Numerical Solutions of Second Order Initial Value Problems of Bratu-Type equation Using Higher Ordered Rungu-Kutta Method

Habtamu Garoma Debela[*], Habtamu Bayissa Yadeta[**], Solomon Bati Kejela[***]

Department of Mathematics, Jimma University, Ethiopia

Abstract

This paper presents the fifth order Runge-Kutta method (RK5) to find the numerical solution of the second order initial value problems of Bratu-type ordinary differential equations. In order to justify the validity and effectiveness of the method, we solve three model examples and compare the exact solutions to numerical solutions. The numerical results in terms of point wise absolute errors presented in tables and the plotted graphs show that the present method approximates the exact solution very well. Besides, the stability of the method had been checked and verified.

Index Terms: Initial-Value Problem; Bratu-type Equation; Numerical Solution; Runge-Kutta method

I. INTRODUCTION

Many problems in science and engineering can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial-value problems (IVP) or boundary-value problems (BVP). Many authors have attempted to solve initial value problems (IVP) to obtain high accuracy rapidly by using numerous methods, such as Taylor's method, Runge-Kutta method, and also some other methods. A more robust and intricate numerical technique is the Runge Kutta method. This method is the most widely used one since it gives reliable starting values and is particularly suitable when the computation of higher derivatives is complicated.

On the other hand, according to Abukhaled, M. *et al*.(2012) the standard Bratu problem is used in a large variety of applications, such as the fuel ignition model of the theory of thermal combustion, the thermal reaction process model, the Chandrasekhar model of the expansion of the universe, radiative heat transfer, nanotechnology and theory of chemical reaction.

The Bratu initial value problems have been studied extensively because of its mathematical and physical properties. Batiha, B (2010) studied a numerical solution of Bratu-type equations by the variational iteration method;

Feng *et al*.(2008) considered Bratu's problems by means of modified homotopy perturbation method; Rashidinia, J. *et al.* (2013) applied Sinc-Galerkin method for numerical solution of the Bratu's problems; Syam and Hamdan(2006) used variational iteration method for numerical solutions of the Bratu-type problems; Wazwaz, A(2005) applied Adomian decomposition method to study the Bratu-type equations. L.Jin (2010) applied modified

ISSN 2250-3153

variational iteration method to Bratu- type problems. Ji-Huan He, *et al.* (2014) considered variational iteration method for Bratu-like equations arising in electrospinning. Saravi, M. *et al.* (2013) studied solution of Bratu's equation by He's variation iteration method. Motivated by the above investigations, the objective of the present study is to investigate numerical solutions of second order initial value problems of Bratu-type equation using higher ordered Rungu-kutta method (RK5).

II. FORMULATION OF THE METHOD

Consider the second order initial value problem of Bratu-Type equation of the form:

$$y''(x)+\lambda e^{y(x)}=g(x), \quad 0<x\le l \tag{1}$$

Subject to the initial conditions

$$y(0)=\alpha, \quad y'(0)=\beta \tag{2}$$

where λ, α *and* β are given constant numbers for $y(x)$ is unknown function.

To reduce the order of Eqn. (1), let we use the substitutions $z(x)=y'(x)$ and $z'(x)=y''(x)$, so that the given second order initial value problem of Eqn. (1) with Eqn. (2) can be re-written as:

$$\begin{cases} y'(x)=z(x)=F(x,y,\ z),\, y(0)=\alpha \\ \quad z'(x)=g(x)-\lambda e^{y(x)}=G(x,y,z), \quad z(0)=y'(0)=\beta \end{cases} \tag{3}$$

Dividing the interval $[0, l]$ into N equal subinterval of mesh length h and the mesh point is given by $x_i=x_0+ih,$ for $i=1,2,\dots,N-1.$ For the sake of simplicity let use the denotation: $y(x_i)=y_i$, $z(x_i)=z_i$, $g(x_i)=g_i$, etc. Thus, at the nodal point x_i Eqn. (3), is written as:

$$\begin{cases} y_i'=F(x_i,y_i,\ z_i), \quad y(0)=\alpha \\ z_i'=G(x_i,y_i,z_i), \quad z(0)=\beta \end{cases} \tag{4}$$

