NUMERICAL SOLUTION OF TWO DIMENSIONAL HEAT EQUATION INPOLAR COORDINATES SYSTEM USING CRNK-NICOLSON WITH HOCKNEY'S METHOD.

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENT OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS.

(NUMERICAL ANALYSIS)

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DECLARATION

I undersigned declare that this thesis entitled **"**Numerical solution of two dimensional Heat equation in polar coordinate systems using Crank-Nicolson with Hockney's Method**.** "is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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ABSTRSCT

In this thesis, Crank-Nicolson with Hockney's method is presented for solving Two dimensional Heat equation in polar coordinates system. First the given partial differential equation of two dimensional heat equation in polar coordinates system is converted into equivalent linear equation using Crank-Nicolson approximation .By extending Hockney's method the linear equation is reduced in to main and sub diagonal system which can be solved using Thomas algorithm. To validate the applicability of the proposed method two model examples with exact solution have been solved. The result presented in tables show that Crank-Nicolson with Hockney's method is faster than Crank-Nicolson method in computational time.

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CHAPTER ONE
INTRODUCTION

1.1. Background of the study

The Limitation of analytical method in practical application is led Mathematicians and other scientist to evolve numerical methods. It is clear that exact method often fail in drawing reasonable inference from a given set of tabulated data or in finding solution for different equation. There are many situations where analytical methods unable to produce desirable results. Even if analytical solutions are available, these are not amenable to direct numerical interpretations (Goyal, 2007).

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications, Wasow, (1942). In real life, we often encounter many problems described by second order elliptic, parabolic and hyperbolic partial differential equations. A wide variety of parabolic partial differential equations are used in engineering and science. Some of the most common ones are the diffusion equation and the convection-diffusion equation. The diffusion equation applies to a problem in mass diffusion and Heat diffusion (conduction),etc. The convection-diffusion equation applies to a problem in which convection occurs in combination with diffusion. For example, fluid mechanics and heat transfer. The heat equation is an important parabolic partial differential equation which describes the distribution of heat (variation in temperature) in a given region over time.

Heat equation in polar Coordinates System.

we want to solve a partial differential equation (PDE) on the domain whose shape is a 2D disk, it is much more convenient to represent the solution in terms of the polar coordinates system than in terms of the usual Cartesian coordinate system. In this case we are going to derive Heat equation in polar coordinates system in detail. Recall that Heat equation in R^2 in terms of the usual (i.e. Cartesian) (x, y)coordinates system is;

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=u_t=u_{xx}+u_{yy}$$

The Cartesian coordinates can be represented by the polar coordinates as follows**;**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

lets us first compute the partial derivatives of x, y w. r. t r,θ

$$\begin{cases} \dfrac{\partial x}{\partial r} = \cos\theta, \dfrac{\partial x}{\partial \theta} = -r\sin\theta \\ \dfrac{\partial y}{\partial r} = \sin\theta, \dfrac{\partial x}{\partial \theta} = r\cos\theta \end{cases}$$

To do so, let's compute $\frac{\partial u}{\partial r}$ first .We will use the Chain rule since (x, y) are functions of (r,θ) as shown in the above**.**

$$\frac{\partial u}{\partial r} = \frac{\partial u \partial x}{\partial x \partial r} + \frac{\partial u \partial y}{\partial y \partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

Now, let's compute $\frac{\partial^2 u}{\partial r^2}$.Noticing that both $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are function of (x, y)

then we have,

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial y} = \cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right)$$
$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}$$

Similarly, let's compute

$$\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta\frac{\partial u}{\partial x} - \sin\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial y}$$
$$= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right) - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right)$$

$$=-r\cos\theta\frac{\partial u}{\partial x}-r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 u}{\partial x\partial y}r\cos\theta\right)-r\sin\theta\frac{\partial u}{\partial y}+r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta)+\frac{\partial^2 u}{\partial y^2}r\cos\theta\right)$$

$$-r(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y})+r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)$$

divides both sides by r^2

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=-\frac{1}{r}\frac{\partial u}{\partial r}+\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}$$

using $\sin^2\theta+\cos^2\theta=1$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=-\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

Then the two dimensional heat equation in polar coordinate system is given by ,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\quad t>0 \text{ on D,}$$

where D is a domain ,we write D in two cases as

$a)\ D_1=\{(r,\theta):R_0\le r<R_1,\theta_0<\theta<\theta_1<,\theta_0<\theta_1<2\pi\}$

$b)\ D_2=\{(r,\theta):R_0\le r<R_1,0\le\theta<2\pi\}$

U describes the temperature at a given location (r,θ), this function will change over a time as heat spread through space.

$\frac{\partial u}{\partial t}$ is the rate of change of temperature at a point over time.

$\frac{\partial u}{\partial r}$ is the first order partial differential derivative of temperature in r direction.

$\frac{\partial^2 u}{\partial r^2}$ *and* $\frac{\partial^2 u}{\partial^2\theta}$ are second order partial differential derivative of temperature (thermal conduction) of temperature in (r, θ) direction respectively.

For time dependent problems considerable progress finite difference method was made during the period of the second world war, when large scale practical applications become possible with the aid of computers. Hence, in the recent time, many researchers have been trying to develop numerical methods for solving two dimensional parabolic (heat) equations, Daoud(2014**),**presented additive splitting up scheme to solve multidimensional parabolic equation. Xiao-Liang(2010) developed in his research the scheme of Iterative method for forward-backward heat equation in two dimension. Jimn Liang Liu (1991)presented A Galerkin method for forward-backward heat equation in two dimension. Shan Zhaho **(**2007**)** presented A matched alternating direction implicitly method to solve heat equation with interface. Shan Zhaho **(**2010) presented A specially second order implicitly method to solve three dimensional heat equation**.** Jim Do galas .J.R (1955**)** presented integration by implicitly method to solve two dimensional heat equation and other several attempts have been made to solve the two dimensional heat equation in particular for physical problems that are related directly or indirectly to this equation. To get alternative numerical solution, this study presents Crank-Nicolson method for solving two dimensional heat equation in polar coordinate system. The method of Crank-Nicolson with Hockney's powerful numerical techniques that has been used to obtain highly accurate numerical approximation of solutions of partial differential equations with small computational effort.Even if many Different methods are used by different researchers to approximate the solution of Heat equation, in this paper we used Crank-Nicolson method to find alternative numerical solution of heat equation in two dimension in polar coordinate system. Since no one goes through this numerical method. .

1.2 Statement of the problem

The numerical solution of heat equation in two dimension has an important applications in many fields of science, engineering and Technology. The increasing desire for the numerical solutions to mathematical problems, which are more difficult or impossible to solve explicitly, has become the present- day scientific research. Thus this shows the importance and application of numerical methods to solve problems in real life. The numerical method used to find approximate solution of systems of linear equations has an impressive importance due to its wide applications in scientific and engineering researchers. Even though many numerical methods were applied to solve these type of equations, still it needs treatment to obtain fast and accurate solution for two dimensional heat equation in polar coordinates system. So, in this case Crank-Nicolson with Hockney's method is used to find approximate solution of the two dimensional Heat equations in polar coordinate system. Hence, the present study attempt to answer the following basic questions.

- the present method over the Cranck-Nicolson method.

1.4. Significance of the Study

1. The results obtained in this study may help: How do the present method be described for Heat Equation?
2. To what extent the proposed method is efficient?
3. What is the advantage of the proposed method over Crank-Nicolson ?

1.3. Objectives of the study

1.3.1. General Objective

The general objective is to find the numerical solutions of two dimensional heat equation in polar coordinates system when r=0 is in the interior or a boundary point of the domain

1.3.2. Specific Objectives

The specific objectives of the present study are:

- To find the numerical solution of two dimensional heat equation in polar coordinates system using Crank-Nicolson with Hockney's method.
- To investigate the efficiency of the proposed method.

  To describe the advantage of

  - To introduce the application of numerical methods in different field of studies.
  - Serve as a reference material for scholars who works on this area.

1.5. Delimitation of the Study

This study is conducted under the stream of numerical analysis and since Heat equation are vast topics and have many applications in the real world. However, this study is delimited to the two dimensional Heat Equation in polar coordinates system of the form;

$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ for $t>0$,on , where D is a domain i.e.

$a)\ D_1=\left\{(r,\theta): R_0\le r<R_1, \theta_0<\theta<\theta_1, 0<\theta<2\pi\right\}$

$b)\ D_2=\left\{(r,\theta): R_0\le r<R_1, 0\le\theta\le 2\pi\right\}$

CHAPTER TWO

REVIEW OF RELATED LITERATURE

Due to the advancement in the field of computational mathematics numerical methods are mostly utilized to solve equations arising in applied science, engineering and technology. Numerical analysis is brunch of mathematics concerned with theoretical foundation of numerical algorithms for solutions of problems arising in scientific applications and that deals with the computational methods which help to find approximate solutions for difficult problems such as finding the roots of non-linear equations, integrations involving complex expressions and solving differential equations for which analytical solution does not exist.

2.1 Finite Difference Method

The Finite Difference theory for general initial value problems and parabolic problems then had an intense period of development during 1950s and 1960s when the concept of stability was explored in the lax equivalent theory and the Kreiss matrix lemmas. Independently of the engineering applications a numbers of papers appeared within mathematical literature in the mid-1960s which were concerned with Rayleigh-Ritz procedure with piecewise linear approximating functions

The Finite Difference Method is a numerical Procedure Which solves a Partial Differential Equations by discretizing the continuous physical domain in to a discrete finite difference grid, approximating the individual exact partial derivatives in the partial differential equation by Algebraic Finite Difference Approximations .substituting the Finite Difference Approximations in to partial differential equation to obtain an algebraic finite difference equation, and solving the resulting algebraic finite difference equation for the dependent variable . This Method is used to discretizing a parabolic partial equation ,Islam,M.R and Alias,N.(2010).They presented a mathematical simulation models using one dimensional parabolic equations.

2.2 Crank-Nicolson method

Crank-Nicolson method for solving parabolic partial differential equations was developed by John Crank and Phyllis Nicolson in (1947).A practical method for numerical evaluation of partial differential equation of Heat conduction way considered. In numerical analysis the Crank-Nicolson method is a finite difference method used for numerical solving heat equation and similar partial differential equations which is a second- order method in time. according to this method the second order partial differential equation is replaced by average of the central difference on the n^{th} and $(n+1)^{th}$ time rows. Crank-Nicolson's main work was on the numerical solution of partial differential equations and in particular, the solution of heat -conduction problems.

2.3 Partial Differential Equation

A partial differential equation is an equation starting a relationship between a function of two or more independent variables and the partial derivatives of this function with respect to these independent variables. Rechard Hambeman(1989). These equations arise in all fields of engineering and science .Most real physical processes are governed by partial differential equations. In many cases, simplifying approximations are made to reduce the governing partial differential equation to ordinary differential equations or even to algebraic equations, Pinsky.M.(1991). engineers and scientist are more and more required to solve the actual partial differential equations that govern the physical problem being investigated. This equation involves derivatives of unknown functions with respect to several variables. It form the basis of many mathematical models of physical, Chemical and Biological phenomena and more recently the use of partial Differential Equation has spread in to Economics, financial forecasting , image processing and other fields Rezzola,(2011).To investigate the predictions of partial differential equation models of such phenomena often necessary to approximate their solution numerically. Commonly in combination with analytical solutions of the simple special cases while in some of recent instance the numerical models play almost independent role. The terminology, elliptic, parabolic and hyperbolic chose to classify partial differential equations reflects the analogy between the form of the discriminates $B^2 - 4AC$ which classify the linear second order partial differential equation given by sections are described by.

