NUMERICAL SOLUTION OF TWO-DIMENSIONAL WAVE EQUATION USING CRANK-NICOLSON SCHEME

A thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

(NUMERICAL ANALYSIS STREAM)

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November, 2018

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DECLARATION

I undersigned declare that, this thesis entitled "numerical solution of two-dimensional wave equation using Crank-Nicolson method" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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ACKNOWLEDGMENT

First of all, I am indebted to my almighty God who gave me long life, strength and helped me to reach this precise time. Next, I would like to express my deepest gratitude to my advisor Alemayehu Shiferaw (Ph. D) for his unreserved support, constructive comments and immediate responses that helped me in developing this research.

ABSTRACT

In this thesis, Crank-Nicholson method is presented for solving two-dimensional wave equation. First the given two-dimensional wave equation is replaced by crank-Nicholson scheme. The resulting large number of algebraic equation was arranged in order to get a block matrix. From block matrix we obtain system of linear algebraic equation and changes to tridiagonal matrices by collecting like terms. Thus the tridiagonal system of equation can solve by Thomas Algorithm. The stability, consistency and convergence of the method have been established. We implement the numerical scheme by computer programming for initial boundary value problem and compare the exact solution with the numerical solution. The results have been presented in Tables 1 to 2 for different values of some mesh points. Then, Crank-Nicolson method is good for solving two-dimensional homogeneous wave equation, since it is easy to solve the resulting tridiagonal system of equations.

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CHAPTER ONE

INTRODUCTION

1.1. Background of the study

Partial differential equations (PDEs) have become enormously successful as models of physical phenomena in all areas of engineering and sciences. The growing need for understanding the partial differential equations modeling of the physical problem has seen an increase in the use of mathematical theory and techniques, and has attracted the interest of many mathematicians. Many interesting progresses have been achieved in the last 60 years with the introduction of numerical methods that allow the use of modern computers to solve PDEs of every kind, in general geometries and under arbitrary external conditions (at least in theory; in practice there are still a large number of hurdles to be overcome).

Especially in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences (particularly in relation to image processing and graphics) and in economics (finance). The primary reason for this interest was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering. The theoretical analysis of PDEs is not merely of academic interest, but rather has many applications that originate from a model of a physical or engineering problem in real life situations like wave equation.

The wave equation is an important second-order linear partial differential equations for the description of waves as they occur in classical physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetics and fluid dynamics.

A variety of problems in scientific computing involve the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \quad \text{in } \mathbb{R} \tag{1.1}$$

subject to the initial condition

$$u(x, y, 0) = f_1(x, y)$$

$$u_t(x, y, 0) = f_2(x, y)$$

and appropriate boundary conditions (BC),

i. $U = f_1$ on $\partial\mathbb{R}$ for a given function f_1, (i.e. U specified on the boundary) is called the Dirichlet problem,

ii. $\frac{\partial U}{\partial n} = f_2$ on $\partial\mathbb{R}$ where f_2 is a given function, $\hat{n}$ denotes the unit outward normal to $\partial\mathbb{R}$, and $\frac{\partial U}{\partial n}$ denotes a differentiation in the direction of $\hat{n}$ (i.e., $\frac{\partial U}{\partial n} = \hat{n} \cdot \nabla$), (i.e. gradient of U normal to the boundary is specified) is called the Neumann problem, and

iii. $U + \alpha \frac{\partial U}{\partial n} = f_3$ on $\partial\mathbb{R}$ where α and f_3 are given functions, (i.e. the BC is in terms of a mixture of the first two types – typically a linear combination) is called a problem of the third kind (it is also sometimes called the Robin problem).

iv. Mixed boundary condition for a PDE; that is, different boundaries are used on different parts of the boundary of the domain of the equation. For example, if U is a solution to a partial differential equation on $\mathbb{R}$ with piecewise-smooth boundary $\partial\mathbb{R}$, and $\partial\mathbb{R}$ is divided into two parts, Γ_1 and Γ_2, one can use a Dirichlet boundary condition on Γ_1 and a Neumann boundary condition on Γ_2, i.e. $\partial\mathbb{R} = \Gamma_1 \cup \Gamma_2$ and U is prescribed on the boundary a $U = g_1$ on Γ_1 and $\frac{\partial U}{\partial n} = g_2$ on Γ_2, where g_1 and g_2 are given functions defined on those portions of the boundary.

Equation (1.1) along with the initial and boundary conditions (i) to (iv) is said to be an initial boundary value problem.

Finding the solution of PDEs numerically, forced people in the area to develop different numerical methods and approaches. Some of the most popular numerical methods are the Finite Difference Method (FDM), the Finite Elements Method (FEM), the Finite Volume

Method (FVM), Fast Fourier Transform Methods, Spline Collocation Methods, Spectral Methods, Multigrid Methods, Galerkin Method, Domain Decomposition Methods, Boundary Element Methods, Wavelet Methods and others.

Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond, d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. In 1746, d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation.

Wave equations are one of the three types of classical partial differential equations of second order. These equations arise in mathematical modeling of the motion of vibrating strings and membranes, and are typical in studying partial differential equations of second order. Wave equations in the sense of classical derivative with smooth boundaries have been investigated extensively.

In recent years, different methods have been applied to find the numerical solution of the hyperbolic one and two-dimensional wave equation. To mention some: Variables-separate methods (Yeow K., 1973). Implicit finite difference scheme (Britta S *et al*, 2017). Accurate discretization schemes (Santosh Konangi *et al*, 2017). Implicit finite difference scheme (Xiaofeng Wang *et al*, 2018). Mimetic finite difference methods (Beirão da Veiga L *et al*, 2017). Method of multiple scales (Huang Guoxiang *et al*, 1989). Galerkin schemes (Luk'a Cov´. M *et al*, 2006). Wave polynomials (Artur Macia and Jorg Wauer, 2005). Finite difference scheme (Nor Syazwani Binti and Mohd Ridzun, 2013). Finite Difference Analysis (Opiyo Richard *et al*, 2015). Crank-Nicholson Method (Sweilam and Nagy, 2011). Compact finite difference time domain schemes (Maarten van and Konrad Kowalczyk, 2008). However, the two-dimensional wave equation still need more work to obtain accurate numerical solutions. Hence, we intended to apply the Crank Nicholson scheme to obtain accurate solutions of the homogeneous two-dimensional wave equation.

When we treated Crank-Nicolson method to wave equation in two-dimensional, because it was one of the finite difference method we obtain a system of algebraic equations and this algebraic equations can be solved. In this study, we applied a numerical method that solves the wave equation in two-dimensional using the finite difference of Crank-Nicolson method.

1.2 Statement of the Problem

Due to the wide range of the application of the two-dimensional wave equation, several numerical methods have been developed to solve this equation subject to initial and boundary conditions. Even though many numerical methods were applied to solve these types of equations, still it need more work to obtain accurate numerical solutions. So, we have been applied a Crank-Nicolson method to obtain the solution of two-dimensional wave equation. Owing to this, the present study attempt to answer the following questions:

1. How does the present method be described the two-dimensional wave equation?
2. To what extent the present method approximate the solution?
3. To what extent the proposed method is convergent?

1.3. Objectives of the study

1.3.1. General Objective

The general objective of this study is to find the numerical solution of the two-dimensional wave equation using Crank-Nicolson scheme.

1.3.1. Specific Objectives

The specific objectives are:

- To describe the Crank-Nicolson method for solving two- dimensional wave equation.
- To investigate the accuracy of the present method.
- To establish the convergence of the present scheme.

1.4. Significance of the study

The results obtained in this study may:

Provide as reference who works in this area and help the graduate students to acquire research skills and scientific procedures.

1.5. Delimitation of the study

Since two-dimensional wave equations have many applications in the real world and many numerical methods are applied to solve the two-dimensional wave equation, this study has been delimited to the two-dimensional wave equation of form:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{ in } \mathbb{R}$$

subject to the initial condition

$$u(x, y, 0) = f_1(x, y)$$

$$u_t(x, y, 0) = f_2(x, y)$$

and boundary conditions:

$$u(0, y, t) = u(a, y, t) = g_1(y, t) \;, 0 \le y \le b, t \ge 0$$

$$u(x, 0, t) = u(x, b, t) = g_2(x, t) \;, 0 \le x \le a, t \ge 0,$$

where c is nonzero constant coefficient $f_1(x, y)$, $f_2(x, y)$ and their derivatives are continuous functions of x, y ; $g_1(y, t), g_2(x, t)$ and their derivatives are continuous functions of t.

1.6. Definition of the Basic Terms

Definition 1: Numerical analysis

Numerical analysis is the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from discrete mathematics).

Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21^{st} century also the life sciences and even the arts have adapted elements of scientific computations. Ordinary differential equations appear in celestial mechanics (planets, stars and galaxies); numerical linear algebra is important for data analysis; stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology.

Before the advent of modern computers numerical methods often depended on hand interpolation in large printed tables. Since the mid-20th century, computers calculate the required functions instead. These same interpolation formulas nevertheless continue to be used as part of the software algorithms for solving differential equations.