where $G(x_i,y_i,z_i)=g_i-\lambda e^{y(x)}$

To solve each initial value problems written in Eqn. (4), we apply the single step methods that require information about the solution at x_i to calculate at x_{i+1}, (Grewal, 2002), (Jain et al, 2007). From one of the single step methods and the family of Runge Kutta methods, the general numerical solution of Eqn. (4) using the fifth Runge Kutta method given as:

$$\begin{cases} y_{i+1} = y_i + \sum_{i=1}^{5} w_i k_i \\ z_{i+1} = z_i + \sum_{i=1}^{5} w_i k_i \end{cases} \tag{5}$$

where $\begin{cases} k_i = hF(x_i + c_i h,\ y_i + \sum_{j=1}^{4} a_{ij} k_j,\ z_i + \sum_{j=1}^{4} a_{ij} m_j) \\ m_i = hG(x_i + c_i h,\ y_i + \sum_{j=1}^{4} a_{ij} k_j,\ z_i + \sum_{j=1}^{4} a_{ij} m_j) \end{cases}$

Christodoulou (2009), was present the fifth order Runge Kutta method to solve a first order initial value problem of the form $\frac{dy}{dt} = f(t, y),\ \ y(t_0) = y_0$, which is given by the following equation:

$$y_{n+1} = y_n + \frac{7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6}{90} \tag{6}$$

where $k_1 = hf(x_n, y_n),\quad k_2 = f(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}),\quad k_3 = f(t_n + \frac{h}{4}, y_n + \frac{3k_1 + k_2}{16})$

$$k_4 = f(t_n + \frac{h}{2}, y_n + \frac{k_3}{2}),\qquad k_5 = f(t_n + \frac{3h}{4}, y_n + \frac{-3k_2 + 6k_3 + 9k_4}{16})$$

$$k_6 = f(t_n + h, y_n + \frac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7})$$

Thus, to solve the system of initial value problem of Eqn. (3), the fifth order Runge Kutta method can be re-written as:

$$\begin{cases} y_{i+1} = y_i + \frac{7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6}{90} \\ z_{i+1} = z_i + \frac{7m_1 + 32m_3 + 12m_4 + 32m_5 + 7m_6}{90} \end{cases} \tag{7}$$

where: $k_1 = F(x_i, y_i, z_i),\qquad m_1 = G(x_i, y_i, z_i),$

$$k_2 = hF(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{m_1}{2})\qquad m_2 = hG(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{m_1}{2})$$

$$k_3 = hF(x_i + \frac{h}{4}, y_i + \frac{3k_1 + k_2}{16}, z_i + \frac{3m_1 + m_2}{16}),\ m_3 = hG(x_i + \frac{h}{4}, y_i + \frac{3k_1 + k_2}{16}, z_i + \frac{3m_1 + m_2}{16})$$

$$k_4 = hF(x_i + \frac{h}{2}, y_i + \frac{k_3}{2}, z_i + \frac{m_3}{2}) \quad , \qquad m_4 = hG(x_i + \frac{h}{2}, y_i + \frac{k_3}{2}, z_i + \frac{m_3}{2})$$

$$k_5 = hF(x_i + \frac{3h}{4}, y_i + \frac{-3k_2 + 6k_3 + 9k_4}{16}, z_i + \frac{-3m_2 + 6m_3 + 9m_4}{16})$$

$$m_5 = hG(x_i + \frac{3h}{4}, y_i + \frac{-3k_2 + 6k_3 + 9k_4}{16}, z_i + \frac{-3m_2 + 6m_3 + 9m_4}{16})$$

$$k_6 = hF(x_i + h, y_i + \frac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7}, z_i + \frac{m_1 + 4m_2 + 6m_3 - 12m_4 + 8m_5}{7})$$

$$m_6 = hG(x_i + h, y_i + \frac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7}, z_i + \frac{m_1 + 4m_2 + 6m_3 - 12m_4 + 8m_5}{7})$$

In the determination of the parameters, since the terms are up to $0(h^5)$ be compared, the truncation error is $O(h^6)$ and the order of method is $O(h^5)$. (Grewal, 2002) and (Jain et al, 2007).

