NUMERICAL SOLUTIONS OF SECOND ORDER INITIAL VALUE PROBLEMS OF BRATU-TYPE EQUATION USING PREDICTOR-CORRECTOR METHOD

A Thesis submitted to department of mathematics, Jimma University in partial fulfillment for the requirements of the degree of masters of Science in Mathematics

By:-Mengistu Mekonnen

Under the Supervision of:

Genanew Gofe (PhD)

Habtamu Garoma (MSc)

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Jimma, Ethiopia

Declaration

I, undersigned declare that, this research entitled "**Numerical Solutions of Second Order Initial Value Problems of Bratu-Type Equation Using Predictor-Corrector Method**" is my own original work and it has not been submitted for the award of any academic degree or the like in this or in any other university and that all the sources I have used or quoted have been indicated and acknowledged.

Name: **Mengistu Mekonnen**

Signature: ------------------------

Date: -----------------------------

The work has been done under the supervision of:

Name: **Genanew Gofe (PhD**)

Signature: -----------------------------

Date: ---------------------------------

Name: **Habtamu Garoma (MSc)**

Signature: ------------------------------

Date: -----------------------------------

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Abstract

This research paper mainly presents predictor-corrector method for solving initial value problems of second order Bratu-type Equations. In order to verify the accuracy, the numerical solutions are compared with the exact solutions. The numerical solutions are in good agreement with the exact solutions.

The stability and convergence of the method have been investigated. Three model examples are considered to demonstrate the reliability and efficiency of the method. Point wise absolute errors are obtained by using MATLAB software. The proposed method also compared with the existing literatures and the proposed method is quite efficient and practically suited for solving these problems.

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CHAPTER ONE

1. INTRODUCTION

1.1. Background of the Study

Many problems in science and engineering can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations which may be evaluated at either initial-value problems or boundary-value problems (Sewell, 2005; King *et al*., 2003). Many authors have attempted to solve initial value problems (IVP) to obtain highly accurate and rapidly convergent to the solution by using numerous methods, such as Taylor's method, Runge-Kutta method, predictor-corrector method and some other methods.

In numerical analysis, predictor–corrector methods belong to a class of algorithms designed to integrate ordinary differential equations to find an unknown function that satisfies a given differential equation. When considering the numerical solution of ordinary differential equations (ODEs), a predictor–corrector method typically uses an explicit method for the predictor step and an implicit method for the corrector step.

The Bratu-type equations arise from a simplification of the solid fuel ignition model in thermal combustion theory. Studies on fuel ignition in thermal combustion theory have been on the increase over the last few years. The reason for the increased study is to ensure the safety of working environment especially when working with combustible fluid in some petro-chemical engineering process. Combustion problems are generally characterized by strong nonlinearity and singularity (Felobello-Nino *et al*., 2013; Zarebnia and Hoshyar, 2014; Adesanya *et al*., 2013).

The standard nonlinear initial value problems of Bratu-Type equation is given by,

$$u''(x) + \lambda e^{u(x)} = 0, \quad 0 \le x \le l, \text{ where } 0 < l \le 1 \tag{1.1}$$

subject to the initial conditions

$$u(0) = \alpha, \;\; u'(0) = \gamma \tag{1.2}$$

where λ, α and γ are constants $(\lambda \neq 0)$ and $u(x)$ is unknown function.

As Jin, (2010) and Feng *et al*., (2008) stated, recently much attention has been given to develop several iterative methods for solving nonlinear equations of the type of Eq. (1.1) with initial condition given on Eq. (1.2). The nonlinear models of real-life problems are still difficult to solve analytically. Batiha, (2010) and Adesanya *et al*., (2013) said that there has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models.

The Bratu-Type problem appears in a large variety of applications such as fuel ignition model of thermal combustion, radiative heat transfer, thermal reaction, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology (Batiha, 2010; Syam and Hamdan, 2006).

Darwish and Kashkari, (2014) and Syam and Hamdan, (2006) proposed that the Bratu-type initial value problems have been studied extensively because of its mathematical and physical properties. Many numerical methods have been successfully applied to solve the Bratu-Type equations. Among these method some of them are Adomian decomposition method by Hasanzadeh and Osgooei, (2017) and Adesanya and Arekete, (2013); optimal homotopy method (Hassan and Semary, 2013); Haar wavelet method (Vankatesh, 2010) and homotopy perturbation method (Ghazanfari and Sepahvandzadeh, 2015; Kashkari *et al*., 2017; Filobello-Nino *et al*., 2013 and Feng *et al*., 2008).

The aim of this study was to apply the predictor-corrector method for solving initial value problems of Bratu-Type equations. We first linearize the given equation using quasi-linearization formula and then used fourth order Adams-Bashforth method as a predictor and Adams-Moulton fourth order method as a corrector. The starting values (u_1, u_2, u_3) were calculated using Runge-Kutta fourth order method.

1.2. Statement of the Problem

The study of nonlinear problem is important not only in mathematics but also in areas of physics, engineering and other disciplines. Since most phenomena in our world are expressed in different differential equations, it is important to find their approximate solution (Wazwaz, 2016). Most of engineering problems are inherently nonlinear, especially those problems arising in fluid mechanics, heat transfer, large deformations, and nonlinear problems are difficult to solve, especially in an analytical manner (Batiha, 2010; Syam and Hamdan, 2006). Bratu-Type equations have fundamental importance in various fields of science and engineering. Thus, it is important to find a better approximate solution. Many researchers approximate the Bratu-Type equations in different years using different methods as clearly discussed in the background of this study.

In most of journals the accuracy of second order initial value problems of Bratu-type equation are still required. Even though the researchers continuous solving this equation, they do not get the most appropriate and accurate method that approximate the solution well.

In this study, we applied a new method which is called Adams Predictor-Corrector Method to find better solution of initial value problems of Bratu-Type equations.

To this end, the present study attempted to answer the following basic questions:

- How does the predictor-corrector method be applied for solving Bratu-Type equation?
- To what extent the predictor-corrector method converges?
- To what extent the predictor-corrector method approximate the solution?
- What is the advantage of the proposed method over some of the other numerical methods?

1.3. Objectives of the Study

1.3.1. General Objective

The general objective of this study is to approximate the initial value problems of Bratu-type equation using the predictor-corrector method.

1.3.2. Specific Objectives

The specific objectives of this study are:

- To apply the predictor-corrector method for solving initial value problems of Bratu-Type equation.
- To establish the convergence of the proposed method.
- To investigate the accuracy and stability of the numerical solutions obtained by the proposed method.
- To show the advantage of the proposed method over the other numerical methods.

1.4. Significance of the Study

The outcome of this study will be to apply the predictor-corrector method to find the numerical solution of initial value problems of Bratu-type equation.

1.5. Delimitation of the Study

This study was delimited to solve initial value problems of Bratu-Type equations of the form in Eq. (1.1) with initial condition given in Eq. (1.2) using predictor-corrector method.

CHAPTER TWO

2. REVIEW OF RELATED LITERATURES

2.1. Differential Equations

A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations which may be evaluated at either initial-value problems (IVP) or boundary-value problems (BVP) (Sewell, 2005; King *et al*., 2003). The distinction between IVP and BVP lies in the location where the extra conditions are specified. For an IVP, the conditions are given at the same value of the variable, whereas in the case of the BVP, they are prescribed at two different values of the variable. Since there are relatively few differential equations arising from practical problems for which analytical solutions are known, one must resort to numerical methods. In this situation it turns out that the numerical methods for each type of problem, IVP or BVP, are quite different and require separate treatment (King *et al*., 2003).

Ordinary differential equations (ODEs) are used in mathematical modeling to describe a variety of real-world problems in science and engineering, such as population growth models, predator-prey models and chemical and biological models (Gholamtabar and Parandin, 2015). The discovery of ODEs goes back to Leibniz, Newton, Bernoulli, and others. Many methods, analytic and numerical, were used to solve ODEs, linear or nonlinear.

The main concern of many researchers is to find a better method that works for almost, but not all, ODEs, linear or nonlinear. Researchers have been attempting to discover new methods for solving differential equations, analytically and numerically (Wazwaz, 2016).