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + F = 0$$

The type of curve represented by the above equation depends on the sign of discriminate $B^2 - 4AC$ .Then if $B^2 - 4AC < 0$ then the equation is Elliptic partial Differential Equation ,if $B^2 - 4AC = 0$ then the equation is parabolic partial Differential Equation and if $B^2 - 4AC > 0$ then the Equation is hyperbolic partial Differential Equation.

The classification of partial differential equation is intimately related to the characteristic of the partial differential equations. Characteristics are (n-1)-dimensional hyper surface inn-dimensional hyperspace that have some feature. The prefix hyper is used to denote spaces of more than three dimensionals,that is xyzt spaces, and curves and surfaces within those space .In two dimension spaces, which is the case considered here, characteristics are path(curved, in general) in the solution domain along which information propagates. In other words, information propagates throughout the solution domain along the characteristics paths.Discontinuites in the derivatives of the dependent variable(if they exist)also propagate along the characteristic path. If a partial differential equation possesses real characteristics, then information passes along this characteristics. if no real characteristics then the information exist then there is no preferred path of information propagation. consequently, the presence or absence of characteristics has a significant impact on the solution of partial differential(by both analytical and numerical method)

2.4 Heat Equation

The Heat equation is fundamental in diverse scientific fields . Jean Baptiste Fourier(1768-1830) was first formulate the transient of Heat conduction described by partial differential equation and presented as a subscript to the institute of France in 1807.At the time this manuscript was prepared ,thermodynamics ,potential theory and differential equations were all in the initial stage of their formulation. By Fourier's law ,the flow rate of heat energy through the surface is proportional to the negative temperature gradient across the surface. Cannon(1984) first derived heat equation from Fourier's Law and conservation of energy. The diffusion equation is the more general of the heat equation arises in connection with the study of chemical diffusion and other related processes.

The Heat equation is parabolic partial differential equation that describes the distribution of heat (variance in temperature) in a given region over time. One of the interesting properties of Heat equation is the maximum principle which says that the maximum value temperature at the given location (u) is either in time than the region of concern or an edge of the region of the concern, This is essentially saying that temperature comes either from some sources or from earlier in time because heat permeates but is not created from nothingness. This is a properties of parabolic differential equation. It predicts that if a hot body is placed in a cylinder of cold water

,the temperature of the body will decrease ,and eventually (after infinite time, and subjected to no external heat source)the temperature in the cylinder will equalized. In other hand if the temperature is constant no heat energy flow, if there are temperature difference, the heat energy flow from hotter region to the cold region, the greater temperature difference(for the same material), the greater is the flow of heat energy, the flow of heat energy will vary for different material even with the same temperature, Dassio.G and Fokas,A.(2008).In many engineering applications finding the solution of various heat conduction problems is fundamental importance, Example including heat exchanger, mathematical finance, in particular after transforming the Black-scholes equations in to heat equation, and various Chemical and Biological systems, including diffusion and transportation problems.Thus,due to its importance, many different numerical techniques have been developed for calculation heat flow.Many researchers have been trying to develop numerical methods for Solving Heat equation..For example, Borjin,M.U and Mbow.C.(1999).developed Numerical analysis of combined radiation and unsteady natural convection with Horizontal annular space .for Heat and fluid flow Fukagata.K and Kasagi.N (2003)presented ,Highly energy- conservation finite Difference method to solve heat equation in the cylindrical coordinate system, Jinn-Liangliu.(1991)found numerical solution by Using A Galerkin Method for a forward-backward two dimensional heat equation, Lyengar S.R.K ,and Manor .R(1988)where developed, High-Order Difference Method to solve Heat Equation in polar and cylindrical coordinates, Verzicco.R and Orland.P.(1996) were presented, A finite Difference scheme for the three-dimensional heat flows in cylindrical coordinates , Zhihan Wel .(2007),was used Second order alternating direction implicitly Method for solving the three dimensional parabolic(heat) interface problems. In mathematics it is parabolic partial differential equation . In statistics, the Heat equation is connected with the study of Brownian motion. It applies to problems in mass diffusion ,heat diffusion (i.e. conduction),neutron diffusion ,electron static, in viscid incompressible fluid flow etc..

CHAPTER THREE
METHODOLOGY

3.1 Study Site

This study is conducted in Jimma University under the department of Mathematics from September 2010 E. to November 2011 E.C. Conceptually, the study focus on Crank-Nicolson with Hockney's Method for solving Two Dimensional Heat Equations in Polar coordinates system

3.2. Study Design

This study employed mixed-design (documentary review design and experimental design) on Numerical solution of the two dimension Heat equation in polar coordinate system.

3.3. Source of Information

The relevant sources of information for this study are books, published articles & related studies from internet and the experimental result will be obtained by writing MATLAB code.

3.4. Study Procedures

In order to achieve the stated objectives, the study is followed the following steps

1. Defining the problem,
2. Discretizing the region/ domain
3. Describing the method used and obtain the scheme.
4. Reduce the obtained scheme by extending Hockney's method in to main and sub diagonal system which can be solved by appropriate method for solving system of equation.
5. Writing MATLAB code for the system obtained,
6. Validating the schemes by using numerical examples.

CHAPTER FOUR
DESCRPTION OF THE METHOD, RESULTS AND DISCUTION

4.1. Description of the Method.

For $r \neq 0$

Consider the two dimensional Heat equation in polar coordinates system of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad t > 0 \text{, on } D \text{, where } D \text{ is the domain i.e.} \tag{4.1}$$

a) $D_1 = \{(r,\theta) : R_0 \leq r < R_1,\ \theta_0 < \theta < \theta_1, 0 < \theta < 2\pi\}$

b) $D_2 = \{(r,\theta)\ R_0 \leq r < R_1\ ,\ 0 \leq \theta \leq 2\pi\}$

Assume that there are m points along the r direction and n points along the θ direction to form mesh points and let the step size along the direction of r be Δr and along the direction of θ be $\Delta\theta$, then

$$r_i = r_0 + i\Delta r, \qquad \theta_j = \theta_0 + j\Delta\theta \tag{4.2}$$

By using Crank-Nicolson scheme, we have the derivative of the given equation is replaced by average central difference approximation on n[th] and(n+1)[th] times row

$$\frac{\partial^2 u}{\partial r^2} \approx \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{2\Delta r^2}$$

$$\frac{\partial^2 u}{\partial \theta^2} \approx \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{2\Delta \theta^2}$$

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t}, and\ \frac{\partial u}{\partial r} \approx \frac{1}{2}\left[\frac{U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}}{2\Delta r} + \frac{U_{i+1,j}^{n} - U_{i-1,j}^{n}}{2\Delta r}\right] \tag{4.3}$$

substituting equation 4.3 in to equation 4.1 we obtain

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} = \frac{1}{2r}\left[\frac{U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}}{2\Delta r} + \frac{U_{i+1,j}^{n} - U_{i-1,j}^{n}}{2\Delta r}\right] + \left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{2(\Delta r)^2}\right)$$

$$+ \frac{1}{2r^2}\left(\frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1} + U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(\Delta\theta)^2}\right)$$

Let, $\frac{\Delta t}{4\Delta r} = a \quad \frac{\Delta t}{2(\Delta r)^2} = b \quad and \quad \frac{\Delta t}{2(\Delta\theta)^2} = c$

substitute in the above equation ,we get

$$\left(\frac{a}{r_i}+b\right)U_{i+1,j}^{n+1}+\frac{c}{r_i^2}(U_{i,j+1}^{n+1}+U_{i,j-1}^{n+1})+(-1-2b-\frac{2c}{r_i^2})U_{i,j}^{n+1}+\left(b-\frac{a}{r_i}\right)U_{i-1,j}^{n+1}+\left(\frac{a}{r_i}+b\right)U_{i+1,j}^{n}$$
$$+(1-2b-\frac{2c}{r_i^2})U_{i,j}^{n}+\frac{c}{r_i^2}(U_{i,j+1}^{n}+U_{i,j-1}^{n})+\left(b-\frac{a}{r_i}\right)U_{i-1,j}^{n}=0 \qquad (4.4)$$

For $r = 0$ intrior or boundary point in the domain.

Consider two dimensional Heat equation in polar coordinate system of the form of

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \text{ for } t>0 \text{ on } D, \qquad (4.5)$$

where D is the domain i.e.

a) $D_1=\{(r,\theta):R_0\le r<R_1,\theta_0<\theta<\theta_1,0<\theta<2\pi\}$
b) $D_2=\{(r,\theta)R_0\le r<R_1,0\le\theta\le 2\pi\}$

Assume that there are m points along r direction and n points along θ direction to form mesh points and let the step size along the direction of r be $\Delta r and along$ the direction of θ be $\Delta\theta$,then

$$r_i = r_0 + i\Delta r, \quad \theta_j = \theta_0 + j\Delta\theta \qquad (4.6)$$

If r=0 is an interior point or boundary point, the numerical solution of this equation because of the factors $\frac{1}{r}$ and $\frac{1}{r^2}$ in the equation needs special attention. So we develop a Crank-Nicolson approximation scheme when $r \neq 0$ and when $r = 0$ is an interior point or a boundary point, we take a different approach to use a finite difference approximation scheme. to solve the result large algebraic system of linear equations. when $r = 0$ in the equation(4.4),the heat equation is singular and to obtain the solution we need a difference equation which is valid at this point.
As $r \to 0$,by L.Hospital rule.

$$\lim_{r\to 0}\frac{u_r}{r}=\lim_{r\to 0}\frac{d(u_r)}{d(r)}=u_{rr}\quad and\quad \lim_{r\to 0}\frac{u_{\theta\theta}}{r^2}=\lim_{r\to 0}\frac{d(u_{\theta\theta})}{d(r^2)}=\quad \frac{U_{rr\theta\theta}}{2} \tag{4.7}$$

substituting equation(4.7) in to (4.5) we get,

$$2u_t = 4u_{rr} + u_{rr\theta\theta} \tag{4.8}$$

By using Crank-Nicolson approximation scheme, we have the derivative of the given equation is replaced by average central -difference approximation on nth and(n+1)th times rows.

$$2u_t \approx 2(\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}) \qquad 4u_{rr} \approx 4(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}+u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{2\left(\Delta r\right)^2})$$

$$u_{rr\theta\theta} \approx \left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{(\Delta r)^2}\right)_{\theta\theta} \tag{4.9}$$

$$\sin ce \qquad r=0, \quad \Rightarrow U_r = 0$$

$$\Rightarrow U_r = \frac{U_{i+1j}^{n}-U_{i-1j}^{n}}{2\Delta r}=0 \quad \Rightarrow U_{i+1j}^{n}=U_{i-1j}^{n}$$

substituting, the above in to equation (4.9) we get,

$$u_{rr\theta\theta} \approx 2\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}}{\left(\Delta r\right)^2}\right)_{\theta\theta}$$

then equation(4.5) can be written us

$$2\left(\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}\right)=4\left(\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}+u_{i+1,j}^{n}-u_{i,j}^{n}}{\left(\Delta r\right)^2}\right)+2\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}}{\left(\Delta r\right)^2}\right)_{\theta\theta} \tag{4.10}$$

Now let apply Crank-Nicolson approximation.