III. STABILITY ANALYSIS

Here, the second order initial value problem Bratu-Type equation of the form of Eqn. (1) is reduced into first order system of equations of the form of Eqn. (4) and let take the second equation from Eqn. (4), then we have:

$$z_i' = G(x_i, y_i, z_i), \quad z(0) = \beta \tag{8}$$

where $G(x_i, y_i, z_i) = g_i - \lambda e^{y_i}$

The nonlinear function Eqn. (8) can be linearized by expanding the function G in Taylor series about the point (x_0, y_0, z_0) and truncating it after the first term as:

$$\begin{aligned} z' = G(x_0, y_0, z_0) + (x - x_0)\frac{\partial G}{\partial x}(x_0, y_0, z_0) + (y - y_0)\frac{\partial G}{\partial y}(x_0, y_0, z_0) \\ + (z - z_0)\frac{\partial G}{\partial z}(x_0, y_0, z_0) \end{aligned} \tag{9}$$

By the differentiation rules of function of several variables Eqn. (9) can be written as:

$$\begin{aligned} z' &= g_0 - \lambda e^{y_0} + (x - x_0)(g_0' - \lambda y_0' e^{y_0}) + \lambda (y_0 - y) e^{y_0} \\ &= C \end{aligned} \tag{10}$$

where $C = g_0 - \lambda e^{y_0} + (x - x_0)(g_0' - \lambda y_0' e^{y_0}) + \lambda (y_0 - y) e^{y_0}$

$\Rightarrow z' = C$, which is linear in the function of z.

Let the test function is $y' = \lambda y$ (11)

which is called the linear test equation for the non-linear Eq. (8). The solution of the test equation, Eqn. (11) is:

$$y_n = y(x_n) = y(x_0) e^{\lambda nh} = y_0 (e^{\lambda h})^n \tag{12}$$

where y_0 is constant.

Now, by applying Eqn. (6) on Eqn. (12), we have:

$$m_1 = \lambda h y_i, \quad m_2 = \lambda h y_i (1 + \frac{1}{2}\lambda h), \quad m_3 = h\lambda y_i + \frac{h^2\lambda^2}{4} y_i + \frac{h^3\lambda^3}{32} y_i$$

$$m_4 = \lambda h y_i (1 + \frac{\lambda h}{2} + \frac{h^2\lambda^2}{8} + \frac{h^3\lambda^3}{64})$$

$$m_5 = \lambda h y_i + \frac{3}{4}\lambda^2 h^2 y_i + \frac{9}{32}\lambda^3 h^3 y_i + \frac{21}{256} h^4 \lambda^4 y_i + \frac{9}{1024}\lambda^5 h^5 y_i$$

$$m_6 = \lambda h y_i + \lambda^2 h^2 y_i + \frac{1}{2}\lambda^3 h^3 y_i + \frac{15}{112} h^4 \lambda^4 y_i + \frac{15}{224} h^5 \lambda^5 y_i + \frac{9}{896}\lambda^6 h^6 y_i$$

By substituting the values of m_1 and $m_3 - m_6$ into $y_{i+1} = y_i + \frac{7m_1 + 32m_3 + 12m_4 + 32m_5 + 7m_6}{90}$, we obtain:

$$y_{i+1} = (1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \frac{1}{6}\lambda^3 h^3 + \frac{1}{24}\lambda^4 h^4 + \frac{1}{120}\lambda^5 h^5 + \frac{1}{1280}\lambda^6 h^6) y_i$$

$$y_{i+1} = E(\lambda h) y_i \tag{13}$$

where: $E(\lambda h) = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \frac{1}{6}\lambda^3 h^3 + \frac{1}{24}\lambda^4 h^4 + \frac{1}{120}\lambda^5 h^5 + \frac{1}{1280}\lambda^6 h^6$

The errors in numerical computation don't grow, if the propagation error tends to zero or if at least bounded, (Jain et al, 2007). Now, from Eqn. (12), it is easily observed, the exact value of $y(x_i)$ increases for the constant $\lambda > 0$ and decreases for $\lambda < 0$ with the factor of $e^{\lambda h}$. While from Eqn. (12) the approximate value of y_i increases or decreases with the factor of $E(\lambda h)$. If $\lambda h > 0$, then $e^{\lambda h} \geq 1$; so the fifth order Runge Kutta method is relatively stable. If $\lambda h < 0$, (*i.e.*, $\lambda < 0$) then the fifth order Runge Kutta method is absolutely stable in the interval of $-5.604 < \lambda h < 0.$

IV. NUMERICAL EXAMPLES AND RESULTS

As discussed above, we are implementing the fifth order Runge Kutta method on three model example of the second order initial value problems of Bratu's - type equation as follows:

Example 1: Consider the Bratu-type initial value problem

$$y'' - 2e^{y} = 0; \quad 0 < x < 1$$

$$y(0) = 0, \quad y'(0) = 0$$

whose exact solution is $y(x) = -2\ln(\cos(x))$

Table 1. Comparison of absolute errors for example 1

	Absolute errors at $h = 0.1$				h = 0.01
x	Darwish and Kashkari, 2014	*Eslam Moradi*, 2015	Sinan and Necdet, 2016	Our Method (RK5)	Our Method (RK5
0.1	6.41021065e-7	1.6674e-5	9.4728e-6	4.9027e-9	4.8904e-015
0.2	9.74693876e-6	3.1000e-7	3.3152e-5	1.0071e-8	9.7697e-015
0.3	4.52998213e-5	1.1310e-6	2.7254e-5	1.5973e-8	1.4542e-014
0.4	1.27118347e-4	21200e-6	4.4563e-6	2.3233e-8	1.9526e-014
0.5	2.68671650e-4	2.9000e-6	5.5511e-8	3.2789e-8	2.4300e-014
0.6	4.83656903e-4	4.1000e-6	7.2047e-5	4.6204e-8	2.9238e-014
0.7	8.36799541e-4	6.5000e-6	7.0044e-5	6.6335e-8	3.4202e-014
0.8	1.60053795e-3	7.5000e-6	1.2821e-4	9.8959e-8	3.9077e-014
0.9	3.64970628e-3	3.3500e-5	4.5236e-4	1.5718e-7	4.4138e-014
1.0	9.39151960e-3	4.3700e-5	4.4409e-8	2.7544e-7	4.9098e-014

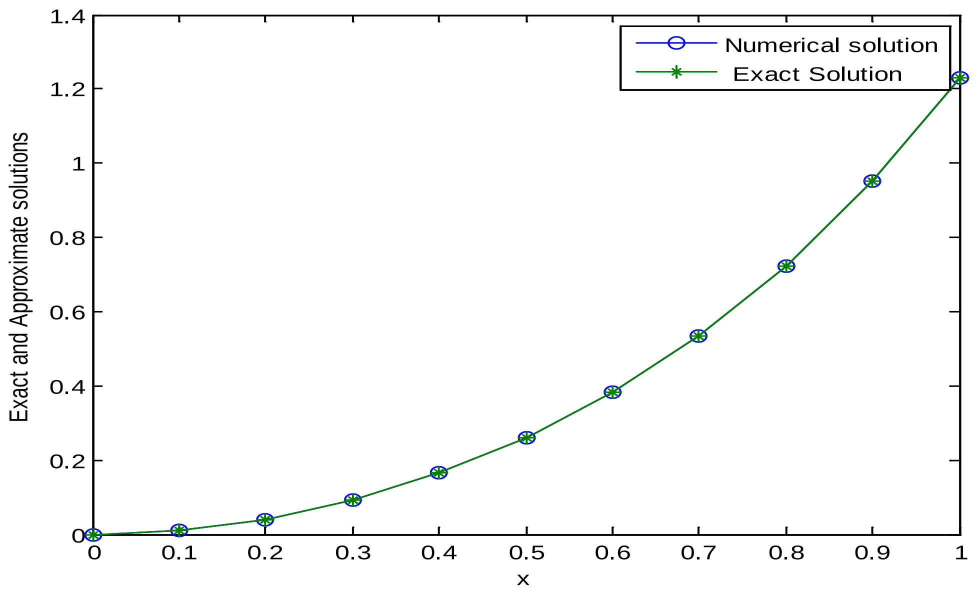

Figure 1: Plot of exact and approximated solutions of Biratu type- Equation using RK5 for $h = 0.1$

