NUMERICAL TREATMENT OF SINGULARLY PERTURBED DELAY REACTION-DIFFUSION EQUATIONS WITH TWIN LAYERS AND OSCILLATORY BEHAVIOUR

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

(NUMERICAL ANALYSIS)

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DECLARATION

I undersigned declare that, this thesis entitled "Numerical Treatment of Singularly Perturbed Delay Reaction-Diffusion Equations with Twin Layers and Oscillatory Behaviour" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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ACRONYMS

ODE – Ordinary differential equation

DDE – Delay differential equation

SPDE – Singularly perturbed differential equation

SPDDE – Singularly perturbed delay differential equation

SPDRDE – Singularly perturbed delay reaction-diffusion equation

RDDE – Retarded delay differential equation

NDDE – Neutral delay differential equation

IVT – Initial Value Technique

ABSTRACT

In this thesis, second and fourth order parametric uniform numerical methods are presented for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour for which a small shift (δ) *is in the reaction term. First, the given singularly perturbed delay reaction-diffusion equation is converted into an asymptotically equivalent singularly perturbed boundary value problem by using the Taylor series expansion for the delay term as the delay parameter is sufficiently small. Using the finite difference approximations the given differential equation is transformed to a three-term recurrence relation, which can easily be solved by using Thomas Algorithm. The stability and* ε*-uniform convergence of the methods have been established. To validate the applicability of the proposed methods, four model examples without exact solution have been considered and solved for different values of parameters* ε *and* δ *and mesh sizes* h*. Both theoretical error bounds and numerical rate of convergence have been established for the methods. The numerical results have been presented in tables and further to examine the effect of delay on the twin boundary layer and oscillatory behavior of the solution, graphs have been given for different values of* δ*. In a nutshell, the present methods gives better results than some existing numerical methods reported in the literature.*

CHAPTER ONE

INTRODUCTION

1.1. Background of the Study

Due to the advancement in the field of computational mathematics numerical methods are most widely utilized to solve the equation arising in the field of applied medical science, engineering and technology. Numerical analysis is the branch of mathematics that deals with the computational methods which helps to find approximate solutions for difficult problems such as finding the roots of non-linear equations, integration involving complex expressions and solving differential equations for which analytical solution is difficult to find. Numerical analysis plays a significant role when difficulties encountered in finding the exact solution of an equation using a direct method and when it becomes very difficult or impossible to apply theoretical methods to find the exact solution.

The problems in which the highest order derivative term is multiplied by a small positive parameter are known to be singularly perturbed problems and the parameter is known as the perturbation parameter. Depending on the solution behavior of the problem in the limiting case when perturbation parameter goes to zero, such type of problems are classified into two classes, namely, (i) regularly perturbed and (ii) singularly perturbed. If the solution of the original problem tends to the solution of the reduced problem (*i.e.*, the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed otherwise, it is known as singularly perturbed.

Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay term, then it is said to be a singularly perturbed delay differential equation. In these problems, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. In the recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling of various physical and biological phenomena, for example, micro scale heat transfer, hydrodynamics of liquid helium, secondsound theory, thermo elasticity, diffusion in polymers, reaction-diffusion equations, stability, control of chaotic systems, a variety of models for physiological processes, Gemechis and Reddy [15]. Hence in the recent times, many researchers have been trying to develop numerical methods for solving these problems.

Ramesh and Kadalbajoo [35] presented the numerical approximation of singularly perturbed linear second order reaction-diffusion boundary value problems with a small shift (δ) in the reaction term (*i.e*., in the undifferentiated term) and the shift depends on the small parameter (ε). Accordingly, the problem is discretized using standard finite difference scheme on a uniform mesh and the retarded arguments are interpolated/extrapolated using the known computational grid points. Phaneendra et al [31] proposed modified upwind finite difference scheme to tackle the delay term which occurs in the convection term. Swamy [44] presented the quantitative analysis of delay differential equations with layer or oscillatory behaviour by employing the numerical integration. Soujanya and Reddy [42] presented a computational method for solving singularly perturbed delay differential equations with twin layer or oscillatory behaviour in which the small delay is in the reaction term. The authors [41, 42] shows, when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour and when the delay is $o(\varepsilon)$, the solution maintains layer behaviour although the coefficients in the equation are of $O(1)$ and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases. If the coefficient of the delay is of $o(1)$, the amplitude of the oscillations increases slowly as the delay increases provided the delay is of $o(\varepsilon)$ and when the solution of the problem exhibits oscillatory behaviour for delay equal to zero, the delay affects the oscillatory behaviour. Also Swamy et al [45] presented a computational technique for solving singularly perturbed delay differential equations with twin layer or oscillatory behaviour in which the small delay is in the reaction term. Phaneendra et al [32] presented a finite difference approach to solve the boundary-value problem for singularly perturbed differential-difference equation, which contains only negative shift in the convection term (*i.e*., in the differentiated term) by using a fourth order finite difference scheme, provided shifts are of $o(\varepsilon)$. Rao and Chakravarthy [37] presented an exponentially fitted tri-diagonal finite difference method for solving boundary value problems for singularly perturbed differential–difference equations containing a small negative shift with shift parameter smaller than the perturbation parameter which is almost second order parameter uniform convergence. The same authors [36] proposed a finite difference method for singularly perturbed differential-difference equations with layer and oscillatory behaviour of convection–diffusion type by using fourth order finite difference method. Sirisha and Reddy [41] presented the numerical solution of singularly perturbed differential-difference equations, which contains the delay and advance in the problem but not in the derivative terms exhibiting dual layer behavior by second order stable central difference scheme. However, some of these methods are not uniformly convergent, simple and more accurate.

Thus, this study presents parametric uniform methods for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour.

1.2. Statement of the Problem

The numerical analysis of singular perturbation problems has always been far from trivial because of the boundary layer behaviour of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the domain of the problem.

The field of delay differential equation (DDE) attracted mathematicians and engineers due to the following reasons. Firstly, we have to find an appropriate approximation of the solution at the delayed arguments $y(x-\delta)$ and/or $y'(x-\delta)$. Secondly, the algorithm has to take care of the jump in the discontinuity due to the delay parameter and thirdly, its solution behavior is very interesting with boundary layers, interior layers and oscillations. However, the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has been growing more and more.

The increasing desire for the numerical solutions to such mathematical problems, which are more difficult or impossible to solve analytically, has become the present-day scientific research. Bellen and Zennaro as cited in Amiraliyev and Erdogan [2] considered some approximating aspects of first order delay differential equations mainly focused on the stability of numerical methods when the boundary layers are absent. However, it is well known that some of standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results in the presence of boundary layers. The treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions, Doolan et al [7], Kadalbajoo and Reddy [18] and Roos, et al [38]. Kadalbajoo and Ramesh [20] states that, the accuracy of the problem increased by increasing the resolution of the grid which might be impractical in some cases like higher dimensions. Pratima and Sharma [34] states that, till date ε-uniformly convergent methods have not been sufficiently developed for a wide class of singularly perturbed delay differential equations. Therefore, it is important to develop simple, more accurate, stable and parameter uniform (*i.e.*, numerical methods whose accuracy does not depend on the parameter ε or the methods which are uniformly convergent with respect to the parameter ε) for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour.

Owing to this, the present study attempt to answer the following questions:

1. How do the present methods be described for singularly perturbed delay reaction-diffusion equation with twin layers and oscillatory behaviour?
2. To what extent the proposed methods approximate the solutions?
3. To what extent are the proposed methods stable and convergent?
4. What is the advantage of the proposed methods over the other numerical methods?

1.3. Objectives of the study

1.3.1. General Objective

The general objective of this study is to present parameter uniform numerical methods for solving singularly perturbed delay reaction-diffusion equation with twin layers and oscillatory behaviour.

1.3.2. Specific Objectives

The specific objectives of the present study are:

- To describe the numerical methods of second and fourth order parametric uniform methods for solving singularly perturbed delay reaction-diffusion equation with twin layers and oscillatory behaviour.
- To investigate the accuracy of the proposed methods.
- To establish the stability and convergence of the proposed methods.
- To describe the advantage of the present methods over the others.

1.4. Significance of the Study

The outcomes of this study may have the following importance:

- Provide some background information for other researchers who work on this area.
- To introduce the application of numerical methods in different field of studies.
- Help graduate students to acquire research skills and scientific procedures.

1.5. Delimitation of the Study

The singularly perturbed delay differential equations perhaps arise in variety of applied mathematics that contributes for the advancement of science and technology. Though, singularly perturbed delay differential equations are vast topics and have many applications in the real world, this study is delimited to singularly perturbed delay reaction-diffusion equation of the form:

$$\varepsilon y''(x)+a(x)y(x-\delta)+b(x)y(x)=f(x),\ 0<x<1$$

with interval and boundary conditions,

$$y(x)=\phi(x),\ -\delta\le x\le 0 \text{ and } y(1)=\beta$$

where, ε is a small parameter, $0<\varepsilon<<1$ and δ is shift parameter; $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are bounded smooth functions in $(0,1)$ and β is given constant. Further, the study is delimited to second and fourth order parametric uniform numerical methods for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour, though there are varieties of methods for solving the problem under the study.

1.6. Important Theorems

Theorem 1.1: For any partition $J\cup K$ of the index set $\{1,2,\cdots,n\}$ of an $n\times n$ matrix A, if there exists $j\in J$ and $k\in K$ such that $a_{jk}\neq 0$, then A is an irreducible matrix, Varga [46].

Theorem 1.2: If A is an L-matrix which is symmetric, irreducible and has weak diagonal dominance, then A is a monotone matrix, Young [49].

CHAPTER TWO
REVIEW OF RELATED LITERATURE

2.1. Singular Perturbation Theory

Singular perturbation problem was first introduced by Prandtl [33] during his talk on fluid motion with small friction in a seven page report presented at the Third International Congress of Mathematicians in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body can be divide in two regions, a boundary layer and outer region. However, the term ‘singular perturbations’ was first used by Friedrichs et al [11] in a paper presented at a seminar on non-linear vibrations at New York University. The solutions of singular perturbation problems typically contain layers. Prandtl [33], originally introduced the term ‘boundary layer’, but this term came into more general following the work of Wasow [48].

The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations. Kadalbajoo, and Gupta [19] these singularly perturbed problems arise in the modeling of various modern complicated processes, such as fluid flow at high Reynolds numbers, water quality problems in rivers networks, convective heat transport problem with large Péclet numbers, drift diffusion equation of semiconductor device modeling, electromagnetic field problem in moving media, financial modeling of option pricing, turbulence model, simulation of oil extraction from under-ground reservoirs, theory of plates and shells, atmospheric pollution, groundwater transport, and chemical reactor theory.

In the modeling of these processes, characterized by dominant convection and/or intensive reactions, one can observe boundary and interior layers whose width, depending on the perturbation parameters, can be arbitrarily small. On the other hand, the domain itself, where the problem in question is considered, can be extremely large, even unbounded, compared to the available computational resources (especially in multidimensional problems for systems of equations). A complicated geometry of the domains, and/or lack of sufficient smoothness (or compatibility) of the problem data may result in singular solutions in which different parts have their own specific scales. Standard numerical methods applied to such multi-scale problems give unsatisfactorily large errors, which make these methods inapplicable for practical use. Thus, it is of considerable scientific interest to develop a solid mathematical theory and specific computational methods for singularly perturbed multi-scale problems and related problems arising from applications [19].