$$U_{i+1,j}^{n} \approx \frac{U_{i+1,j+1}^{n+1}-2U_{i+1,j}^{n+1}+U_{i+1.j-1}^{n+1}}{2(\Delta\theta)^2}+\frac{U_{i+1,j+1}^{n}-2U_{i+1,j}^{n}+U_{i+1,j-1}^{n}}{2(\Delta\theta)^2}$$

$$U_{i,j}^{n} \approx \frac{U_{i,j+1}^{n+1} - 2U_{i.j}^{n+1} + U_{i,j-1}^{n+1}}{2(\Delta\theta)^2} + \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{2(\Delta\theta)^2} \tag{4.11}$$

then substituting (4.11)in to equation (4.10) and multiplying both sides by $\frac{\Delta t}{2}$ we get,

$$U_{i,j}^{n+1} - U_{i,j}^{n} = \frac{4\Delta t}{(\Delta r)^2}\left(U_{i+1,j}^{n+1} - U_{i,j}^{n+1} + U_{i+1,j}^{n} - U_{i,j}^{n}\right) +$$
$$\frac{\Delta t}{(\Delta r)^2(\Delta\theta)^2}\left(U_{i+1,j+1}^{n+1} - 2U_{i+1,j}^{n+1} + U_{i+1,j-1}^{n+1} + U_{i+1,j+1}^{n} - 2U_{i+1,j}^{n} + U_{i+1,j-1}^{n} - U_{i,j+1}^{n+1} + 2U_{i,j}^{n+1} - U_{i,j-1}^{n+1} - U_{i,j+1}^{n} + 2U_{i,j}^{n} - U_{i,j-1}^{n}\right)$$

Let $\frac{4\Delta t}{(\Delta r)^2} = a_0$ *and* $\frac{\Delta t}{(\Delta r)^2(\Delta\theta)^2} = b_0$ the above equation become,

$$U_{i,j}^{n+1} - U_{i,j}^{n} = a_0\left(U_{i+1,j}^{n+1} - U_{i,j}^{n+1} + U_{i+1,j}^{n} - U_{i,j}^{n}\right) +$$
$$b_0\left(U_{i+1,j+1}^{n+1} - 2U_{i+1,j}^{n+1} + U_{i+1,j-1}^{n+1} + U_{i+1,j+1}^{n} - 2U_{i+1,j}^{n} + U_{i+1,j-1}^{n} - U_{i,j+1}^{n+1} + 2U_{i,j}^{n+1} - U_{i,j-1}^{n+1} - U_{i,j+1}^{n} + 2U_{i,j}^{n} - U_{i,j-1}^{n}\right)$$

collecting like terms and rearranging we obtain

$$\left(a_0 - 2b_0\right)U_{1,j}^{n+1} + \left(2b_0 - a_0 - 1\right)U_{0.j}^{n+1} + b_0\left(U_{1,j+1}^{n+1} + U_{0,j-1}^{n+1}\right) - b_0\left(U_{0,j+1}^{n+1} + U_{0,j-1}^{n+1}\right) \tag{4.12}$$
$$+\left(2b_0 - a_0 + 1\right)U_{0,j}^{n} + \left(a_0 - 2b_0\right)U_{1,j}^{n} + b_0\left(U_{1,j+1}^{n} + U_{1,j-1}^{n}\right) - b_0\left(U_{0,i+1}^{n} + U_{0,j-1}^{n}\right) = 0$$

combining equation (4.4) for r ≠ 0, and equation(4.12) for r = 0 is interi pointor boundary pint

we obtain,

$$\begin{cases} b(U_{i+1,j}^{n+1} + U_{i-1,j}^{n+1}) + \frac{c}{r_i^2}(U_{i,j+1}^{n+1} + U_{i,j-1}^{n+1}) + (-1 - 2b - \frac{2c}{r_i^2})U_{i,j}^{n+1} + \frac{a}{r_i}(U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}) \\ +(1 - 2b - \frac{2c}{r_i^2})U_{i,j}^{n} + \frac{c}{r_i^2}(U_{i,j+1}^{n} + U_{i,j-1}^{n}) + b(U_{i+1j}^{n} + U_{i-1,j}^{n}) + \frac{a}{r_i}(U_{i+1j}^{n} - U_{i-1j}^{n}) = 0 \\ (a_0 - 2b_0)U_{1,j}^{n+1} + (2b_0 - a_0 - 1)U_{O,j}^{n+1} + b_0(U_{1,j+1}^{n+1} + U_{0,j-1}^{n+1}) - b_0(U_{0,j+1}^{n+1} + U_{0,j-1}^{n+1}) \\ +(2b_0 - a_0 + 1)U_{0,j}^{n} + (a_0 - 2b_0)U_{1,j}^{n} + b_0(U_{1,j+1}^{n} + U_{1,j-1}^{n}) - b_0(U_{0,j+1}^{n} + U_{0,j+1}^{n}) = 0 \end{cases} \tag{4.13}$$

equation(4.13)can be put in matrix form as follow.

AU=B (4.14)

$$A = \begin{bmatrix} R_0 & S_0 & & & & \\ & R & S & & & \\ & & R & S & & \\ & & & \cdots & \cdots & \\ & & & & R & S \\ & & & & & \end{bmatrix} \qquad (4.15)$$

Matrix A has P+1 blocks and each blocks is order of M by N.

$$R_0 = \begin{bmatrix} B & C & & & & \\ C & B & C & & & \\ & C & B & C & & \\ & & C & B & C & \\ & & & & \cdots & \\ & & & & C & B \end{bmatrix}$$

$$B = diag(2b_0 - a_0 - 1 \quad 2b_0 - a_0 - 1 \quad 2b_0 - a_0 - 1 \ldots \; 2b_0 - a_0 - 1)$$
$$C = diag(1 + 2b_0 - a_0 \quad 1 + 2b_0 - a_0 \quad 1 + 2b_0 - a_0 \quad . \; . \; . \quad 1 + 2b_0 - a_0)$$

$S_0 = diag(D \quad D \quad D \quad \ldots D)$ where D is given by

$$D = \begin{bmatrix} a_0 - 2b_0 & b_0 & & & & \\ b_0 & a_0 - 2b_0 & b_0 & & & \\ & b_0 & a_0 - 2b_0 & b_0 & & \\ & & b_0 & a_{0-} 2b_0 & b_0 & \\ & & & & \cdots & \\ & & & & a_0 & a_0 - 2b_0 \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 & R_2 & & & & \\ R_2 & R_1 & R_2 & & & \\ & R_2 & R_1 & R_2 & & \\ & & R_2 & R_1 & R_2 & \\ & & & & \cdots & \\ & & & & R_2 & R_1 \end{bmatrix} S = \begin{bmatrix} S_1 & S_2 & & & & \\ S_2 & S_1 & S_2 & & & \\ & S_2 & S_1 & S_2 & & \\ & & S_2 & S_1 & S_2 & \\ & & & & \cdots & \\ & & & & S_2 & S_1 \end{bmatrix} \qquad (4.16)$$

R and S have N blocks and each blocks is order MxM .

where

$$S_1 = \begin{bmatrix} -1-2b-\frac{2c}{r_i^2} & b & & & & \\ b & -1-2b-\frac{2c}{r_i^2} & b & & & \\ & b & -1-2b-\frac{2c}{r_i^2} & b & & \\ & & b & -1-2b-\frac{2c}{r_i^2} & b & \\ & & & \dots & \dots & \dots \\ & & & & b & -1-2b-\frac{2c}{r_i^2} \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1-2b-\frac{2c}{r_i^2} & b & & & \\ b & 1-2b-\frac{2c}{r_i^2} & b & & \\ & b & 1-2b-\frac{2c}{r_i^2} & b & \\ & & b & 1-2b-\frac{2c}{r_i^2} & b \\ & & & \dots & \dots \\ & & & b & 1-2b-\frac{2c}{r_i^2} \end{bmatrix}$$

$$R_2 = S_2 = \begin{bmatrix} \frac{c}{r_i^2} & & & & \\ & \frac{c}{r_i^2} & & & \\ & & \frac{c}{r_i^2} & & \\ & & & \dots & \\ & & & & \frac{c}{r_i^2} \end{bmatrix} \quad (4.17)$$

Where R_1 *and* S_1 are square matrixes of order M and R_2 *and* S_2 are diagonal matrix of order M.

$$U = (U_1\ U_2\ \ ...U_{P-1}\ U_P)^T \text{ and } B = (B_1\ B_2\ \ ...B_{p-1}\ B_p)^T$$

where $U_n = [U_{1n} U_{2n}\ U_{3n}...U_{Nn}]^T$ *and* $U_{jn} = [U_{1jn}\ U_{2jn}\ U_{3jn} ... U_{mjn}]^T$

$$B_n = (d_{1n}\, d_{2n}\, d_{3n}...d_{Nn})^T \text{ and } \quad d_{jn} = (d_{1jn}\, d_{2jn}\, d_{3jn} ...d_{mjn})^T$$

where n=1 2 3 . . .p. U is the known column vector such that each d_{ijn} represents known boundary values of U.

Thus equation(4.16) can be written as

$$\begin{bmatrix} R_0 & S_0 & & & & \\ & R & S & & & \\ & & R & S & & \\ & & & ... & ... & \\ & & & & R & S \\ & & & & & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ ... \\ U_p \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ ... \\ B_P \end{bmatrix} \tag{4.18}$$

Equation (4.18) again can be written as

$$\begin{aligned} &R_0U_1 + S_0U_2 = B_1 \\ &RU_2 + SU_3 = B_2 \\ &RU_3 + SU_4 = B_3 \\ &\quad ... \\ &RU_{p-1} + SU_p = B_p \end{aligned} \tag{4.19}$$

We obtain the solution of the system of linear equations(4.19)by applying extended Hockney's method to two dimensional heat equations.

4.2.Extended Hockney's Method

As we can see the matrix R_1 *and* S_1 are areal diagonal symmetric matrix and hence the Eigen value and eigenvectors can easily be obtained.

Theorem; The eigenvectors of matrix R_1 and S_1 with eigenvalues η_i *and* τ_i respectively given by ; Smith G.D.(1985)

$$\eta_i = -1 - 2b - \frac{2c}{r_i^2} + 2b\cos\left(\frac{i\pi}{M+1}\right) \text{ and}$$
$$\tau_i = 1 - 2b - \frac{2c}{r_i^2} + 2b\cos\left(\frac{i\pi}{M+1}\right) \qquad \text{i= 1, 2, 3, . . . M}$$

Let q_i be an eigenvector of S_1 *and* R_1 corresponding the eigenvalues of η_i *and* τ_i respectively. and $= Q = [q_1\ q_2\ q_3\ . . .\ q_n\ ]$ be the modal matrix of the matrices S_1 and R_1 of order M such that,

$$Q^T Q = I$$
$$Q^T S_1 Q = \text{diag}(\eta_1 \eta_2\ \eta_3\ . . .\ \eta_M) = (\text{say})H$$
$$Q^T R_1 Q = \text{diag}\left(\tau_1\ \tau_2\ \tau_3\ . . .\ \tau_M\right) = (\text{say})T \qquad (4.20)$$
$$Q^T R_2 Q = R_2 \text{ since } R_2 \text{ is a diagonal matrix and}$$
$$Q^T S_2 Q = S_2 \sin ce\ S_2\ is\, adiagonal\ matrix.$$

The MxM modal matrix Q is defined by,

$$q_{ij} = \sqrt{\frac{2}{M+1}}\sin\left(\frac{ij\pi}{M+1}\right) \text{i , j =1,2,3,. . . M} \ .$$