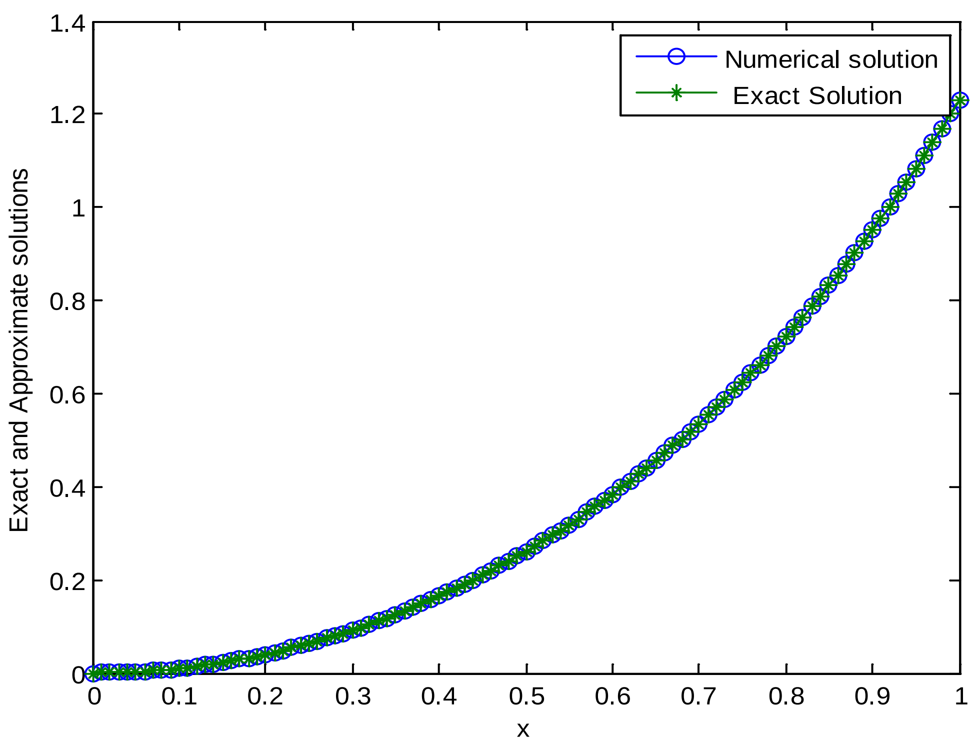

Figure 2: Plot of exact and approximated solution of Biratu type- Equation using RK5 for $h = 0.1$

Example 2: Consider the Bratu-type initial value problem

$$\frac{d^2y}{dx^2} = -\pi^2 e^{-y}; \quad y(0)=0, \quad y'(0)=\pi$$

Whose exact solution is $y(x) = \ln(1+\sin(\pi x))$

Table 2: Comparison of absolute errors, for example 2

	Absolute errors at $h=0.1$		$h=0.01$
x	*Eslam Moradi*, 2015	Our Method (RK5)	Our Method (RK5)
0.1	3.20777e-4	8.3507e-008	2.2668e-013
0.2	2.37600e-5	4.4458e-008	4.1313e-013
0.3	3.58700e-5	2.6380e-007	5.6410e-013
0.4	8.01000e-5	5.3872e-007	6.8325e-013
0.5	1.19500e-4	8.6014e-007	7.7396e-013
0.6	1.66200e-4	1.2299e-006	8.3944e-013
0.7	2.20200e-4	1.6559e-006	8.8199e-013
0.8	2.85100e-4	2.1494e-006	9.0389e-013
0.9	4.03400e-4	2.7194e-006	9.0730e-013
1.0	5.37400e-4	3.3454e-006	8.9412e-013