Perturbation theory is a subject which studies the effect of small parameter in the mathematical model problems in ordinary differential equations. In Mathematics, more precisely in perturbation theory, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero.

During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. The main concern with singular perturbation problems is the rapid growth or decay of the solution in one or more narrow "layer region(s)".

2.2. Singularly Perturbed Delay Differential Equation

The theory and numerical solution of singularly perturbed delay differential equations are still at the initial stage. In the past, only very few people had worked in the area of numerical methods on singularly perturbed delay differential equations (SPDDEs). But in the recent years, there has been a growing interest in this area. In fact, Erdogan [10] proposed an exponentially fitted operator method for singularly perturbed first order delay differential equation, Kadalbajoo and Sharma [20, 22, 23] and Mohapatra and Natesan [29] proposed some numerical methods for SPDDEs with a small delay. It may be noted that Lange and Miura [27] gave an asymptotic approximation to solve singularly perturbed second order delay differential equations. In the present work a numerical method named as Initial Value Technique (IVT) is suggested to solve the boundary value problems for second order ordinary differential equations of reaction-diffusion type with a negative shift in the differentiated term. The initial value method was introduced by the authors Gasparo and Macconi [13]. In fact they applied this method to solve singularly perturbed boundary value problems for differential equations without negative shift/delay. Chakravarthy et al [6] presented an exponentially fitted finite difference scheme to solve singularly perturbed delay differential equation of second order with a large delay $\delta = 1$.

Gemechis and Reddy [14] presented a numerical method that does not depend on the asymptotic expansion and matching of the coefficients for solving a class of singularly perturbed delay differential equations with negative shift in the differentiated term. Awoke and Reddy [3] provided a parameter fitted scheme and effect of small shifts on the boundary layer solution of the problem to solve singularly perturbed delay differential equations in the differentiated term of second order with left or right boundary. Accordingly, when the delay parameter is smaller than the perturbation parameter, the layer behaviour is maintained.

A delay differential equation (DDE) is an equation where the evolution of the system at a certain time, depends on the state of the system at an earlier time. This is distinct from ordinary differential equations (ODEs) where the derivatives depend only on the current value of the independent variable. A DDE is said to be of retarded delay differential equation (RDDE) if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equation (NDDE). If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we obtain singularly perturbed delay differential equations of the retarded type. Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means of first order accurate Taylor's series expansions of the terms that involve delay and the resulting differential equations have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not [24]. When the delay argument is sufficiently small, to tackle the delay term Kadalbajoo and Sharma [21], used Taylor's series expansion and presented an asymptotic as well as numerical approach to solve such type boundary value problem. But the existing methods in the literature fail in the case when the delay argument is bigger one because in this case, the use of Taylor's series expansion for the term containing delay may lead to a bad approximation.

2.2.1. Boundary Value Problem

Finding the numerical solution of a boundary value problem is more difficult than that of corresponding initial value problem. The boundary-value problems for singularly perturbed delay-differential equations arise in various practical problems in biomechanics and physics such as in variational problem in control theory. These problems mainly depend on a small positive parameter and a delay parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. Moreover, this class of problems possesses boundary layers, *i.e.* regions of rapid change in the solution near one of the boundary points.

There is a wide class of asymptotic expansion methods available for solving the above type problems. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions, which are not routine exercises but require skill, insight and experimentation. Gülsu and Öztürk [16] the numerical treatment of singularly perturbed problems presents some major computational difficulties and in recent years a large number of special purpose methods have been proposed to provide accurate numerical solutions. This type of problem has been intensively studied analytically and it is known that its solution generally has boundary layers where the solution varies rapidly. The outer solution corresponds to the reduced problem, *i.e.*, that obtained by setting the small perturbation parameter to zero. In recent years, the Chebyshev, cubic-spline, B-spline, finite difference, stable central difference methods [36, 41, 5, 25] etc has been used to find the approximate solutions of differential, difference, integral and integro-differential-difference equations. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem**.**

Lange and Miura [26] gave asymptotic approaches in the study of class of boundary value problems for linear second order differential difference equations in which the highest order derivative is multiplied by small parameter. Accordingly, there are BVPs for DDEs exhibiting solutions with rapid oscillations all across the interval for shifts that are sufficiently small. Also, there are BVPs with solutions in which oscillations were previously confined to layer regions when the shifts are sufficiently small, but where the oscillations can extend into the outer region when the shifts are increased. The oscillatory solutions are treated using the method of matched asymptotic expansions and the WKB method, which account for the small shifts.

2.2.2. Uniformly Convergent Methods

The singularly perturbed boundary-value problems cannot be solved numerically in a satisfactory manner by standard finite difference methods on uniform mesh. This encourages the need for the methods that behave uniformly well, *i.e.*, which converges independent of the singular perturbation parameter ε. Such methods are referred as ε-uniform of parameter uniform methods, where ε is the singular perturbation parameter. In the construction of an ε-uniform method, there are mainly two approaches. The first are the fitted operator methods which comprise of specially designed finite difference operator which reflects the singularly perturbed nature of the solution. Such fitted operator methods were first suggested by Allen and Southwell [1] for solving the problem of viscous fluid flow past a cylinder. An extensive account of ε-uniform fitted operator methods is discussed in Doolan et al [7]. The second are the fitted mesh methods which comprise of standard finite difference operators on fitted piecewise-uniform meshes condensing in the boundary layers [28].

The fitted mesh methods have probably received less detailed attention in the literature, than the construction of an appropriate finite difference fitted operator or finite element subspace methods. In 1996 [28], Miller et al. established the great importance of fitted mesh methods for solving singular perturbation problems. There are some problems for which no ε-uniform method can be constructed using a fitted operator approach on a uniform mesh while for such problem an ε-uniform fitted mesh method can be constructed.

In [23, 25], an ε-uniform numerical scheme is constructed for a class of boundary value problems for singularly perturbed differential–difference equations with small shifts. The numerical method comprises a standard upwind finite difference operator on a fitted piecewise-uniform mesh which is condensed in the boundary layers by approximating the terms containing small shift by Taylor series and then apply the fitted mesh method, provided shifts are of $o(\varepsilon)$.

Pratima and Sharma [34] states that, till date ε-uniformly convergent methods have not been sufficiently developed for a wide class of singularly perturbed delay differential equations. There is still a lot to be explored in the study of boundary value problems for singularly perturbed delay differential equations with bigger delay and in particular, the case when the convection coefficient vanishes inside the domain for such type of differential equations is still to be investigated for big as well as small delay.

2.3. Double Mesh Principle

According to Doolan et al as cited in Pratima and Sharma [34] the double-mesh principle used to calculate the maximum absolute error and rate of convergence of the numerical scheme, when the differential equation has no exact solution. So the accuracy of their numerical solutions will be computed using double mesh principle. For any value of N, the maximum point wise errors $\|E\|$ for the solution y_i, will be calculated by $E_h = \max_i \left| y_i^h - y_i^{h/2} \right|,\ i = 1, 2, \ldots, N-1$, where y_i^h is the computed solution with N number of mesh intervals and $y_i^{h/2}$ is the numerical solution on a mesh, obtained by bisecting the original mesh with N number of mesh intervals (*i.e.* $2N$ mesh intervals). And rate of convergence ρ are computed using the double-mesh principle as,

$$\rho = \frac{\left(\log\left(E_h\right) - \log\left(E_{h/2}\right) \right)}{\log 2}$$

2.4. Finite Difference Method

Finite difference methods were made during the period of, and immediately following, the Second World War, when large-scale practical applications became possible with the aid of computers. A major role was played by the work of von Neumann, partly reported in O'Brien, Hyman and Kaplan (1951).

Finite difference methods are always a convenient choice for solving boundary value problems because of their simplicity. Finite difference methods are one of the most widely used numerical schemes to solve differential equations and their application in sciences and technology. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of linear equations to be solved by Thomas Algorithm or other methods in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward difference method, backward difference method, central difference method, etc.

The finite difference method as cited in Vasil'eva [47] and Prandtl [33] is widely used by the scientific community for the numerical solution of reaction–diffusion equations; however, there are comparatively few studies that give stability and convergence results see Beckett, et al [4], Hoff [17]. For a unified treatment of how and when the finite difference method for reaction–diffusion equations breaks down see Stuart [43], Elliott et al [9], and Ruuth [39]. A uniform higher order difference schemes for singularly perturbed two-point boundary value problem is presented by Gartland [12].

Present-day scientific research concerns on the methods of numerical solutions to mathematical problems which are simpler to use and solve difficult problems. Accordingly, obtaining stable, accurate, uniformly convergent and fast numerical solutions for singularly perturbed delay differential equations has a great importance due to its wide applications in science and engineering research, since they are difficult or impossible to solve analytically. Owing to this, this study presents parametric uniform numerical methods for solving singularly perturbed delay reaction-diffusion equation with twin layers and oscillatory behaviour by the methods of second and fourth order.

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted in Jimma University under the department of Mathematics from September 2015 to June 2016 G.C. Conceptually the study focus on parametric uniform numerical methods for solving singularly perturbed delay reaction-diffusion equation with twin layers and oscillatory behaviour, particularly by second and fourth order methods.

3.2. Study Design

This study was employed mixed-design (documentary review design and experimental design) on singularly perturbed delay boundary value problems of reaction-diffusion equation type.

3.3. Source of Information

The relevant sources of information for this study are books, published articles & related studies from internet and the experimental result was obtained by writing MATLAB code for the present numerical methods. The proposed methods are programmed using MATLAB ver. 8.1.0.604 (R2013a).

3.4. Study Procedures

Necessary materials and data for the study were collected by means of documentary review and algorithm development. Hence, in order to achieve the stated objectives, the study procedures followed were:

1. Defining the problem,
2. Discretizing the domain/interval and formulating the methods.
3. Replacing the differential equation by the finite difference approximations and obtaining the systems of equations.
4. Reduce the obtained systems of equations into tri-diagonal systems which can be easily solved by Thomas Algorithm.
5. Establishing the stability and convergence of the methods.
6. Writing MATLAB code for the tri-diagonal systems obtained.
7. Validating the schemes using numerical examples.
8. Comparing the results of the present methods with existing methods.

3.5. Ethical Considerations

Ethical clearance was obtained from Research and Post Graduate program coordinator Office of College of Natural Sciences, Jimma University and any concerned body were informed about the purpose of the study.

CHAPTER FOUR

DESCRIPTION OF THE METHODS, RESULTS AND DISCUSSION

4.1. Description of the Methods

In this section, the description of second and fourth order finite difference methods and their stability and convergence analysis is discussed. Consider singularly perturbed delay reaction-diffusion equation of the standard form:

$$\varepsilon y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x),\ 0 < x < 1 \tag{4.1}$$

with the interval and boundary conditions,

$$y(x) = \phi(x),\ -\delta \le x \le 0 \text{ and } y(1) = \beta \tag{4.2}$$

where ε is small parameter, $0 < \varepsilon << 1$ and δ is also small delay parameter, $0 < \delta << 1$; $a(x), b(x), f(x)$ and $\phi(x)$ are bounded smooth functions in $(0,1)$ and β is a given constant. For $\delta = 0$, the solution of the boundary value problem in Eqs. (4.1) and (4.2) exhibits layer or oscillatory behaviour depending on the sign of $a(x)+b(x)$, for all $x \in (0,1)$. If $a(x)+b(x) < 0$, the solution of the problem in Eqs. (4.1) and (4.2) exhibits layer behaviour, and if $a(x)+b(x) > 0$, it exhibits oscillatory behaviour, Swamy et al [45].

