x}10^{-2}$	$3.76\text{x}10^{-2}$

Table 4.3: Maximum absolute errors of example 2 for different order of base function

ε	Order 3	Order 5	Order 7	Order 9
2^{-2}	$1.7321\text{x}10^{-4}$	$9.7792\text{x}10^{-7}$	$3.4644\text{x}10^{-9}$	$3.1349\text{x}10^{-7}$
2^{-4}	$2.4999\text{x}10^{-3}$	$5.4650\text{x}10^{-5}$	$7.1434\text{x}10^{-7}$	$5.9456\text{x}10^{-8}$
2^{-6}	$1.9313\text{x}10^{-2}$	$1.4944\text{x}10^{-3}$	$7.3215\text{x}10^{-5}$	$2.5381\text{x}10^{-6}$
2^{-8}	$1.0077\text{x}10^{-1}$	$1.7969\text{x}10^{-2}$	$2.4117\text{x}10^{-3}$	$2.6089\text{x}10^{-4}$
2^{-10}	$2.8574\text{x}10^{-1}$	$1.0700\text{x}10^{-1}$	$2.3994\text{x}10^{-2}$	$7.1512\text{x}10^{-3}$
2^{-12}	$4.9550\text{x}10^{-1}$	$2.6125\text{x}10^{-1}$	$8.6681\text{x}10^{-2}$	$4.3854\text{x}10^{-2}$

Table 4.4: Maximum absolute errors for example 2 for $n=16$ (using ninth order base function)

ε	Present method	Terefe et al., 2016
1/8	$5.9456\text{x}10^{-8}$	$1.424\text{x}10^{-6}$
1/16	$5.9456\text{x}10^{-8}$	$4.148\text{x}10^{-6}$
1/32	$5.4286\text{x}10^{-7}$	$9.622\text{x}10^{-6}$
1/64	$2.5381\text{x}10^{-6}$	$3.074\text{x}10^{-4}$
1/128	$3.0480\text{x}10^{-6}$	$1.301\text{x}10^{-4}$
1/256	$2.6089\text{x}10^{-4}$	$5.910\text{x}10^{-4}$

Table 4.5: Maximum absolute errors for example 2 for $n=32$ (using ninth order base function)

ε	Present method	Khalid et al., 2018
1	$8.36\text{x}10^{-7}$	$3.92\text{x}10^{-9}$
1/10	$7.9975\text{x}10^{-8}$	$8.86\text{x}10^{-7}$
1/100	$1.2940\text{x}10^{-5}$	$1.77\text{x}10^{-5}$
1/1000	$6.8420\text{x}10^{-3}$	$4.05\text{x}10^{-4}$