Definition 2: Numerical Methods

In numerical analysis, a numerical method is a mathematical tool designed to solve numerical problems. The implementation of a numerical method with an appropriate convergence check in a programming language is called a numerical algorithm.

Definition 3: Differential Equation

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

Definition 4: Partial Differential Equation

In mathematics, a partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to

formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. A special case is ordinary differential equations (ODE), which deal with functions of a single variable and their derivatives.
Partial Differential Equation can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalization in stochastic partial differential equations.

Numerical partial differential equation is the branch of numerical analysis that studies the numerical solution of partial differential equations (PDEs).

A PDE of the form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + D\left(x, y, u, u_x, u_y\right) = 0$$

Where A, B and C are constants, called quasilinear. There are three types of quasilinear equations:

If $B^2 - 4AC < 0$, the equation is called elliptic,

If $B^2 - 4AC = 0$, the equation is called parabolic,

If $B^2 - 4AC > 0$, the equation is called hyperbolic. Wave equation is a classic example of hyperbolic PDE.

Definition 5: Boundary value problem

In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Definition 6: Matrix

A matrix is a collection of numbers arranged into a fixed number of rows and columns. Usually the numbers are real numbers. In general, matrices contain complex numbers.

A matrix is a way to organize data in columns and rows. A matrix is written inside brackets. Also a matrix is a two-dimensional array of numbers that can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Where a_{ij} is called the element or entry in the i^{th} row and j^{th} column. An alternative notation is $A = (a_{ij})n \times m$. A column vector is also a $n \times 1$ matrix and a row vector is also a $1 \times m$ matrix.

Definition 7: Thomas Algorithm

In numerical linear algebra, the tridiagonal matrix algorithm, also known as the Thomas algorithm (named after Llewellyn Thomas), is a simplified form of Gaussian elimination that can be used to solve tridiagonal systems of equations. The Thomas Algorithm is a special form of Gauss elimination that can be used to solve tridiagonal systems of equations. The form of the equation is:

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i, \; i = 1, k, n.$$ Where a_1 and c_n are zero.

Definition 8: Stability

A numerical method is said to be *stable* if the cumulative effect of all the errors is bounded independent of the number of mesh points.

Definition 10: Consistency

A numerical method is called consistent if the local error decays sufficiently fast as $h \to 0$ (mesh size tends to zero). Consistency is the study of the local error.

Definition 11: Convergence

Convergence is the numerical solution should approach the exact solution of the PDE and converge to it as the mesh size tends to zero. Convergence is the study of the global error.

CHAPTER TWO

LITRETURE REVIEW

2.1. Partial Differential Equations

Involving one or more partial derivatives of a function of two or more independent variables are called Partial differential Equations (PDEs). Historically, partial differential equations originated from the study of surfaces in geometry and a wide variety of problems in mechanics. During the second half of the nineteenth century, a large number of famous mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations (Debnath, 2011). The primary reason for this research was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering. The next phase of the development of linear partial differential equations was characterized by efforts to develop the general theory and various methods of solution of linear equations (Myint-u, and Debnath, 2007). Almost all physical phenomena obey mathematical laws that can be formulated by differential equations. This striking fact was first discovered by Isaac Newton (1642– 1727) when he formulated the laws of mechanics and applied them to describe the motion of the planets.

During the three centuries since Newton's fundamental discoveries, many partial differential equations that govern physical, chemical, and biological phenomena have been found and successfully solved by numerous methods. These equations include Euler's equations for the dynamics of rigid bodies and for the motion of an ideal fluid, Lagrange's equations of motion, Hamilton's equations of motion in analytical mechanics, Fourier's equation for the diffusion of heat, Cauchy's equation of motion and Navier's equation of motion in elasticity, the Navier–Stokes equations for the motion of viscous fluids, the Cauchy Riemann equations in complex function theory, the Cauchy Green equations for the static and dynamic behavior of elastic solids, Kirchhoff's equations for electrical circuits, Maxwell's equations for electromagnetic fields, and the Schrödinger equation and the Dirac equation in quantum mechanics.

In its early stages of development, the theory of second-order linear partial differential equations was concentrated on applications to mechanics and physics. All such equations can be classified into three basic categories: the wave equation, the heat equation, and the Laplace equation or potential equation. Thus, a study of these three different kinds of equations yields much information about more general second-order linear partial differential equations.

Jean d'Alembert (1717–1783) first derived the one dimensional wave equation for vibration of an elastic string and solved this equation in 1746. His solution is now known as the d'Alembert solution. The wave equation is one of the oldest equations in mathematical physics. Some form of this equation, or its various generalizations, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. In fact, the studies of water waves, acoustic waves, elastic waves in solids, and electromagnetic waves are all based on this equation. A technique known as the method of separation of variables is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation (Aubert and Kornprobst, 2006).The wave equation and its methods of solution attracted the attention of many famous mathematicians including Leonhard Euler (1707–1783), James Bernoulli (1667–1748), Daniel Bernoulli (1700–1782), J.L. Lagrange (1736–1813), and Jacques Hadamard (1865–1963).

Hence, hyperbolic wave equation is a significant class of the partial differential equation due to its wide range of applications in many areas of science and engineering as mentioned in the introduction part.

2.2. The wave equation

A wave is a disturbance from a normal or equilibrium condition that propagates without the transport of matter. Wave equation is an important second-order linear partial differential equation for the description of waves as they occur in classical physics such as sound waves, light waves and water waves. It arises in fields like acoustics electromagnetics, and fluid dynamics.

Wave equations are one of the three types of classical partial differential equations of second order. These equations arise in mathematical modeling of the motion of vibrating strings and membranes, and are typical in studying partial differential equations of second order. Wave equations in the sense of classical derivative with smooth boundaries have been investigated extensively.

To deal with such equation, various mathematical methods have been proposed for obtaining exact and approximate analytic solutions. For instance, (Yeow K. W, 1973) used variables-separate methods to Webster wave equation in two dimensions. (Shuonan Dong http://mit.edu/dongs/Public/18.086/Project1) solved with finite difference methods for the Hyperbolic Wave Partial Differential Equations. (Santosh Konangi *et al*, 2017) used von Neumann stability analysis of first-order accurate discretization schemes for one-dimensional and two-dimensional fluid flow equations. (Britta S *et al*, 2017) numerical solution of the wave equation with variable wave speed on nonconforming domains by high-order difference potentials using an implicit finite difference scheme. (Xiaofeng Wang *et al*, 2018) a conservative linear difference scheme for the two-dimensional regularized long-wave equation using implicit finite difference scheme. (Beirão da Veiga. *et al*, 2017) used mimetic finite difference methods for Hamiltonian wave equations in 2D. (Luk'a ˇCov´a *et al*, 2006) on the stability of evolution Galerkin schemes applied to a two-dimensional wave equation system. (Artur Macia and Jorg Wauer, 2005) found solution of the two-dimensional wave equation used wave polynomials. (Nor Syazwani and Mohd Ridzun, 2013) done numerical modeling of one-dimensional wave equation using finite difference scheme. (Opiyo Richard *et al*, 2015) used finite difference analysis of two-dimensional Acoustic Wave with a Signal Function. (Sweilam and Nagy, 2011) done Numerical Solution of Fractional Wave Equation using Crank-Nicholson Method. (Maarten Van and Konrad Kowalczyk, 2008) done on the Numerical Solution of the two-dimensional Wave Equation with Compact Finite Difference Time Domain schemes. Solutions of the wave equation are still an attractive and interesting topic. Due to this, we are interested in finding the numerical solution of hyperbolic wave equation using finite difference of Crank-Nicolson scheme.

2.3. Finite Difference Method

A computational solution of a partial differential equation (PDE) involves a discretization procedure by which the continuous equation is replaced by a discrete algebraic equation. The discretization procedure consists of an approximation of the derivatives in the governing PDE by differences of the dependent variables, which are computed only at discrete points (grid or mesh points) in different geometries. In general, one starts with a given PDE and uses a discretization procedure for developing a finite-difference equation (FDE) that is a linear relation between discrete values of the unknown function computed on grid point.

Thus, a finite difference solution basically involves three steps:

1. Dividing the solution domain into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary and/or initial conditions.

When approximating the given PDE by its finite difference approximation, we have to consider some factors, for instance, the order of accuracy of an approximation, stability, consistency and convergence of the difference scheme having a potential impact on the approximate solution. Here, one of the methods is the Crank-Nicolson method.

2.4. Crank-Nicolson Scheme

Crank-Nicolson scheme is a finite difference method based on two time steps used for solving numerically a partial differential equation. It is either fully or semi implicit in time and transform the given PDE into algebraic equations that can be solved by any existing method. The method was developed by John Crank and Phyllis Nicolson in the mid-20th century (Crank and Nicolson, 1947).

CHAPTER THREE

METHODOLOGY

3.1. Study Site

This study was conducted in Jimma University department of Mathematics under the numerical Analysis stream from September 2017 G.C to November 2018 G.C. The study focus on two-dimensional wave equation.

3.2. Study Design

This study was employed mixed-design (documentary review design and experimental design) on two- dimensional wave equation.