Let $\Omega = \text{diag}(Q\ Q\ \ldots Q)$ be a matrix of order MxN. Thus,

Ω satsfies $\Omega^T \Omega = I$

$$\Omega^T R\, \Omega = \begin{bmatrix} T & R_2 & & & & \\ R_2 & T & R_2 & & & \\ & R_2 & T & R_2 & & \\ & & \ldots & \ldots & \ldots & \\ & & & R_2 & T & R_2 \\ & & & & R_2 & T \end{bmatrix} = R^*$$

$$\Omega^T S\, \Omega = \begin{bmatrix} H & S_2 & & & & \\ S_2 & H & S_2 & & & \\ & S_2 & H & S_2 & & \\ & & \ldots & \ldots & \ldots & \\ & & & S_2 & H & S_2 \\ & & & & S_2 & H \end{bmatrix} = S^* \tag{4.21}$$

Let $\Omega^T U_k = V_k \Rightarrow U_k = \Omega V_k$, $\Omega^T \mathrm{B}_k = B_k^{'} \Rightarrow \mathrm{B}_k = \Omega B_k^{'}$

where $V_n = \left[V_{1n}\ V_{2n}\ \ldots V_{Nn}\right]^T$ *and* $V_{jn} = \left[V_{1jn}\ \ V_{2jn}\ \ldots\ V_{Mjn}\right]^T$

$B_n^{'} = \left[b_{1n} b_{2n} \ldots b_{Nn}\right]^T$ and $B_{jn}^{'} = \left[b_{1jn}\ b_{2jn} \ldots b_{Mjn}\right]^T$

Consider the equation (4.21)can be written in matrix form as,

$$\begin{bmatrix} R^* & S* & & \\ & R^* & S^* & \\ & & \ldots & \ldots \\ & & & R^* \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \ldots \\ V_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix} \tag{4.22}$$

pre multiplying(4.18) by using (4.22) we get,

$$\begin{aligned} &R^*V_1 + S^*V_2 = B_1^{'} \\ &R^*V_2 + S^*V_3 = B_2^{'} \\ &R^*V_3 + S^*V_4 = B_3^{'} \\ &\qquad \ldots \\ &R^*V_{p-1} + S^*V_p = B_p^{'} \end{aligned} \tag{4.23}$$

consider the first equation of(4.23)i.e. $R^*V_1 + S^*V_2 = B_1^{'}$ which can be written as,

$$\begin{bmatrix} T & R_2 & & & & \\ R_2 & T & R_2 & & & \\ & R_2 & T & R_2 & & \\ & & & T & & \\ & & R_2 & & \ldots & R_2 \\ & & & \ldots. & & \\ & & & R_2 & T & R_2 \\ & & & & R_2 & T \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \\ \ldots \\ V_{(N-1)1} \\ V_{N1} \end{bmatrix} + \begin{bmatrix} H & S_2 & & & & \\ S_2 & H & S_2 & & & \\ & S_2 & H & S_2 & & \\ & & & \ldots & & \\ & & & S_2 & H & S_2 \\ & & & & S_2 & H \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \\ V_{32} \\ \ldots \\ V_{(N-1)2} \\ V_{N2} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \ldots \\ b_{(N-1)1} \\ b_{N1} \end{bmatrix} \tag{4.24}$$

again we can write equation (4.24)as follow.

$$\begin{aligned} &TV_{11} + R_2V_{21} + HV_{12} + S_2V_{22} = b_{11} \\ &R_2V_{11} + TV_{21} + R_2V_{31} + S_2V_{12} + HV_{22} + S_2V_{32} = b_{21} \\ &R_2V_{21} + TV_{31} + R_2V_{41} + S_2V_{22} + HV_{32} + S_2V_{32} = b_{31} \\ &\qquad\qquad \ldots \\ &R_2V_{(N-1)1} + TV_{N1} + S_2V_{(N-1)2} + HV_{N2} = b_{N1} \end{aligned} \tag{4.25}$$

Now collecting the first equations from each equation(4.25) and consider as one group of equation.

$$\begin{aligned} &\tau_1V_{111} + \frac{c}{r_i^2}V_{121} + \eta_1V_{112} + \frac{c}{r_i^2}V_{122} = b_{111} \\ &\frac{c}{r_i^2}V_{111} + \tau_1V_{121} + \frac{c}{r_i^2}V_{131} + \frac{c}{r_i^2}V_{112} + \eta_1V_{122} + \frac{c}{r_i^2}V_{132} = b_{121} \\ &\frac{c}{r_i^2}V_{121} + \tau_1V_{131} + \frac{c}{r_i^2}V_{141} + \frac{c}{r_i^2}V_{122} + \eta_1V_{132} + \frac{c}{r_i^2}V_{142} = b_{131} \\ &\qquad\qquad \cdots \\ &\frac{c}{r_i^2}V_{1(N-1)1} + \tau_1V_{1N1} + \frac{c}{r_i^2}V_{1(N-1)2} + \eta_1V_{1N2} = b_{1N1} \end{aligned} \tag{4.25a}$$

again we collecting the second equation from each (4.25) and consider as second group of equations.

$$
\begin{aligned}
&\tau_2 V_{211} + \frac{c}{r_i^2} V_{221} + \eta_2 V_{212} + \frac{c}{r_i^2} V_{222} = b_{211} \\
&\frac{c}{r_i^2} V_{211} + \tau_2 V_{221} + \frac{c}{r_i^2} V_{231} + \frac{c}{r_i^2} V_{212} + \eta_2 V_{222} + \frac{c}{r_i^2} V_{232} = b_{221} \\
&\frac{c}{r_i^2} V_{221} + \tau_2 V_{231} + \frac{c}{r_i^2} V_{241} + \frac{c}{r_i^2} V_{222} + \eta_2 V_{232} + \frac{c}{r_i^2} V_{232} + \frac{c}{r_i^2} V_{243} = b_{231} \\
&\qquad \cdots \qquad \cdots \qquad \cdots \\
&\frac{c}{r_i^2} V_{2(N-1)1} + \tau_2 V_{2N1} + \frac{c}{r_i^2} V_{2(N-1)2} + \eta_2 V_{2N2} = b_{2N1}
\end{aligned}
\tag{4.25b}
$$

collect the last equations from each equations of(4.25) and consider as a last group of equations.

$$
\begin{aligned}
&\tau_M V_{M11} + \frac{c}{r_i^2} V_{M21} + \eta_M V_{M12} + \frac{c}{r_i^2} V_{M22} = b_{M11} \\
&\frac{c}{r_i^2} V_{M11} + \tau_M V_{M21} + \frac{c}{r_i^2} V_{M31} + \frac{c}{r_i^2} V_{M31} + \frac{c}{r_i^2} V_{M12} + \eta_M V_{M22} + \frac{c}{r_i^2} V_{M32} = b_{M21} \\
&\frac{c}{r_i^2} V_{M21} + \tau_M V_{M31} + \frac{c}{r_i^2} V_{M41} + \frac{c}{r_i^2} V_{M22} + \eta_M V_{M32} + \frac{c}{r_i^2} V_{M42} = b_{M31} \\
&\frac{c}{r_i^2} V_{M(N-1)1} + \tau_M V_{M31} + \frac{c}{r_i^2} V_{M(N-1)2} + \eta_M V_{MN2} = b_{MN1}
\end{aligned}
\tag{4.25c}
$$

Equation (4.25a) to (4.25c) can be written in matrix form as follow,

$$
\begin{bmatrix}
\tau_i & \frac{c}{r_i^2} & & & \\
\frac{c}{r_i^2} & \tau_i & \frac{c}{r_i^2} & & \\
& \frac{c}{r_i^2} & \tau_i & \frac{c}{r_i^2} & \\
& & \cdots & \cdots & \\
& & & \frac{c}{r_i^2} & \tau_i
\end{bmatrix}
\begin{bmatrix} V_{i11} \\ V_{i21} \\ V_{i31} \\ \cdots \\ V_{iN1} \end{bmatrix}
+
\begin{bmatrix}
\eta_i & \frac{c}{r_i^2} & & & \\
\frac{c}{r_i^2} & \eta_i & \frac{c}{r_i^2} & & \\
& \frac{c}{r_i^2} & \eta_i & \frac{c}{r_i^2} & \\
& & \cdots & \cdots & \\
& & & \frac{c}{r_i^2} & \eta_i
\end{bmatrix}
\begin{bmatrix} V_{i12} \\ V_{i22} \\ V_{i32} \\ \cdots \\ V_{iN2} \end{bmatrix}
=
\begin{bmatrix} b_{i11} \\ b_{i21} \\ b_{i21} \\ \cdots \\ b_{iN1} \end{bmatrix}
\tag{4.26}
$$

$$Let \quad F_i = \begin{bmatrix} \eta_i & \frac{c}{r_i^2} & & & & \\ \frac{c}{r_i^2} & \eta_i & \frac{c}{r_i^2} & & & \\ & \frac{c}{r_i^2} & \eta_i & \frac{c}{r_i^2} & & \\ & & \cdots & \cdots & \cdots & \\ & & & \frac{c}{r_i^2} & \eta_i & \frac{c}{r_i^2} \\ & & & & \frac{c}{r_i^2} & \eta_i \end{bmatrix} \qquad G_i = \begin{bmatrix} \tau_i & \frac{c}{r_i^2} & & & & \\ \frac{c}{r_i^2} & \tau_i & \frac{c}{r_i^2} & & & \\ & \frac{c}{r_i^2} & \tau_i & \frac{c}{r_i^2} & & \\ & & \cdots & \cdots & \cdots & \\ & & & \frac{c}{r_i^2} & \tau_i & \frac{c}{r_i^2} \\ & & & & \frac{c}{r_i^2} & \tau_i \end{bmatrix} \text{i=1,2,..M}$$

$$W_{ik} = \begin{bmatrix} V_{i1n} \\ V_{i2n} \\ V_{i3n} \\ \cdots \\ V_{iNn} \end{bmatrix} \qquad \overline{B_n} = \begin{bmatrix} b_{i1n} \\ b_{i2n} \\ b_{i3n} \\ \cdots \\ b_{iNk} \end{bmatrix}$$

we can write equation(4.25a),(4.25b) and (4.25c) as follow

$$F_i W_{i1} + G_i W_{i2} = \overline{B_{i1}} \tag{4.27}$$

$$Let\ F = \begin{bmatrix} F_i & & & & \\ & F_i & & & \\ & & F_i & & \\ & & & \cdots & \\ & & & & F_i \end{bmatrix} \quad and \quad G = \begin{bmatrix} G_i & & & & \\ & G_i & & & \\ & & G_i & & \\ & & & \cdots & \\ & & & & G_i \end{bmatrix}$$

$W_n = (W_{i1}\ W_{i2}\ .\ .\ .\ W_{ip})^T$ and $\overline{B_n} = (\overline{B_{i1}}\ \overline{B_{i2}}\ .\ .\ .\ \overline{B_{ip}})^T$

Thus the first equation of (4.23) be written as using a matrices F, G, W_n *and* $\overline{B_n}$ as

$$
\begin{aligned}
&FW_1 + GW_2 = \overline{B_1} \\
&FW_2 + GW_3 = \overline{B_2} \\
&FW_3 + GW_4 = \overline{B_3} \\
&\quad \cdots \\
&FW_{N-1} + GW_N = \overline{B_N}
\end{aligned} \tag{4.28}
$$

observe that

$\Omega^T F\Omega = diag(\phi_1\ \phi_2\ \ldots\ \phi_N) = X\,(\text{say})$ where $\phi_i = diag(\delta_1\ \delta_2\ \ldots\ \delta_M)$