2.2. Bratu-Type Equations

Bratu equations named from the name of person, Bratu, who first proposed and solved the equation of the form Eq. (1.1) in 1914. The Bratu-Type equations are employed in the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology (El Hajaji *et al*., 2013). Combustion problems are generally characterized by strong nonlinearity and singularity, as such in most cases exact solution of combustion problems are very difficult to obtain (Hassan and Erturk, 2007; Filobello-Nino *et al.*, 2013; Zarebnia and Hoshyar, 2014 and Adesanya *et al.*, 2013). Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve analytically. There has recently been much attention devoted to the search for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models (El Hajaji *et al.*, 2013 and Batiha, 2010). Bratu-Type equation is widely used in science and engineering to describe complicated physical and chemical models (Kashkari and Abbas, 2017). Studies on fuel ignition in thermal combustion theory have been on the increase over the last few years. The reason for the increased study is to ensure the safety of working environment especially when working with combustible fluid in some petro-chemical engineering process.

Jin, (2010) forwarded that a substantial amount of research work has been directed for the study of the Bratu problem. Several numerical techniques, such as the Adomian Decomposition Method by Hasanzadeh and Osgooei, (2017) and Adesanya and Arekete, (2013), Optimal Homotopy Method (Hassan and Semary, 2013), Haar wavelet method (Vankatesh *et al.*, 2010) and Homotopy perturbation method (Ghazanfari and Sepahvandzadeh, 2015; Kashkari *et al.*, 2017; Filobello-Nino *et al.*, 2013 and Feng *et al.*, 2008) have been implemented to handle the Bratu-type model numerically. In addition, Wazwaz, (2016) has been established the successive differentiation method to determine the solutions of Bratu-type equations; Hilal *et al.*, (2013) developed a cubic spline collocation method for solving Bratu's problem; Hassan and Erturk, (2007) had applied differential transformation method for solving Bratu-type problem; Noor and Mohyud-Din, (2008) established the variational iteration method (VIM) for solving Bratu-type problem. In all of these methods Bratu-type equations approximated well. But, still there is a lack of accuracy in this equation.

2.3. Quasi-linearization Method

The quasi-linearization method (QLM) whose origins are in the theory of dynamic programming was first proposed by Bellman and Kalaba, in 1965. This method can be viewed as the Newton-Raphson method applied to nonlinear differential equations. It is a very powerful method for approximating solutions of nonlinear differential equations and makes use of the Taylor series expansion of first order to linearize a nonlinear differential equation. The solution is then approximated as a sequence of the linear equations. Originally, the method was restricted to twice differentiable and strictly concave (or convex) functions. However, great work was done by Lakshmikantham who presented the QLM with the concavity assumption relaxed. This made the QLM applicable to a wider variety of problems (Muzara, 2015, in press; Eman *et al*. 2013).

2.4. Predictor-Corrector Method

The predictor–corrector method belongs to a class of algorithms designed to integrate ordinary differential equations to find an unknown function that satisfies a given differential equation. All such algorithms proceed in two steps: First, the initial predictor step, starts from a function fitted to the function-values and derivative-values at a preceding set of points to extrapolate this function's value at a subsequent, new point. Second, the corrector step refines the initial approximation by using the predicted value of the function and another method to interpolate that unknown function's value at the same subsequent point.

The major advantage of the multistep methods is that fewer functional evaluations are usually required per integration step (Fujii, 1991, Sehnalova, 2011).We obtain different types by the combinations of explicit and implicit methods. Usually the predictor is an Adams-Bashforth formula and it predicts first approximation value of the solution. The derivative evaluated from this approximation is used in Adams-Moulton corrector formula in the next step. Apart from the better stability of the predictor-corrector formulae over the explicit formulae, the predictor-corrector formulae are generally more accurate and provide reasonable and adequate error estimators (Fatunla, 1988).

When considering the numerical solution of ordinary differential equations (ODEs), a predictor–corrector method typically uses an explicit method for the predictor step and an implicit method for the corrector step.

In the calculation of predictor-corrector pairs there are three stages:

1.Predict the starting value for the dependent variable u_{n+k} as u_{n+k}^{p}.

2. Evaluate the derivative at (x_{n+k}, u_{n+k}^{p}).

3. Correct the evaluated predicted value.

A combination of three stages is called PEC (predict–evaluate–correct) mode (Sehnalova, 2011).

As listed above many researchers were contributed different methods to solve Bratu-Type equations. In this study, we applied the predictor-corrector method to find better solution by computing Bratu-Type equations point wise by varying step sizes.

CHAPTER THREE

3. METHODOLOGY

3.1. Study Area and Period

This study is conducted under mathematics department, college of Natural sciences of Jimma University from September 2016 to October 2017.

3.2. The Study Design

The study design is a mixed design. That is experimental and documentary review design method.

3.3. Sources of Information

The relevant sources of information for this study are books, different journals and internet access.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the study follows the following procedures:

- Defining the problem.
- Linearizing the given nonlinear second order Bratu-type equation using the Quasi-linearization method.
- Reducing the linearized second order equation into first order equations.
- Solving the reduced first order equations using Adams-Bashforth fourth order as a predictor and Adams-Moulton fourth order as a corrector.
- Writing MATLAB code for the proposed method.
- Validating the scheme using numerical examples.

CHAPTER FOUR

4. FORMULATION OF THE METHOD, RESULT AND DISCUSSION

4.1. Formulation of the Method

In this section we consider nonlinear second order initial value problem of Bratu-Type equation of the form Eq. (1.1) with initial condition given in Eq. (1.2).

The given nonlinear second order initial value problem of Eq. (1.1) can be linearized using the quasi-linearization method.

4.1.1. Derivation of the Quasi-linearization Method (QLM) formula

Let us consider an nth order nonlinear differential equation of the form

$$F[u(x)] = 0, \ \ x \in [a, b] \, , \tag{4.1}$$

where x is independent variable and $u(x) = (u, u^{'}, u^{''}, ..., u^{(n)})$ is a vector solutions of Eq. (4.1)

Let $u' = \dfrac{du}{dx}$ and $u^{(n)} = \dfrac{d^n u}{dx^n}$, for $n = 2, 3, 4, ...$ Let assume that $z = (z, z^{'}, z^{"}, ..., z^{(n)})$ is an approximate solution of Eq. (4.1) which is sufficiently close to the true solution u. Assuming that all the partial derivatives of F exist, applying Taylor's theorem we get:

$$F[u] = F(z) + \nabla F(z).(u - z) + \text{(higher order terms)} \tag{4.2}$$

Ignoring the higher terms and simplifying Eq. (4.2) we obtain:

$$\nabla F(z).u = \nabla F(z).z - F(z) \tag{4.3}$$

The solution from Eq. (4.3) will not be the exact solution of Eq. (4.1) because of the discarded higher order terms. We use the initial approximate solution z as a calculated solution to iteratively compute the new solution u. Denoting z and u by u_k and u_{k+1} respectively, we get the iterative formula

$$\nabla F(u_k).u_{k+1} = \nabla F(u_k).u_k - F(u_k) \, , \tag{4.4}$$

where $k = 1, 2, 3, ...$

Since

$$\nabla F(u_k).u_{k+1} = \frac{\partial F(u_k)}{\partial u_k} u_{k+1} + \frac{\partial F(u_k)}{\partial u_k^{'}} \frac{d}{dx}(u_{k+1}) + \dots + \frac{\partial F(u_k)}{\partial u_k^{(n-1)}} \frac{d^{n-1}}{dx^{n-1}}(u_{k+1}) + \frac{\partial F(u_k)}{\partial u_k^{(n)}} \frac{d^n}{dx^n}(u_{k+1})$$

then Eq. (4.4) can be written in operator form as

$$\mathcal{L} u_{k+1} = \mathcal{L} u_k - F(u_k) \tag{4.5}$$

where $\mathcal{L} = b_0 \frac{d^n}{dx^n} + b_1 \frac{d^{n-1}}{dx^{n-1}} + \dots + b_{n-1} \frac{d}{dx} + b_n$ (4.6)

And $b_0 = \frac{\partial F(u_k)}{\partial u_k^{(n)}},\ b_1 = \frac{\partial F(u_k)}{\partial u_k^{(n-1)}}, \dots,\ b_{n-1} = \frac{\partial F(u_k)}{\partial u_k^{'}},\ b_n = \frac{\partial F(u_k)}{\partial u_k}$

The iterative scheme Eq. (4.5) is the standard QLM formula used to obtain the $(k+1)^{th}$ iterative approximation $u_{k+1}(x)$ of the solution of Eq. (4.1) (Muzara; in press, Eman *et al.*, 2013).