3.3. Source of Information

The relevant sources of information for this study were Journals, books, published articles & related studies from internet and the experimental result has be obtained by writing MATLAB code.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the study was followed the following procedures:

1. Defining the problem of the two-dimensional wave equation,
2. Discretizing the domain for the defined problem,
3. Replace the Partial Differential Equation by finite difference approximation method of Crank-Nicolson scheme.
4. The result system of equations has been solved by Thomas Algorithm.
5. MATLAB code for the systems obtained and
6. Validating the schemes by using numerical examples.

CHAPTER FOUR

DISCRPTION OF METHODS, NUMERICAL RESULTS AND DISCUSSION

4.1. Description of the methods

Consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{4.1}$$

$$\Rightarrow c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial^2 u}{\partial t^2} = 0$$

subject to the initial condition

$$u(x, y, 0) = f_1(x, y)$$

$$u_t(x, y, 0) = f_2(x, y) \tag{4.2}$$

and boundary conditions:

$$u(0, y, t) = u(a, y, t) = g_1(y, t)\ , 0 \le y \le b, t \ge 0$$

$$u(x, 0, t) = u(x, b, t) = g_2(x, t)\ , 0 \le x \le a, t \ge 0\,. \tag{4.3}$$

where c is nonzero constant coefficient $f_1(x, y), f_2(x, y)$ and their derivatives are continuous functions of x, y ; $g_1(y,t), g_2(x,t)$ and their derivatives are continuous functions of t The domain $\mathbb{R}$ with a two-dimensional grid in a rectangular region in length a and b , $\mathbb{R} = [0, a] \times [0, b]$, we first partition the intervals $[0, a]$ and $[0, b]$ into respective finite grids as follows.

Let $x_i = i\Delta x, i = 1, 2, ..., M$ where $\Delta x = \frac{a}{M}$,

$y_j = j\Delta y, j = 1, 2, ..., N$ where $\Delta y = \frac{b}{N}$

Similarly we partition $[0, T]$ as $t_k = k\Delta t, k = 0, 1, 2, ..., P$ where $\Delta t = \frac{T}{P}$

Where $i, j =$ location (node numbers), $k =$ time (time step number)

Further, to find the more accurate numerical solution of Eq. (4.1) with respect to the given conditions in Eqs. (4.2) and (4.3), we will apply finite difference method of Crank-Nicolson.

Assume that there are M , N and P mesh points along the X , Y and time in t directions respectively, and let $U(x,y,t)$ be discretized at the mesh point (i,j,k) and we adopt writing $U_{i,j}^{k}$ for $U(x_i,y_j,t_k)$, where $i=1(1)M$, $j=1(1)N$ and $k=0(1)P$.

Let Δx and Δy be the step sizes in the X and Y directions respectively. We transform equation (4.1) into its equivalent finite difference approximation by Crank – Nicolson method.

$$\Rightarrow \frac{c^2}{2(\Delta x)^2}\left(U_{i+1,j}^{k+1}-2U_{i,j}^{k+1}+U_{i-1,j}^{k+1}+U_{i+1,j}^{k}-2U_{i,j}^{k}+U_{i-1,j}^{k}\right)+$$

$$\frac{c^2}{2(\Delta y)^2}\left(U_{i,j+1}^{k+1}-2U_{i,j}^{k+1}+U_{i,j-1}^{k+1}+U_{i,j+1}^{k}-2U_{i,j}^{k}+U_{i,j-1}^{k}\right)-\frac{1}{(\Delta t)^2}\left(U_{i,j}^{k+1}-2U_{i,j}^{k}+U_{i,j}^{k-1}\right)=0 \quad (4.4)$$

Where $r=\frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2$ and $e=\frac{1}{2}\left(\frac{c\Delta t}{\Delta y}\right)^2$

We can write the above equation as

$$\Rightarrow (-1-2r-2e)U_{i,j}^{k+1}+r\left(U_{i+1,j}^{k+1}+U_{i-1,j}^{k+1}\right)+e\left(U_{i,j+1}^{k+1}+U_{i,j-1}^{k+1}\right)+$$

$$(2-2r-2e)U_{i,j}^{k}+r\left(U_{i+1,j}^{k}+U_{i-1,j}^{k}\right)+e\left(U_{i,j+1}^{k}+U_{i,j-1}^{k}\right)-U_{i,j}^{k-1}=0 \quad (4.5)$$

In (4.4) we put $k=0$ and $j=1$, by taking i from 1 to M we get M set of equations along the x direction. Again putting $j=2$ still we get another M set of equations, and so on until $j=N$ getting a total of MN set of equations on the plane parallel to the XY plane. Again for $k=1(1)P$ we follow the same pattern as $k=0$ and finally we have P block of equations and each block has a set of MN equations. Thus, in general, (4.1) can be written in matrix form as

$$AU=B \quad (4.6)$$

where

$$A=\begin{pmatrix} R & S & & & & \\ Y & R & S & & & \\ & Y & R & S & & \\ & & \ddots & \ddots & \ddots & \\ & & & Y & R & S \\ & & & & Y & R \end{pmatrix} \quad (4.7)$$

Matrix A has p blocks and each block is of order $\mathrm{MN} \times \mathrm{MN}$,

$$R = \begin{pmatrix} R_1 & R_2 & & & & \\ R_2 & R_1 & R_2 & & & \\ & R_2 & R_1 & R_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & R_2 & R_1 & R_2 \\ & & & & R_2 & R_1 \end{pmatrix} \qquad S = \begin{pmatrix} S_1 & S_2 & & & & \\ S_2 & S_1 & S_2 & & & \\ & S_2 & S_1 & S_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & S_2 & S_1 & S_2 \\ & & & & S_2 & S_1 \end{pmatrix}$$

$$And\ \ Y = \begin{pmatrix} Y_1 & & & & \\ & Y_1 & & & \\ & & Y_1 & & \\ & & & \ddots & \\ & & & & Y_1 \end{pmatrix} \tag{4.8}$$

R, S and Y have N blocks and each block is of order $\mathrm{M} \times \mathrm{M}$,

$$R_1 = \begin{pmatrix} -1-2r-2e & r & & & & \\ r & -1-2r-2e & r & & & \\ & r & -1-2r-2e & r & & \\ & & \ddots & \ddots & \ddots & \\ & & & r & -1-2r-2e & r \\ & & & & r & -1-2r-2e \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 2-2r-2e & r & & & & \\ r & 2-2r-2e & r & & & \\ & r & 2-2r-2e & r & & \\ & & \ddots & \ddots & \ddots & \\ & & & r & 2-2r-2e & r \\ & & & & r & 2-2r-2e \end{pmatrix}$$

$$R_2 = \begin{pmatrix} e & & & & \\ & e & & & \\ & & e & & \\ & & & \ddots & \\ & & & & e \end{pmatrix} = S_2 \quad \text{And} \quad Y_1 = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} \tag{4.9}$$

Where R_1 And S_1 are square matrices of order M and R_2, S_2 and Y_1 are diagonal matrices of order M,

$$U = \begin{pmatrix} U_0 & U_1 & \dots & U_{p-1} & U_p \end{pmatrix}^T \quad And \; B = \begin{pmatrix} B_0 & B_1 & \dots & B_{p-1} & B_p \end{pmatrix}^T$$

$$U_k = \begin{bmatrix} U_{1k} & U_{2k} & \cdots & U_{Nk} \end{bmatrix}^T \quad And \; U_{jk} = \begin{bmatrix} U_{1jk} & U_{2jk} & \cdots & U_{Mjk} \end{bmatrix}^T$$

$$B_k = \begin{bmatrix} d_{1k} & d_{2k} & \cdots & d_{Nk} \end{bmatrix}^T \quad And \; d_{jk} = \begin{bmatrix} d_{1jk} & d_{2jk} & \cdots & d_{Mjk} \end{bmatrix}^T$$

$k = 0,1,2,...,p$ is the known column vector such that each d_{ijk} represents known boundary values of U. Thus (4.6) can be written as

$$\begin{pmatrix} R & S & & & & \\ Y & R & S & & & \\ & Y & R & S & & \\ & & \ddots & \ddots & \ddots & \\ & & & Y & R & S \\ & & & & Y & R \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_{p-1} \\ U_p \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_{p-1} \\ B_p \end{pmatrix} \tag{4.10}$$

Equation (4.10) once again can be written as

$$\begin{aligned} RU_0 + SU_1 &= B_0 \\ YU_0 + RU_1 + SU_2 &= B_1 \\ YU_1 + RU_2 + SU_3 &= B_2 \\ &\dots \\ YU_{P-1} + RU_P &= B_P \end{aligned} \tag{4.11}$$

Now we can solve equation (4.11) directly by using Crank-Nicolson method depending on initial and boundary conditions.