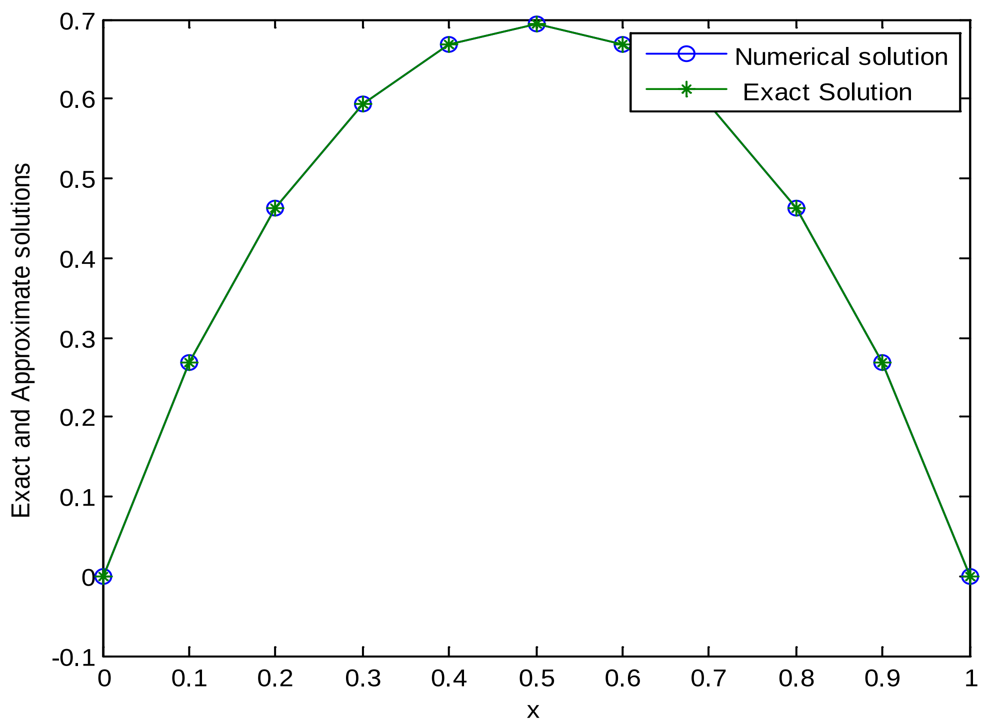

Figure 3: Plot of exact and approximated solutions of Biratu type- Equation using RK5 for $h=0.1$

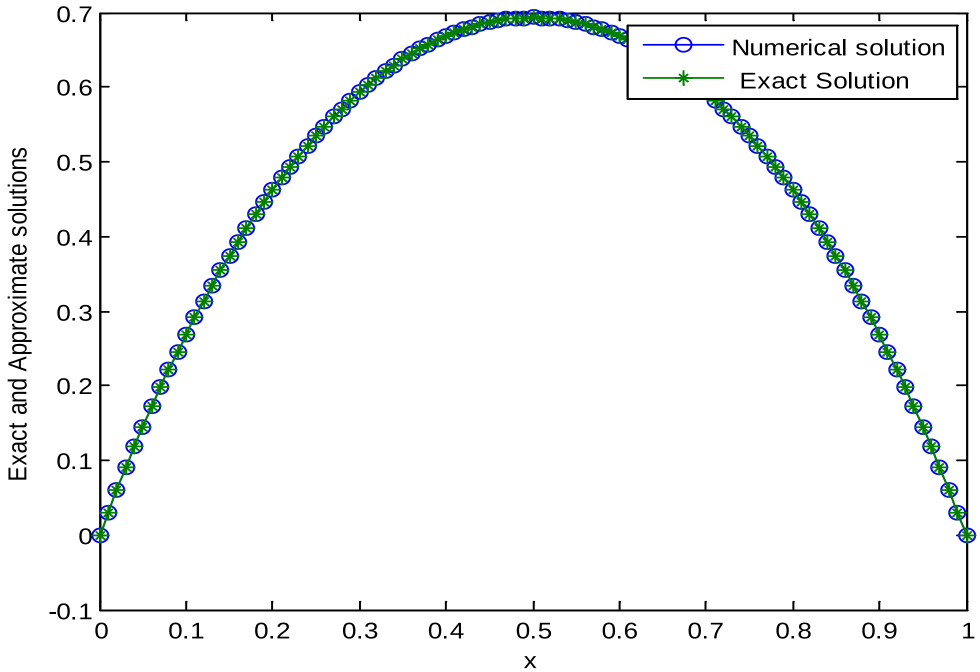

Figure 4: Plot of exact and approximated solution of Biratu type- Equation using RK5 for $h = 0.01$

Example 3: Consider the Bratu-type initial value problem

$$y'' - e^{2y} = 0; \qquad 0 < x < 1$$

$$y(0) = 0, \qquad y'(0) = 0$$

whose exact solution is $y(x) = \ln(\sec(x))$:

Table 3: Comparison of absolute errors, for example 3

	Absolute errors at $h = 0.1$		$h = 0.01$
x	*Eslam Moradi*, 2015	Our Method (RK5)	Our Method (RK5)
0.1	8.3330e-6	2.4514e-009	2.3599e-015
0.2	6.4000e-7	5.0353e-009	4.8279e-015
0.3	6.4000e-7	7.9863e-009	7.3091e-015
0.4	1.0200e-6	1.1616e-008	9.6700e-015
0.5	1.3500e-6	1.6395e-008	1.2101e-014
0.6	1.9000e-6	2.3102e-008	1.4654e-014
0.7	3.2000e-6	3.3168e-008	1.7124e-014
0.8	3.5000e-6	4.9480e-008	1.9546e-014
0.9	1.6500e-5	7.8591e-008	2.2018e-014
1.0	2.1400e-5	1.3772e-007	2.4441e-014