The layer or oscillatory behaviour of the problem under consideration (*i.e.*, reaction-diffusion type) is maintained for $\delta \ne 0$, but sufficiently small. Therefore, if the solution exhibits layer behaviour, there will be two boundary layers which will be at both the end points. *i.e.*, at $x = 0$ and $x = 1$. In general, the solution of the problem in Eqs. (4.1) and (4.2) exhibits layer or oscillatory behaviour depending on the sign of $a(x)+b(x)$. The solution $y(x)$ must be continuous on $[0,1]$, continuously differentiable on $(0,1)$ and also satisfies Eqs. (4.1) and (4.2), Ramesh and Kadalbajoo [35].

By using Taylor series expansion in the neighborhood of the point x, we have:

$$y(x-\delta) \approx y(x) - \delta y'(x) + \mathrm{o}(\delta^2) \tag{4.3}$$

Substituting Eq. (4.3) into Eq. (4.1), we obtain an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$Ly(x) \equiv y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{4.4}$$

under the boundary conditions,

$$y(0) = \phi_0 \text{ and } y(1) = \beta . \tag{4.5}$$

where, $p(x) = \dfrac{-\delta a(x)}{\varepsilon}$, $q(x) = \dfrac{a(x)+b(x)}{\varepsilon}$ and $r(x) = \dfrac{f(x)}{\varepsilon}$.

The transition from Eq. (4.1) to Eq. (4.4) is admitted, because of the condition that $0 < \delta << 1$ is sufficiently small. Further details on the validity of this transition can be found in Elsgolt's and Norkin [8].

Now, divide the interval $[0,1]$ into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, ..., x_N = 1$ be the mesh points. Then, we have $x_i = x_0 + ih$, $i = 0,1, 2, ..., N$.

4.1.1. Method I (Second Order Method)

Assuming that $y(x)$ has continuous derivatives on $[0,1]$ and making use of Taylor's series expansion, we have:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + O(h^5) \tag{4.6}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i^{(3)} + \frac{h^4}{4!}y_i^{(4)} + O(h^5) \tag{4.7}$$

Subtracting Eq. (4.7) from Eq. (4.6), we get:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6}y_i''' + T_1 \tag{4.8}$$

where, $T_1 = -\dfrac{h^4}{120}y^{(5)}(\xi_1)$, for $\xi_1 \in [x_{i-1}, x_i]$.

Again, adding Eq. (4.6) and Eq. (4.7), we get:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + T_2 \tag{4.9}$$

where, $T_2 = -\frac{h^4}{360} y^{(6)}(\xi_2)$, for $\xi_2 \in [x_{i-1}, x_i]$.

Substituting Eqs. (4.8) and (4.9) into Eq. (4.4), we obtain:

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{p_i}{2h}(y_{i+1} - y_{i-1}) - \frac{h^2}{6} p_i y_i''' + q_i y_i = r_i + T \tag{4.10}$$

where, $T = \frac{h^2}{12} y^{(4)}(\xi_2) - p_i T_1 - T_2$ is the local truncation error and $p(x_i) = p_i$, $q(x_i) = q_i$, $r(x_i) = r_i$, $y(x_i) = y_i$.

Rewriting Eq. (4.4), we have:

$$y''(x) = r(x) - p(x)y'(x) - q(x)y(x) \tag{4.11}$$

Differentiating both sides of Eq. (4.11) with respect to x and evaluating at x_i, we get:

$$y_i''' = r_i' - p_i y_i'' - (p_i' + q_i) y_i' - q_i' y_i \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.10) for y_i''' and using central difference approximation for y_i'' and y_i', we obtain:

$$\begin{aligned}&\left(\frac{1}{h^2} - \frac{p_i}{2h} + \frac{p_i^2}{6} - \frac{h}{12} p_i \left(p_i' + q_i\right)\right) y_{i-1} - \left(\frac{2}{h^2} + \frac{p_i^2}{3} - q_i - \frac{h^2}{12} p_i q_i'\right) y_i \\ &+ \left(\frac{1}{h^2} + \frac{p_i}{2h} + \frac{p_i^2}{6} + \frac{h}{12} p_i \left(p_i' + q_i\right)\right) y_{i+1} = r_i + \frac{h^2}{6} p_i r_i'\end{aligned} \tag{4.13}$$

Eq. (4.13) can be written as the three term recurrence relation of the form:

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \text{, for } i = 1, 2, \ldots, N-1 \tag{4.14}$$

where,

$$E_i = \frac{1}{h^2} - \frac{p_i}{2h} + \frac{p_i^2}{6} - \frac{h}{12} p_i \left(p_i' + q_i \right)$$

$$F_i = \frac{2}{h^2} + \frac{p_i^2}{3} - q_i - \frac{h^2}{6} p_i q_i'$$

$$G_i = \frac{1}{h^2} + \frac{p_i}{2h} + \frac{p_i^2}{6} + \frac{h}{12} p_i \left(p_i' + q_i \right)$$

$$H_i = r_i + \frac{h^2}{6} p_i r_i'$$

The tri-diagonal system in Eq. (4.14) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

Stability and Convergence Analysis for Method I

To present the minimum principle and stability of the Eqs. (4.4), (4.5) and (4.14), we followed the procedure given by Sirisha and Reddy [40].

Case 1: Layer Behaviour $\left(i.e.\ a(x) + b(x) < 0, \text{ for } x \in (0,1). \text{ Thus } q(x) < 0, \text{ since } \varepsilon > 0 \right)$.

Lemma 4.1: (Continuous Minimum Principle)

If $y(0) \geq 0$ and $Ly(x) \leq 0$, for all $x \in \left(0,1\right)$, then the solution $y(x) \geq 0$ for all $x \in \left(0,1\right)$ for Eqs. (4.4) and (4.5).

Proof:

We prove this Lemma by contradiction.

Suppose $t \in \left(0,1\right)$, such that $y(t) = \min\limits_{x \in (0,1)} y(x)$ and $y(t) < 0$. Since, $t \notin \left\{0,1\right\}$ and is a point of minima, then $y'(t) = 0$ and $y''(t) \geq 0$.

Therefore, we have:

$Ly(t) \equiv y''(t) + p(t) y'(t) + q(t) y(t) > 0$, since $y(t) < 0$ (by the assumption) and $q(t) < 0$. But, this is a contradiction.

It follows that $y(t) \geq 0$ and therefore, $y(x) \geq 0$ for all $x \in \left(0,1\right)$.

Theorem 4.1: (Stability)

The solution of the problem in Eqs. (4.4) and (4.5) satisfies $|y(x)| \le C \max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\}$, for some constant $C \ge 1$.

Proof:

Define two functions, $\psi^{\pm} = C \max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\} \pm y(x)$. Then,

(i) $\psi^{\pm}(0) = C \max\{|y(0)|, \max|Ly(0)|\} \pm y(0)$

Case I: if $\max\{|y(0)|, \max|Ly(0)|\} = |y(0)|$, we have;

$\psi^{\pm}(0) = C|y(0)| \pm y(0) \ge 0$ as $C \ge 1$.

Case II: if $\max\{|y(0)|, \max|Ly(0)|\} = \max|Ly(0)|$, then:

$\max|Ly(0)| \ge |y(0)| \Rightarrow C\max|Ly(0)| \ge C|y(0)|$ as $C \ge 1$.

Thus, $\psi^{\pm}(0) = C \max\{|y(0)|, \max|Ly(0)|\} \pm y(0)$

$$= C\max|Ly(0)| \pm y(0)$$
$$\ge C|y(0)| \pm y(0) \ge 0.$$

Hence, $\psi^{\pm}(0) \ge 0$.

(ii) $L\psi^{\pm}(x) \equiv C\,q(x) \max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\} \pm Ly(x)$

Case I: if $\max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\} = \max\limits_{x\in(0,1)}|Ly(x)|$, we have:

$$L\psi^{\pm}(x) \equiv C\,q(x) \max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\} \pm Ly(x)$$

$= C\,q(x) \max\limits_{x\in(0,1)}|Ly(x)| \pm Ly(x) \le 0$, since $q(x) < 0$ and for suitable choice of C.

Case II: if $\max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\} = |y(0)|$, then,

$|y(0)| \ge \max\limits_{x\in(0,1)}|Ly(x)| \Rightarrow C\,q(x)|y(0)| \le C\,q(x)\max\limits_{x\in(0,1)}|Ly(x)|$ as $C \ge 1$ and $q(x) < 0$.

Thus,

$$
\begin{aligned}
L\psi^{\pm}(x) &\equiv C\,q(x)\max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\} \pm Ly(x) \\
&= C\,q(x)|y(0)| \pm Ly(x) \\
&\le C\,q(x)\max_{x\in(0,1)}|Ly(x)| \pm Ly(x) \le 0\ ,\ \text{ since } q(x) < 0.
\end{aligned}
$$

Hence, $L\psi^{\pm}(x) \le 0.$

Therefore, by Lemma 4.1, we get, $\psi^{\pm}(x) \ge 0$, for all $x \in (0,1)$. So,

$$
\begin{aligned}
&\psi^{\pm} = C\max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\} \pm y(x) \ge 0 \\
&\Rightarrow C\max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\} \ge \mp y(x) \\
&\Rightarrow |y(x)| \le C\max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\}.
\end{aligned}
$$

Hence, the stability of the solutions of the problem in Eqs. (4.4) and (4.5) is proved for the case of layer behaviour.

Lemma 4.2: (Discrete Minimum Principle)

The finite difference operator L^N in Eq. (4.14) has the discrete minimum principle, if w_i is any mesh function such that $w_0 \ge 0$ and $L^N w_i \le 0$, for all $x_i \in (0,1)$, then $w_i \ge 0$ for all $x \in (0,1)$.

Proof:

Suppose that there exists a positive integer k such that $w_k < 0$ and $w_k = \min\limits_{0\le i\le N} w_i$.

Then from Eq. (4.14), we have:

$$
\begin{aligned}
L^N w_k &\equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1} \\
&= \left(\frac{1}{h^2} - \frac{p_k}{2h} + \frac{p_k^2}{6} - \frac{h}{12}p_k\left(p_k' + q_k\right)\right)w_{k-1} - \left(\frac{2}{h^2} + \frac{p_k^2}{3} - q_k - \frac{h^2}{6}p_k q_k'\right)w_k \\
&\quad + \left(\frac{1}{h^2} + \frac{p_k}{2h} + \frac{p_k^2}{6} + \frac{h}{12}p_k\left(p_k' + q_k\right)\right)w_{k+1}
\end{aligned}
$$

$$=\left(\frac{1}{h^2}+\frac{p_k^2}{6}\right)\left(w_{k-1}-w_k\right)+\left(\frac{1}{h^2}+\frac{p_k^2}{6}\right)\left(w_{k+1}-w_k\right)+\left(\frac{p_k}{2h}+\frac{h}{12}p_k\left(p_k'+q_k\right)\right)\left(w_{k+1}-w_{k-1}\right)$$
$$+\left(q_k+\frac{h^2}{6}p_kq_k'\right)w_k$$

For sufficiently small h (*i.e.*, as $h\to 0$) and for suitable value of p_k, we obtain:

$L^N w_k>0$. Since, $w_k<0$ (by the assumption) and $\left(q_k+\frac{h^2}{6}p_kq_k'\right)\to q_k<0$. But, this is a contradiction.