$\Omega^T G\Omega = diag(\beta_1\ \beta_2\ \ \ldots\ \beta_N) = Z(say)$ where $Z_i = diag(\lambda_1\ \ \lambda_2\ \ \ldots\ \lambda_M)$

$$
\begin{aligned}
&Here \quad \delta_i \ = \ \eta_i + \frac{2c}{r_i^2}\cos\left(\frac{i\pi}{M+1}\right) \\
&\lambda_i = \tau_i + \frac{2c}{r_i^2}\cos\left(\frac{i\pi}{M+1}\right) \quad i = 1(1)M \\
&Let \quad \Omega^T W_n = \psi_n \Rightarrow \ W_n = \Omega\psi_n \\
&\Omega^T \overline{B_k} = \Gamma_k \Rightarrow \overline{B_k} = \Omega\Gamma_k \\
&\text{where} \quad \psi_n = (\psi_{1n}\ \psi_{2n} \ldots\ \psi_{Mn})^T \ and \ \ \psi_{jn} = (\psi_{1jn}\psi_{2jn} \ldots \psi_{Mjn})^T \\
&\Gamma_n = (\alpha_{1n}\ \alpha_{2n}\ \ \ldots\ \ \alpha_{jn})^T \ \ and \ \ \Gamma_{jn} = (\alpha_{1jn}\ \ \alpha_{2jn}\ \ \ldots \alpha_{Mjn})^T
\end{aligned} \tag{4.29}
$$

Now pemultipling equation (4.28) by Ω^T and make use of (4.29)we get,

$$
\begin{aligned}
&X\psi_1 + Z\psi_2 = \Gamma_1 \\
&X\psi_2 + Z\psi_3 = \Gamma_2 \\
&X\psi_3 + Z\psi_4 = \Gamma_3 \\
&\quad \cdots \\
&X\psi_p + Z\psi_p = \Gamma_p
\end{aligned} \tag{4.30}
$$

Now we write those sets of equations (4.30)turn by turn starting from the first row,i.e

$$X\psi_1 + Z\psi_2 = \Gamma_1 \ as,$$

$$
\begin{aligned}
&\delta_1\psi_{111} + \lambda_1\psi_{112} = \alpha_{111}\\
&\delta_2\psi_{211} + \lambda_2\psi_{212} = \alpha_{211}\\
&\delta_3\psi_{311} + \lambda_3\psi_{312} = \alpha_{311}\\
&\qquad \cdots\\
&\delta_M\psi_{M11} + \lambda_M\psi_{M12} = \alpha_{M11}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1\psi_{121} + \lambda_1\psi_{122} = \alpha_{121}\\
&\delta_2\psi_{221} + \lambda_2\psi_{222} = \alpha_{221}\\
&\delta_3\psi_{321} + \lambda_3\psi_{322} = \alpha_{321}\\
&\qquad \cdots\\
&\delta_M\psi_{M21} + \lambda_M\psi_{M22} = \alpha_{M21}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1\psi_{1N1} + \lambda_1\psi_{1N2} = \alpha_{1N1}\\
&\delta_2\psi_{2N1} + \lambda_2\psi_{2N2} = \alpha_{2N1}\\
&\delta_3\psi_{3N1} + \lambda_3\psi_{3N2} = \alpha_{3N1}\\
&\qquad \cdots\\
&\delta_M\psi_{MN1} + \lambda_M\psi_{MN2} = \alpha_{MN1}
\end{aligned}
\tag{4.30a}
$$

for the second of equation (4.30) *i.e,* $Z\psi_2 + X\psi_3 = \Gamma_2$ we get the second group of equations.

$$
\begin{aligned}
&\delta_1\psi_{112} + \lambda_1\psi_{113} = \alpha_{112}\\
&\delta_2\psi_{212} + \lambda_2\psi_{213} = \alpha_{212}\\
&\delta_3\psi_{312} + \lambda_3\psi_{313} = \alpha_{312}\\
&\qquad \cdots\\
&\delta_M\psi_{M12} + \lambda_M\psi_{M13} = \alpha_{M12}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1\psi_{122} + \lambda_1\psi_{123} = \alpha_{122}\\
&\delta_2\psi_{222} + \lambda_2\psi_{223} = \alpha_{222}\\
&\delta_3\psi_{322} + \lambda_3\psi_{322} = \alpha_{322}\\
&\qquad \cdots\\
&\delta_M\psi_{M22} + \lambda_M\psi_{M23} = \alpha_{M22}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_{1N2} + \lambda_1 \psi_{1N3} = \alpha_{1N2} \\
&\delta_2 \psi_{2N2} + \lambda_2 \psi_{2N3} = \alpha_{2N2} \\
&\delta_3 \psi_{3N2} + \lambda_3 \psi_{3N3} = \alpha_{3N2} \\
&\qquad \cdots \\
&\delta_M \psi_{MN2} + \lambda_M \psi_{MN2} = \alpha_{MN2}
\end{aligned} \tag{4.30b}
$$

For the last equation of (4.30),i.e. $Z\psi_{p-1} + X\psi_p = \Gamma_p$ we obtain

$$
\begin{aligned}
&\delta_1 \psi_{11p-1} + \lambda_1 \psi_{11p} = \alpha_{11p} \\
&\delta_2 \psi_{21p-1} + \lambda_2 \psi_{21p} = \alpha_{21p} \\
&\delta_3 \psi_{31p-1} + \lambda_3 \psi_{31p} = \alpha_{31p} \\
&\cdots \qquad \cdots \qquad \cdots \\
&\delta_M \psi_{M1p-1} + \lambda_M \psi_{M1p} = \alpha_{M1p}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1 \psi_{12p-1} + \lambda_1 \psi_{12p} = \alpha_{12p} \\
&\delta_2 \psi_{22p-1} + \lambda_2 \psi_{22p} = \alpha_{22p} \\
&\delta_3 \psi_{32p-1} + \lambda_3 \psi_{32p} = \alpha_{32p} \\
&\cdots \qquad \cdots \qquad \cdots \\
&\delta_M \psi_{M2p-1} + \lambda_M \psi_{M2p} = \alpha_{M2p}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1 \psi_{1Np-1} + \lambda_1 \psi_{1Np} = \alpha_{1Np} \\
&\delta_2 \psi_{2Np-1} + \lambda_2 \psi_{2Np} = \alpha_{2Np} \\
&\delta_3 \psi_{3Np-1} + \lambda_3 \psi_{3Np} = \alpha_{3Np} \\
&\cdots \qquad \cdots \qquad \cdot\,\cdot \\
&\delta_M \psi_{MNp-1} + \lambda_M \psi_{MNp} = \alpha_{MNp}
\end{aligned} \tag{4.30c}
$$

Now from each set of equation of(4.30a)to(4.30c),we select the first equations from(4.30a)(4.30b)and(4.30c) and put together as one group of equation, again we take the second equations from each of(4.30a),(4.30b)and(4.30c) and put together as a second group of equations; consider the third equations and put together as a third group of equations and so on and finally we consider the last equations and put together .In doing these we obtain the following sets of equations, each set being of order p .

$$
\begin{aligned}
&\delta_1\psi_{111} + \lambda_1\psi_{112} = \alpha_{111} \\
&\delta_1\psi_{112} + \lambda_1\psi_{113} = \alpha_{112} \\
&\delta_1\psi_{113} + \lambda_1\psi_{114} = \alpha_{113} \\
&\quad \cdots \\
&\delta_1\psi_{11p-1} + \lambda_1\psi_{11p} = \alpha_{11p}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_2\psi_{211} + \lambda_2\psi_{212} = \alpha_{121} \\
&\delta_2\psi_{212} + \lambda_2\psi_{213} = \alpha_{212} \\
&\delta_2\psi_{213} + \lambda_2\psi_{214} = \alpha_{213} \\
&\quad \cdots \\
&\delta_2\psi_{21p-1} + \lambda_2\psi_{21p} = \alpha_{21p}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_M\psi_{M11} + \lambda_M\psi_{M12} = \alpha_{M11} \\
&\delta_M\psi_{M12} + \lambda_M\psi_{M13} = \alpha_{M12} \\
&\delta_M\psi_{M13} + \lambda_M\psi_{M14} = \alpha_{M13} \\
&\quad \cdots \\
&\delta_M\psi_{M1P-1} + \lambda_M\psi_{M1P} = \alpha_{M1P}
\end{aligned}
\qquad (4.31)
$$

Observe that the above set of equations (4.31),for j=1 ,and for i=1(1)M the coofficient matrix of the left hand side is main diagonal and sub diagonal matrix of order p.and has the form of

contineouning for the other groups of equations as above for j=2,3,...N, we get

$$
\begin{aligned}
&\delta_1\psi_{1j1} + \lambda_1\psi_{1j2} = \alpha_{1j1} \\
&\delta_1\psi_{1j2} + \lambda_1\psi_{1j3} = \alpha_{1j2} \\
&\delta_1\psi_{1j3} + \lambda_1\psi_{1j4} = \alpha_{1j3} \\
&\quad \cdots \\
&\delta_1\psi_{1jp-1} + \lambda_1\psi_{1jp} = \alpha_{1jp}
\end{aligned}
$$

$$
\begin{aligned}
&\delta_2\psi_{2j1} + \lambda_2\psi_{2j2} = \alpha_{2j1} \\
&\delta_2\psi_{2j2} + \lambda_2\psi_{2j3} = \alpha_{2j2} \\
&\delta_2\psi_{2j3} + \lambda_2\psi_{2j4} = \alpha_{2j3} \\
&\quad \cdots \\
&\delta_2\psi_{2jp-1} + \lambda_2\psi_{2jp} = \alpha_{2jp}
\end{aligned}
$$

$$\begin{aligned}
&\delta_M \psi_{Mj1} + \lambda_M \psi_{Mj2} = \alpha_{Mj1} \\
&\delta_M \psi_{Mj2} + \lambda_M \psi_{Mj3} = \alpha_{Mj2} \\
&\delta_M \psi_{Mj3} + \lambda_M \psi_{Mj4} = \alpha_{Mj3} \\
&\quad \cdots \\
&\delta_M \psi_{MjP-1} + \lambda_M \psi_{MjP} = \alpha_{MjP}
\end{aligned} \tag{4.32}$$

In this case also for each j=2,3,...N the cooefficient matrix of the left hand side (4.32) is main diagonal and sub diagonal matrix similar to that (4.31) for j=1

$$L_i = \begin{bmatrix} \lambda_i & & & & & \\ \delta_i & \lambda_i & & & & \\ & \delta_i & \lambda_i & & & \\ & & \delta_i & \lambda_i & & \\ & & & \cdots & \cdots & \\ & & & & \delta_i & \lambda_i \end{bmatrix}, \quad i = 1(1)M,\ j = 2,3,\ldots N \tag{4.33}$$

We can easly observe that equation (4.19)reduced to the matrix (4.33)which is main diagonal and sub diagonal matrix for j=1,2,3,...N and hence we solve this sets of equations (4.33) for ψ_{ij} by the use of Thomas algorithm.. Once after getting each ψ_{ij} and hence ψ_n by the help of (4.28) we get W_n and again

by the help of (4.23) we obtain U_n and this means that each U_{ijn} are obtained .