Applying the Quasi-linearization method on Bratu-Type Problem

The Bratu-type equation of the form Eq. (1.1) can be transformed to a linear differential problem using the QLM Eq. (1.1) is of second order, thus we have

$F(u,u^{'},u^{''}) = u^{''}(x) + \lambda e^{u(x)}$ and $\mathcal{L} = b_0 \frac{d^2}{dx^2} + b_1 \frac{d}{dx} + b_2$

Calculating the coefficients $b_0,\ b_1$ and b_2 and substituting these values into Eq. (4.5) we get the iterative scheme,

$$b_0 = 1,\ \ b_1 = 0 \ \text{ and } \ b_2 = \lambda e^{u_k(x)}$$

$$u_{k+1}^{''}(x) + \lambda e^{u_k(x)} u_{k+1}(x) = \lambda e^{u_k(x)} (u_k(x) - 1) \tag{4.7}$$

$$u_{k+1}(0) = \beta \text{ and } u_{k+1}^{'}(0) = \gamma\ , \tag{4.8}$$

where $k = 1, 2, 3, \dots$.

Eq. (4.7) can be used to compute $u_{k+1}(x)$ provided $u_k(x)$ is known. In particular, the initial approximation $u_0(x)$ must be specified so that we compute $u_1(x)$. Once $u_1(x)$ is known, we compute $u_2(x)$ using Eq. (4.7) and so on.

Eqs. (4.7) and (4.8) can be reduced to the equations

$$\mathcal{L}(u) = u''(x) + a(x)u(x) = b(x), \quad 0 \le x \le l \tag{4.9}$$

$$\Rightarrow \quad u''(x) = -a(x)u(x) + b(x)$$

with initial conditions

$$u(0) = \alpha \text{ and } u'(0) = \gamma, \tag{4.10}$$

where $a(x) = \lambda e^{u(x)}$ and $b(x) = \lambda e^{u(x)}(u(x) - 1)$

Therefore the given second order IVP of Bratu-type equation is linearized to Eq. (4.9) with initial condition given in Eq. (4.10). This equation with the given initial conditions can be solved by Adams-Bashforth-Moulton predictor-corrector method.

The second order initial value problem of Eq. (4.9) can be reduced to first order system of equations using the substitutions $v(x) = u'(x)$ and $v'(x) = u''(x)$. Then the given second order initial value problem of Eq. (4.9) with Eq. (4.10) can be re-written as:

$$\begin{cases} u'(x) = v(x) = F(x, u, v), u(0) = \alpha \\ v'(x) = b(x) - a(x)u(x) = G(x, u, v), \quad v(0) = \gamma \end{cases} \tag{4.11}$$

Dividing the interval $[0, l]$ into N equal subintervals of mesh length h and the mesh point is given by $x_n = x_0 + nh,$ for $n = 1, 2, ..., N-1.$ For the sake of simplicity, let use the notation: $u(x_n) = u_n$, $v(x_n) = v_n$, etc.

Thus, at the nodal point x_n Eq. (4.11), can be written as:

$$\begin{cases} u'_n = F(x_n, u_n, v_n), \quad u(0) = \alpha \\ v'_n = G(x_n, u_n, v_n), \quad v(0) = \gamma, \end{cases} \tag{4.12}$$

where $G(x_n, u_n, v_n) = -a(x_n)u(x_n) + b(x_n)$

To solve the system of equations given in Eq. (4.12) we may use multistep methods that require information about the solution at x_n to calculate at x_{n+1} from the solution at a number of previous solutions. From one of the single step methods we use the fourth order Runge-Kutta method since it is self-starter and then we use the fourth order Adams-Bashforth-Moulton method to solve the Bratu-type equations.

For the general case, let's consider the first order initial value problem of the form

$$u'(x) = f(x, u(x)), \quad u(x_0) = \alpha \tag{4.13}$$

and $x_n = x_0 + nh$

4.1.2. The fourth order Runge-Kutta Method

The initial value problem of the form of Eq. (4.13) can be solved by using fourth order Runge-Kutta method. The general fourth order Runge-Kutta method of Eq. (4.13) is given by (Butcher, 2008):

$$\begin{aligned} u_{n+1} &= u_n + h\sum_{n=1}^{4} w_n k_n, \\ \text{where } k_n &= f(x_n + c_n h, u_n + \sum_{j=1}^{4} a_{n,j} k_j) \end{aligned} \tag{4.14}$$

The values of w_n, c_n and $a_{n,j}$ are given by the Butcher tableau (Butcher, pp. 175)

$$\begin{array}{c|cccc} c_1 & & & & \\ c_2 & a_{2,1} & & & \\ c_3 & a_{3,1} & a_{3,2} & & \\ c_4 & a_{4,1} & a_{4,2} & a_{4,3} & \\ \hline & w_1 & w_2 & w_3 & w_4 \end{array}$$

The corresponding values of variables of the above tableau are given for the classical runge-kutta method (Butcher, pp.180).

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Using these values for particular fourth order classical Runge-Kutta method the scheme is given by:

$$u_{n+1} = u_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \tag{4.15}$$

where $K_1 = f(x_n, u_n)$

$$k_2 = f(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1)$$
$$k_3 = f(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2)$$
$$k_4 = f(x_n + h, u_n + k_3)$$

For the fourth order Runge-Kutta method of the system of two equations of the form of Eq. (4.12) can also be expressed as:

$$\begin{cases} u_{n+1} = u_n + \sum_{n=1}^{4} w_n k_n \\ v_{n+1} = v_n + \sum_{n=1}^{4} w_n m_n, \end{cases} \tag{4.16}$$

where

$$\begin{cases} k_n = hF(x_n + c_n h,\ u_n + \sum_{j=1}^{4} a_{nj} k_j,\ v_n + \sum_{j=1}^{4} a_{nj} m_j) \\ m_n = hG(x_n + c_n h,\ u_n + \sum_{j=1}^{4} a_{nj} k_j,\ v_n + \sum_{j=1}^{4} a_{nj} m_j) \end{cases}$$

Particularly, Eq. (4.16) can also be simplified to the fourth order of classical Runge-Kutta method as:

$$\begin{cases} u_{n+1} = u_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\ v_{n+1} = v_n + \frac{1}{6}h(m_1 + 2m_2 + 2m_3 + m_4), \end{cases} \tag{4.17}$$

where $k_1 = F(x_n, u_n, v_n)$ $\qquad m_1 = G(x_n, u_n, v_n)$

$k_2 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1)$ $\qquad m_2 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1)$

$k_3 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2)$ $\qquad m_3 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2)$

$k_4 = F(x_n + h, u_n + k_3, v_n + m_3)$ $\qquad m_4 = G(x_n + h, u_n + k_3, v_n + m_3)$

Using Eq. (4.17) we can derive the Runge-kutta general formula of the linearized Bratu-type equation given in Eq. (4.12). Let calculate the values of k_i and m_i for $i = 1, 2, 3$ and 4 as follow:

$$k_1 = F(x_n, u_n, v_n) = u_n^{'}$$

$$m_1 = G(x_n, u_n, v_n) = \text{-}a_n u_n + b_n$$

$$k_2 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1)$$

$$= F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}u_n^{'} \;, v_n + \frac{1}{2}(\text{-}a_n u_n + b_n))$$

$$= u_n^{'} + \frac{1}{2}u_n^{''}$$

$$m_2 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1)$$

$$= G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}u_n^{'}, v_n + \frac{1}{2}(\text{-}a_n u_n + b_n))$$

$$= \text{-}a_n(u_n + \frac{1}{2}u_n^{'}) + b_n$$

$$k_3 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2)$$

$$= F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}(u_n^{'} + \frac{1}{2}u_n^{''}), v_n + \frac{1}{2}(-a_n(u_n^{'} + u_n^{''}) + b_n))$$

$$= u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{4}u_n^{'''}$$

$$m_3 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2)$$

$$= G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}(u_n^{'} + \frac{1}{2}u_n^{''}), v_n + \frac{1}{2}(-a_n(u_n^{'} + u_n^{''}) + b_n))$$

$$= -a_n(u_n + \frac{1}{2}u_n^{'} + \frac{1}{4}u_n^{''}) + b_n$$

$$k_4 = F(x_n + h, u_n + k_3, v_n + m_3)$$

$$= F(x_n + h, u_n + u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{4}u_n^{'''}, v_n + (-a_n(u_n + \frac{1}{2}u_n^{'} + \frac{1}{4}u_n^{''}) + b_n))$$

$$= u_n^{'} + u_n^{''} + \frac{1}{2}u_n^{'''} + \frac{1}{4}u_n^{(4)}$$

$$m_4 = G(x_n + h, u_n + k_3, v_n + m_3)$$

$$= G(x_n + h, u_n + u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{4}u_n^{'''}, v_n + (-a_n(u_n + \frac{1}{2}u_n^{'} + \frac{1}{4}u_n^{''}) + b_n))$$

$$= -a_n(u_n + u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{4}u_n^{'''}) + b_n$$