Hence, $w_i\geq 0$ for all $x_i\in(0,1)$.

Theorem 4.2: The finite difference operator L^N in Eq. (4.14) is stable for $a(x)+b(x)<0$, if w_i is any mesh function, then $|w_i|\leq C\max\left\{|w_0|,\max_{x_i\in(0,1)}|Lw_i|\right\}$, for some constant $C\geq 1$.

Proof:

We define two functions, $\psi_i^{\pm}\equiv C\max\left\{|w_0|,\max_{x_i\in(0,1)}|Lw_i|\right\}\pm w_i$. Then, similar to Theorem 4.1, we get:

$\psi_0^{\pm}\geq 0$ and

$L\psi_i^{\pm}\equiv Cq_i\max\left\{|w_0|,\max_{x_i\in(0,1)}|Lw_i|\right\}\pm Lw_i\leq 0$, since $a_i+b_i<0\Rightarrow q_i<0$ and $C\geq 1$.

Therefore by Lemma 4.2 we get:

$\psi_i^{\pm}\geq 0,\ \text{for all}\ x_i\in(0,1).$

$\Rightarrow\psi_i^{\pm}\equiv C\max\left\{|w_o|,\max_{x_i\in(0,1)}|Lw_i|\right\}\pm w_i\geq 0$.

Thus, $|w_i|\leq C\max\left\{|w_o|,\max_{x_i\in(0,1)}|Lw_i|\right\}$.

This proves the stability of the scheme for the case of layer behaviour.

Case 2: Oscillatory Behaviour $\left(i.e.\ a(x)+b(x)>0, \text{ for } x\in(0,1). \text{ Thus } q(x)>0, \text{ as } \varepsilon>0 \right)$.

The continuous maximum principle and stability of the solution of Eqs. (4.4) and (4.5) are presented as follows for the case of oscillatory behaviour.

Lemma 4.3: (Continuous Maximum Principle)

If $y(0)\geq 0$ and $Ly(x)\geq 0$, for all $x\in(0,1)$, then the solution $y(x)\geq 0$ for all $x\in(0,1)$ for Eqs. (4.4) and (4.5).

Proof:

We prove this Lemma by contradiction.

Suppose $t\in(0,1)$, such that $y(t)=\max\limits_{x\in(0,1)} y(x)$ and $y(t)<0$. Since, $t\notin\{0,1\}$ and is a point of maxima, therefore $y'(t)=0$ and $y''(t)\leq 0$.

Therefore, we have:

$Ly(t)\equiv y''(t)+p(t)y'(t)+q(t)y(t)<0$, since $y(t)<0$ (by the assumption) and $q(t)>0$. But, this is a contradiction.

Hence, $y(x)\geq 0$, for all $x\in(0,1)$.

Theorem 4.3: (Stability)

The solution of the problem in Eqs. (4.4) and (4.5) satisfies $|y(x)|\leq C\max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\}$, for some constant $C\geq 1$.

Proof:

Define two functions, $\psi^{\pm}=C\max\left\{|y(0)|, \max\limits_{x\in(0,1)}|Ly(x)|\right\}\pm y(x)$. Then, similar to Theorem 4.1, we obtain:

$\psi^{\pm}(0)\geq 0$ and

$$L\psi^{\pm}(x) \equiv C\,q(x)\max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\} \pm Ly(x) \geq 0 \text{, since } q(x) > 0 \text{ and } C \geq 1.$$

Therefore by Lemma 4.3 we get, $\psi^{\pm}(x) \geq 0$ for all $x \in (0,1)$.

Thus, $|y(x)| \leq C\max\left\{|y(0)|, \max_{x\in(0,1)}|Ly(x)|\right\}$.

Hence, the stability of the solutions of the problem in Eqs. (4.4) and (4.5) is proved for the case of oscillatory behaviour.

Now, we present the maximum principle and stability of the discrete problem in Eq. (4.14) for the case of oscillatory behaviour.

Lemma 4.4: (Discrete Maximum Principle)

The finite difference operator L^N in Eq. (4.14) has the discrete maximum principle, if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \geq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in (0,1)$.

Proof:

Suppose that there exists a positive integer k such that $w_k < 0$ and $w_k = \max_{0\leq i\leq N} w_i$.

Then from Eq. (4.14), we have:

$$\begin{aligned}
L^N w_k &\equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1} \\
&= \left(\frac{1}{h^2} - \frac{p_k}{2h} + \frac{p_k^2}{6} - \frac{h}{12}p_k\left(p_k' + q_k\right)\right)w_{k-1} - \left(\frac{2}{h^2} + \frac{p_k^2}{3} - q_k - \frac{h^2}{6}p_k q_k'\right)w_k \\
&\quad + \left(\frac{1}{h^2} + \frac{p_k}{2h} + \frac{p_k^2}{6} + \frac{h}{12}p_k\left(p_k' + q_k\right)\right)w_{k+1}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{h^2} + \frac{p_k^2}{6}\right)\left(w_{k-1} - w_k\right) + \left(\frac{1}{h^2} + \frac{p_k^2}{6}\right)\left(w_{k+1} - w_k\right) + \left(\frac{p_k}{2h} + \frac{h}{12}p_k\left(p_k' + q_k\right)\right)\left(w_{k+1} - w_{k-1}\right) \\
&\quad + \left(q_k + \frac{h^2}{6}p_k q_k'\right)w_k
\end{aligned}$$

For sufficiently small h and for suitable value of p_k, we obtain:

$L^N w_k < 0$. Since, $w_k < 0$ (by the assumption) and $\left(q_k + \frac{h^2}{6} p_k q_k' \right) \to q_k > 0$. But, this is a contradiction.

Hence, $w_i \geq 0$ for all $x_i \in (0,1)$.

Theorem 4.4: The finite difference operator L^N in Eq. (4.14) is stable for $a(x)+b(x)>0$, $\left(i.e.\ q(x)>0\right)$, if w_i is any mesh function, then $\left|w_i\right| \leq C \max\left\{\left|w_0\right|, \max\limits_{x_i \in (0,1)} \left|Lw_i\right|\right\}$, for some constant $C \geq 1$.

Proof:

We define two functions, $\psi_i^{\pm} \equiv C \max\left\{\left|w_0\right|, \max\limits_{x_i \in (0,1)} \left|Lw_i\right|\right\} \pm w_i$. Then, similar to Theorem 4.1, we get:

$$\psi_0^{\pm} \geq 0 \text{ and } L\psi_i^{\pm} \equiv C\, q_i \max\left\{\left|w_0\right|, \max_{x_i \in (0,1)} \left|Lw_i\right|\right\} \pm Lw_i \geq 0 \text{ since } q_i > 0 \text{ and } C \geq 1.$$

Therefore, by Lemma 4.4, we get:

$$\psi_i^{\pm} \geq 0, \text{ for all } x_i \in (0,1).$$

$$\Rightarrow \psi_i^{\pm} \equiv C \max\left\{\left|w_o\right|, \max_{x_i \in (0,1)} \left|Lw_i\right|\right\} \pm w_i \geq 0.$$

Thus, $\left|w_i\right| \leq C \max\left\{\left|w_o\right|, \max\limits_{x_i \in (0,1)} \left|Lw_i\right|\right\}$.

This proves the stability of the scheme for the case of oscillatory behaviour.

Definition 4.1 (Uniformly Convergence): Let y be a solution of Eqs. (4.1) and (4.2). Consider a difference scheme for solving Eqs. (4.1) and (4.2). If the scheme has a numerical solution y^N that satisfies

$$\left\| y - y^N \right\| \le C\, h^p \,,$$

where $C > 0$ and $p > 0$ are independent of ε and of the mesh size h, then we say the scheme uniformly converges to y with respect to the norm $\|.\|$, O'Riordan and Stynes [30].

Theorem 4.5: Let $y(x)$ be the analytical solution of the problem in Eqs. (4.4) and (4.5) and $y^N(x)$ be the numerical solution of the discretized problem of Eq. (4.14). Then, $\left\| y - y^N \right\| \le C\, h^2$ for sufficiently small h and C is positive constant.

Proof:

Multiplying both sides of Eq. (4.13) by $-h^2$, we get:

$$\begin{aligned}&\left(-1+\frac{h}{2}p_i-\frac{h^2}{6}p_i^2+\frac{h^3}{12}p_i\left(p_i'+q_i\right)\right)y_{i-1}+\left(2+\frac{h^2}{3}p_i^2-h^2q_i-\frac{h^4}{6}p_iq_i'\right)y_i\\&+\left(-1-\frac{h}{2}p_i-\frac{h^2}{6}p_i^2-\frac{h^3}{12}p_i\left(p_i'+q_i\right)\right)y_{i+1}+h^2\left(r_i+\frac{h^2}{6}p_ir_i'\right)+T_i\left(h\right)=0\end{aligned} \tag{4.15}$$

where, $T_i(h)=\dfrac{h^4}{12}y^{(4)}(\xi_2)+O\left(h^6\right)$ is a local truncation error, for $i=1,2,\ldots,N-1$.

Simplifying Eq. (4.15), we get:

$$\left(-1+u_i\right)y_{i-1}+\left(2+v_i\right)y_i+\left(-1+w_i\right)y_{i+1}+g_i+T_i=0 \tag{4.16}$$

where,

$$\begin{aligned}u_i&=\frac{h}{2}p_i-\frac{h^2}{6}p_i^2+\frac{h^3}{12}p_i\left(p_i'+q_i\right)\\v_i&=\frac{h^2}{3}p_i^2-h^2q_i-\frac{h^4}{6}p_iq_i'\\w_i&=-\frac{h}{2}p_i-\frac{h^2}{6}p_i^2-\frac{h^3}{12}p_i\left(p_i'+q_i\right)\end{aligned}$$

$$g_i = h^2\left(r_i + \frac{h^2}{6} p_i r_i'\right)$$

Incorporating the boundary conditions $y_0 = \phi(x_0) = \phi_0$, $y_N = y(1) = \beta$ in Eq. (4.16), we get the systems of equations of the form:

$$\begin{bmatrix} (2+v_1) & (-1+w_1) & 0 & \cdots & 0 \\ (-1+u_2) & (2+v_2) & (-1+w_2) & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & (-1+u_{N-1}) & (2+v_{N-1}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \end{bmatrix} + \begin{bmatrix} g_1 + (-1+u_1)\phi(0) \\ g_2 \\ g_3 \\ \vdots \\ g_{N-1} + (-1+w_{N-1})\beta \end{bmatrix} + \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-1} \end{bmatrix} = \overline{0}$$

$$\Rightarrow (D+P)y + M + T(h) = 0 \tag{4.17}$$

where,

$$D = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & -1 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} v_1 & w_1 & 0 & \cdots & 0 \\ u_2 & v_2 & w_2 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & u_{N-1} & v_{N-1} \end{bmatrix}$$ are tri-diagonal matrices of order $N-1$, and

$M = \left[\left(g_1 + (-1+u_1)\phi(0)\right),\ g_2,\ g_3, \cdots, \left(g_{N-1} + (-1+w_{N-1})\beta\right)\right]^T$, $T(h) = O\left(h^4\right)$ and $y = [y_1, y_2, \cdots, y_{N-1}]^T$, $T(h) = [T_1, T_2, \cdots, T_{N-1}]^T$, $\overline{0} = [0, 0, \cdots, 0]^T$ are the associated vectors of Eq. (4.17).