From Eq. (4.5) and Eq. (4.11) we get the following algebraic equations

For $k = 0,1,2,...,P, \;\; j = 1, \;\; i = 1,2,...,M$

$$\begin{aligned}
&(-1-2r-2e)u_{1,1}^{k+1} + r\left(u_{2,1}^{k+1} + u_{0,1}^{k+1}\right) + e\left(u_{1,2}^{k+1} + u_{1,0}^{k+1}\right) + (2-2r-2e)u_{1,1}^{k} + r\left(u_{2,1}^{k} + u_{0,1}^{k}\right) + e\left(u_{1,2}^{k} + u_{1,0}^{k}\right) - u_{1,1}^{k-1} = 0 \\
&(-1-2r-2e)u_{2,1}^{k+1} + r\left(u_{3,1}^{k+1} + u_{1,1}^{k+1}\right) + e\left(u_{2,2}^{k+1} + u_{2,0}^{k+1}\right) + (2-2r-2e)u_{2,1}^{k} + r\left(u_{3,1}^{k} + u_{1,1}^{k}\right) + e\left(u_{2,2}^{k} + u_{2,0}^{k}\right) - u_{2,1}^{k-1} = 0 \\
&(-1-2r-2e)u_{3,1}^{k+1} + r\left(u_{4,1}^{k+1} + u_{2,1}^{k+1}\right) + e\left(u_{3,2}^{k+1} + u_{3,0}^{k+1}\right) + (2-2r-2e)u_{3,1}^{k} + r\left(u_{4,1}^{k} + u_{2,1}^{k}\right) + e\left(u_{3,2}^{k} + u_{3,0}^{k}\right) - u_{3,1}^{k-1} = 0 \\
&(-1-2r-2e)u_{4,1}^{k+1} + r\left(u_{5,1}^{k+1} + u_{3,1}^{k+1}\right) + e\left(u_{4,2}^{k+1} + u_{4,0}^{k+1}\right) + (2-2r-2e)u_{4,1}^{k} + r\left(u_{5,1}^{k} + u_{3,1}^{k}\right) + e\left(u_{4,2}^{k} + u_{4,0}^{k}\right) - u_{4,1}^{k-1} = 0 \\
&\qquad \cdots \; \cdots \; \cdots
\end{aligned}$$

For $j=2$

$$\left(-1-2r-2e\right)u_{1,2}^{k+1}+r\left(u_{2,2}^{k+1}+u_{0,2}^{k+1}\right)+e\left(u_{1,3}^{k+1}+u_{1,1}^{k+1}\right)+\left(2-2r-2e\right)u_{1,2}^{k}+r\left(u_{2,2}^{k}+u_{0,2}^{k}\right)+e\left(u_{1,3}^{k}+u_{1,1}^{k}\right)-u_{1,2}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{2,2}^{k+1}+r\left(u_{3,2}^{k+1}+u_{1,2}^{k+1}\right)+e\left(u_{2,3}^{k+1}+u_{2,1}^{k+1}\right)+\left(2-2r-2e\right)u_{2,2}^{k}+r\left(u_{3,2}^{k}+u_{1,2}^{k}\right)+e\left(u_{2,3}^{k}+u_{2,1}^{k}\right)-u_{2,2}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{3,2}^{k+1}+r\left(u_{4,2}^{k+1}+u_{2,2}^{k+1}\right)+e\left(u_{3,3}^{k+1}+u_{3,1}^{k+1}\right)+\left(2-2r-2e\right)u_{3,2}^{k}+r\left(u_{4,2}^{k}+u_{2,2}^{k}\right)+e\left(u_{3,3}^{k}+u_{3,1}^{k}\right)-u_{3,2}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{4,2}^{k+1}+r\left(u_{5,2}^{k+1}+u_{3,2}^{k+1}\right)+e\left(u_{4,3}^{k+1}+u_{4,1}^{k+1}\right)+\left(2-2r-2e\right)u_{4,2}^{k}+r\left(u_{5,2}^{k}+u_{3,2}^{k}\right)+e\left(u_{4,3}^{k}+u_{4,1}^{k}\right)-u_{4,2}^{k-1}=0$$
$$\cdots\ \cdots\ \cdots$$

Again for $j=3$

$$\left(-1-2r-2e\right)u_{1,3}^{k+1}+r\left(u_{2,3}^{k+1}+u_{0,3}^{k+1}\right)+e\left(u_{1,4}^{k+1}+u_{1,2}^{k+1}\right)+\left(2-2r-2e\right)u_{1,3}^{k}+r\left(u_{2,3}^{k}+u_{0,3}^{k}\right)+e\left(u_{1,4}^{k}+u_{1,2}^{k}\right)-u_{1,3}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{2,3}^{k+1}+r\left(u_{3,3}^{k+1}+u_{1,3}^{k+1}\right)+e\left(u_{2,4}^{k+1}+u_{2,2}^{k+1}\right)+\left(2-2r-2e\right)u_{2,3}^{k}+r\left(u_{3,3}^{k}+u_{1,3}^{k}\right)+e\left(u_{2,4}^{k}+u_{2,2}^{k}\right)-u_{2,3}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{3,3}^{k+1}+r\left(u_{4,3}^{k+1}+u_{2,3}^{k+1}\right)+e\left(u_{3,4}^{k+1}+u_{3,2}^{k+1}\right)+\left(2-2r-2e\right)u_{3,3}^{k}+r\left(u_{4,3}^{k}+u_{2,3}^{k}\right)+e\left(u_{3,4}^{k}+u_{3,2}^{k}\right)-u_{3,3}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{4,3}^{k+1}+r\left(u_{5,3}^{k+1}+u_{3,3}^{k+1}\right)+e\left(u_{4,4}^{k+1}+u_{4,2}^{k+1}\right)+\left(2-2r-2e\right)u_{4,3}^{k}+r\left(u_{5,3}^{k}+u_{3,3}^{k}\right)+e\left(u_{4,4}^{k}+u_{4,2}^{k}\right)-u_{4,3}^{k-1}=0$$
$$\cdots\ \cdots\ \cdots$$

Continuing for $j=4$

$$\left(-1-2r-2e\right)u_{1,4}^{k+1}+r\left(u_{2,4}^{k+1}+u_{0,4}^{k+1}\right)+e\left(u_{1,5}^{k+1}+u_{1,3}^{k+1}\right)+\left(2-2r-2e\right)u_{1,4}^{k}+r\left(u_{2,4}^{k}+u_{0,4}^{k}\right)+e\left(u_{1,5}^{k}+u_{1,3}^{k}\right)-u_{1,4}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{2,4}^{k+1}+r\left(u_{3,4}^{k+1}+u_{1,4}^{k+1}\right)+e\left(u_{2,5}^{k+1}+u_{2,3}^{k+1}\right)+\left(2-2r-2e\right)u_{2,4}^{k}+r\left(u_{3,4}^{k}+u_{1,4}^{k}\right)+e\left(u_{2,5}^{k}+u_{2,3}^{k}\right)-u_{2,4}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{3,4}^{k+1}+r\left(u_{4,4}^{k+1}+u_{2,4}^{k+1}\right)+e\left(u_{3,5}^{k+1}+u_{3,3}^{k+1}\right)+\left(2-2r-2e\right)u_{3,4}^{k}+r\left(u_{4,4}^{k}+u_{2,4}^{k}\right)+e\left(u_{3,5}^{k}+u_{3,3}^{k}\right)-u_{3,4}^{k-1}=0$$
$$\left(-1-2r-2e\right)u_{4,4}^{k+1}+r\left(u_{5,4}^{k+1}+u_{3,4}^{k+1}\right)+e\left(u_{4,5}^{k+1}+u_{4,3}^{k+1}\right)+\left(2-2r-2e\right)u_{4,4}^{k}+r\left(u_{5,4}^{k}+u_{3,4}^{k}\right)+e\left(u_{4,5}^{k}+u_{4,3}^{k}\right)-u_{4,4}^{k-1}=0$$
$$\cdots\ \cdots\ \cdots$$

Collecting the like term we obtain such likes of tridiagonal linear algebraic equations

For $k=0,\ j=1, i=1,2,3,...,M$

$$\left(-1-2r-e\right)u_{1,1}^{k+1}+ru_{2,1}^{k+1}=-\left(2-2r-3e\right)u_{1,1}^{k}-ru_{2,1}^{k}+u_{1,1}^{k-1}$$
$$ru_{1,1}^{k+1}+\left(-1-2r-e\right)u_{2,1}^{k+1}+ru_{3,1}^{k+1}=-ru_{1,1}^{k}-\left(2-2r-3e\right)u_{2,1}^{k}-ru_{3,1}^{k}+u_{2,1}^{k-1}$$
$$ru_{2,1}^{k+1}+\left(-1-2r-e\right)u_{3,1}^{k+1}+ru_{4,1}^{k+1}=-ru_{2,1}^{k}-\left(2-2r-3e\right)u_{3,1}^{k}-ru_{4,1}^{k}+u_{3,1}^{k-1}$$
$$ru_{3,1}^{k+1}+\left(-1-2r-e\right)u_{4,1}^{k+1}+ru_{5,1}^{k+1}=-ru_{3,1}^{k}-\left(2-2r-3e\right)u_{4,1}^{k}-ru_{5,1}^{k}+u_{4,1}^{k-1}$$
$$\vdots\quad = \quad\vdots$$

Continuing in this way, we get the same linear algebraic equations and solve by Thomas Algorithm.