Then substituting these values of k_i's and m_i's $\left(\mathrm{i}=1,\ 2,\ 3,\ 4\right)$ in Eq. (4.17) and simplifying the equations separately for u_{n+1} and v_{n+1} we get:

$$
\begin{aligned}
u_{n+1} &= u_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\
&= u_n + \frac{1}{6}h(u_n^{'} + 2(u_n^{'} + \frac{1}{2}u_n^{''}) + 2(u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{4}u_n^{'''}) + (u_n^{'} + u_n^{''} + \frac{1}{2}u_n^{'''} + \frac{1}{4}u_n^{(4)})) \\
&= u_n + \frac{1}{6}h(u_n^{'} + 2u_n^{'} + u_n^{''} + 2u_n^{'} + u_n^{''} + \frac{1}{2}u_n^{'''} + u_n^{'} + u_n^{''} + \frac{1}{2}u_n^{'''} + \frac{1}{4}u_n^{(4)}) \\
&= u_n + \frac{1}{6}h((u_n^{'} + 2u_n^{'} + u_n^{'} + 2u_n^{'}) + (u_n^{''} + u_n^{''} + u_n^{''}) + (\frac{1}{2}u_n^{'''} + \frac{1}{2}u_n^{'''}) + \frac{1}{4}u_n^{(4)}) \\
&= u_n + \frac{1}{6}h(6u_n^{'} + 3u_n^{''} + u_n^{'''} + \frac{1}{4}u_n^{(4)}) \\
&= u_n + h(u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{6}u_n^{'''} + \frac{1}{24}u_n^{(4)})
\end{aligned}
$$

And the values of v_{n+1} can also be calculated as follows:

$$
\begin{aligned}
v_{n+1} &= v_n + \frac{1}{6}h(m_1 + 2m_2 + 2m_3 + m_4) \\
&= v_n + \frac{1}{6}h((\text{-}a_n u_n + b_n) + 2(\text{-}a_n(u_n + \frac{1}{2}u_n^{'}) + b_n) + 2(-a_n(u_n + \frac{1}{2}u_n^{'} + \frac{1}{4}u_n^{''}) + b_n) \\
&\qquad + (-a_n(u_n + u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{4}u_n^{'''}) + b_n))
\end{aligned}
$$

Simplifying this equation we get:

$$
\begin{aligned}
v_{n+1} &= v_n + \frac{1}{6}h(-6a_n u_n - 3a_n u_n^{'} - a_n u_n^{''} - \frac{1}{4}a_n u_n^{'''} + 6b_n) \\
&= v_n - h(a_n u_n + \frac{1}{2}a_n u_n^{'} + \frac{1}{6}a_n u_n^{''} + \frac{1}{24}a_n u_n^{'''} + b_n)
\end{aligned}
$$

Therefore the system of Eq. (4.17) is simplified to:

$$
\begin{cases}
u_{n+1} = u_n + h(u_n^{'} + \frac{1}{2}u_n^{''} + \frac{1}{6}u_n^{'''} + \frac{1}{24}u_n^{(4)}) \\
v_{n+1} = v_n - h(a_n u_n + \frac{1}{2}a_n u_n^{'} + \frac{1}{6}a_n u_n^{''} + \frac{1}{24}a_n u_n^{'''} + b_n)
\end{cases}
\tag{4.18}
$$

This equation, Eq. (4.18), is Runge-Kutta fourth order formula used to approximate the values of u_n and v_n for $\mathrm{n}=1,\ 2,\ 3$ since the Adams-Bashforth-Moulton predictor-corrector method requires these values.

4.1.3. The Fourth order Adams-Bashforth Method

To solve the general form of initial value problem of equation given in Eq. (4.13), we can apply the multistep method that requires information about the solution at x_{n+1} from the solution at a number of previous solutions.

To begin the derivation of the multistep methods, if we integrate the initial-value problem of Eq. (4.13) from x_n to x_{n+1}, then the following property exists (Chiou and Wu, 1999):

$$u(x_{n+1}) = u(x_n) + \int_{x_n}^{x_{n+1}} f(x, u(x))dx, \tag{4.19}$$

where $f(x,u(x))$ is the first derivative of $u(x)$.

Replace $f(x, u)$ of Eq. (4.19) by the polynomial $p_{k-1}(x)$ of degree k-1, which interpolates $f(x, u)$ at k points and Newton backward interpolation formula gives polynomial of degree k-1. To derive Adams-Bashforth method, Newton backward difference formula with a set of equal spacing points, $x_n,\ x_{n-1}, ...,\ x_{n-k+1}$, is used to approximate the integral and the fourth order Adams-Bashforth method is given by (Jain et al. 2007):

$$u_{n+1} = u_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right] + T_k, \tag{4.20}$$

where T_k is the truncation error of the fourth order Adams-Bashforth method and is given by:

$$T_k = \frac{251}{720} h^5 u^{(5)}\left(\xi\right) = O(h^5) \tag{4.21}$$

To use Eq. 4.20, we require the starting values u_n, u_{n-1}, u_{n-2} and u_{n-3} which are calculated by self-starting single step method, Runge-Kutta fourth order method for our case. The fourth order Adams-Bashforth method for the system of equations given in Eq. (4.12), can be solved using Eq. 4.20 and it becomes

$$\begin{cases} u_{n+1} = u_n + \dfrac{h}{24}\left[55F_n - 59F_{n-1} + 37F_{n-2} - 9F_{n-3}\right] \\ v_{n+1} = v_n + \dfrac{h}{24}\left[55G_n - 59G_{n-1} + 37G_{n-2} - 9G_{n-3}\right] \end{cases} \tag{4.22}$$

Using Eq. (4.22) we can formulate the general form of the system of Eq. (4.12) for $n \geq 4$. Therefore, Eq. (4.22) can be derived as follow:

$$u_{n+1} = u_n + \frac{h}{24}(55F_n - 59F_{n-1} + 37F_{n-2} - 9F_{n-3})$$

But, since the values of F_n, F_{n-1}, F_{n-2} and F_{n-3}, for $n \geq 4$, can be calculated using the linearized system of Eq. (4.12), we have

$F_n = u_n^{'}, F_{n-1} = u_{n-1}^{'}, F_{n-2} = u_{n-2}^{'}, F_{n-3} = u_{n-3}^{'},$ then

$$u_{n+1} = u_n + \frac{h}{24}(55u_n^{'} - 59u_{n-1}^{'} + 37u_{n-2}^{'} - 9u_{n-3}^{'})$$

For $v_{n+1} = v_n + \frac{h}{24}(55G_n - 59G_{n-1} + 37G_{n-2} - 9G_{n-3})$, where the values of

G_n, G_{n-1}, G_{n-2}, G_{n-3} are given by:

$$\begin{aligned} G_n &= -a_n u_n + b_n, \\ G_{n-1} &= -a_{n-1}u_{n-1} + b_{n-1}, \\ G_{n-2} &= -a_{n-2}u_{n-2} + b_{n-2}, \\ G_{n-3} &= -a_{n-3}u_{n-3} + b_{n-3} \end{aligned}$$

$$\begin{aligned} v_{n+1} = v_n + \frac{h}{24}(&55(-a_n u_n + b_n) - 59(-a_{n-1}u_{n-1} + b_{n-1}) + 37(-a_{n-2}u_{n-2} + b_{n-2}) \\ &- 9(-a_{n-3}u_{n-3} + b_{n-3})) \end{aligned}$$

Then the system of equation becomes

$$\begin{cases} u_{n+1} = u_n + \frac{h}{24}(55u_n^{'} - 59u_{n-1}^{'} + 37u_{n-2}^{'} - 9u_{n-3}^{'}) \\ v_{n+1} = v_n + \frac{h}{24}(55(-a_n u_n + b_n) - 59(-a_{n-1}u_{n-1} + b_{n-1}) + 37(-a_{n-2}u_{n-2} + b_{n-2}) \\ \qquad - 9(-a_{n-3}u_{n-3} + b_{n-3})) \end{cases} \tag{4.23}$$

This equation, Eq. (4.23), is the fourth order Adams-Bashforth predictor method for the given system of equations in Eq. (4.12).