Let $y^N = \left[y_1^N, y_2^N, \cdots, y_{N-1}^N\right]^T \cong y$ be the solution which satisfies the Eq. (4.17), we have:

$$(D+P)y^N + M = 0 \tag{4.18}$$

Let $e_i = y_i - y_i^N$, for $i = 1, 2, \cdots, N-1$ be the discretization error, then,

$$y - y^N = [e_1, e_2, \cdots, e_{N-1}]^T.$$

Subtracting Eq. (4.17) from Eq. (4.18), we get:

$$\left(D+P\right)\left(y^N-y\right)=T(h) \tag{4.19}$$

Let $\left|p_i\right|\le C_1$, $\left|p_i'\right|\le C_2$, $\left|q_i\right|\le K_1$, $\left|q_i'\right|\le K_2$

Let t_{ij} be the $\left(i,\, j\right)^{th}$ element of the matrix P, then:

$$\begin{aligned}\left|t_{i,i+1}\right|=\left|w_i\right|&=\left|-\frac{h}{2}p_i-\frac{h^2}{6}p_i^2-\frac{h^3}{12}p_i\left(p_i'+q_i\right)\right|\\&\le h\left\{\frac{C_1}{2}+\frac{h}{6}C_1^2+\frac{h^2}{12}C_1\left(C_2+K_1\right)\right\}\end{aligned},\qquad i=1,2,\cdots,N-2$$

$$\begin{aligned}\left|t_{i,i-1}\right|=\left|u_i\right|&=\left|\frac{h}{2}p_i-\frac{h^2}{6}p_i^2+\frac{h^3}{12}p_i\left(p_i'+q_i\right)\right|\\&\le h\left\{\frac{C_1}{2}+\frac{h}{6}C_1^2+\frac{h^2}{12}C_1\left(C_2+K_1\right)\right\}\end{aligned},\qquad i=2,3,\,\cdots,N-1.$$

Thus, for sufficiently small h, we have:

$$-1+\left|t_{i,i+1}\right|<0,\ i=1,2,\cdots,N-2$$

$-1+\left|t_{i,i-1}\right|<0$, $i=2,3,\,\cdots,N-1$, since the $\left(i,i+1\right)^{th}$ and $\left(i,i-1\right)^{th}$ of the matrix D is -1.

Hence, the matrix $\left(D+P\right)$ is irreducible, Varga [46].

Let S_i be the sum of the elements of the i^{th} row of the matrix $\left(D+P\right)$, then:

$$\begin{aligned}S_i&=1+v_i+w_i,\quad \text{for } i=1\\&=1+\left(\frac{h^2}{3}p_i^2-h^2q_i-\frac{h^4}{6}p_iq_i'\right)+\left(-\frac{h}{2}p_i-\frac{h^2}{6}p_i^2-\frac{h^3}{12}p_i\left(p_i'+q_i\right)\right)\\&=1+h\left(-\frac{p_i}{2}\right)+h^2\left(\frac{p_i^2}{6}-q_i\right)+h^3\left(-\frac{p_i}{12}\left(p_i'+q_i\right)\right)+h^4\left(-\frac{p_iq_i'}{6}\right)\end{aligned}$$

$$S_i = u_i + v_i + w_i\,, \quad \text{for } i = 2,3,\cdots,N-2$$

$$= \left(\frac{h}{2}p_i - \frac{h^2}{6}p_i^2 + \frac{h^3}{12}p_i\left(p_i' + q_i\right)\right) + \left(\frac{h^2}{3}p_i^2 - h^2 q_i - \frac{h^4}{6}p_i q_i'\right)$$

$$+ \left(-\frac{h}{2}p_i - \frac{h^2}{6}p_i^2 - \frac{h^3}{12}p_i\left(p_i' + q_i\right)\right)$$

$$= h^2\left(-q_i\right) + h^4\left(-\frac{p_i q_i'}{6}\right)$$

$$S_i = 1 + u_i + v_i\,, \quad \text{for } i = N-1$$

$$= 1 + \left(\frac{h}{2}p_i - \frac{h^2}{6}p_i^2 + \frac{h^3}{12}p_i\left(p_i' + q_i\right)\right) + \left(\frac{h^2}{3}p_i^2 - h^2 q_i - \frac{h^4}{6}p_i q_i'\right)$$

$$= 1 + h\left(\frac{p_i}{2}\right) + h^2\left(\frac{p_i^2}{6} - q_i\right) + h^3\left(\frac{p_i}{12}\left(p_i' + q_i\right)\right) + h^4\left(-\frac{p_i q_i'}{6}\right)$$

Let $C_{1*} = \min_{1\le i\le N-1}\left|p_i\right|,\ C_1^* = \max_{1\le i\le N-1}\left|p_i\right|,\ K_{1*} = \min_{1\le i\le N-1}\left|q_i\right|,\ K_1^* = \max_{1\le i\le N-1}\left|q_i\right|$, then:

$$0 < C_{1*} \le C_1 \le C_1^* \text{ and } 0 < K_{1*} \le K_1 \le K_1^*$$

For sufficiently small h, $\left(D+P\right)$ is monotone. Since, $\left(D+P\right) \to D$ which is symmetric and has weak diagonal dominance, Varga [46] and Young [49].

Hence, $\left(D+P\right)^{-1}$ exists and $\left(D+P\right)^{-1} \ge 0$.

From the error Eq. (4.19), we have:

$$\left\|y - y^N\right\| \le \left\|\left(D+P\right)^{-1}\right\|\left\|T(h)\right\| \tag{4.20}$$

For sufficiently small h, we have:

$$S_i > h^2 \mathrm{K}_{1*}\,, \text{ for } i = 1, 2, \cdots, N-1$$

Let $\left(D+P\right)_{i,k}^{-1}$ be the $\left(i,k\right)^{th}$ element of $\left(D+P\right)^{-1}$ and we define,

$$\left\|\left(D+P\right)^{-1}\right\| = \max_{1\le i\le N-1}\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} \text{ and } \left\|T(h)\right\| = \max_{1\le i\le N-1}\left|T_i\right| \tag{4.21}$$

Since $\left(D+P\right)_{i,k}^{-1} \geq 0$, then from the theory of matrices, we have:

$$\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1}.S_k = 1, \quad i = 1, 2, \cdots, N-1.$$

Hence,

$$\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} \leq \frac{1}{\min\limits_{1\leq k\leq N-1} S_k} \leq \frac{1}{h^2 K_{1^*}} = \frac{\varepsilon}{h^2 Q} < \frac{1}{h^2 Q} \ , \quad \text{since } 0 < \varepsilon << 1 \tag{4.22}$$

where, $Q = \min\limits_{1\leq i\leq N-1}\left|a_i + b_i\right|$, since $q(x_i) = \left(\dfrac{a(x_i)+b(x_i)}{\varepsilon}\right)$.

Now, from Eqs. (4.20), (4.21) and (4.22), we get:

$$\left\|y - y^N\right\| \leq \left(\frac{1}{h^2 Q}\right)\left|\frac{h^4}{12} y^{(4)}(\xi_2)\right| = \left(\frac{y^{(4)}(\xi_2)}{12Q}\right) h^2 = Ch^2 \tag{4.23}$$

where $C = \dfrac{y^{(4)}(\xi_2)}{12Q}$.

This establishes the convergence of the finite difference scheme of Eq. (4.14) and its rate of convergence is 2. From Eq. (4.23), one can observed that the proposed method is ε-uniform convergent, since the error is of the form $\left\|y - y^N\right\| \leq C h^2$, where C is independent of perturbation parameter ε and mesh size h.

4.1.2. Method II (Fourth Order Method)

Assuming that $y(x)$ has continuous derivatives on $[0,1]$ and making use of Taylor's series expansion, we have:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{(4)} + \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} + O(h^7) \tag{4.24}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{(4)} - \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} + O(h^7) \tag{4.25}$$

Subtracting Eq. (4.25) from Eq. (4.24), we get:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' - \frac{h^4}{120} y_i^{(5)} + \tau_1 \tag{4.26}$$

where, $\tau_1 = -\frac{h^6}{7!} y^{(7)}(\xi_1)$, for $\xi_1 \in [x_{i-1}, x_i]$.

Again, adding Eq. (4.24) and Eq. (4.25), we get:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} - \frac{h^4}{360} y_i^{(6)} + \tau_2 \tag{4.27}$$

where, $\tau_2 = -\frac{h^6}{8!} y^{(8)}(\xi_2)$, for $\xi_2 \in [x_{i-1}, x_i]$.

Substituting Eqs. (4.26) and (4.27) into Eq. (4.4) and simplifying, we obtain:

$$\frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + \frac{p_i}{2h}\left(y_{i+1} - y_{i-1}\right) - \frac{h^2}{6} p_i y_i''' - \frac{h^2}{12} y_i^{(4)} - \frac{h^4}{120} p_i y_i^{(5)} + q_i y_i = r_i + \tau \tag{4.28}$$

where, $\tau = \frac{h^4}{360} y^{(6)}(\xi_2) - p_i \tau_1 - \tau_2$ is the local truncation error and

$$p(x_i) = p_i,\ q(x_i) = q_i,\ r(x_i) = r_i,\ y(x_i) = y_i.$$

By successively differentiating both sides of Eq. (4.11) and evaluating at x_i, we have:

$$y_i''' = r_i' - p_i y_i'' - \left(p_i' + q_i\right) y_i' - q_i' y_i \tag{4.29}$$

$$y_i^{(4)} = r_i'' - p_i r_i' + \left(p_i^2 - 2p_i' - q_i\right) y_i'' + \left(p_i\left(p_i' + q_i\right) - p_i'' - 2q_i'\right) y_i' + \left(p_i q_i' - q_i''\right) y_i \tag{4.30}$$

$$\begin{aligned} y_i^{(5)} = {} & r_i''' - p_i r_i'' + \left(p_i^2 - q_i - 3p_i'\right) r_i' + \left(2p_i p_i' - 3p_i'' - 3q_i' + p_i\left(p_i' + q_i\right) - p_i\left(p_i^2 - 2p_i' - q_i\right)\right) y_i'' \\ & + \left(p_i'\left(p_i' + q_i\right) + p_i\left(p_i'' + q_i'\right) - p_i''' - 3q_i'' + p_i q_i' - \left(p_i' + q_i\right)\left(p_i^2 - 2p_i' - q_i\right)\right) y_i' \\ & + \left(p_i' q_i' + p_i q_i'' - q_i''' - q_i'\left(p_i^2 - 2p_i' - q_i\right)\right) y_i \end{aligned} \tag{4.31}$$