4.2. Convergence Analysis

4.2.1. Stability of Crank-Nicolson scheme

A numerical method is said to be *stable* if the cumulative effect of all the errors is bounded independent of the number of mesh points. The von Neumann stability analysis is a way to determine when a particular numerical method is stable. This tells us whether the amplitude of the wave is less than or equal to one. If the amplitude is greater than one, then the amplitude is increasing and will therefore eventually become unstable. Thus the method is stable at the values $|\rho| \leq 1$. In general, the Von Neumann's procedure introduces an error represented by a finite Fourier series and examines how this error propagates during the solution. To get stability of Crank Nicolson via this method, since stability is independent of source term then substitute $u_{i,j}^{k} = \rho^{k} e^{i\beta mh} e^{i\gamma nh}$ in the homogeneous equation

Where

β is time index in x

γ is time index in y

h is spatial step size in x and y

m is spatial index in x and

n is spatial index in y

$$-ru_{i+1,j}^{k+1} + (1+4r)u_{i,j}^{k+1} - ru_{i-1,j}^{k+1} - ru_{i,j+1}^{k+1} - ru_{i,j-1}^{k+1} = ru_{i+1,j}^{k} + (2-4r)u_{i,j}^{k} + ru_{i-1,j}^{k} + ru_{i,j+1}^{k} + ru_{i,j-1}^{k} - u_{i,j}^{k-1}$$

Where $r = \frac{1}{2}\left(\frac{c\Delta t}{h}\right)^2$ (4.12)

Which yields,

$$\begin{aligned}
&-r\rho^{k+1}e^{i\beta(m+1)h}e^{i\gamma nh} + (1+4r)\rho^{k+1}e^{i\beta mh}e^{i\gamma nh} - r\rho^{k+1}e^{i\beta(m-1)h}e^{i\gamma nh} \\
&-r\rho^{k+1}e^{i\beta mh}e^{i\gamma(n+1)h} - r\rho^{k+1}e^{i\beta mh}e^{i\gamma(n-1)h} = r\rho^{k}e^{i\beta(m+1)h}e^{i\gamma nh} + (2-4r)\rho^{k}e^{i\beta mh}e^{i\gamma nh} \\
&+r\rho^{k}e^{i\beta(m-1)h}e^{i\gamma nh} + r\rho^{k}e^{i\beta mh}e^{i\gamma(n+1)h} + r\rho^{k+1}e^{i\beta mh}e^{i\gamma(n-1)h} - \rho^{k-1}e^{i\beta mh}e^{i\gamma nh}
\end{aligned} \quad (4.13)$$

Again dividing the above equation by $\rho^{k} e^{i\beta mh} e^{i\gamma nh}$, we obtain

$$(1+4r)\rho - r\rho\left(e^{i\beta h} + e^{-i\beta h}\right) - r\rho\left(e^{i\gamma h} + e^{-i\gamma h}\right) = (2-4r) + r\left(e^{i\beta h} + e^{-i\beta h}\right) + r\left(e^{i\gamma h} + e^{-i\gamma h}\right) - \rho^{-1} \quad (4.14)$$

Recall that $\cos\theta = 1-2\sin^2\frac{\theta}{2} = \frac{e^{i\theta}+e^{-i\theta}}{2}$, therefore using this fact in (4.14), yields

$$(1+4r)\rho - r\rho\left(2\left(1-2\sin^2\frac{\beta h}{2}\right)+2\left(1-2\sin^2\frac{\gamma h}{2}\right)\right) = (2-4r)+r\left(2\left(1-2\sin^2\frac{\beta h}{2}\right)+2\left(1-2\sin^2\frac{\gamma h}{2}\right)\right)-\frac{1}{\rho}$$

After rearrangement, we get

$$\rho^2\left[1+4r\left(\sin^2\frac{\beta h}{2}+\sin^2\frac{\gamma h}{2}\right)\right]-\rho\left[2-4r\left(\sin^2\frac{\beta h}{2}+\sin^2\frac{\gamma h}{2}\right)\right]+1=0$$

Which has a non-trivial solution when $-1 \le \rho_i \le 1$, where ρ_i is the magnification factor corresponding to eigenvalue, thus;

$$\rho_i = \frac{1-4r\left(\sin^2\frac{\beta h}{2}+\sin^2\frac{\gamma h}{2}\right)}{1+4r\left(\sin^2\frac{\beta h}{2}+\sin^2\frac{\gamma h}{2}\right)} \le 1$$

Now for $|\rho_i| \le 1$, we have $\sin^2\frac{\beta h}{2}=1$ and $\sin^2\frac{\gamma h}{2}=1$, therefore

$$\rho_i = \frac{1-8r}{1+8r} \le 1$$

Hence for stability $r > 0$, which makes ρ_i less than unity for all values of r implying unconditional stability throughout: from (Opiyo Richard *et al*, 2015).

Therefore, Crank-Nicolson scheme is unconditionally stable.

4.2.2. Stability of time

If $\Delta x = \Delta y = h$ and $u_t(x,y,0)=0$

$$2U_{i,j}^{k+1} = 2U_{i,j}^{k} + \frac{\Delta t^2 c^2}{2h^2}\left(U_{i+1,j}^{k+1}+U_{i-1,j}^{k+1}-4U_{i,j}^{k+1}+U_{i,j+1}^{k+1}+U_{i,j-1}^{k+1}+U_{i+1,j}^{k}+U_{i-1,j}^{k}-4U_{i,j}^{k}+U_{i,j+1}^{k}U_{i,j-1}^{k}\right)$$

$$\left(1+\frac{2\Delta t^2c^2}{h^2}\right)U_{i,j}^{k+1}-\frac{\Delta t^2c^2}{2h^2}\left(U_{i+1,j}^{k+1}+U_{i-1,j}^{k+1}+U_{i,j+1}^{k+1}+U_{i,j-1}^{k+1}\right) = \left(1-\frac{2\Delta t^2c^2}{h^2}\right)U_{i,j}^{k}+\frac{\Delta t^2c^2}{2h^2}\left(U_{i+1,j}^{k}+U_{i-1,j}^{k}+U_{i,j+1}^{k}U_{i,j-1}^{k}\right)$$

Coefficient on $U_{i,j}^{k}$ must be non-negative for stability.

Hence, $\left(1-\frac{2\Delta t^2c^2}{h^2}\right) \ge 0$

So $\frac{2\Delta t^2 c^2}{h^2} \le 1 \quad \Rightarrow \Delta t^2 \le \frac{h^2}{2c^2} \quad \Rightarrow \Delta t \le \frac{h}{c\sqrt{2}}$

4.2.3. Truncation error and order of the method

The local truncation error (LTE) of a numerical scheme $T_{Error} = F$ is the error made when evaluating the numerical scheme with the exact solution $u(x_j)$ in place of the numerical solution U_j. The local truncation is directly obtained from the truncation error of the finite difference scheme.

Taylor's series expansion for time dependent

$$U_{i,j}^{k+1} = U_{i,j}^{n} + \frac{\Delta t}{1!}\frac{\partial U_{i,j}^{n}}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 U_{i,j}^{n}}{\partial t^2} + \frac{\Delta t^3}{3!}\frac{\partial^3 U_{i,j}^{n}}{\partial t^3} + \frac{\Delta t^4}{4!}\frac{\partial^4 U_{i,j}^{n}}{\partial t^4} + \dots \tag{4.15}$$

$$U_{i,j}^{k-1} = U_{i,j}^{n} - \frac{\Delta t}{1!}\frac{\partial U_{i,j}^{n}}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 U_{i,j}^{n}}{\partial t^2} - \frac{\Delta t^3}{3!}\frac{\partial^3 U_{i,j}^{n}}{\partial t^3} + \frac{\Delta t^4}{4!}\frac{\partial^4 U_{i,j}^{n}}{\partial t^4} - \dots \tag{4.16}$$

Adding (4.15) and (4.16) we get

$$\frac{\partial^2 U_{i,j}^{k}}{\partial t^2} = \frac{U_{i,j}^{k+1} - 2U_{i,j}^{k} + U_{i,j}^{k-1}}{\Delta t^2} - 2\left(\frac{\left(\Delta t^2\right)}{4!}\frac{\partial^4 U_{i,j}^{k}}{\partial t^4} + \frac{\left(\Delta t^4\right)}{6!}\frac{\partial^6 U_{i,j}^{k}}{\partial t^6} + \dots\right)$$

$$\Rightarrow \frac{\partial^2 U_{i,j}^{k}}{\partial t^2} \approx \frac{U_{i,j}^{k+1} - 2U_{i,j}^{k} + U_{i,j}^{k-1}}{\Delta t^2},\ T_{ErrorZ} = -2\left(\frac{\left(\Delta t^2\right)}{4!}\frac{\partial^4 U_{i,j}^{k}}{\partial t^4} + \frac{\left(\Delta t^4\right)}{6!}\frac{\partial^6 U_{i,j}^{k}}{\partial t^6} + \dots\right) \tag{4.17}$$