4.3 Convergence Analysis

4.3.1 Truncation Error.

Consider the Heat equation in the polar form,

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} \Rightarrow \frac{\partial U}{\partial t} - \left[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right] = 0 \tag{4.34}$$

by Crank-Nicolson approximation

$$\left[\begin{array}{l}\dfrac{U_{ij}^{n+1}-U_{ij}^{n}}{\Delta t}-\dfrac{1}{2}\left[\dfrac{U_{i+1}^{n+1}-2U_{ij}^{n}+U_{i-1j}^{n}+U_{i+1j}^{n}-2U_{ij}^{n}+U_{i-1j}^{n}+}{(\Delta r)^2}\right]+\dfrac{1}{2r}\left[\dfrac{U_{i+1j}^{n+1}-U_{i-1j}^{n+1}}{2\Delta r}+\dfrac{U_{i+1j}^{n}-U_{i-1j}^{n}}{2\Delta r}\right]\\ \qquad +\dfrac{1}{r^2}\left[\dfrac{U_{ij+1}^{n+1}-2U_{ij}^{n+1}+U_{ij-1}^{n+1}+U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{(\Delta\theta)^2}\right]\end{array}\right]$$

Using Taylor's expansion we have the following**,**

$$U_{ij}^{n+1}=U_{ij}^{n}+\Delta t\frac{\partial(U_{ij}^{n})}{\partial t}+\frac{(\Delta t)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial t^2}+\frac{(\Delta t)^3}{6}\frac{\partial^3(U_{ij}^{n})}{\partial t^3}+\frac{(\Delta t)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial t^4}\ \ldots \tag{4.34.1}$$

$$\begin{aligned}U_{i-1j}^{n+1}&=U_{ij}^{n}-\Delta r\frac{\partial(U_{ij}^{n})}{\partial r}+\Delta t\frac{\left(\partial U^{n}{}_{ij}\right)}{\partial t}+\frac{(\Delta r)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial r^2}+\frac{(\Delta t)^2}{2}\frac{(\partial^2U_{ij}^{n})}{\partial t^2}-\Delta r\Delta t\frac{\partial^2(U_{ij}^{n})}{\partial r\partial t}\\&-\frac{(\Delta r)^2}{6}\frac{(\partial^3U_{ij}^{n})}{\partial r^3}+\frac{(\Delta t)^3}{6}\frac{(\partial^3U_{ij}^{n})}{\partial t^3}+\frac{(\Delta r)^2}{2}\Delta t\frac{(\partial^3U_{ij}^{n})}{\partial r^2\partial t}-\Delta r\frac{(\Delta t)^2}{2}\frac{\partial^3(U_{ij}^{n})}{\partial r^2\partial t}+\frac{(\Delta r)^2(\Delta t)^2}{4}\frac{\partial^4(U_{ij}^{n})}{\partial r^2\partial t^2}\\&-\frac{(\Delta t)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial t^4}-\frac{\Delta r(\Delta t)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial t^4}\ldots\end{aligned} \tag{3.34.2}$$

$$\begin{aligned}U_{ij+1}^{n+1}&=U_{ij}^{n}+\Delta\theta\frac{\partial(U_{ij}^{n})}{\partial\theta}+\Delta t\frac{\left(\partial U^{n}{}_{ij}\right)}{\partial t}+\frac{(\Delta\theta)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial\theta^2}+\frac{(\Delta t)^2}{2}\frac{(\partial^2U_{ij}^{n})}{\partial t^2}+\Delta\theta\Delta t\frac{\partial^2(U_{ij}^{n})}{\partial\theta\partial t}\\&+\frac{(\Delta\theta)^3}{6}\frac{(\partial^3U_{ij}^{n})}{\partial\theta^3}+\frac{(\Delta t)^3}{6}\frac{(\partial^3U_{ij}^{n})}{\partial t^3}+\frac{(\Delta\theta)^2}{2}\Delta t\frac{(\partial^3U_{ij}^{n})}{\partial\theta^2\partial t}+\Delta\theta\frac{(\Delta t)^2}{2}\frac{\partial^3(U_{ij}^{n})}{\partial\theta\partial t^2}+\frac{(\Delta\theta)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial\theta^4}+\frac{(\Delta t)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial t^4}+\\&\frac{(\Delta r)^2(\Delta\theta)^2}{4}\frac{\partial^4(U_{ij}^{n})}{\partial\theta^2\partial t^2}+\frac{(\Delta r)^3}{6}\Delta t\frac{\partial^4(U_{ij}^{n})}{\partial\theta^3\partial t}+\Delta r\frac{(\Delta t)^3}{6}\frac{\partial^4(U_{ij}^{n})}{\partial r\partial t^3}\ldots\end{aligned} \tag{4.34.3}$$

$$\begin{aligned}U_{ij-1}^{n+1}&=U_{ij}^{n}-\Delta\theta\frac{\partial(U_{ij}^{n})}{\partial\theta}+\Delta t\frac{\left(\partial U^{n}{}_{ij}\right)}{\partial t}+\frac{(\Delta\theta)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial\theta^2}+\frac{(\Delta t)^2}{2}\frac{(\partial^2U_{ij}^{n})}{\partial t^2}-\Delta\theta\Delta t\frac{\partial^2(U_{ij}^{n})}{\partial\theta\partial t}\\&-\frac{(\Delta\theta)^3}{6}\frac{(\partial^3U_{ij}^{n})}{\partial\theta^3}+\frac{(\Delta t)^3}{6}\frac{(\partial^3U_{ij}^{n})}{\partial t^3}+\frac{(\Delta\theta)^2}{2}\Delta t\frac{(\partial^3U_{ij}^{n})}{\partial\theta^2\partial t}-\Delta\theta\frac{(\Delta t)^2}{2}\frac{\partial^3(U_{ij}^{n})}{\partial\theta^2\partial t}+\frac{(\Delta\theta)^2(\Delta t)^2}{4}\frac{\partial^4(U_{ij}^{n})}{\partial\theta^2\partial t^2}\\&-\frac{(\Delta t)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial t^4}-\frac{\Delta\theta(\Delta t)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial t^4}\ldots\end{aligned} \tag{4.34.4}$$

$$U_{i+1j}^{n}=U_{ij}^{n}+\Delta r\frac{\partial(U_{ij}^{n})}{\partial r}+\frac{(\Delta r)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial r^2}+\frac{(\Delta r)^3}{6}\frac{\partial^3(U_{ij}^{n})}{\partial r^3}+\frac{(\Delta r)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial r^4}\ \ldots \tag{4.34.5}$$

$$U_{i-1j}^{n}=U_{ij}^{n}-\Delta r\frac{\partial(U_{ij}^{n})}{\partial r}+\frac{(\Delta r)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial r^2}-\frac{(\Delta r)^3}{6}\frac{\partial^3(U_{ij}^{n})}{\partial r^3}+\frac{(\Delta r)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial r^4}\ \ldots \tag{4.34.6}$$

$$U_{ij+1}^{n}=U_{ij}^{n}+\Delta\theta\frac{\partial(U_{ij}^{n})}{\partial\theta}+\frac{(\Delta\theta)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial\theta^2}+\frac{(\Delta\theta)^3}{6}\frac{\partial^3(U_{ij}^{n})}{\partial\theta^3}+\frac{(\Delta\theta)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial\theta^4}\ \ldots \tag{4.34.7}$$

$$U_{ij-1}^{n}=U_{ij}^{n}-\Delta\theta\frac{\partial(U_{ij}^{n})}{\partial\theta}+\frac{(\Delta\theta)^2}{2}\frac{\partial^2(U_{ij}^{n})}{\partial\theta^2}-\frac{(\Delta\theta)^3}{6}\frac{\partial^3(U_{ij}^{n})}{\partial\theta^3}+\frac{(\Delta\theta)^4}{24}\frac{\partial^4(U_{ij}^{n})}{\partial\theta^4}\ \ldots \tag{4.34.8}$$

substituting equation(4.34.1-4.36.8) in to equation 4.34 then gives,

$$\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial^2 U_{ij}^n}{\partial r^2}\right)+\frac{\Delta t}{2}\frac{\partial}{\partial t}\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial^2 U_{ij}^n}{\partial r^2}\right)+\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-\frac{(\Delta r)^2}{12}\left(\frac{\partial^4 U_{ij}^n}{\partial r^4}\right)+\frac{1}{r}\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial^2 U_{ij}^n}{\partial r^2}\right)+$$

$$+\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial U_{ij}^n}{\partial r}\right)+\frac{\Delta t}{2}\frac{\partial}{\partial t}\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial U_{ij}^n}{\partial r}\right)+\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-\frac{(\Delta r)^2}{12}\left(\frac{\partial U_{ij}^n}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial U_{ij}^n}{\partial r}\right)$$

$$+\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{\partial^2 U_{ij}^n}{\partial r^2}\right)+\frac{\Delta t}{2}\frac{\partial}{\partial t}\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{1}{r^2}\frac{\partial^2 U_{ij}^n}{\partial \theta^2}\right)+\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-\frac{1}{r^2}\frac{(\Delta\theta)^2}{12}\left(\frac{\partial^4 U_{ij}^n}{\partial \theta^4}\right)+\frac{1}{r}\left(\frac{\partial U_{ij}^n}{\partial t}-\frac{1}{r^2}\frac{\partial^2 U_{ij}^n}{\partial \theta^2}\right) \quad (3.35)$$

$$\Rightarrow\left[\frac{\partial U_{ij}^n}{\partial t}-\left(\frac{\partial^2 U_{ij}^n}{\partial r^2}+\frac{1}{r}\frac{\partial U_{ij}^n}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_{ij}^n}{\partial \theta^2}\right)\right]+\frac{\Delta t}{2}\frac{\partial}{\partial t}\left(\frac{\partial U_{ij}^n}{\partial t}-\left(\frac{\partial^2 U_{ij}^n}{\partial r^2}+\frac{1}{r}\frac{\partial U_{ij}^n}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_{ij}^n}{\partial \theta^2}\right)\right)+\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-$$

$$\left(\frac{(\Delta r)^2}{12}\frac{\partial^4 U_{ij}^n}{\partial r^4}\right)-\left(\frac{(\Delta\theta)^2}{12r^2}\frac{\partial^4 U_{ij}^n}{\partial \theta^4}\right)=\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-\left(\frac{(\Delta r)^2}{12r}\frac{\partial^4 U_{ij}^n}{\partial r^4}\right)-\left(\frac{(\Delta\theta)^2}{12r^2}\frac{\partial^4 U_{ij}^n}{\partial \theta^4}\right)+o((\Delta t)^3)+o((\Delta r)^3)+o((\Delta\theta)^3)$$

Hence the trunctional error is

$$TE=\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-\left(\frac{(\Delta r)^2}{12r}\frac{\partial^4 U_{ij}^n}{\partial r^4}\right)-\left(\frac{(\Delta\theta)^2}{12r^2}\frac{\partial^4 U_{ij}^n}{\partial \theta^4}\right) \quad (4.36)$$

4.3.2 Consistency

Finite difference approximation is consistent with partial differential equation if the difference between partial differential equation(i.e. the truncation error) vanishes as the size of the grid spacing go to zero.