Using Eqs. (4.29), (4.30) and (4.31) into Eq. (4.28) and simplifying, we obtain:

$$\frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) + \frac{p_i}{2h}\left(y_{i+1} - y_{i-1}\right) + A_i y_i'' + B_i y_i' + C_i y_i = H_i \text{, for } i = 1, 2, \cdots, N-1 \tag{4.32}$$

where,

$$\begin{aligned} A_i &= \frac{h^2}{6} p_i^2 - \frac{h^2}{12}\left(p_i^2 - 2p_i' - q_i\right) - \frac{h^4}{120} p_i\left(2p_i p_i' - 3p_i'' - 3q_i' + p_i\left(p_i' + q_i\right) - p_i\left(p_i^2 - 2p_i' - q_i\right)\right) \\ B_i &= \frac{h^2}{6} p_i\left(p_i' + q_i\right) - \frac{h^2}{12}\left(p_i\left(p_i' + q_i\right) - p_i'' - 2q_i'\right) - \frac{h^4}{120} p_i\left\{p_i'\left(p_i' + q_i\right) + p_i\left(p_i'' + q_i'\right) - p_i''' \right. \\ &\quad \left. - 3q_i'' + p_i q_i' - \left(p_i' + q_i\right)\left(p_i^2 - 2p_i' - q_i\right)\right\} \\ C_i &= \frac{h^2}{6} p_i q_i' - \frac{h^2}{12}\left(p_i q_i' - q_i''\right) - \frac{h^4}{120} p_i\left(p_i' q_i' + p_i q_i'' - q_i''' - q_i'\left(p_i^2 - 2p_i' - q_i\right)\right) + q_i \\ H_i &= r_i + \left(\frac{h^2}{12} p_i + \frac{h^4}{120} p_i\left(p_i^2 - 3p_i' - q_i\right)\right) r_i' + \left(\frac{h^2}{12} - \frac{h^4}{120} p_i^2\right) r_i'' + \frac{h^4}{120} p_i r_i''' \end{aligned}$$

Now, using central difference approximation for y_i'' and y_i' in Eq. (4.32) and further simplifying, we get:

$$\left(\frac{1}{h^2} - \frac{p_i}{2h} + \frac{A_i}{h^2} - \frac{B_i}{2h}\right) y_{i-1} - \left(\frac{2}{h^2} + \frac{2A_i}{h^2} - C_i\right) y_i + \left(\frac{1}{h^2} + \frac{p_i}{2h} + \frac{A_i}{h^2} + \frac{B_i}{2h}\right) y_{i+1} = H_i \tag{4.33}$$

Eq. (4.33) can be written as the three term recurrence relation of the form:

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \text{, for } i = 1, 2, \ldots, N-1 \tag{4.34}$$

where,

$$E_i = \frac{1}{h^2} - \frac{p_i}{2h} + \frac{A_i}{h^2} - \frac{B_i}{2h}$$

$$F_i = \frac{2}{h^2} + \frac{2A_i}{h^2} - C_i$$

$$G_i = \frac{1}{h^2} + \frac{p_i}{2h} + \frac{A_i}{h^2} + \frac{B_i}{2h}$$

$$H_i = r_i + \left(\frac{h^2}{12} p_i + \frac{h^4}{120} p_i \left(p_i^2 - 3p_i' - q_i \right) \right) r_i' + \left(\frac{h^2}{12} - \frac{h^4}{120} p_i^2 \right) r_i'' + \frac{h^4}{120} p_i r_i'''$$

The tri-diagonal system in Eq. (4.34) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

Stability and Convergence Analysis for Method II

Case 1: Layer Behaviour $\left(i.e.\ a(x) + b(x) < 0, \text{ for } x \in (0,1). \text{ Thus } q(x) < 0, \text{ since } \varepsilon > 0 \right)$.

The discrete minimum principle and stability of the scheme given in Eq. (4.34) are presented as follows. But, for the continuous problems it is analogous to the first method.

Lemma 4.5: (Discrete Minimum Principle)

The finite difference operator L^N in Eq. (4.34) satisfies the discrete minimum principle, if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \leq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in (0,1)$.

Proof:

Suppose there exists a positive integer k such that $w_k < 0$ and $w_k = \min\limits_{0 \leq i \leq N} w_i$.

Then, from Eq. (4.34), we have:

$$\begin{aligned} L^N w_k &\equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1} \\ &= \left(\frac{1}{h^2} + \frac{A_k}{h^2} \right) \left(w_{k-1} - w_k \right) + \left(\frac{1}{h^2} + \frac{A_k}{h^2} \right) \left(w_{k+1} - w_k \right) + \left(\frac{p_k}{2h} + \frac{B_k}{2h} \right) \left(w_{k+1} - w_{k-1} \right) + C_k w_k \end{aligned}$$

$$
\begin{aligned}
&= \left\{ \left(\frac{1}{h^2} + \frac{p_k^2}{12} + \frac{p_k'}{6} + \frac{q_k}{12} \right) + \frac{h^2 p_k}{120} \left(3p_k'' + 3q_k' - p_k \left(p_k' + q_k \right) + p_k \left(p_k^2 - 2p_k' - q_k \right) - 2p_k p_k' \right) \right\} \left(w_{k-1} - w_k \right) \\
&+ \left\{ \left(\frac{1}{h^2} + \frac{p_k^2}{12} + \frac{p_k'}{6} + \frac{q_k}{12} \right) + \frac{h^2 p_k}{120} \left(3p_k'' + 3q_k' - p_k \left(p_k' + q_k \right) + p_k \left(p_k^2 - 2p_k' - q_k \right) - 2p_k p_k' \right) \right\} \left(w_{k+1} - w_k \right) \\
&+ \left\{ \frac{p_k}{2h} + h \left(\frac{1}{24} \left(p_k'' + 2q_k' - p_k p_k' - p_k q_k \right) + \frac{p_k}{12} \left(p_k' + q_k \right) \right) \right. \\
&\left. + \frac{h^3 p_k}{240} \left(-p_k' \left(p_k' + q_k \right) - p_k \left(p_k'' + q_k' \right) + p_k''' + 3q_k'' - p_k q_k' + \left(p_k' + q_k \right) \left(p_k^2 - 2p_k' - q_k \right) \right) \right\} \left(w_{k+1} - w_{k-1} \right) \\
&+ C_k w_k
\end{aligned}
$$

For sufficiently small h and for suitable value of p_k, we obtain:

$L^N w_k > 0$. Since, $w_k < 0$ (by the assumption) and $C_k \to q_k < 0$. But, this is a contradiction.

Hence, $w_i \geq 0$ for all $x_i \in (0,1)$.

Theorem 4.6: The finite difference operator L^N in Eq. (4.34) is stable for $a(x) + b(x) < 0$, if w_i is any mesh function, then $|w_i| \leq K \max \left\{ |w_0|, \max_{x_i \in (0,1)} |Lw_i| \right\}$, for some constant $K \geq 1$.

Proof: The proof is analogous to Theorem 4.2.

Case 2: Oscillatory Behaviour $\left(i.e.\ a(x) + b(x) > 0, \text{ for } x \in (0,1). \text{ Thus, } q(x) > 0, \text{ as } \varepsilon > 0 \right)$

By the same approach in the first method, here we use the maximum principles for discrete problems, since for the continuous problems it has already been presented in the first method.

Lemma 4.6: (Discrete Maximum Principle)

The finite difference operator L^N in Eq. (4.34) satisfies the discrete maximum principle, if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \geq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in (0,1)$.

Proof:

Suppose that there exists a positive integer k such that $w_k < 0$ and $w_k = \max_{0\le i\le N} w_i$.

Then, from Eq. (4.34), we have:

$$L^N w_k \equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1}$$

$$\begin{aligned}
&= \left\{\left(\frac{1}{h^2}+\frac{p_k^2}{12}+\frac{p_k'}{6}+\frac{q_k}{12}\right)+\frac{h^2 p_k}{120}\left(3p_k''+3q_k'-p_k\left(p_k'+q_k\right)+p_k\left(p_k^2-2p_k'-q_k\right)-2p_k p_k'\right)\right\}\left(w_{k-1}-w_k\right)\\
&\quad+\left\{\left(\frac{1}{h^2}+\frac{p_k^2}{12}+\frac{p_k'}{6}+\frac{q_k}{12}\right)+\frac{h^2 p_k}{120}\left(3p_k''+3q_k'-p_k\left(p_k'+q_k\right)+p_k\left(p_k^2-2p_k'-q_k\right)-2p_k p_k'\right)\right\}\left(w_{k+1}-w_k\right)\\
&\quad+\left\{\frac{p_k}{2h}+h\left(\frac{1}{24}\left(p_k''+2q_k'-p_k p_k'-p_k q_k\right)+\frac{p_k}{12}\left(p_k'+q_k\right)\right)\right.\\
&\quad\left.+\frac{h^3 p_k}{240}\left(-p_k'\left(p_k'+q_k\right)-p_k\left(p_k''+q_k'\right)+p_k'''+3q_k''-p_k q_k'+\left(p_k'+q_k\right)\left(p_k^2-2p_k'-q_k\right)\right)\right\}\left(w_{k+1}-w_{k-1}\right)\\
&\quad+C_k w_k
\end{aligned}$$

For sufficiently small h and for suitable value of p_k, we obtain:

$L^N w_k < 0$. Since, $w_k < 0$ (by the assumption) and $C_k \to q_k > 0$. But, this is a contradiction.

Hence, $w_i \ge 0$ for all $x_i \in (0,1)$.

Theorem 4.7: The finite difference operator L^N in Eq. (4.34) is stable for $a(x)+b(x)>0$, if w_i is any mesh function, then $|w_i| \le K \max\left\{|w_0|, \max_{x_i\in(0,1)} |Lw_i|\right\}$, for some constant $K \ge 1$.

Proof: The proof is similar to Theorem 4.4.

Therefore, we conclude that the stability of the scheme in Eq. (4.34) is proved for both cases (*i.e.*, layer behaviour and oscillatory behaviour).

Theorem 4.8: Let $y(x)$ be the analytical solution of the problem in Eq. (4.4) and Eq. (4.5) and $y^N(x)$ be the numerical solution of the discretized problem of Eq. (4.34). Then, $\left\| y - y^N \right\| \le C^* h^4$ for sufficiently small h and C^* is positive constant.

Proof:

Multiplying both sides of Eq. (4.33) by $-h^2$, we obtain:

$$\left(-1+\frac{h}{2}p_i - A_i + \frac{h}{2}B_i\right)y_{i-1} + \left(2+2A_i - h^2C_i\right)y_i + \left(-1-\frac{h}{2}p_i - A_i - \frac{h}{2}B_i\right)y_{i+1} + h^2H_i + \tau_i\left(h\right) = 0 \tag{4.35}$$

where, $\tau_i(h) = \dfrac{h^6}{360}y^{(6)}(\xi_2) + O\left(h^8\right)$ is a local truncation error, for $i = 1, 2, ..., N-1$.

Rewriting Eq. (4.35), we get:

$$\left(-1+u_i\right)y_{i-1} + \left(2+v_i\right)y_i + \left(-1+w_i\right)y_{i+1} + g_i + \tau_i = 0 \tag{4.36}$$

where,

$$u_i = \frac{h}{2}p_i - A_i + \frac{h}{2}B_i$$
$$v_i = 2A_i - h^2C_i$$
$$w_i = -\frac{h}{2}p_i - A_i - \frac{h}{2}B_i$$
$$g_i = h^2H_i$$

Incorporating the boundary conditions $y_0 = \phi(x_0) = \phi_0$, $y_N = y(1) = \beta$ in Eq. (4.36), we get the systems of equations of the form:

$$\begin{bmatrix} (2+v_1) & (-1+w_1) & 0 & \cdots & 0 \\ (-1+u_2) & (2+v_2) & (-1+w_2) & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & (-1+u_{N-1}) & (2+v_{N-1}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \end{bmatrix} + \begin{bmatrix} g_1+(-1+u_1)\phi(0) \\ g_2 \\ g_3 \\ \vdots \\ g_{N-1}+(-1+w_{N-1})\beta \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \vdots \\ \tau_{N-1} \end{bmatrix} = \overline{0}$$

$$\Rightarrow (D+R)\,y + Z + \tau(h) = 0 \tag{4.37}$$

where;

$D = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & -1 & 2 \end{bmatrix}$, $R = \begin{bmatrix} v_1 & w_1 & 0 & \cdots & 0 \\ u_2 & v_2 & w_2 & \cdots & 0 \\ 0 & - & - & & - \\ \vdots & & & \ddots & \vdots \\ 0 & - & - & u_{N-1} & v_{N-1} \end{bmatrix}$ are tri-diagonal matrices of order $N-1$, and

$Z = \left[\left(g_1+(-1+u_1)\phi(0)\right),\ g_2,\ g_3,\cdots,\left(g_{N-1}+(-1+w_{N-1})\beta\right)\right]^T$, $\tau(h) = O\left(h^4\right)$ and $y = [y_1, y_2, \cdots, y_{N-1}]^T$, $\tau(h) = [\tau_1,\ \tau_2,\cdots,\tau_{N-1}]^T$, $\overline{0} = [0,\ 0,\cdots,0]^T$ are the associated vectors of Eq. (4.37).