The same way we get independent variable x and y

$$\frac{\partial^2 U_{i,j}^{k}}{\partial x^2} = \frac{U_{i+1,j}^{k} - 2U_{i,j}^{k} + U_{i-1,j}^{k}}{\Delta x^2} - 2\left(\frac{\left(\Delta x^2\right)}{4!}\frac{\partial^4 U_{i,j}^{k}}{\partial x^4} + \frac{\left(\Delta x^4\right)}{6!}\frac{\partial^6 U_{i,j}^{k}}{\partial x^6} + \dots\right)$$

$$\Rightarrow \frac{\partial^2 U_{i,j}^{k}}{\partial x^2} \approx \frac{U_{i,j+1}^{k} - 2U_{i,j}^{k} + U_{i,j-1}^{k}}{\Delta x^2},\ T_{ErrorX} = -2\left(\frac{\left(\Delta x^2\right)}{4!}\frac{\partial^4 U_{i,j}^{k}}{\partial x^4} + \frac{\left(\Delta x^4\right)}{6!}\frac{\partial^6 U_{i,j}^{k}}{\partial x^6} + \dots\right) \tag{4.18}$$

$$\frac{\partial^2 U_{i,j}^{k}}{\partial y^2} = \frac{U_{i,j+1}^{k} - 2U_{i,j}^{k} + U_{i,j-1}^{k}}{\Delta y^2} - 2\left(\frac{\left(\Delta y^2\right)}{4!}\frac{\partial^4 U_{i,j}^{k}}{\partial y^4} + \frac{\left(\Delta y^4\right)}{6!}\frac{\partial^6 U_{i,j}^{k}}{\partial y^6} + \dots\right)$$

$$\Rightarrow \frac{\partial^2 U_{i,j}^{k}}{\partial y^2} \approx \frac{U_{i,j+1}^{k} - 2U_{i,j}^{k} + U_{i,j-1}^{k}}{\Delta y^2},\ T_{ErrorY} = -2\left(\frac{\left(\Delta y^2\right)}{4!}\frac{\partial^4 U_{i,j}^{k}}{\partial y^4} + \frac{\left(\Delta y^4\right)}{6!}\frac{\partial^6 U_{i,j}^{k}}{\partial y^6} + \dots\right) \tag{4.19}$$

Substituting Eqs. (4.17), (4.18) and (4.19) in the two dimensions wave equation we obtain

$$\frac{U_{i,j}^{k+1}-2U_{i,j}^{k}+U_{i,j}^{k-1}}{\Delta t^2}+T_{ErrorZ}=c^2\left(\frac{U_{i+1,j}^{k}-2U_{i,j}^{k}+U_{i-1,j}^{k}}{\Delta x^2}+T_{ErrorX}+\frac{U_{i,j+1}^{k}-2U_{i,j}^{k}+U_{i,j-1}^{k}}{\Delta y^2}+T_{ErrorY}\right)$$

$$\frac{U_{i,j}^{k+1}-2U_{i,j}^{k}+U_{i,j}^{k-1}}{\Delta t^2}=c^2\left(\frac{U_{i+1,j}^{k}-2U_{i,j}^{k}+U_{i-1,j}^{k}}{\Delta x^2}+\frac{U_{i,j+1}^{k}-2U_{i,j}^{k}+U_{i,j-1}^{k}}{\Delta y^2}+\left(T_{ErrorX}+T_{ErrorY}\right)\right)-T_{ErrorZ}$$

By Crank-Nicolson scheme:

$$\frac{U_{i,j}^{k+1}-2U_{i,j}^{k}+U_{i,j}^{k-1}}{\Delta t^2}=\frac{c^2}{2}\left(\begin{array}{l}\dfrac{U_{i+1,j}^{k+1}-2U_{i,j}^{k+1}+U_{i-1,j}^{k+1}+U_{i+1,j}^{k}-2U_{i,j}^{k}+U_{i-1,j}^{k}}{\Delta x^2}+\\ \dfrac{U_{i,j+1}^{k}-2U_{i,j}^{k}+U_{i,j-1}^{k}+U_{i,j+1}^{k}-2U_{i,j}^{k}+U_{i,j-1}^{k}}{\Delta y^2}\end{array}\right)+\left(T_{ErrorX}+T_{ErrorY}\right)-T_{ErrorZ} \tag{4.20}$$

Therefore, the truncation error is $\left(T_{ErrorX}+T_{ErrorY}\right)+T_{ErrorZ}$

And the order of this method $\Rightarrow O\left(\Delta t^2,\Delta x^2,\Delta y^2\right)$

4.2.4. Consistency of Crank-Nicolson scheme

A numerical method is called consistent if the local error decays sufficiently fast as $h\to 0$ (mesh size tends to zero). Consistency is the study of the local error. Convergence is the study of the global error. So the two-dimensional wave equation on Eq. (4.1)

At $\Delta x=\Delta y=h$ by Crank-Nicolson scheme we get

$$\frac{U_{i,j}^{k+1}-2U_{i,j}^{k}+U_{i,j}^{k-1}}{\Delta t^2}-\frac{c^2}{2}\left(\frac{U_{i+1,j}^{k+1}+U_{i-1,j}^{k+1}-4U_{i,j}^{k+1}+U_{i,j+1}^{k+1}+U_{i,j-1}^{k+1}+U_{i+1,j}^{k}+U_{i-1,j}^{k}-4U_{i,j}^{k}+U_{i,j+1}^{k}U_{i,j-1}^{k}}{h^2}\right)$$

By performing a Taylor series expansion around the point x_i,y_j,t_k

$$u_{i,j}^{k+1}=\left(u+\Delta t\frac{\partial u}{\partial t}+\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}+\frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}+\frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t^4}+\ldots\right)_{i,j}^{k} \tag{4.21}$$

$$u_{i,j}^{k+1}=\left(u-\Delta t\frac{\partial u}{\partial t}+\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}-\frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}+\frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t^4}-\ldots\right)_{i,j}^{k} \tag{4.22}$$

Adding the Taylor series of Eqs. (4.21) and (4.22)

$$\left(\frac{\partial^2 u}{\partial t^2}+\frac{\Delta t^2}{12}\frac{\partial^4 u}{\partial t^4}+\ldots\right) \tag{4.23}$$

The same way we obtain for x and y

$$u_{i+1,j}^{k}=\left(u+\Delta x\frac{\partial u}{\partial x}+\frac{\Delta x^{2}}{2}\frac{\partial^{2}u}{\partial x^{2}}+\frac{\Delta x^{3}}{6}\frac{\partial^{3}u}{\partial x^{3}}+\frac{\Delta x^{4}}{24}\frac{\partial^{4}u}{\partial x^{4}}+\ldots\right)_{i,j}^{k} \tag{4.24}$$

$$u_{i-1,j}^{k}=\left(u-\Delta x\frac{\partial u}{\partial x}+\frac{\Delta x^{2}}{2}\frac{\partial^{2}u}{\partial x^{2}}-\frac{\Delta x^{3}}{6}\frac{\partial^{3}u}{\partial x^{3}}+\frac{\Delta x^{4}}{24}\frac{\partial^{4}u}{\partial x^{4}}-\ldots\right)_{i,j}^{k} \tag{4.25}$$

$$u_{i,j+1}^{k}=\left(u+\Delta y\frac{\partial u}{\partial y}+\frac{\Delta y^{2}}{2}\frac{\partial^{2}u}{\partial y^{2}}+\frac{\Delta y^{3}}{6}\frac{\partial^{3}u}{\partial y^{3}}+\frac{\Delta y^{4}}{24}\frac{\partial^{4}u}{\partial y^{4}}+\ldots\right)_{i,j}^{k} \tag{4.26}$$

$$u_{i,j-1}^{k}=\left(u-\Delta y\frac{\partial u}{\partial y}+\frac{\Delta y^{2}}{2}\frac{\partial^{2}u}{\partial y^{2}}-\frac{\Delta y^{3}}{6}\frac{\partial^{3}u}{\partial y^{3}}+\frac{\Delta y^{4}}{24}\frac{\partial^{4}u}{\partial y^{4}}-\ldots\right)_{i,j}^{k} \tag{4.27}$$

$$u_{i+1,j}^{k+1}=\left(\begin{array}{l}
u+\Delta t\left(\frac{\partial u}{\partial t}\right)+\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta t)^{2}}{2}\left(\frac{\partial^{2}u}{\partial t^{2}}\right)+\frac{(\Delta x)^{2}}{2}\left(\frac{\partial^{2}u}{\partial x^{2}}\right)+\Delta t\Delta x\left(\frac{\partial^{2}u}{\partial t\partial x}\right)+\\
\frac{(\Delta t)^{3}}{6}\left(\frac{\partial^{3}u}{\partial t^{3}}\right)+\frac{(\Delta x)^{3}}{6}\left(\frac{\partial^{3}u}{\partial x^{3}}\right)+\frac{(\Delta t)^{2}}{2}\Delta x\left(\frac{\partial^{3}u}{\partial t^{2}\partial x}\right)+\Delta t\frac{(\Delta x)^{2}}{2}\left(\frac{\partial^{3}u}{\partial t\partial x^{2}}\right)+\\
\frac{(\Delta t)^{4}}{24}\left(\frac{\partial^{4}u}{\partial t^{4}}\right)+\frac{(\Delta x)^{4}}{24}\left(\frac{\partial^{4}u}{\partial x^{4}}\right)+\frac{(\Delta t)^{2}(\Delta x)^{2}}{4}\left(\frac{\partial^{4}u}{\partial t^{2}\partial x^{2}}\right)+\frac{(\Delta t)^{3}}{6}\Delta x\left(\frac{\partial^{4}u}{\partial t^{3}\partial x}\right)+\\
\Delta t\frac{(\Delta x)^{3}}{6}\left(\frac{\partial^{4}u}{\partial t\partial x^{3}}\right)+\ldots
\end{array}\right)_{i,j}^{k} \tag{4.28}$$