4.1.4. The Fourth order Adams-Moulton Method

Again to solve the given differential equation using fourth order Adams-Moulton method, first let's consider the first order initial value problem of the form Eq. (4.13) and the method is derived by using the set of equal spacing points, $x_{n+1},\ x_n,\ ...,\ x_{n-k+1}$.

Integrating both sides of Eq. (4.13) with respect to x from x_n to x_{n+1} we have, (Chiou and Wu, 1999)

$$u(x_{n+1}) = u(x_n) + \int_{x_n}^{x_{n+1}} f(x, u(x))dx \tag{4.24}$$

Replace $f(x, u)$ in Eq. (4.24) by the polynomial $p_k(x)$ of degree k, which interpolates $f(x, u)$ at k + 1 points and Newton backward interpolation formula, gives polynomial of degree k and the fourth order Adams-Moulton method is given by (Jain et al. 2007):

$$u_{n+1} = u_n + \frac{h}{24}\left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right] + T_l, \tag{4.25}$$

where the truncation error T_l is given by:

$$T_l = \frac{-19}{720} h^5 u^{(5)}(\zeta) = O(h^5) \tag{4.26}$$

The system of equations given in Eq. (4.12) is then given by:

$$\begin{cases} u_{n+1} = u_n + \dfrac{h}{24}\left[9F_{n+1} + 19F_n - 5F_{n-1} + F_{n-2}\right] \\ v_{n+1} = v_n + \dfrac{h}{24}\left[9G_{n+1} + 19G_n - 5G_{n-1} + G_{n-2}\right] \end{cases} \tag{4.27}$$

To apply Eq. (4.27) on Bratu-type equation, we simplify this equation using the same procedures as we have done for the Adams-Bashforth predictor method.

That is, the values of $F_{n+1},\ F_n,\ F_{n-1},\ F_{n-2}$ and $G_{n+1},\ G_n,\ G_{n-1}, G_{n-2}$ are as calculated for the predictor method. Therefore, the system of Eq. (4.27) can be written as:

$$\begin{cases} u_{n+1} = u_n + \dfrac{h}{24}(9u'_{n+1} + 19u'_n - 5u'_{n-1} + u'_{n-2}) \\ v_{n+1} = v_n + \dfrac{h}{24}(9(-a_{n+1}u_{n+1} + b_{n+1}) + 19(-a_n u_n + b_n) - 5(-a_{n-1}u_{n-1} + b_{n-1}) \\ \qquad\qquad + (-a_{n-2}u_{n-2} + b_{n-2})) \end{cases} \tag{4.28}$$

This equation, Eq. (4.28), is the Adams-Moulton corrector formula.

4.1.5. Predictor-Corrector Method

Here we combine the Adams-Bashforth and Adams-Moulton fourth order method. We use the fourth order Adams-Bashforth method as a predictor and Adams-Moulton method as a corrector and we have the following equations.

$$\begin{cases} u^p_{n+1} = u_n + \dfrac{h}{24}\left[55F_n - 59F_{n-1} + 37F_{n-2} - 9F_{n-3}\right] \\ v^p_{n+1} = v_n + \dfrac{h}{24}\left[55G_n - 59G_{n-1} + 37G_{n-2} - 9G_{n-3}\right] \end{cases} \tag{4.29}$$

$$\begin{cases} y^c{}_{n+1} = y_n + \dfrac{h}{24}\left[9F^*{}_{n+1} + 19F_n - 5F_{n-1} + F_{n-2}\right] \\ z^c{}_{n+1} = z_n + \dfrac{h}{24}\left[9G^*{}_{n+1} + 19G_n - 5G_{n-1} + G_{n-2}\right], \end{cases} \tag{4.30}$$

where $F^*{}_{n+1} = F(x_{n+1}, u^p{}_{n+1}, v^p{}_{n+1})$

$G^*{}_{n+1} = G(x_{n+1}, u^p{}_{n+1}, v^p{}_{n+1})$

$u^p{}_{n+1}$ and $v^p{}_{n+1}$ are calculated from Eq. 4.29

Applying these equations, Eqs. (4.29) and (4.30), on the Bratu-type equations is the same as combining Eq. (4.23) and Eq. (4.28); using Eq. (4.23) as a predictor and Eq. (4.28) as a corrector and it becomes:

Predictor Formula

$$\begin{cases} u_{n+1}^{p} = u_n + \dfrac{h}{24}(55u_n^{'} - 59u_{n-1}^{'} + 37u_{n-2}^{'} - 9u_{n-3}^{'}) \\ v_{n+1}^{p} = v_n + \dfrac{h}{24}(55(-a_n u_n + b_n) - 59(-a_{n-1}u_{n-1} + b_{n-1}) + 37(-a_{n-2}u_{n-2} + b_{n-2}) \\ \qquad\qquad - 9(-a_{n-3}u_{n-3} + b_{n-3})) \end{cases} \tag{4.31}$$

And corrector formula

$$\begin{cases} u_{n+1}^{c} = u_n + \dfrac{h}{24}(9(u_{n+1}^{p})^{'} + 19u_n^{'} - 5u_{n-1}^{'} + u_{n-2}^{'}) \\ v_{n+1}^{c} = v_n + \dfrac{h}{24}(9(-a_{n+1}u_{n+1}^{p} + b_{n+1}) + 19(-a_n u_n + b_n) - 5(-a_{n-1}u_{n-1} + b_{n-1}) \\ \qquad\qquad + (-a_{n-2}u_{n-2} + b_{n-2})) \end{cases} \tag{4.32}$$

4.2. Truncation Error, Stability and Convergence Analysis

Let's consider the more general multistep method of the following (Sewell, 2005)

$$\begin{aligned} &\frac{\left[U(t_{k+1}) + \alpha_1 U(t_k) + \alpha_2 U(t_{k-1}) + \ldots + \alpha_m U(t_{k+1-m})\right]}{h} \\ &= \beta_0 f(t_{k+1}, U(t_{k+1})) + \beta_1 f(t_k, U(t_k)) + \ldots + \beta_m f(t_{k+1-m}, U(t_{k+1-m})), \end{aligned} \tag{4.33}$$

where α_i and β_j, (for $i = 1, 2, 3, \ldots, m$ and $j = 0, 1, 2, 3, \ldots., m$) are constants.

Before we go to the actual analysis of the truncation error, stability and convergence let state the definitions of these words and some related theorems without proofs.

Definition 4.1: The truncation error is the amount by which the solution of the differential equation fails to satisfy the approximate equation (Sewell, 2005 pp. 45).

Definition 4.2: An approximate method is consistent with the differential equation if the truncation error goes to zero as the step size h goes to zero.

Definition 4.3: An approximate method is stable if the error goes to zero as the truncation error goes to zero.

Theorem 4.1: If a sequence of numbers e_k satisfies

$$e_{k+1} + \rho_1 e_k + \rho_2 e_{k-1} + ... + \rho_m e_{k+1-m} = hT_k \tag{4.34}$$

for $k \geq m-1 (m \geq 1)$ and if all the roots of the corresponding characteristic polynomial

$$\lambda^m + \rho_1 \lambda^{m-1} + \rho_2 \lambda^{m-2} + ... + \rho_m \tag{4.35}$$

are less than or equal to 1 in absolute value, and all multiple roots are strictly less than 1 in absolute value, then

$|e_k| \leq M_\rho \left[\max\{|e_0|,...,|e_{m-1}|\} + t_k T \right]$, where $t_k = kh$, $T = \max|T_j|$, and M_ρ is a constant depending only on the ρ_i.

Theorem 4.2. The multistep method (4.33) is stable provided all roots of

$$\lambda^m + \alpha_1 \lambda^{m-1} + \alpha_2 \lambda^{m-2} + ... + \alpha_m \tag{4.36}$$

are less than or equal to 1 in absolute value, and all multiple roots are strictly less than 1 in absolute value.

4.2.1. Adams-Bashforth-Moulton Method

Error Estimation

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of order $O(h^5)$ (Eq. 4.21 and Eq.4.26). Therefore, the local truncation errors (L.T.E.) for equations (4.20 and 4.25) are

$$u(x_{n+1}) - p_{n+1} = \frac{251}{720} h^5 u^{(5)}(\xi) \quad \text{(L.T.E. for the predictor)} \tag{4.37}$$

$$u(x_{n+1}) - u_{n+1} = \frac{-19}{720} h^5 u^{(5)}(\zeta) \quad \text{(L.T.E. for the corrector),} \tag{4.38}$$

where $u(x_{n+1})$ is given by Eq. (4.19) for the predictor and Eq. (4.24) for corrector and p_{n+1} and u_{n+1} are calculated values for Adams-Bashforth predictor and Adams-Moulton corrector given by Eq. (4.20) and Eq. (4.24) respectively.