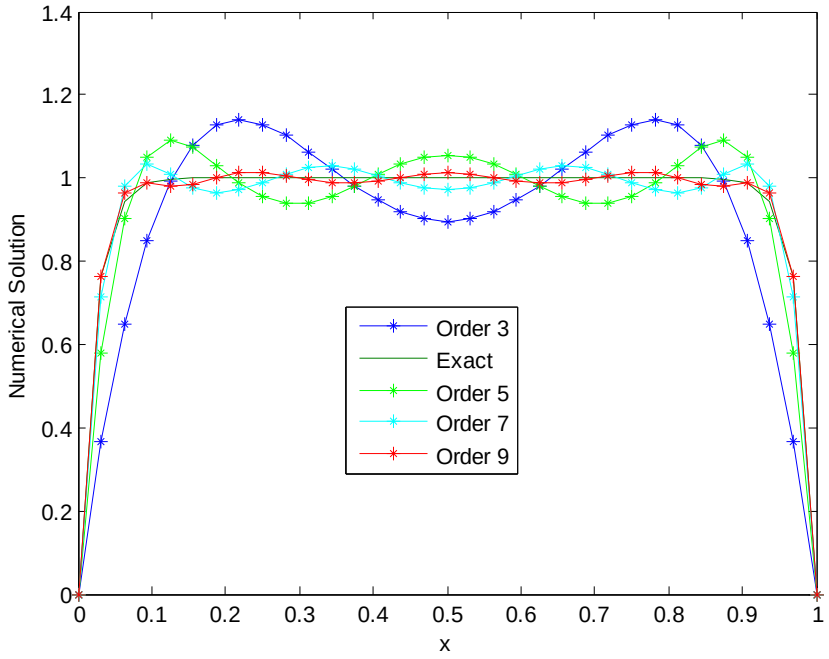

Figure 4.1: Numerical solution of Example 1 when $\varepsilon = 2^{-11}$ and $n = 32$.

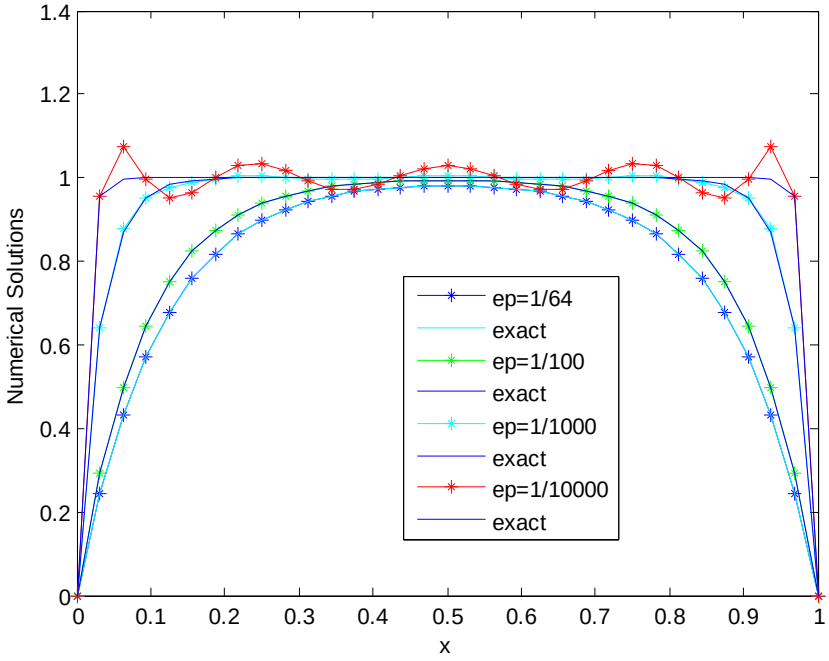

Figure 4.2: Numerical solution of Example 1 when $\varepsilon = \frac{1}{64}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}$ and $n = 16$.

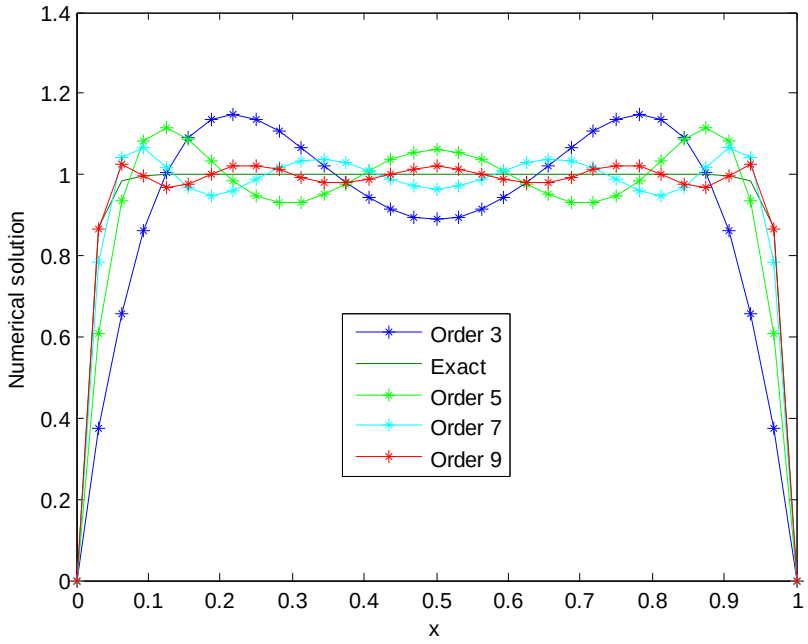

Figure 4.3: Numerical solution of Example 2 with $\varepsilon = 2^{-12}$ and $n = 32$.