From the above equation(4.36)

$$\lim_{\Delta t\to 0}\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 U_{ij}^n}{\partial t^3}\right)-\lim_{\Delta r\to 0}\left(\frac{(\Delta r)^2}{12r}\frac{\partial^4 U_{ij}^n}{\partial r^4}\right)-\lim_{\Delta\theta\to 0}\left(\frac{(\Delta\theta)^2}{12r^2}\frac{\partial^4 U_{ij}^n}{\partial \theta^4}\right)=0$$

Hence the scheme is consistency

4.4.3 Stability

A finite difference scheme is stable if the error stays constant or decreases as the iterative process is go on .contrary if the error growth with time ,the scheme is said to be unstable.

consider the two dimensional heat equation in polar coordinates system ,

by Crank-Nicolson approximation, we get,

$$U_{ij}^{n+1}-U_{ij}^{n}=\frac{\Delta t}{2\Delta r^{2}}\left(U_{i+1j}^{n+1}-2U_{ij}^{n+1}+U_{i-1j}^{n+1}+U_{i+1j}^{n}-2U_{ij}^{n}+U_{i-1j}^{n}\right)+\frac{\Delta t}{4\Delta rr}\left(U_{i+1j}^{n+1}-U_{i-1j}^{n+1}+U_{i+1j}^{n}+U_{i-1j}^{n}\right)$$
$$\frac{\Delta t}{2\Delta\theta^{2}r^{2}}\left(U_{ij+1}^{n+1}-2U_{ij}^{n+1}+U_{ij-1}^{n+1}+U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}\right)$$

Let $\frac{\Delta t}{2\Delta r^{2}}=r_1 \quad \frac{\Delta t}{4\Delta r\, r}= r_2 \quad and \quad \frac{\Delta t}{2r^{2}\Delta\theta^{2}}= r_3$

substituting in to the above ,we obtain,

$$-r_1\left(U_{i+1j}^{n+1}+U_{i-1j}^{n+1}\right)+(1+2r_1)U_{ij}^{n+1}-r_2\left(U_{i+1j}^{n+1}-U_{i-1j}^{n+1}\right)-\left(r_3(U_{ij+1}^{n+1}+U_{ij-1}^{n+1})\right)+2r_3U_{ij}^{n+1}=$$
$$r_1\left(U_{i+1j}^{n}+U_{i-1j}^{n}\right)+(1-2r_1)U_{ij}^{n}+r_2\left(U_{i+1j}^{n}-U_{i-1j}^{n}\right)+\left(r_3(U_{ij+1}^{n}+U_{ij-1}^{n})\right)-2r_3U_{ij}^{n} \quad (4.37)$$

Applying Van Neumann stability analysis,

$$-r_1\left(e^{i\xi}+e^{-i\xi}\right)U_{ij}^{n+1}+\left(1+2r_1\right)U_{ij}^{n+1}+\left(r_2(e^{i\alpha}+e^{-i\alpha})\right)U_{ij}^{n+1}+-r_3\left(e^{i\beta}+e^{i\beta}\right)U_{ij}^{n+1}+2r_3U_{ij}^{n+1}=$$
$$r_1\left(e^{i\xi}+e^{-i\xi}\right)U_{ij}^{n}+\left(1-2r_1\right)U_{ij}^{n}+\left(r_2(e^{i\alpha}+e^{-i\alpha})\right)U_{ij}^{n}+r_3\left(e^{i\beta}+e^{-i\beta}\right)U_{ij}^{n}-2r_3U_{ij}^{n} \quad (4.38)$$
$$\Rightarrow -r_1\left(\cos\xi+i\sin\xi+\cos\xi-i\sin\xi\right)U_{ij}^{n+1}+\left(1+2r_1\right)U_{ij}^{n+1}-r_2\left(\cos\alpha+i\sin\alpha-\cos\alpha+i\sin\alpha\right)U_{ij}^{n+1}$$
$$-r_3\left(\cos\beta+i\sin\beta+\cos\beta-i\sin\beta\right)+2r_3U_{ij}^{n+1}=r_1\left(\cos\xi+i\sin\xi+\cos\xi-i\sin\xi\right)U_{ij}^{n}+\left(1-2r_1\right)U_{ij}^{n}+$$
$$r_2\left(\cos\alpha+i\sin\alpha-\cos\alpha+i\sin\alpha\right)U_{ij}^{n}+r_3\left(\cos\beta+i\sin\xi+\cos\beta-i\sin\beta\right)-2r_3U_{ij}^{n}$$
$$\Rightarrow\left[1+4r_1\sin^2\frac{\xi}{2}+2r_2 i\sin^2\frac{\alpha}{2}+4r_3\sin^2\frac{\beta}{2}\right]U_{ij}^{n+1}=\left[1-4r_1\sin^2\frac{\xi}{2}-2r_2 i\sin^2\frac{\alpha}{2}-4r_3\sin^2\frac{\beta}{2}\right]U_{ij}^{n}$$
$$\left[1+4r_1\sin^2\frac{\xi}{2}+2r_2 i\sin^2\frac{\alpha}{2}+4r_3\sin^2\frac{\beta}{2}\right]U_{ij}^{n+1}=\left[1-4r_1\sin^2\frac{\xi}{2}-2r_2 i\sin^2\frac{\alpha}{2}-4r_3\sin^2\frac{\beta}{2}\right]U_{ij}^{n}$$
$$U_{ij}^{n+1}=\left(\frac{1-4r_1\sin^2\frac{\xi}{2}-2r_2 i\sin^2\frac{\alpha}{2}-4r_3\sin^2\frac{\beta}{2}}{1+4r_1\sin^2\frac{\xi}{2}+2r_2 i\sin^2\frac{\alpha}{2}+4r_3\sin^2\frac{\beta}{2}}\right)U_{ij}^{n}$$

$$\Rightarrow \rho = \left(\frac{1 - 4r_1 \sin^2\frac{\xi}{2} - 2r_2 i \sin^2\frac{\alpha}{2} - 4r_3 \sin^2\frac{\beta}{2}}{1 + 4r_1 \sin^2\frac{\xi}{2} + 2r_2 i \sin^2\frac{\alpha}{2} + 4r_3 \sin^2\frac{\beta}{2}} \right) \tag{4.39}$$

the above equation 3.39 is amplification factor, then stability satisfy that,

$$|\rho(\xi)| \le 1 \tag{4.40}$$

for uniform mesh point,M=N and h= $\frac{1}{N} = \frac{1}{M}$

$\xi = \beta = \alpha = \frac{N\pi h}{2} = \frac{M\pi h}{2} = \frac{N\pi}{2N} = \frac{M\pi}{2M} = \frac{\pi}{2}$

Then equation 4.39 above can be written as follow,

$$\Rightarrow -1 \le \left(\frac{1 - 4r_1 \sin^2\frac{\pi}{2} - 2r_2 \sin^2\frac{\pi}{2} - 4r_3 \sin^2\frac{\pi}{2}}{1 + 4r_1 \sin^2\frac{\pi}{2} + 2r_2 \sin^2\frac{\pi}{2} + 4r_3 \sin^2\frac{\pi}{2}} \right) \le 1$$

$\Rightarrow -1 \le \left(\frac{1 - 4r_1 - 2r_2 - 4r_3}{1 + 4r_1 + 2r_2 + 4r_3} \right) \le 1$ since all r_1, r_2 *and* r_3 are greater than zero.

Cranck-Nicolson scheme is unconditionally stable. A numerical scheme is convergent if the computed solution of the discretized equation leads to the exact solution of the differential equations as the time and grid spacing leads to zero. satisfying the following convergence conditions .

$\lim_{\Delta t, \Delta r, \Delta\theta \to 0} |\varepsilon_{ij}| \to 0$ a fixed $r_i = i\Delta r, \theta_j = j\Delta\theta$ and $t_n = n\Delta t$ where, ε_{ij} is an error.

Lax theorem; states that for a well posed initial and boundary value problems, if a finite difference scheme is consistence with the partial differential equations ,then the stability is the necessary and sufficient condition for convergence that is,

consistence + stability $\leftrightarrow$ convergence

$\Rightarrow$ The scheme is convergence

4.4 .Numerical Results

In order to test the efficiency and adaptabilities of the proposed method two selected problems that may arise in practice for which the analytic solutions of U are known .The computed solutions are found for all grid points in Crank-Nicolson scheme with Hockney's. The results are reported in terms of absolute error and are shown in table1 and table2.

This computational result is made on personal computer (Laptop) with processor: intel(R) Core i3 CPU@2.40GHz and RAM memory 2.00GB

Example1:

Consider the two dimensional heat equation in polar coordinates system is given by

$$U_t = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}, 0 < r < 1, \ and \ 0 < \theta < 2\pi$$

subject to the initial condition;

$$U(r,\theta,0) = \sin 2\pi\left(r\cos\theta\right)\sin 2\pi(r\sin\theta)$$

with boundary condition

$$U(0,\theta,t) = U(r,0,t) = 0$$
$$U(1,\theta,t) = \sin 2\pi(\cos\theta)\sin 2\pi(\sin\theta)$$
$$U(r,2\pi,t) = 0$$

and exact solution

$$U(r,\theta,t) = \sin 2\pi(r\cos\theta)2\pi\sin(r\sin\theta)e^{-8\pi^2 t}$$

Table 4.1: MaximumAbsolute errors and CPU computational time for Crank-Nicolson and Crank-Nicolson with Hockney's method

Scheme / N=M	**Crank Nicolson Scheme**		**Crank Nicolson with Hockney's**	
	Max.abs.error $\max_{i,j,n}\left\|U_{ij}^{n}-u(x_i,y_j,t_n)\right\|$	**CPU Comp. Time in sec.**	**Max. abs. error** $\max_{i,j,n}\left\|U_{ij}^{n}-u(x_i,y_j,t_n)\right\|$	**CPU Comp. Time in sec.**
10	1.4300e-02	8.204946	1.4300e-02	9.531230
20	9.8561e-04	15.635776	9.8561e-04	12.752585
30	3.4418e-04	52.771701	3.4418e-04	32.330180
40	9.7899e-05	240.986369	9.7899e-05	52.541178
50	2.9530e-05	954.767625	2.9530e-05	82.873367
60	------------	------------	9.2119e-06	120.5373595
70	------------	------------	2.8230e-06	182.890577
80	------------	-------------	7.9206e-07	227.166939

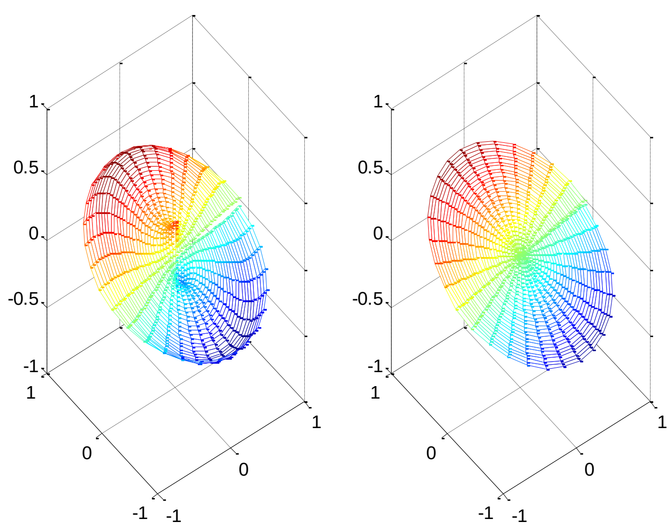

Figure1: On the left hand side the numerical solution using Crank Nicolson with Hockeny's and on the right hand side the exact solution at time T=0.01 second