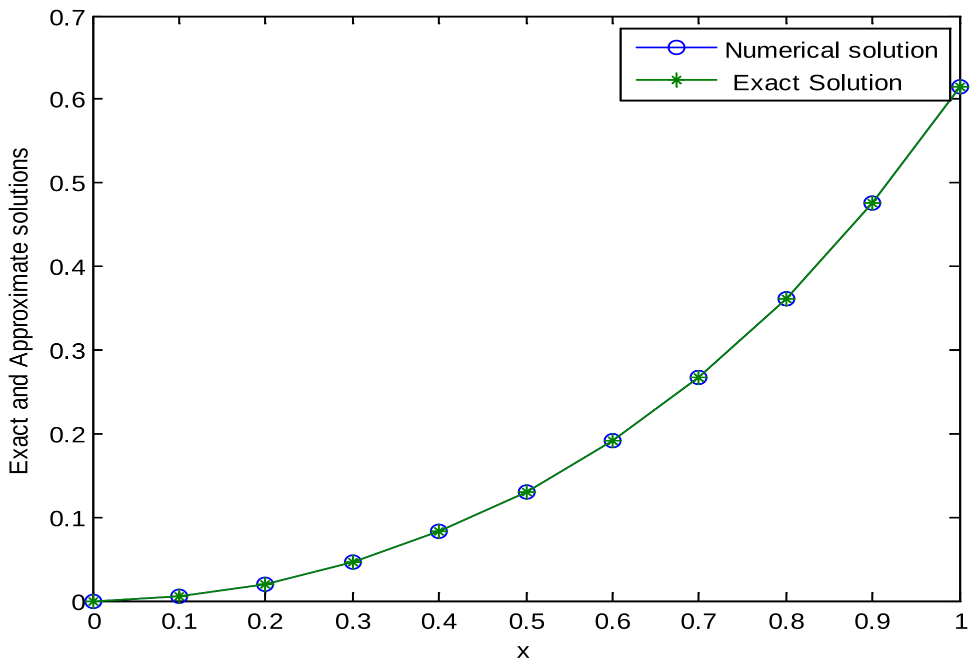

Figure 5: Plot of exact and approximated solutions of Biratu type- Equation using RK5 for $h = 0.1$

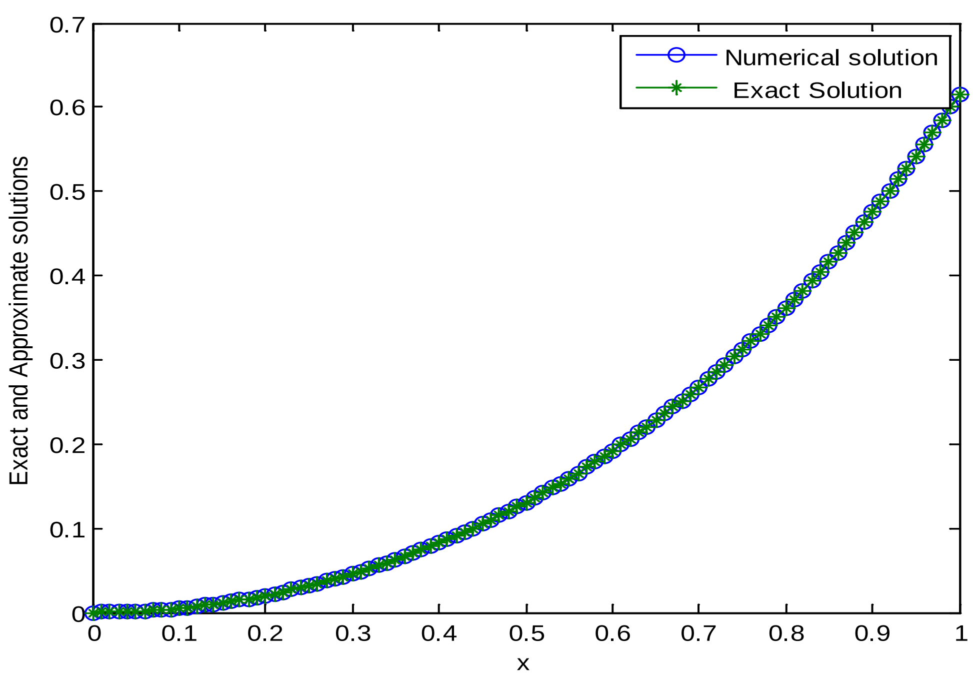

Figure 6: Plot of exact and approximated solution by Biratu type- Equation using RK for $h = 0.01$