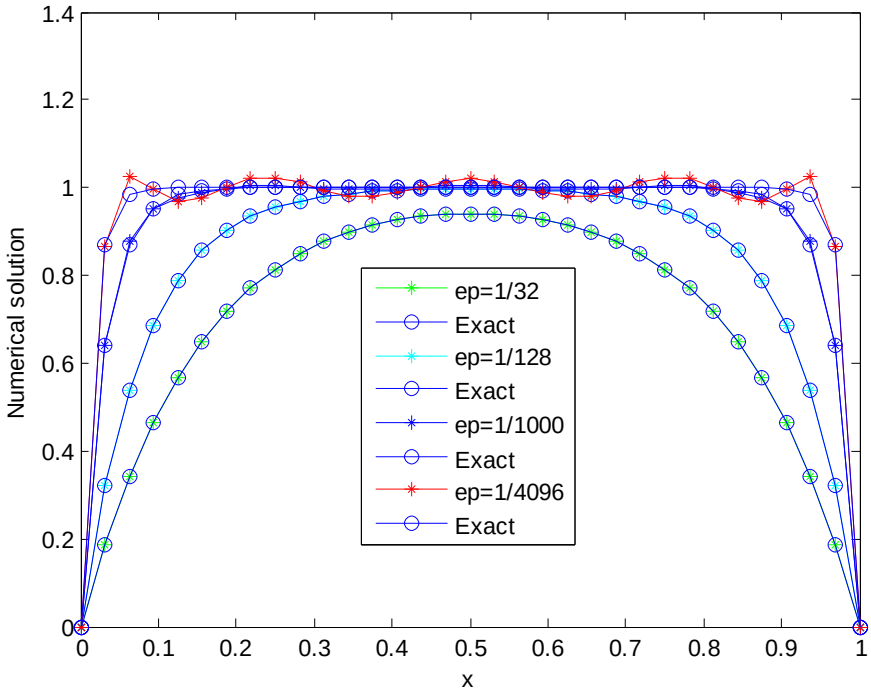

Figure 4.4: Numerical solution of Example 2 when $\varepsilon = \frac{1}{32}, \frac{1}{128}, \frac{1}{1000}, \frac{1}{4096}$ and $n = 16$.

4.4. Discussion

In this thesis, numerical solution of self–adjoint problems using Galerkin method have been presented for solving singularly perturbed second order boundary value problems. The numerical results have been presented in Tables 4.1 – 4.5 for different order of the basis of the polynomial and different values of perturbation parameter ε. The results obtained by the present method have been compared with the numerical results obtained by Terefe Asrat et al., 2016 and Kalid et al., 2018 from literature. As it can be observed from the tables that the methods presented in this thesis approximates the exact solution better than the methods proposed by Terefe Asrat et al., 2016 and and Kalid et al., 2018. Similarly, the figures (fig. 4.1– 4.4) shows that as the order of the basis function increases, the proposed method approximates the exact solution very well independent of h which makes the present method different from other existing methods. Moreover, all the maximum absolute errors decrease rapidly as the order of the basis function increases. Finally, the theoretical error bounds have been established for Galerkin methods in energy norm.