Suppose that h is small and let $u^{(5)}$ is nearly constant over the interval, then the terms involving the fifth derivative in Eqs. (4.37) and (4.38) can be eliminated, and the result becomes (Sewell, 2005),

$$u(x_{n+1}) - u_{n+1} = \frac{-19}{270}(u_{n+1} - p_{n+1}) \tag{4.39}$$

The importance of the predictor-corrector method should now be evident. This formula, Eq. (4.39), gives an approximate error estimate based on the two computed values p_{n+1} and u_{n+1} and does not use $u^{(5)}(x)$ (Sewell, 2005).

Stability Analysis

Some of the most popular higher-order, stable, multistep methods are the Adams methods, which ensure stability by choosing $\alpha_1 = -1$ and $\alpha_2 = \alpha_3 = ... = \alpha_m = 0$. The characteristic polynomial corresponding to theorems 4.1 and 4.2 is $\lambda^m - \lambda^{m-1}$ which has 1 as a simple root and 0 as a multiple root. Thus these methods are stable regardless of the values chosen for the β_i's. The values of β_i's are determined in order to maximize the order of the truncation error (Sewell, 2005).

For Adams-Bashforth and Adams-Moulton method the value of β_i are calculated and given on Sewell (2005) and Jain *et al*. (2007). Since for all Adams methods the values of $\alpha_1 = -1$ and $\alpha_2 = \alpha_3 = ... = \alpha_m = 0$, the fourth order Adams-Bashforth method (Eq. 4.22) and fourth order Adams-Moulton method (Eq. 4.27) have the characteristic equation of (theorem 4.2):

$$\rho(\lambda) = \lambda^4 - \lambda^3 = 0$$
$$\Rightarrow \lambda^3(\lambda - 1) = 0$$

Then, $\lambda = 1$ is a simple root and 0 is a multiple root with multiplicity 3.

Therefore, since the simple root is 1, and multiple roots are 0 which is strictly less than 1, by theorem 4.2 Adams-Bashforth and Adams-Moulton methods are stable.

Convergence Analysis

As stated in definition 4.2, the method is said to be consistence if the truncation error goes to zero as the step size *h* goes to zero. If the truncation error goes to zero as *h* decreases (goes to zero), then the method is stable. However, consistency does not automatically guarantee convergence. Fourth order Adams-methods are consistence and also stable. Therefore, our method is convergent, since convergency is the sum of consistency and stability (Sewell, pp. 46-49).

4.3. Numerical Examples and Results

To show the applicability and efficiency of the proposed method, we have considered three model examples of Bratu-type equation and compared the numerical solutions with different other numerical methods considered in this study and exact solution as follow.

Example 1: Consider the Bratu-type initial value problem

$$u'' - 2e^{u} = 0, \quad 0 < x < 1$$

$$u(0) = 0, \quad u'(0) = 0$$

whose exact solution is $u(x) = -2\ln(\cos(x))$

Table 4.1: Comparison of numerical approximations and absolute errors for Example 4.1 with step size h= 0.1 and h=0.01 with different numerical methods

x	Absolute errors at $h=0.1$				Our Method (PC) at $h=0.01$
	Aksoy and Pakdemirli, 2010	Darwish and Kashkari, 2014	Sinan and Necdet, 2016	Our Method (PC)	
0.1	6.71e-6	6.41021065e-7	9.4728e-6	2.8436e-9	2.8073e-15
0.2	9.55e-6	9.74693876e-6	3.3152e-5	1.2788e-7	1.2232e-13
0.3	3.31e-6	4.52998213e-5	2.7254e-5	3.9593e-7	3.5863e-13
0.4	8.04e-6	1.27118347e-4	4.4563e-6	3.4141e-6	2.3147e-12
0.5	8.48e-6	2.68671650e-4	5.5511e-8	6.4578e-7	1.8502e-12
0.6	2.03e-5	4.83656903e-4	7.2047e-5	6.3262e-7	1.5055e-12
0.7	7.15e-5	8.36799541e-4	7.0044e-5	1.5328e-6	1.2795e-12
0.8	2.91e-4	1.60053795e-3	1.2821e-4	1.5028e-6	1.1715e-12
0.9	1.05e-3	3.64970628e-3	4.5236e-4	1.2844e-5	1.1814e-12
1.0	3.53e-3	9.39151960e-3	4.4409e-8	3.6627e-5	1.3084e-12

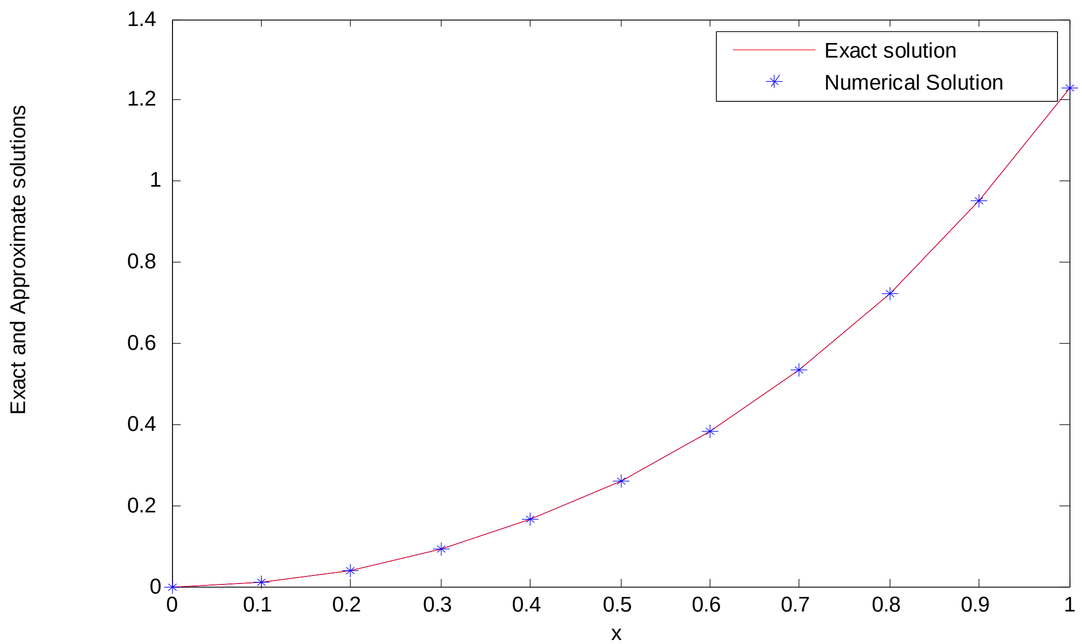

Figure 4.1. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for example 4.1 with mesh length h=0.1

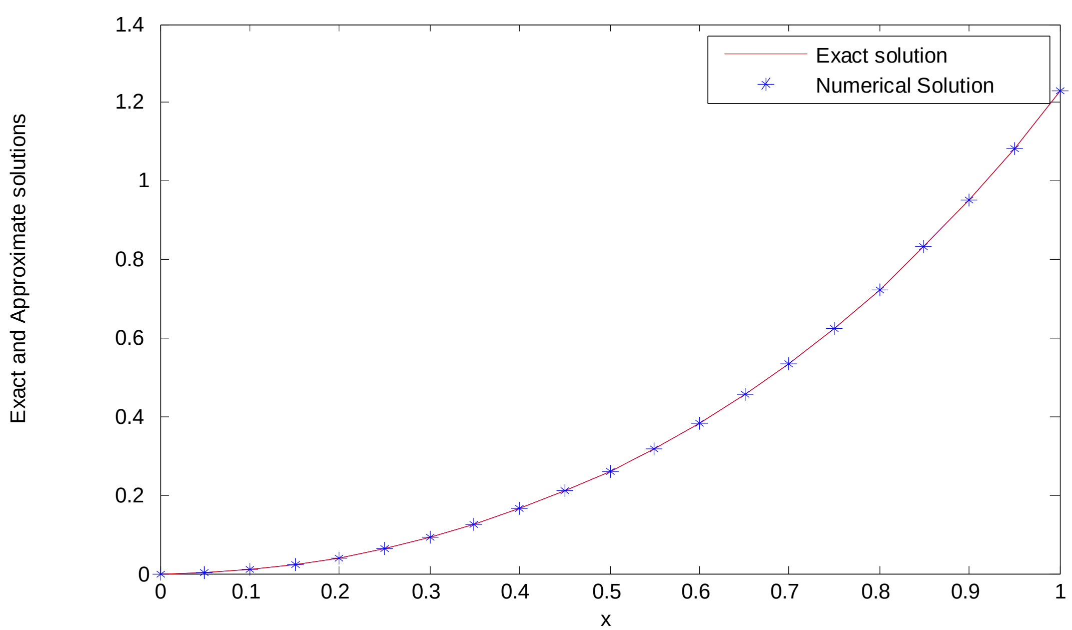

Figure 4.2. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for example 4.1 with mesh length h=0.01