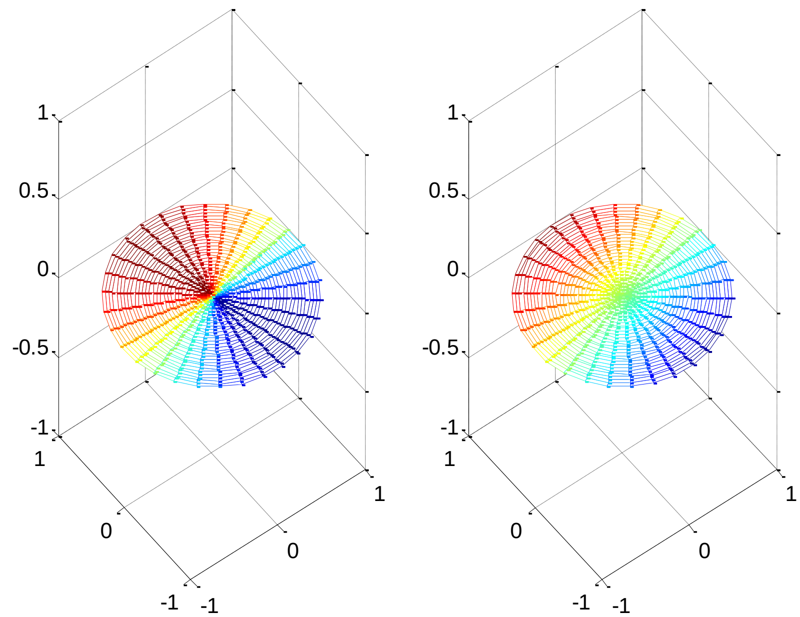

Figure2: On the left hand side the numerical solution using Crank Nicolson with Hockeny and on the right hand side the exact solution at time T=0.1 second

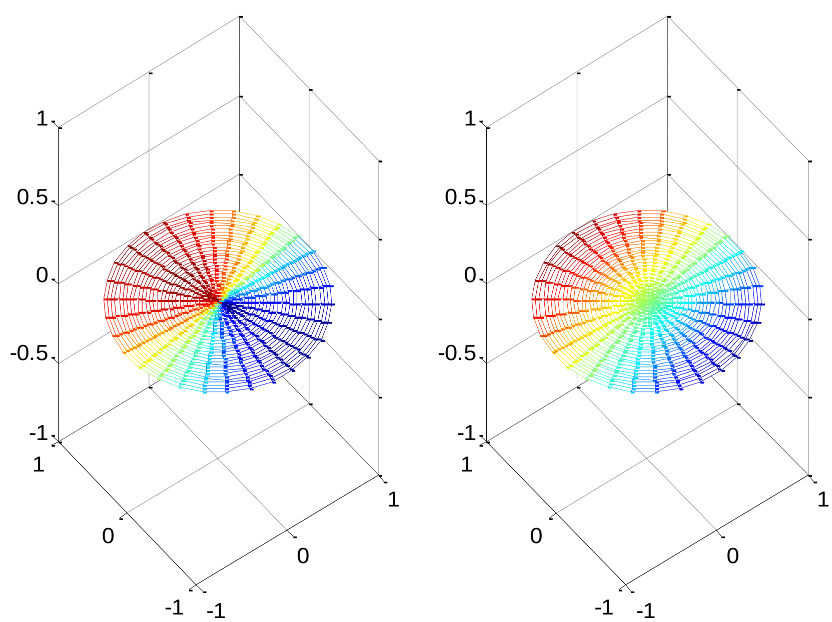

Figure3: On the left hand side the numerical solution using Crank Nicolson with Hockney's and on the right hand side the exact solution at time T=1 second.

Example 2: Consider the two dimensional heat equation in polar coordinates system is given by

$$U_t = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}, 0 < r < 1, \text{ and } 0 < \theta < 2\pi$$

subject to the initial condition ;

$$U(r,\theta,0) = 10\sin 2\pi(r\cos\theta)\sin 2\pi(r\sin\theta)$$

with boundary condition

$$U(0,\theta,t) = U(r,0,t) = 0$$
$$U(1,\theta,t) = 10\sin 2\pi(\cos\theta)\sin 2\pi(\sin\theta)$$
$$U(r,2\pi,t) = 0$$

and exact solution;

$$U(r,\theta,t) = 10\sin 2\pi(r\cos\theta)2\pi\sin(r\sin\theta)e^{-8\pi^2 t}$$

Table 4.2: Maximum Absolute errors and computational time for Crank-Nicolson and Crank-Nicolson with Hockney's method.

Scheme / N=M	Crank Nicolson Scheme		Crank- Nicolson with Hockney's	
	Max.abs.error $\max_{i,j,n}\left\|U_{ij}^n - u(x_i, y_j, t_n)\right\|$	**CPU Comp. Time in sec.**	**Max. abs. error** $\max_{i,j,n}\left\|U_{ij}^n - u(x_i, y_j, t_n)\right\|$	**CPU Comp. Time in sec.**
10	1.4300e-01	6.986180	1.4300e-01	6.344063
20	9.9000e-03	13.166935	9.9000e-03	11.770737
30	7.8000e-03	23.3468170	7.8000e-03	21.846837
40	2.9530e-04	206.716720	2.9530e-03	38.071393
50	2.9530e-04	862.982759	2.9530e-04	60.989153
60	------------	------------	9.2119e-05	88.973377
70	------------	------------	2.8230e-05	120.735133
80	------------	-------------	7.9206e-06	176.222124

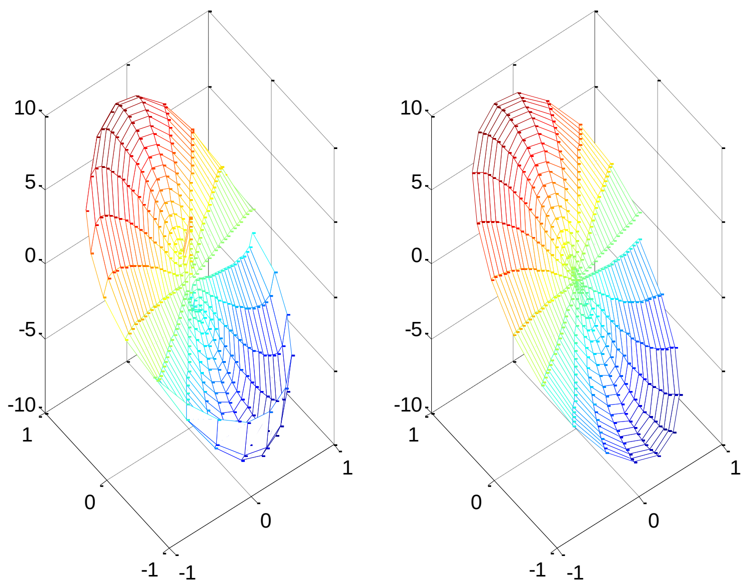

Figure1: On the left hand side the numerical solution using Crank Nicolson with Hockney's and on the right hand side the exact solution at time T=0.001 second

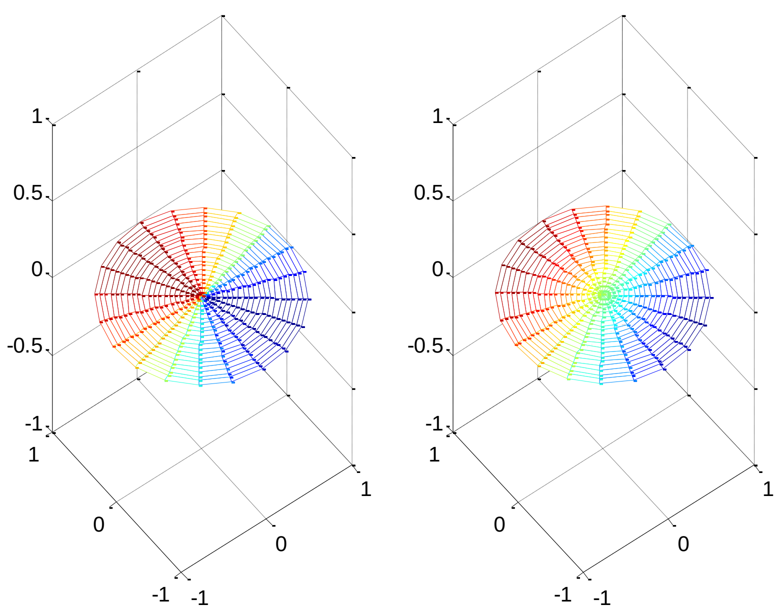

Figure2: On the left hand side the numerical solution using Crank Nicolson with Hockeny and on the right hand side the exact solution at time T=1 second

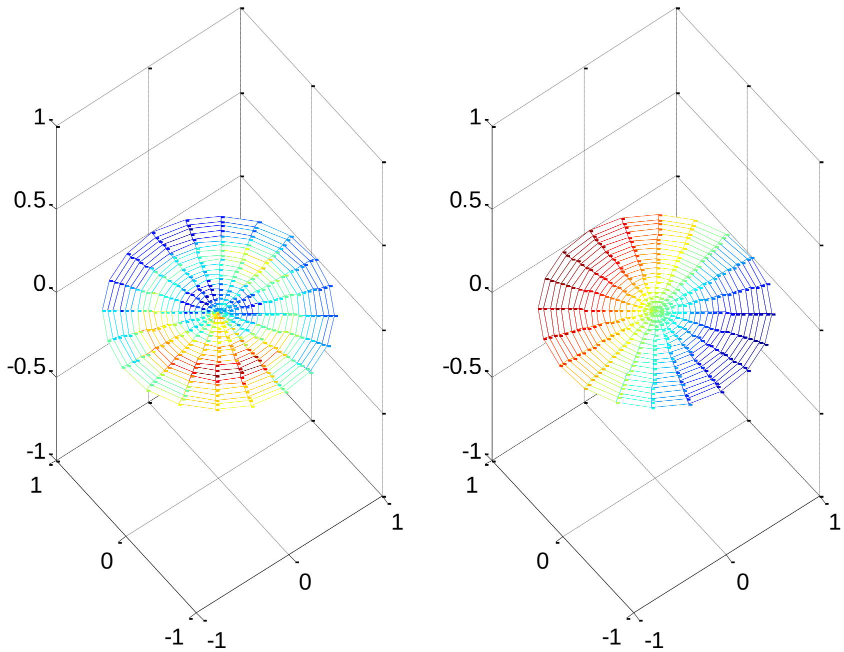

Figure 3: On the left hand side the numerical solution using Crank Nicolson with Hockeny and on the right hand side the exact solution at time T=1.5 sec

4.5. Discussion

In this thesis, Crank-Nicholson Scheme and Crank-Nicolson with Hockney's method have been compared for CPU computational time and convergence accuracy for solving two dimensional Heat equation in polar coordinate domain. According to numerical result presented on table 4.1and table 4.2 the accuracy of the two methods are equivalent but the Crank-Nicolson with Hockney's splitting is very fast compared to the Crank-Nicolson method in computational time When the number of mesh point increases while Crank-Nicolson with Hockney's giving result. The Crank-Nicolson fails to compute the solution due to shortage of memory storage capacity of the computer and processing ability . So we conclude that Crank-Nicolson with Hockney' splitting is fast solver method with equivalent accuracy of Crank(i.e. $O\left((\Delta t)^2+(\Delta r)^2+(\Delta\theta)^2\right)$

CHAPTER FIVE
CONCLUSION AND SCOPE FOR FUTURE WORK
5.1. Conclusion

In this thesis, we have presented Crank-Nicolson with Hockney's method for solving two dimensional Heat equation in polar coordinates system .Two examples have been used to compared with the present numerical method with Cranck-Nicolson. the accuracy of the two methods are equivalent but the Crank-Nicolson with Hockney's splitting is very fast compared to the Crank-Nicolson method in computational time When the number of mesh point increases the error of the method is decresing.The numerical results obtained in this method is well.

5.2. Scope for Future Work

In this thesis, the numerical method based on Finite Difference(Crank-Nicolson) with Hockney's method is introduced for solving two dimensional Heat equation in polar coordinate system. Hence, the scheme proposed in this thesis can also be extended to three dimensional Heat and wave equation in polar coordinate system. We recommend researchers and peoples working on computational area for using this algorithm for fast and accurate computational requiring purpose.

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