CHAPTER FIVE

CONCLUSION AND FUTURE WORK

5.1 Conclusion

The Galerkin method is developed for the approximate solution of a second order singularly perturbed self-adjoint boundary value problems with Dirichlet boundary conditions. The method has been proved to be first order convergent in energy norm. Two examples are considered for numerical illustration of the method. Our numerical result is better than other method proposed Terefe Asrat et al., 2016 and Kalid et al., 2018 from literature. As the order increase the numerical solution approach to the exact solution and as paraturbation parameter is very small numerical soluton is far from the exact solution. In concise manner, the present methods are conceptually simple, easy to use and readily adaptable for computer implementation for solving singularly perturbed self-adjoint boundary value problems.

5.2. Scope for Future Work

In the present thesis, the numerical methods using Galerkin method with polynomial basis function was constructed for solving singularly perturbed second order self-adjoint boundary value problems. Hence, the schemes proposed in this thesis can also be extended to higher order for singularly perturbed boundary value problems or one can change the basis function in to trigonometric, Fourier or other to increase the accuracy.

REFFERENCES

Boglave, I. P.(1981). A variation difference scheme for boundary value problems with a small parameter in the highest derivative, *Journal of Computational Mathematics and mathematical Physics.* **21** (4): 71–81.

Byme,C.andMishra,H.K.(2009).**Notesonself-adjointdifferentialequations.**Computer technology and Application. **2** (5); 1426-1431**.**

Carey, G.C.and Oden, J.T. (1983) Finite Elements: a Second Course, volume II. Prentice-Hall, englewood Cliffs.

Doolan. (1980) Uniform numerical methods for problems with initial and boundary layers, Dublin, Boole Press.

Farrell, A. F., Miller, J. J. H., O'Riordan, E. and Shishkin, G. I. (2000).**Robust Computational Techniques for Boundary Layers,** Chapman-Hall/CRC, New York**.**

Firdous, A.S., Abass .R. and Iqbal, J. (2016). Numerical solution of singularly perturbed problems Using Haar Wavelet collection method, *Journal: Cogent Mathematics.***3**, *113.http://dx.doi.org/10.1080/23311835.2016.1202504*

Gupta,Y., Kumar,P., Manoj Kumar. (2011). Application of B-spline to Numerical Solution of a system of Singularly Perturbed Problems, *MathematicaAeterna*, **1**(6), 405-415.

Gupta, Y., and Pankaj, S. A. (2011). Computational method for solving two points Boundary problems of order four, *International Journal of Computer .Technology and Application, **2** (5): 1426-1431.*

Hrennikoff,A. (*1941).* "Solution of problems in elasticity by the framework method", *J. Appl. Mech. Vol. **8**: pp. 169-175.*

Kadalbajoo, M. K., and Kumar, D. (2008).Parameter-uniform fitted operator B-spline collocation method for self-adjoint singularly perturbed two-point boundary value

problems, *Electronic Transactions on Numerical Analysis.* **30**: *346-358.*

Kadalbajoo, M.K., and K.C., Patidar, (2002). Spline techniques for the numerical solution of singularperturbation problems, *J. Optim. Theory Appl.* ***112****:575–594.*

Kadalbajoo, M. K., and Kumar, D. (2010).Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems, *Journal of Computational Mathematics.* ***28*** *(5): 711-724.*

Kadalbajoo, M.K., and V.K., Aggarwal.(2005). Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems, *Appl. Math Comput.****161****:973–987.*

Khalid K. Ali, A.R., Hadhoud, M.A.and Shaalan. (2018) Numerical study of self-adjoint singularly perturbed two-point boundary value problems using collocation method with error estimation.*Journal of Ocean Engineering and Science.* ***3****, 237–243.*

Kuma, D. (2012).Finite difference scheme for singularly perturbed convection – diffusion problem with two small parameters, *Mathematical Aeterna.* ***2****(5), 441- 458.*

Miller, J.J.H. (*1979)* on the convergence, uniformly in ε, of difference schemes for a two point boundary singular perturbation problem, Numerical Analysis of Singular Perturbation Problems (*Pr Academic Press, New York, pp. 467–474. oc. Conf., Math.Inst., Catholic Univ., Nijmegen).*

Mishra, K. (*2009).* Initial value technique for self-adjoint singular perturbation boundary values problems, *Computational Mathematics and modeling. Vol.* ***29,*** *No.2.*

Reddy, J.N.(2006).**An introduction the finite element method** .3rd edition.