$$u_{i-1,j}^{k+1}=\left(\begin{array}{l}
u+\Delta t\left(\frac{\partial u}{\partial t}\right)-\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta t)^{2}}{2}\left(\frac{\partial^{2}u}{\partial t^{2}}\right)+\frac{(\Delta x)^{2}}{2}\left(\frac{\partial^{2}u}{\partial x^{2}}\right)-\Delta t\Delta x\left(\frac{\partial^{2}u}{\partial t\partial x}\right)+\\
\frac{(\Delta t)^{3}}{6}\left(\frac{\partial^{3}u}{\partial t^{3}}\right)-\frac{(\Delta x)^{3}}{6}\left(\frac{\partial^{3}u}{\partial x^{3}}\right)+\frac{(\Delta t)^{2}}{2}\Delta x\left(\frac{\partial^{3}u}{\partial t^{2}\partial x}\right)-\Delta t\frac{(\Delta x)^{2}}{2}\left(\frac{\partial^{3}u}{\partial t\partial x^{2}}\right)+\\
\frac{(\Delta t)^{4}}{24}\left(\frac{\partial^{4}u}{\partial t^{4}}\right)+\frac{(\Delta x)^{4}}{24}\left(\frac{\partial^{4}u}{\partial x^{4}}\right)+\frac{(\Delta t)^{2}(\Delta x)^{2}}{4}\left(\frac{\partial^{4}u}{\partial t^{2}\partial x^{2}}\right)-\frac{(\Delta t)^{3}}{6}\Delta x\left(\frac{\partial^{4}u}{\partial t^{3}\partial x}\right)\\
-\Delta t\frac{(\Delta x)^{3}}{6}\left(\frac{\partial^{4}u}{\partial t\partial x^{3}}\right)+\ldots
\end{array}\right)_{i,j}^{k} \tag{4.29}$$

$$u_{i,j+1}^{k+1}=\begin{pmatrix} u+\Delta t\left(\frac{\partial u}{\partial t}\right)+\Delta y\left(\frac{\partial u}{\partial y}\right)+\frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)+\frac{(\Delta y)^2}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)+\Delta t\Delta y\left(\frac{\partial^2 u}{\partial t\partial y}\right)+ \\ \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)+\frac{(\Delta y)^3}{6}\left(\frac{\partial^3 u}{\partial y^3}\right)+\frac{(\Delta t)^2}{2}\Delta y\left(\frac{\partial^3 u}{\partial t^2\partial y}\right)+\Delta t\frac{(\Delta y)^2}{2}\left(\frac{\partial^3 u}{\partial t\partial y^2}\right)+ \\ \frac{(\Delta t)^4}{24}\left(\frac{\partial^4 u}{\partial t^4}\right)+\frac{(\Delta y)^4}{24}\left(\frac{\partial^4 u}{\partial y^4}\right)+\frac{(\Delta t)^2(\Delta y)^2}{4}\left(\frac{\partial^4 u}{\partial t^2\partial y^2}\right)+\frac{(\Delta t)^3}{6}\Delta y\left(\frac{\partial^4 u}{\partial t^3\partial y}\right)+ \\ \Delta t\frac{(\Delta y)^3}{6}\left(\frac{\partial^4 u}{\partial t\partial y^3}\right)+\ldots \end{pmatrix}_{i,j}^{k} \tag{4.30}$$

$$u_{i,j-1}^{k+1}=\begin{pmatrix} u+\Delta t\left(\frac{\partial u}{\partial t}\right)-\Delta y\left(\frac{\partial u}{\partial y}\right)+\frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)+\frac{(\Delta y)^2}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)-\Delta t\Delta y\left(\frac{\partial^2 u}{\partial t\partial y}\right)+ \\ \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)-\frac{(\Delta y)^3}{6}\left(\frac{\partial^3 u}{\partial y^3}\right)+\frac{(\Delta t)^2}{2}\Delta y\left(\frac{\partial^3 u}{\partial t^2\partial y}\right)-\Delta t\frac{(\Delta y)^2}{2}\left(\frac{\partial^3 u}{\partial t\partial y^2}\right)+ \\ \frac{(\Delta t)^4}{24}\left(\frac{\partial^4 u}{\partial t^4}\right)+\frac{(\Delta y)^4}{24}\left(\frac{\partial^4 u}{\partial y^4}\right)+\frac{(\Delta t)^2(\Delta y)^2}{4}\left(\frac{\partial^4 u}{\partial t^2\partial y^2}\right)-\frac{(\Delta t)^3}{6}\Delta y\left(\frac{\partial^4 u}{\partial t^3\partial y}\right) \\ -\Delta t\frac{(\Delta y)^3}{6}\left(\frac{\partial^4 u}{\partial t\partial y^3}\right)+\ldots \end{pmatrix}_{i,j}^{k} \tag{4.31}$$

From Eq. (4.24) to Eq. (4.31) the local truncation error is

$$T_{error}=\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)+\frac{\Delta t}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)+\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)-\frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)-$$
$$\frac{(\Delta y)^2}{6}\left(\frac{\partial^3 u}{\partial y^3}\right)+O\left(\left(\Delta t^3\right)+O\left(\Delta x^3\right)+O\left(\Delta y^3\right)\right)$$

So that $\lim\limits_{\Delta t\to 0,h\to 0} T_{error}=0$

Due to (Lax's Equivalence Theorem): Given a properly posed initial- value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence. Therefore Crank-Nicolson method is consistent.

Thus, when we conclude the Crank-Nicolson method is convergent for two-dimensional wave equation. Because of

Stability and Consistency $\Leftrightarrow$ Convergence

Generally:

- Stability is numerical errors which are generated during the solution of discretized equations should not be magnified.
- Consistency is the discretization of a Partial Differential Equation should become exact as the mesh size tends to zero (truncation error should vanish).
- Convergence is the numerical solution should approach the exact solution of the Partial Differential Equation and converge to it as the mesh size tends to zero.

4.3. Numerical Results

In order to test the efficiency and adaptability of this proposed method, a computational experiment is done on two examples for which the exact solutions of are known to us. The computed solutions are displayed in terms of exact solution, numerical solution and absolute error (i.e. the error taken between the exact value and the computed value using this method) for some grid points. The results for these test problems are reported in Tables 1 to 2.

Example 1: Consider the two-dimensional wave equation of the form:

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}\ ,\ \ x,y\in[0,1]\times[0,1],t\geq 0\quad ,c=2$$

The initial conditions

$$u(x,y,0)=\sin(\pi x)\sin(2\pi y)\ \ ,\ x,y\in(0,1)\times(0,1)$$

$$u_t(x,y,0)=0$$

And the boundary conditions

$$u(0,y.t)=u(1,y,t)=0,\quad u(x,0,t)=u(x,1,t)=0,$$

The exact solution is:

$$u(x,y,t)=\sin(\pi x)\sin(2\pi y)\cos(\sqrt{5}\pi t)$$

Example 2: Consider the two-dimensional wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),x,y\in[0,1]\times[0,1],c=1,t\geq 0$$

Initial conditions:

$$u(x,y,0)=\sin\left(\frac{1}{c}\pi x\right)\sin\left(\frac{1}{c}\pi y\right),x,y\in(0,1)$$

$$u_t(x,y,0)=0$$

Boundary conditions:

$$u(0,y.t)=u(1,y,t)=0,\quad u(x,0,t)=u(x,1,t)=0.$$

The exact solution is:

$$u(x,y,t)=\sin\left(\frac{1}{c}\pi x\right)\sin\left(\frac{1}{c}\pi y\right)\cos\left(\sqrt{2}\pi t\right)$$