Let $y^N = \left[y_1^N, y_2^N, \cdots, y_{N-1}^N\right]^T \cong y$ be the solution which satisfies the Eq. (4.37), then we have:

$$(D+R)\,y^N + Z = 0 \tag{4.38}$$

Let $e_i = y_i - y_i^N$, for $i = 1, 2, \cdots, N-1$ be the discretization error, then,

$$y - y^N = [e_1, e_2, \cdots, e_{N-1}]^T.$$

Subtracting Eq. (4.37) from Eq. (4.38), we get:

$$(D+R)\left(y^N - y\right) = \tau(h) \tag{4.39}$$

Let $|p_i| \le M_1$, $|p_i'| \le M_2$, $|p_i''| \le M_3$, $|q_i| \le K_1$, $|q_i'| \le K_2$, $|q_i''| \le K_3$

Let r_{ij} be the $(i, j)^{th}$ element of the matrix R, then:

For $i = 1, 2, \cdots, N-2$

$$
\begin{aligned}
\left|r_{i,i+1}\right| = \left|w_i\right| &= \left|-\frac{h}{2}p_i - A_i - \frac{h}{2}B_i\right| \\
&\leq h\left\{\frac{M_1}{2} + \frac{h}{6}M_1^2 + \frac{h}{12}\left(M_1^2 - 2M_2 - K_1\right)\right. \\
&\quad \left. + \frac{h^3 M_1}{120}\left(2M_1M_2 - 3M_3 - 3K_2 + M_1\left(M_2 + K_1\right) - M_1\left(M_1^2 - 2M_2 - K_1\right)\right) + \frac{B_i}{2}\right\}
\end{aligned}
$$

For $i = 2, 3, \cdots, N-1$

$$
\begin{aligned}
\left|r_{i,i-1}\right| = \left|u_i\right| &= \left|\frac{h}{2}p_i - A_i + \frac{h}{2}B_i\right| \\
&\leq h\left\{\frac{M_1}{2} + \frac{h}{6}M_1^2 + \frac{h}{12}\left(M_1^2 - 2M_2 - K_1\right)\right. \\
&\quad \left. + \frac{h^3 M_1}{120}\left(2M_1M_2 - 3M_3 - 3K_2 + M_1\left(M_2 + K_1\right) - M_1\left(M_1^2 - 2M_2 - K_1\right)\right) + \frac{B_i}{2}\right\}
\end{aligned}
$$

Thus, for sufficiently small h, we have:

$-1 + \left|r_{i,i+1}\right| < 0$, $i = 1, 2, \cdots, N-2$

$-1 + \left|r_{i,i-1}\right| < 0$, $i = 2, 3, \cdots, N-1$, since the $(i, i+1)^{th}$ and $(i, i-1)^{th}$ of the matrix D is -1.

Hence, the matrix $(D + R)$ is irreducible, Varga [46].

Let S_i be the sum of the elements of the i^{th} row of the matrix $(D + R)$, then;

$$S_i = 1 + v_i + w_i\,, \quad \text{for } i = 1$$
$$= 1 + 2A_i - h^2 C_i - \frac{h}{2} p_i - A_i - \frac{h}{2} B_i$$
$$= 1 + A_i - h^2 C_i - \frac{h}{2} p_i - \frac{h}{2} B_i$$
$$= 1 + h\left(\frac{-p_i}{2}\right) + h^2\left(\frac{p_i^2}{12} + \frac{p_i'}{6} - \frac{11}{12} q_i\right) + h^3\left(\frac{1}{24}\left(p_i p_i' + p_i q_i - p_i'' - 2q_i'\right) - \frac{1}{12} p_i\left(p_i' + q_i\right)\right) + O\left(h^4\right)$$

$$S_i = u_i + v_i + w_i\,, \quad \text{for } i = 2, 3, \cdots, N-2$$
$$= \frac{h}{2} p_i - A_i + \frac{h}{2} B_i + 2A_i - h^2 C_i - \frac{h}{2} p_i - A_i - \frac{h}{2} B_i$$
$$= h^2\left(-q_i\right) + O\left(h^4\right)$$

$$S_i = 1 + u_i + v_i\,, \quad \text{for } i = N-1$$
$$= 1 + \frac{h}{2} p_i + A_i + \frac{h}{2} B_i - h^2 C_i$$
$$= 1 + h\left(\frac{p_i}{2}\right) + h^2\left(\frac{p_i^2}{12} + \frac{p_i'}{6} - \frac{11}{12} q_i\right) + h^3\left(\frac{1}{24}\left(-p_i p_i' - p_i q_i + p_i'' + 2q_i'\right) + \frac{1}{12} p_i\left(p_i' + q_i\right)\right) + O\left(h^4\right)$$

Let $M_{1*} = \min_{1 \le i \le N-1} |p_i|$, $M_1^* = \max_{1 \le i \le N-1} |p_i|$, $K_{1*} = \min_{1 \le i \le N-1} |q_i|$, $K_1^* = \max_{1 \le i \le N-1} |q_i|$, then:

$$0 < M_{1*} \le M_1 \le M_1^* \text{ and } 0 < K_{1*} \le K_1 \le K_1^*$$

For sufficiently small h, $(D+R)$ is monotone, Varga [46] and Young [49].

Hence, $(D+R)^{-1}$ exists and $(D+R)^{-1} \ge 0$.

From the error Eq. (4.39), we have:

$$\left\| y - y^N \right\| \le \left\| (D+R)^{-1} \right\| \left\| \tau(h) \right\| \tag{4.40}$$

For sufficiently small h, we have:

$$S_i > \frac{11}{12} h^2 \mathrm{K}_{1*}\,, \ \text{for } i = 1$$

$S_i > \ h^2 K_{1*}$, for $i = 2, 3, \cdots, N-2$

$S_i > \dfrac{11}{12} h^2 K_{1*}$, for $i = N-1$,   where, $K_{1*} = \min_{1 \le i \le N-1} |q_i|$

Let $(D+R)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+R)^{-1}$ and we define,

$$\left\| (D+R)^{-1} \right\| = \max_{1 \le i \le N-1} \sum_{k=1}^{N-1} (D+R)^{-1}_{i,k} \text{ and } \|\tau(h)\| = \max_{1 \le i \le N-1} |\tau_i| \tag{4.41}$$

Since $(D+R)^{-1}_{i,k} \ge 0$, then from the theory of matrices, we have:

$$\sum_{k=1}^{N-1} (D+R)^{-1}_{i,k} . S_k = 1, \ i = 1, 2, \cdots, N-1.$$

Hence, $(D+R)^{-1}_{i,1} \le \dfrac{1}{S_1} < \dfrac{12}{11}\left(\dfrac{1}{h^2 K_{1*}}\right) = \dfrac{12\varepsilon}{11 h^2 Q^*} < \dfrac{12}{11 h^2 Q^*}$, for $k = 1$, since $0 < \varepsilon << 1$.   (4.42)

$$(D+R)^{-1}_{i,N-1} \le \frac{1}{S_{N-1}} < \frac{12}{11}\left(\frac{1}{h^2 K_{1*}}\right) = \frac{12\varepsilon}{11 h^2 Q^*} < \frac{12}{11 h^2 Q^*}, \text{ for } k = N-1 \tag{4.43}$$

Further, $\displaystyle\sum_{k=2}^{N-2} (D+R)^{-1}_{i,k} \le \frac{1}{\min\limits_{2 \le k \le N-2} S_k} \le \frac{1}{h^2 K_{1*}} = \frac{\varepsilon}{h^2 Q^*} < \frac{1}{h^2 Q^*}$, *for* $k = 2, 3, \cdots, N-2$ (4.44)

where, $Q^* = \min_{1 \le i \le N-1} |a_i + b_i|$, since $q(x_i) = \left(\dfrac{a(x_i) + b(x_i)}{\varepsilon}\right)$.

Now, from Eqs. (4.40) − (4.44), we get:

$$\left\| y - y^N \right\| \le \left(\frac{12}{11 h^2 Q^*} + \frac{1}{h^2 Q^*} + \frac{12}{11 h^2 Q^*} \right) \left| \frac{h^6}{360} y^{(6)}(\xi_2) \right| = \frac{7}{792} \left(\frac{y^{(6)}(\xi_2)}{Q^*} \right) h^4 = C^* h^4 \tag{4.45}$$

where, $C^* = \dfrac{7}{792}\left(\dfrac{y^{(6)}(\xi_2)}{Q^*}\right)$, which is independent of perturbation parameter ε and mesh size h .

This establishes that the method is of fourth order uniformly convergent.

4.2. Numerical Examples

To demonstrate the applicability of the methods, we implement the methods on four numerical examples, two with twin boundary layers and two with oscillatory behaviour. We present the graphs of the computed solution of the problem for different values of ε and δ of $o(\varepsilon)$. Since those examples have no exact solution, so the numerical solutions are computed using double mesh principle. The maximum absolute errors are computed using double-mesh principle given by:

$$Z_h = \max_i \left| y_i^h - y_i^{h/2} \right|, \quad i = 1, 2, \ldots, N-1 \tag{4.46}$$

where y_i^h is the numerical solution on the mesh $\{x_i\}_1^{N-1}$ at the nodal point x_i and $x_i = x_0 + ih$, $i = 1, 2, ..., N-1$ and $y_i^{h/2}$ is the numerical solution at the nodal point x_i on the mesh $\{x_i\}_1^{2N-1}$ where, $x_i = x_0 + i\,h/2$, $i = 1, 2, ..., 2N-1$ (*i.e.*, the numerical solution on a mesh, obtained by bisecting the original mesh with N number of mesh intervals), Doolan et al [7].

Example 4.1:

Consider the singularly perturbed delay reaction-diffusion equation with layer behaviour, Swamy et al [45].

$$\varepsilon y''(x) + 0.25y(x-\delta) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \le x \le 0 \text{ and } y(1) = 0.$$

The maximum absolute errors are presented for both present methods, in Tables (4.1) and (4.5) for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Figs. (4.1) and (4.5).

Example 4.2:

Consider the singularly perturbed delay reaction-diffusion equation with layer behaviour, Swamy et al [45].

$$\varepsilon y''(x) - 2y(x-\delta) - y(x) = 1$$

under the interval and boundary conditions

$$y(x)=1,\ -\delta \le x \le 0 \text{ and } y(1)=0.$$

The maximum absolute errors are presented for both present methods, in Tables (4.2) and (4.6) for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Figs. (4.2) and (4.6).