Example 2: Consider the Bratu-type initial value problem

$$\frac{d^2u}{dx^2} = -\pi^2 e^{-u}; u(0) = 0, \quad u'(0) = \pi$$

Whose exact solution is $u(x) = \ln(1+\sin(\pi x))$

Table 4.2: Comparison of numerical approximations and absolute errors for Example 4.2 with step size h= 0.1 and h=0.01 with RKM

x	Exact value	Absolute errors at $h=0.1$		Our Method (PC) at $h=0.01$
		Eslam et al. 2015, (RKM)	Our Method (PC)	
0.1	0.26928	3.20777e-4	3.4129e-5	4.1477e-10
0.2	0.46234	2.37600e-5	5.7752e-5	8.0185e-10
0.3	0.59278	3.58700e-5	7.9099e-5	1.1645e-09
0.4	0.66837	8.01000e-5	2.7368e-4	3.5093e-09
0.5	0.69315	1.19500e-4	4.2841e-5	2.7071e-09
0.6	0.66837	1.66200e-4	6.8607e-5	1.9756e-09
0.7	0.59278	2.20200e-4	1.3754e-4	1.3059e-09
0.8	0.46234	2.85100e-4	1.8845e-4	6.9034e-10
0.9	0.26928	4.03400e-4	2.2350e-4	1.2237e-10
1.0	2.2204e-16	5.37400e-4	2.1737e-4	4.0377e-10

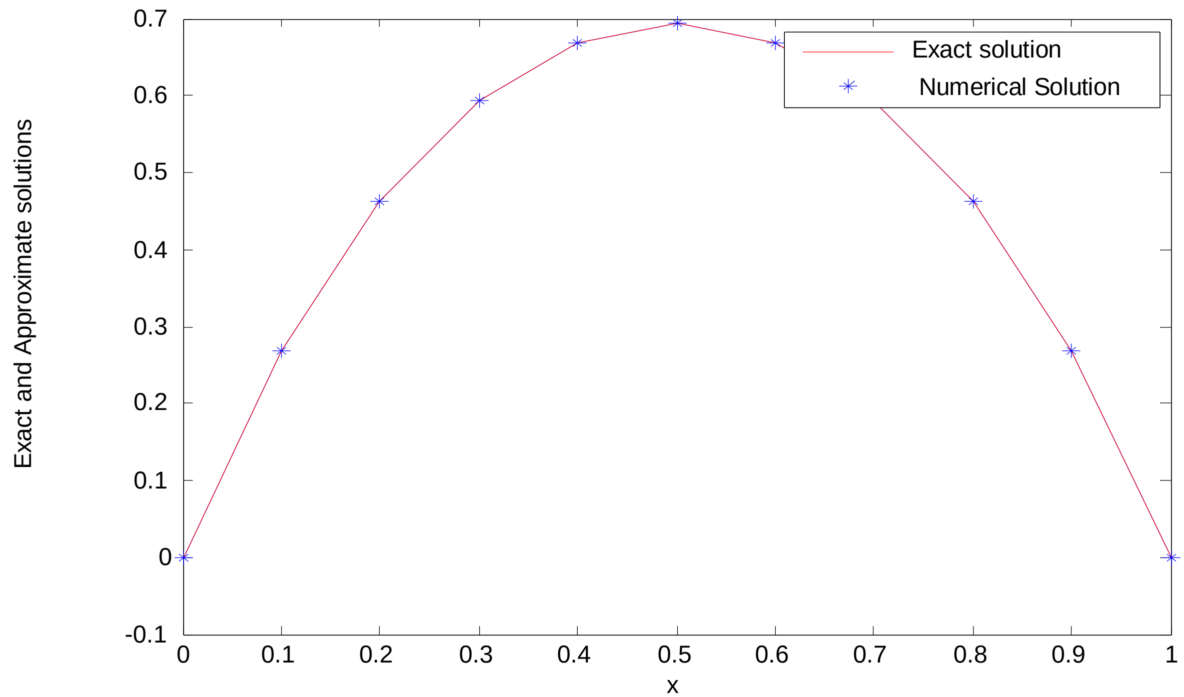

Figure 4.3. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for example 4.2 with mesh size h=0.1

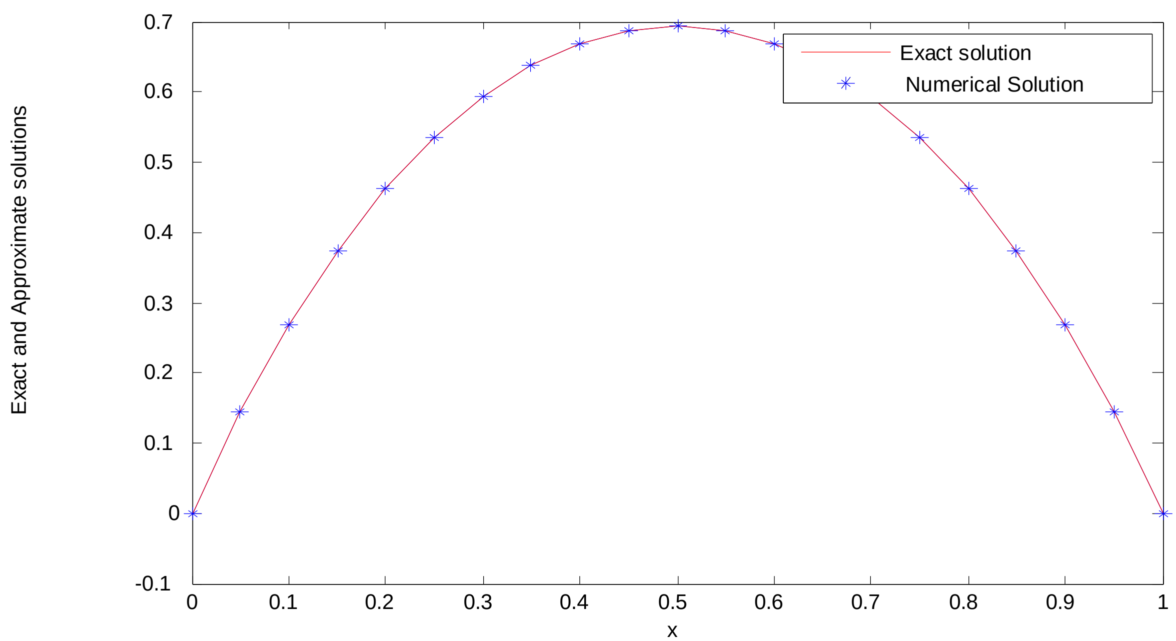

Figure 4.4. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for example 4.2 with mesh size h=0.01

Example 3: (**Eslam *et al.*, 2015**) Consider the Bratu-type initial value problem

$u'' - e^{2u} = 0$; $0 < x < 1$

$u(0) = 0, \quad u'(0) = 0$

whose exact solution is $u(x) = \ln(\sec(x))$:

Table 4.3: Comparison of numerical approximations and absolute errors for Example 4.3 with step size h= 0.1 and h=0.01 with the exact solution

x	Exact solution	Numerical solution at		Absolute errors at	
		$h=0.1$	$h=0.01$	$h=0.1$	$h=0.01$
0.1	5.0084e-3	5.0084e-3	5.0084e-3	1.4218e-9	6.5411e-13
0.2	2.0135e-2	2.0135e-2	2.0135e-2	6.3940e-8	4.5335e-12
0.3	4.5692e-2	4.5691e-2	4.5692e-2	1.9796e-7	1.4653e-11
0.4	8.2229e-2	8.2227e-2	8.2229e-2	1.7071e-6	3.2161e-11
0.5	1.3058e-1	1.3058e-1	1.3058e-1	2.2452e-6	5.9328e-11
0.6	1.9197e-1	1.9196e-1	1.9197e-1	3.4494e-6	1.0021e-10
0.7	2.6809e-1	2.6808e-1	2.6809e-1	5.8706e-6	1.6204e-10
0.8	3.6319e-1	3.6138e-1	3.6319e-1	1.0790e-5	2.5830e-10
0.9	4.7544e-1	4.7542e-1	4.7544e-1	2.1378e-5	4.1603e-10
1.0	6.1563e-1	6.1558e-1	6.1563e-1	4.6258e-5	6.9674e-10