Table 1: The absolute error of example 1

Δt	$(\Delta x, \Delta y)$	Exact solution	Numerical solution	Absolute error
0.035	(0.1, 0.1)	1.76173e-01	1.74814e-01	1.35903e-03
	(0.05, 0.1)	8.91846e-02	8.26268e-02	6.55777e-03
	(0.03, 0.1)	5.36519e-02	5.21268e-02	1.52515e-03
	(0.025, 0.1)	4.47302e-02	4.51871e-02	4.56955e-04
0.017	(0.1, 0.05)	9.48114e-02	9.35176e-02	1.29375e-03
	(0.05, 0.05)	4.79966e-02	4.57689e-02	2.22772e-03
	(0.03, 0.05)	2.88739e-02	2.95612e-02	6.87222e-04
	(0.025, 0.05)	2.40725e-02	2.53616e-02	1.28907e-03
0.012	(0.1, 0.03)	5.76984e-02	5.81287e-02	4.30265e-04
	(0.05, 0.03)	2.92088e-02	2.68622e-02	2.34657e-03
	(0.03, 0.03)	1.75715e-02	1.61873e-02	1.38424e-03
	(0.025, 0.03)	1.46496e-02	1.31232e-02	1.52640e-03
0.0088	(0.1, 0.025)	4.82486e-02	4.10551e-02	7.19351e-03
	(0.05, 0.025)	2.44250e-02	1.92489e-02	5.17614e-03
	(0.03, 0.025)	1.46937e-02	1.42847e-02	4.08998e-04
	(0.025, 0.025)	1.22503e-02	1.11034e-02	1.14686e-03

Table 2: The absolute error of example 2

Δt	$(\Delta x, \Delta y)$	Exact solution	Numerical solution	Absolute error
0.07	(0.1, 0.1)	9.09105e-02	9.01872e-02	7.23304e-04
	(0.05, 0.1)	4.60219e-02	4.40901e-02	1.93176e-03
	(0.03, 0.1)	2.76860e-02	2.24453e-02	5.24071e-03
	(0.025, 0.1)	2.30821e-02	1.73397e-02	5.74239e-03
0.035	(0.1, 0.05)	4.77576e-02	4.11927e-02	6.56492e-03
	(0.05, 0.05)	2.41765e-02	2.42718e-02	9.53398e-05
	(0.03, 0.05)	1.45442e-02	1.42899e-02	2.54204e-04
	(0.025, 0.05)	1.21256e-02	1.14968e-02	6.28811e-04
0.021	(0.1, 0.03)	2.89546e-02	2.91683e-02	2.13678e-04
	(0.05, 0.03)	1.46578e-02	1.46081e-02	4.96992e-05
	(0.03, 0.03)	8.81786e-03	8.63809e-03	1.79769e-04
	(0.025, 0.03)	7.35154e-03	7.31780e-03	3.37355e-05
0.017	(0.1, 0.025)	2.41761e-02	2.45731e-02	3.96989e-04
	(0.05, 0.025)	1.22387e-02	1.21268e-02	1.11963e-04
	(0.03, 0.025)	7.36260e-03	7.32752e-03	3.50868e-05
	(0.025, 0.025)	6.13828e-03	6.17461e-03	3.63311e-05

4.4. Discussion

In this thesis, Crank-Nicholson Scheme has been presented for solving two-dimensional wave equation in rectangular coordinate system with the given boundary condition. The numerical results have been presented in Tables 1 to 2 for different values of mesh points. The results obtained by the present methods have been compared with the exact solution. As we can observe the results from the tables, this methods was approximates to the exact solution very well. So, Crank-Nicolson method is good for two-dimensional wave equation, since it is easy to solve the resulting tridiagonal system of equations. Thus the tridiagonal system of equation was solved by Thomas Algorithm. Finally we concluded that the two dimensional wave equation by Crank-Nicholson scheme applied in the present paper gives good approximation results.

CHAPTER FIVE

CONCLUSION AND FETURE WORK

5.1. Conclusions

In this thesis, we have transformed the two dimensional wave equation in rectangular coordinates system in to a system of linear equations using Crank-Nicholson scheme. The resulting large number of algebraic equation was arranged in order to get a block matrix. From block matrix we obtain system of linear algebraic equation and changes to tridiagonal matrices by collecting like terms. We have implemented Crank-Nicholson method to find the solution of the two-dimensional wave equation. And we can be easily concluded that from Tables 1 to 2 that produces good approximation results in comparing with exact solution.

5.2. Scope for future work

In the present thesis, the numerical methods based on Crank-Nicholson scheme were constructed for solving two dimensional wave equation in rectangular coordinates system. Hence, the schemes proposed in this thesis can also be extended to solve three dimensional wave equation by Crank-Nicholson scheme in rectangular coordinates system.

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APPENDIX

6.1. Matrix Generation

This matrix depend on the solved examples

Set one, $k=0,1,2,...,P,\ j=1, i=1,2,3,...,M$

$$\begin{aligned}
&(-1-2r-e)u_{1,1}^{k+1}+ru_{2,1}^{k+1}=-(2-2r-3e)u_{1,1}^{k}-ru_{2,1}^{k}+u_{1,1}^{k-1}\\
&ru_{1,1}^{k+1}+(-1-2r-e)u_{2,1}^{k+1}+ru_{3,1}^{k+1}=-ru_{1,1}^{k}-(2-2r-3e)u_{2,1}^{k}-ru_{3,1}^{k}+u_{2,1}^{k-1}\\
&ru_{2,1}^{k+1}+(-1-2r-e)u_{3,1}^{k+1}+ru_{4,1}^{k+1}=-ru_{2,1}^{k}-(2-2r-3e)u_{3,1}^{k}-ru_{4,1}^{k}+u_{3,1}^{k-1}\\
&ru_{3,1}^{k+1}+(-1-2r-e)u_{4,1}^{k+1}+ru_{5,1}^{k+1}=-ru_{3,1}^{k}-(2-2r-3e)u_{4,1}^{k}-ru_{5,1}^{k}+u_{4,1}^{k-1}\\
&\qquad\vdots \quad = \quad \vdots
\end{aligned}$$

In the set two, we set $j=2$ to generate the system of equations

$$\begin{aligned}
&(-1-2r)u_{1,2}^{k+1}+ru_{2,2}^{k+1}=-(2-2r-4e)u_{1,2}^{k}-ru_{2,2}^{k}+u_{1,2}^{k-1}\\
&ru_{1,2}^{k+1}+(-1-2r)u_{2,2}^{k+1}+ru_{3,2}^{k+1}=-ru_{1,2}^{k}-(2-2r-4e)u_{2,2}^{k}-ru_{3,2}^{k}+u_{2,2}^{k-1}\\
&ru_{2,2}^{k+1}+(-1-2r)u_{3,2}^{k+1}+ru_{4,2}^{k+1}=-ru_{2,2}^{k}-(2-2r-4e)u_{3,2}^{k}-ru_{4,2}^{k}+u_{3,2}^{k-1}\\
&ru_{3,2}^{k+1}+(-1-2r)u_{4,2}^{k+1}+ru_{5,2}^{k+1}=-ru_{3,2}^{k}-(2-2r-4e)u_{4,2}^{k}-ru_{5,2}^{k}+u_{4,2}^{k-1}\\
&\qquad\vdots \quad = \quad \vdots
\end{aligned}$$

Continuing in the same trend, we set $j=3$ to give

$$\begin{aligned}
&(-1-2r)u_{1,3}^{k+1}+ru_{2,3}^{k+1}=-(2-2r-4e)u_{1,3}^{k}-ru_{2,3}^{k}+u_{1,3}^{k-1}\\
&ru_{1,3}^{k+1}+(-1-2r)u_{2,3}^{k+1}+ru_{3,3}^{k+1}=-ru_{1,3}^{k}-(2-2r-4e)u_{2,3}^{k}-ru_{3,3}^{k}+u_{2,3}^{k-1}\\
&ru_{2,3}^{k+1}+(-1-2r)u_{3,3}^{k+1}+ru_{4,3}^{k+1}=-ru_{2,3}^{k}-(2-2r-4e)u_{3,3}^{k}-ru_{4,3}^{k}+u_{3,3}^{k-1}\\
&ru_{3,3}^{k+1}+(-1-2r)u_{4,3}^{k+1}+ru_{5,3}^{k+1}=-ru_{3,3}^{k}-(2-2r-4e)u_{4,3}^{k}-ru_{5,3}^{k}+u_{4,3}^{k-1}\\
&\qquad\vdots \quad = \quad \vdots
\end{aligned}$$

Setting $j=4$ yields

$$\begin{aligned}
&(-1-2r)u_{1,4}^{k+1}+ru_{2,4}^{k+1}=-(2-2r-4e)u_{1,4}^{k}-ru_{2,4}^{k}+u_{1,4}^{k-1}\\
&ru_{1,4}^{k+1}+(-1-2r)u_{2,4}^{k+1}+ru_{3,4}^{k+1}=-ru_{1,4}^{k}-(2-2r-4e)u_{2,4}^{k}-ru_{3,4}^{k}+u_{2,4}^{k-1}\\
&ru_{2,4}^{k+1}+(-1-2r)u_{3,4}^{k+1}+ru_{4,4}^{k+1}=-ru_{2,4}^{k}-(2-2r-4e)u_{3,4}^{k}-ru_{4,4}^{k}+u_{3,4}^{k-1}\\
&ru_{3,4}^{k+1}+(-1-2r)u_{4,4}^{k+1}+ru_{5,4}^{k+1}=-ru_{3,4}^{k}-(2-2r-4e)u_{4,4}^{k}-ru_{5,4}^{k}+u_{4,4}^{k-1}\\
&\qquad\vdots \quad = \quad \vdots
\end{aligned}$$

$$\cdots \quad \cdots \quad \cdots$$