Roos, H. G., Stynes, M. and Tobiska, L. (1996).**Numerical Methods for Singularly perturbed Differential equation:convection-diffusion and flow proplem.** SpringerVerlag.

Sakai, R.A. and Usmani, (1986) on exponential splines, *J. Approx. Theory.***47**: 122–131

Sakai, R.A. Usmani, (1989). A class of simple exponential B-splines and their applications to

numerical solution to singular perturbation problems, *Numer.Math*.**55**:493–500.

Stojanovic.M. (2006).Spline collocation method for singular perturbation problem, *Glas. Mat*.**37**: 393–403. ´

Terefe Asrat, Gemechis File and Tesfaye Aga. (2016).Fourth-order stable central divergence method for self-adjoint singular perturbation problems, Ethiop. *J. Sci. &Technol*.**9**(1) 53-68.

Turner, M.J., Clough.R.W., Martin.H.C.and Topp.L.T. (1956) "Stiffness and deflection Analysis of complex structures", *J. Aero.Sci. Vol. **23**: pp. 805-823.*

Uzelac, Z. K., Surla. (2003). A uniformly accurate collocation method for a singularly perturbed sproblem, *Novi Sad J. Math.**33**:133–143.*

Vasil'eva, A. B.(1976) "The Development of the Theory of Ordinary Differential Equations with a Small Parameter Multiplying the Highest Derivatives in the Years 1966-1976," *Russian Mathematical Surveys. Vol. **31**, pp. 109-131.*

Vukoslavcevic,K.,and Surla, V. (1996):, Finite element method for solving self-adjoint singularly perturbed boundary value problems, *Math.Montisnigri.**7:**79–86.*

Wasow, W. (1942).**On Boundary Layer Problems in the Theory of Ordinary Differential Equations**, Dissertation, New York University.

Appendix

For any differential equations of the form:

$$Ay = z, \quad \text{in } \Omega.$$

Linearity in A: Let S be a space containing functions that are admissible over the domain $\Omega = (a,b)$. Then, $\forall \phi, \psi \in S$ and $\forall \alpha, \beta \in R$, we must show that

$$A(\alpha\phi + \beta\psi) = \alpha A(\phi) + \beta A(\psi).$$

Proof: We show linearity: $A(\alpha\phi + \beta\psi) = \alpha(A\phi) + \beta(A\psi)$

$$A(\alpha\phi + \beta\psi) = \left(-\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)\right)(\alpha\phi + \beta\psi)$$

$$= \left(-\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)\right)(\alpha\phi) + \left(-\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)\right)(\beta\psi)$$

$$= \alpha\left(-\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)\right)(\alpha\phi) + \beta\left(-\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)\right)(\psi)$$

$$= \alpha\left(-\varepsilon \frac{d}{dx}(p(x)\frac{d\phi}{dx}) + r(x)\phi\right) + \beta\left(-\varepsilon \frac{d}{dx}(p(x)\frac{d\psi}{dx}) + r(x)\psi\right)$$

$$= \alpha(A\phi) + \beta(A\psi)$$

Hence the operator A defined by $A = -\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)$ is linear.