V. DISCUSSION AND CONCLUSION

In this paper we presented higher order Rang – Kutta (RK5) method to investigate numerical solutions of second order initial value problems of Bratu-type equation. To further justify the applicability of the proposed method; tables of point wise absolute errors and graphs have been plotted for the three model examples to compare the exact solutions and numerical solutions at different mesh size *h*. Tables 1, 2 and 3 depicted that the fifth order Rung – Kutta method improves the findings of (Darwish and Kashkari, 2014), (*Eslam Moradi*, 2015), (Sinan and Necdet, 2016). Moreover, it is evident that all the absolute errors decrease rapidly as the mesh size *h* decreases, which in turn shows that the smaller mesh size provides the better approximate value. Figures (1 – 6) show that the present method approximates the exact solution in an excellent manner.

VI. REFERENCES

[1] Abukhaled, M., Khuri, S. and Sayfy, A. Spline-Based Numerical Treatments of Bratu-Type Equations. *Pales- tine Journal of Mathematics*, **1**, 63-70.

[2] Wazwaz, A. (2012) A Reliable Study for Extensions of the Bratu Problem with Boundary Conditions. *MathematicalMethods in the Applied Sciences*, **35**, 845-856. http://dx.doi.org/10.1002/mma.1616

[3] Batiha, B. (2010) Numerical Solution of Bratu-Type Equations by the Variational Iteration Method. *Hacettepe Journal of Mathematics and Statistics*, **39**, 23-29.

[4] Feng, X., He, Y. and Meng, J. (2008) Application of Homotopy Perturbation Method to the Bratu-Type Equations.*Topological Methods in Nonlinear Analysis*, **31**, 243-252

[5] Rashidinia, J., Maleknejad, K. and Taheri, N. (2013) Sinc-Galerkin Method for Numerical Solution of the Bratu's Problems. *Numerical Algorithms*, **62**, 1-11. http://dx.doi.org/10.1007/s11075-012-9560-3

[6] Syam, M.I. and Hamdan, A. (2006) An Efficient Method for Solving Bratu Equations. *Applied Mathematics andComputation*, **176**, 704-713. http://dx.doi.org/10.1016/j.amc.2005.10.021

[7] Wazwaz, A. (2005) Adomian Decomposition Method for a Reliable Treatment of the Bratu-Type Equations. *Applied Mathematics and Computation*, **166**, 652-663

[8] Grewal B'S (2002),Numerical method in engineering and science with programs in FORTRAN 77, C and C++ ,khanna publisher sixth edition

[9] Jain, M. K., Iyengar, S.R.K, and Jain, R.K.(2007). Numerical methods for scientific and Engineering Computations; equations, 3rd edition.

[10] A. Islam.2015. Accurate Solutions of Initial Value Problems for Ordinary Differential Equations with Fourth Order Runge Kutta Method. Journal of Mathematics Research, 7, 41-45.

[11] Nikolaos S. Christodoulou. 2009. An Algorithm Using Runge-Kutta Methods OfOrders 4 And 5 For Systems Of Odes, International Journal of Numerical Methods and Applications, 2(1), 47 – 57.

[12] L. Jin, "Application of Modified Variational Iteration Me- thod to the Bratu-Type Problems," *International Jour- nal of Engineering, Contemporary Mathematics and Sci- ences*, Vol. 4, 2010, pp. 153-158.

[13] Saravi, M., Hermann, M., & Kaiser, D. (2013). Solution of Bratu's equation by He's variational iteration method. American Journal of Computational and Applied Mathematics, 3, 46–48

VII. AUTHORS

First Author- Habtamu Garoma Debela, Jimma University, Ethiopia, E-mail: habte200@gmail.com

Second Author - Habtamu Bayissa Yadeta, Jimma University, Ethiopia, E – mail: habtamubayissa@gmail.com

Third Author- Solomon Bati Kejela, Jimma University, Ethiopia, E – mail: solomonbati@yahoo.com