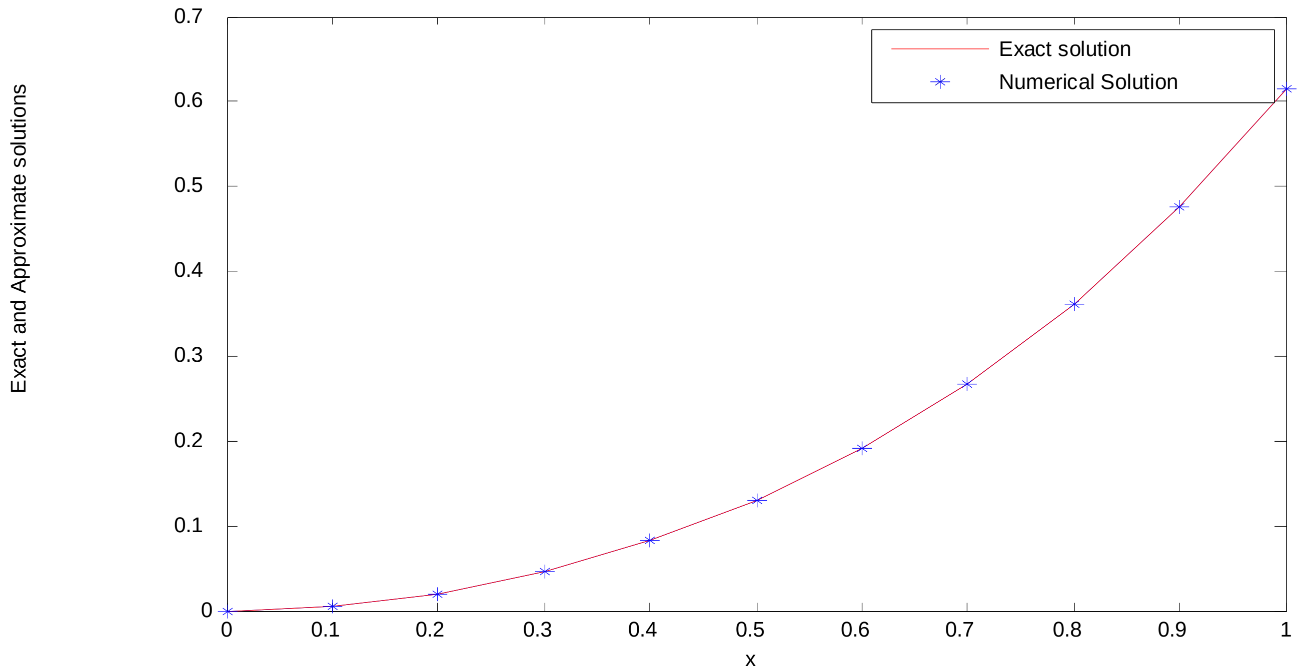

Figure 4.5. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for example 4.3 with mesh size h=0.1

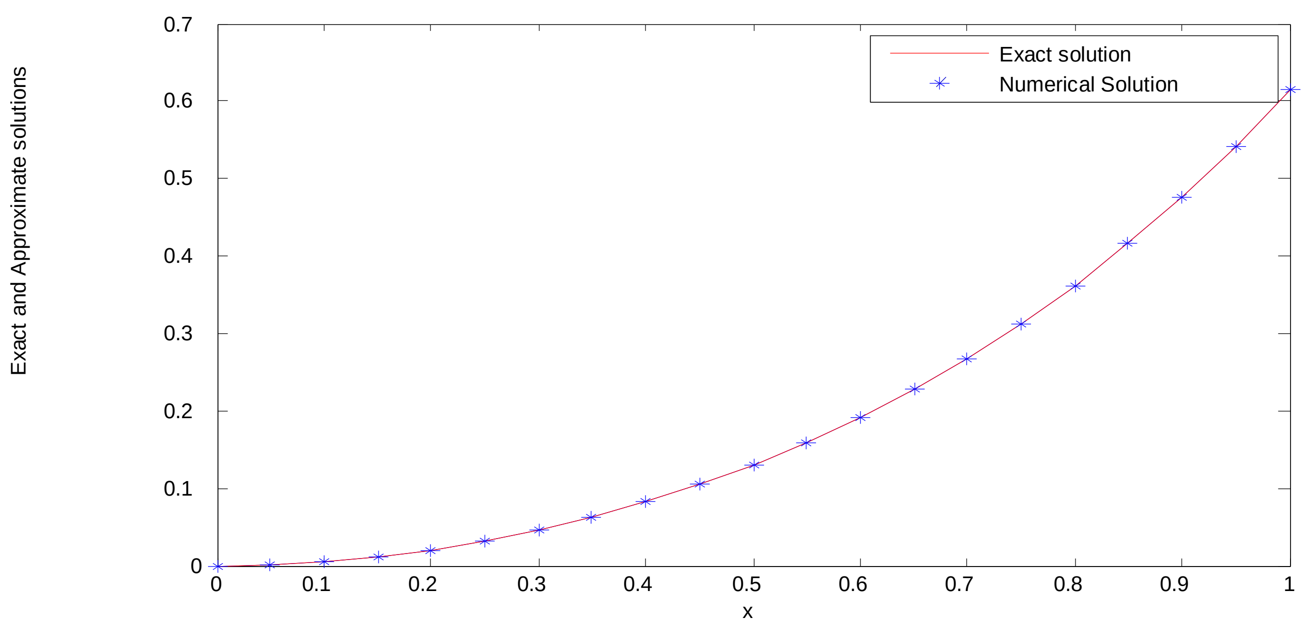

Figure 4.6. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for example 4.3 with mesh size h=0.01

4.4. Discussion

In this study, we presented the Predictor-corrector method for solving second order initial value problems of Bratu-Type equation. In order to verify the accuracy of the method, three model examples were considered and the point wise absolute error displayed using tables. Graphs have been plotted for the model examples with the exact and approximate solution for different step size h.

Table 1 shows that the results of the point wise absolute errors obtained by predictor-corrector method perform better when compared with other numerical methods such as; Perturbation Iteration Algorithm (PIA) by Aksoy and Pakdemirli, (2010), Adomian Decomposition Method (ADM) by Wazwaz, (2012) and Optimal Perturbation Iteration Asymptote method (OPIA) by Sinan and Necdet, (2016).

Table 2 shows the comparison of point wise absolute error of the proposed method with the Reproducing Kernels Hilbert method (RKM), (Eslam *et al.*, 2015). As indicated in the table, the proposed method is better approximates than the indicated numerical method.

Table 3 shows the comparison of exact and numerical solution obtained by predictor-corrector method and also it indicates the point wise absolute errors of the method. As it can be observed from the tables, as the mesh size h decreases the numerical solution approximate the given equations well and the point wise absolute error becomes decreased which shows the method is consistent and convergent by definition 4.1 and 4.2.

Moreover, according to the plotted graphs (figures 4.1-4.6) one can clearly observe that the numerical and exact solutions agree very well. This shows that the proposed method approximates the exact solution well. Generally, the present method is stable, efficient, convergent and accurate than some previously existing methods.

CHAPTER FIVE

5. CONCLUSION AND RECOMMENDATION

5.1. Conclusion

This study was devoted to apply the new method which is called Adams Predictor-corrector method to the nonlinear second order IVP of Bratu-Type equations. First QLM was applied on the nonlinear equation to linearize the given Bratu-type equation and the fourth order Adams-Bashforth was used as a predictor and the fourth order Adams-Moulton method as a corrector to solve the Bratu-type equation. The starting values for the fourth order Adams-Bashforth method were calculated using the fourth order Runge-Kutta method.

The method was validated by considering three model examples. The point wise absolute errors of example 1 and 2 were compared with the results of other numerical methods, and the show that the present method has a better accuracy.

The third example was compared with the exact value of the problem at different mesh size h .

For each example the tables were given and also the graphs were plotted.

In general, the proposed method was easy to use, more accurate and more efficient as compared to the other methods considered in this study.

5.2. Recommendation

In the present study, the numerical method called Adams-Bashforth-Moulton Predictor-corrector method was constructed for solving nonlinear second order initial value problem of Bratu-type equation. Hence, the proposed scheme can also be extended to the nonlinear higher order differential equations other than Bratu-type equations.

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