Adjoint of an operator: Let A be a linear differential operator and let Ω be its domain of definition. If

$$(A\phi, \psi) = (\phi, A^*\psi) + (A\phi, \psi)_\Gamma, \qquad \forall \phi, \psi \in S$$

holds, then A^* is called the adjoint of A

Proof. $A = -\varepsilon \frac{d}{dx}(p(x)\frac{d}{dx}) + r(x)$

$$(A\phi,\psi)=\int_a^b(-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})+r(x)\phi)\psi dx$$

$$=\int_a^b(-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})+r(x)\phi)\psi dx$$

$$=\int_a^b(-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})\psi)dx+\int_a^b r(x)\phi\psi dx \qquad (1)$$

Using integration by part $\int_a^b(-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})\psi)dx$

let $u=\psi,\quad du=\frac{d\psi}{dx}dx,\ dv=-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx}),\ \text{v}=-\varepsilon p(x)\frac{d\phi}{dx}$

$$\int_a^b(-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})\psi)dx=\int_a^b\varepsilon p(x)\frac{d\phi}{dx}\frac{d\psi}{dx}dx-\left(\varepsilon p(x)\psi\frac{d\phi}{dx}\right)\Big|_a^b \qquad (2)$$

Again using integration by part:

$$\int_a^b\varepsilon p(x)\frac{d\phi}{dx}\frac{d\psi}{dx}dx=\int_a^b\phi\left(-\varepsilon\frac{d}{dx}(p(x)\frac{d\psi}{dx})\right)dx+\left(\varepsilon p(x)\phi\frac{d\psi}{dx}\right)\Big|_a^b \qquad (3)$$

Substitute Eq.(3) in to Eq.(2) we get:

$$\int_a^b(-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})\psi)dx=\int_a^b\phi\left(-\varepsilon\frac{d}{dx}(p(x)\frac{d\psi}{dx})\right)dx+\left(\varepsilon p(x)\phi\frac{d\psi}{dx}-\varepsilon p(x)\psi\frac{d\phi}{dx}\right)\Big|_a^b \qquad (4)$$

Again substitute Eq.(4) in to Eq.(1) we get:

$$=\int_a^b\phi\left(-\varepsilon\frac{d}{dx}(p(x)\frac{d\psi}{dx})+r(x)\psi\right)dx+\left(\varepsilon p(x)\phi\frac{d\psi}{dx}-\varepsilon p(x)\psi\frac{d\phi}{dx}\right)\Big|_a^b$$

$$=(\phi,A^*\psi)+(A\phi,\psi)_\Gamma,\qquad\forall\phi,\psi\in S\ \text{where}\ A^*=-\varepsilon\frac{d}{dx}(p(x)\frac{d}{dx})+r(x)$$

$(A\phi,\psi)_\Gamma=\left(\varepsilon p(x)\phi\frac{d\psi}{dx}-\varepsilon p(x)\psi\frac{d\phi}{dx}\right)\Big|_a^b$ and $A^*=A$ then A^* is adjoint of A.

Here $(A\phi,\psi)_\Gamma$ is called the concomitant, and it is only a symbol that represents the boundary terms that are obtained in the process of moving operator A from ϕ to operator A^* on ψ.

Thus, the concomitant is a collection of boundary terms obtained as aconsequence of integration by parts.

Symmetric operator: Let A be a linear operator with its domain of definition Ω, then A is symmetric if

$$(A\phi,\psi)=(\phi,A\psi),\qquad \forall\phi,\psi\in S$$

If we use the definition of the scalar product, the Symmetry of A implies:

$$\int_\Omega (A\phi)\psi\, d\Omega=\int_{\bar{\Omega}}\phi(A\psi)d\Omega$$

Symmetric property; $(A\phi,\psi)=\int_a^b (A\phi)\psi\, dx$

$$=\int_a^b (-\varepsilon\frac{d}{dx}(p(x)\frac{d\phi}{dx})+r(x)\phi)\psi\, dx$$

$$=\int_a^b \phi(-\varepsilon\frac{d}{dx}(p(x)\frac{d\psi}{dx})+r(x)\psi)dx$$

$$=\int_a^b \phi(A\psi)dx$$

$$=(\phi,A\psi)$$

Hence the operator A is symmetric.