Example 4.3:

Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behaviour, Swamy et al [45].

$$\varepsilon y''(x) + 0.25 y(x-\delta) + y(x) = 1$$

under the interval and boundary conditions

$$y(x)=1,\ -\delta \le x \le 0 \text{ and } y(1)=0.$$

The maximum absolute errors are presented for both present methods, in Table (4.3) for different values of δ. The graph of the computed solution for $\varepsilon = 0.001$ and different values of δ is also given in Figs. (4.3) and (4.7).

Example 4.4:

Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behaviour, Swamy et al [45].

$$\varepsilon y''(x) + y(x-\delta) + 2y(x) = 1$$

under the interval and boundary conditions

$$y(x)=1,\ -\delta \le x \le 0 \text{ and } y(1)=0.$$

The maximum absolute errors are presented for both present methods, in Table (4.4) for different values of δ. The graph of the computed solution for $\varepsilon = 0.001$ and different values of δ is also given in Figs. (4.4) and (4.8).

4.3. Numerical Results

Table 4.1: The maximum absolute errors of **Example 4.1**, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Fourth Order Method (Our Method)					
0.03	1.2007e-09	7.5051e-10	1.4815e-11	4.6660e-12	1.9661e-12
0.05	1.2135e-09	7.5860e-10	1.4980e-11	4.7337e-12	1.9339e-12
0.09	1.2290e-09	7.6818e-10	1.5168e-11	4.7632e-12	2.0450e-12
Second Order Method (Our Method)					
0.03	2.4645e-05	6.1616e-06	2.7385e-06	1.5404e-06	9.8587e-07
0.05	2.4393e-05	6.0986e-06	2.7105e-06	1.5247e-06	9.7581e-07
0.09	2.3947e-05	5.9872e-06	2.6611e-06	1.4969e-06	9.5799e-07
Results in Swamy et al [45]					
0.03	2.1999e-03	1.1041e-03	7.3705e-04	5.5315e-04	4.4269e-04
0.05	2.2012e-03	1.1049e-03	7.3749e-04	5.5345e-04	4.4293e-04
0.09	2.1999e-03	1.1038e-03	7.3676e-04	5.5289e-04	4.4247e-04

Table 4.2: The maximum absolute errors of **Example 4.2**, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Fourth Order Method (Our Method)					
0.03	5.9892e-09	3.7452e-10	7.3976e-11	2.3404e-11	9.5863e-12
0.05	3.3028e-09	2.0657e-10	4.0807e-11	1.2909e-11	5.2809e-12
0.09	4.6352e-09	2.8949e-10	5.7180e-11	1.8085e-11	7.4190e-12
Second Order Method (Our Method)					
0.03	5.5262e-05	1.3819e-05	6.1422e-06	3.4551e-06	2.2112e-06
0.05	6.1292e-05	1.5325e-05	6.8113e-06	3.8314e-06	2.4521e-06
0.09	7.5050e-05	1.8764e-05	8.3405e-06	4.6916e-06	3.0026e-06
Results in Swamy et al [45]					
0.03	3.1674e-03	1.6058e-03	1.0754e-03	8.0837e-04	6.4760e-04
0.05	3.1437e-03	1.5949e-03	1.0685e-03	8.0338e-04	6.4367e-04
0.09	3.0784e-03	1.5660e-03	1.0502e-03	7.9000e-04	6.3310e-04

Table 4.3: The maximum absolute errors of **Example 4.3**, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Fourth Order Method (Our Method)					
0.03	3.9856e-08	2.4916e-09	4.9143e-10	1.5603e-10	6.1932e-11
0.05	3.8949e-08	2.4343e-09	4.8003e-10	1.5358e-10	7.0907e-11
0.09	3.7554e-08	2.3446e-09	4.6287e-10	1.6033e-10	6.1303e-11
Second Order Method (Our Method)					
0.03	5.2227e-04	1.3061e-04	5.8052e-05	3.2655e-05	2.0899e-05
0.05	5.1649e-04	1.2916e-04	5.7409e-05	3.2293e-05	2.0668e-05
0.09	5.0518e-04	1.2634e-04	5.6156e-05	3.1588e-05	2.0217e-05
Results in Swamy et al [45]					
0.03	2.5991e-03	1.2872e-03	8.5528e-04	6.4039e-04	5.1179e-04
0.05	2.6270e-03	1.3013e-03	8.6474e-04	6.4750e-04	5.1749e-04
0.09	2.6813e-03	1.3289e-03	8.8320e-04	6.6139e-04	5.2863e-04

Table 4.4: The maximum absolute errors of **Example 4.4**, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$
Fourth Order Method (Our Method)					
0.03	1.5497e-07	9.6846e-09	1.9131e-09	6.0394e-10	2.4770e-10
0.05	1.5900e-07	9.9375e-09	1.9630e-09	6.2120e-10	2.5444e-10
0.09	1.7208e-07	1.0754e-08	2.1244e-09	6.7226e-10	2.7451e-10
Second Order Method (Our Method)					
0.03	8.3415e-04	2.0833e-04	9.2578e-05	5.2071e-05	3.3324e-05
0.05	8.8299e-04	2.2050e-04	9.7980e-05	5.5110e-05	3.5269e-05
0.09	9.7538e-04	2.4370e-04	1.0828e-04	6.0909e-05	3.8980e-05
Results in Swamy et al [45]					
0.03	1.5929e-02	7.4850e-03	4.8816e-03	3.6202e-03	2.8764e-03
0.05	1.5470e-02	7.2782e-03	4.7473e-03	3.5209e-03	2.7975e-03
0.09	2.1396e-02	1.0097e-02	6.5922e-03	4.8916e-03	3.8879e-03

Table 4. 5: The maximum absolute errors of **Example 4.1**, for different values of ε with $\delta = 0.5\varepsilon$.

$\varepsilon \downarrow$	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$
Fourth Order Method (Our Method)					
2^{-4}	4.7163e-06	2.9533e-07	1.8473e-08	1.1546e-09	7.2184e-11
2^{-5}	1.6851e-05	1.0582e-06	6.6233e-08	4.1407e-09	2.5883e-10
2^{-6}	6.1305e-05	3.9010e-06	2.4413e-07	1.5281e-08	9.5513e-10
2^{-7}	2.3541e-04	1.5098e-05	9.4835e-07	5.9419e-08	3.7143e-09
2^{-8}	9.2982e-04	5.9195e-05	3.7478e-06	2.3512e-07	1.4703e-08
2^{-9}	3.5840e-03	2.3115e-04	1.4856e-05	9.3248e-07	5.8449e-08
2^{-10}	1.1856e-02	9.1935e-04	5.8565e-05	3.7066e-06	2.3261e-07
Second Order Method (Our Method)					
2^{-4}	1.5070e-03	3.7828e-04	9.4715e-05	2.3685e-05	5.9215e-06
2^{-5}	2.6509e-03	6.6781e-04	1.6749e-04	4.1894e-05	1.0475e-05
2^{-6}	4.8151e-03	1.2413e-03	3.1158e-04	7.8047e-05	1.9517e-05
2^{-7}	9.2994e-03	2.4334e-03	6.1499e-04	1.5434e-04	3.8604e-05
2^{-8}	1.8030e-02	4.8019e-03	1.2303e-03	3.0956e-04	7.7486e-05
2^{-9}	3.3607e-02	9.3674e-03	2.4542e-03	6.1966e-04	1.5557e-04
2^{-10}	5.2477e-02	1.8177e-02	4.8372e-03	1.2385e-03	3.1168e-04
Results in Swamy et al [45]					
2^{-4}	1.8632e-02	9.6189e-03	4.8865e-03	2.4643e-03	1.2376e-03
2^{-5}	2.8161e-02	1.4818e-02	7.6255e-03	3.8713e-03	1.9509e-03
2^{-6}	3.7958e-02	2.0967e-02	1.0977e-02	5.6273e-03	2.8498e-03
2^{-7}	5.0640e-02	2.8316e-02	1.5267e-02	7.9105e-03	4.0287e-03
2^{-8}	6.3580e-02	3.7706e-02	2.0984e-02	1.1012e-02	5.6555e-03
2^{-9}	8.3843e-02	5.0477e-02	2.8297e-02	1.5261e-02	7.9111e-03
2^{-10}	9.9137e-02	6.3529e-02	3.7660e-02	2.0974e-02	1.1011e-02

Table 4. 6: The maximum absolute errors of **Example 4.2**, for different values of ε with $\delta = 0.5\varepsilon$.

$\varepsilon \downarrow$	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$
Fourth Order Method (Our Method)					
2^{-4}	1.7218e-05	1.0980e-06	6.9308e-08	4.3372e-09	2.7116e-10
2^{-5}	8.6267e-05	5.7179e-06	3.5965e-07	2.2514e-08	1.4086e-09
2^{-6}	4.0309e-04	2.6120e-05	1.6483e-06	1.0385e-07	6.4944e-09
2^{-7}	1.6675e-03	1.1001e-04	7.1717e-06	4.5007e-07	2.8201e-08
2^{-8}	5.7218e-03	4.6571e-04	2.9880e-05	1.8861e-06	1.1867e-07
2^{-9}	1.5760e-02	1.8472e-03	1.2042e-04	7.7976e-06	4.8901e-07
2^{-10}	3.3872e-02	6.2077e-03	4.9356e-04	3.1554e-05	1.9940e-06
Second Order Method (Our Method)					
2^{-4}	3.5264e-03	8.9037e-04	2.2369e-04	5.5986e-05	1.4001e-05
2^{-5}	6.2964e-03	1.6598e-03	4.1737e-04	1.0450e-04	2.6149e-05
2^{-6}	1.1914e-02	3.1276e-03	7.9216e-04	1.9981e-04	4.9993e-05
2^{-7}	2.1388e-02	5.8351e-03	1.5338e-03	3.8613e-04	9.6851e-05
2^{-8}	3.2782e-02	1.1174e-02	2.9520e-03	7.5112e-04	1.8935e-04
2^{-9}	4.1139e-02	2.0396e-02	5.6170e-03	1.4743e-03	3.7135e-04
2^{-10}	4.1585e-02	3.1521e-02	1.0818e-02	2.8673e-03	7.3159e-04
Results in Swamy et al [45]					
2^{-4}	2.1118e-02	1.1692e-02	6.1941e-03	3.1887e-03	1.6178e-03
2^{-5}	2.7872e-02	1.6023e-02	8.6367e-03	4.4957e-03	2.2948e-03
2^{-6}	3.5711e-02	2.1293e-02	1.1869e-02	6.2731e-03	3.2240e-03
2^{-7}	4.6679e-02	2.8350e-02	1.6107e-02	8.6728e-03	4.5120e-03
2^{-8}	5.4895e-02	3.6018e-02	2.1373e-02	1.1929e-02	6.2847e-03
2^{-9}	5.7371e-02	4.7254e-02	2.8581e-02	1.6140e-02	8.6961e-03
2^{-10}	5.7878e-02	5.5695e-02	3.6153e-02	2.1406e-02	1.1956e-02

4.3.1. Illustration of the Effect of Delay for Method I

The following graphs (Figs. (4.1) – (4.4)) show the numerical solutions obtained by the present method (*i.e.*, second order) for different values of delay parameter δ .

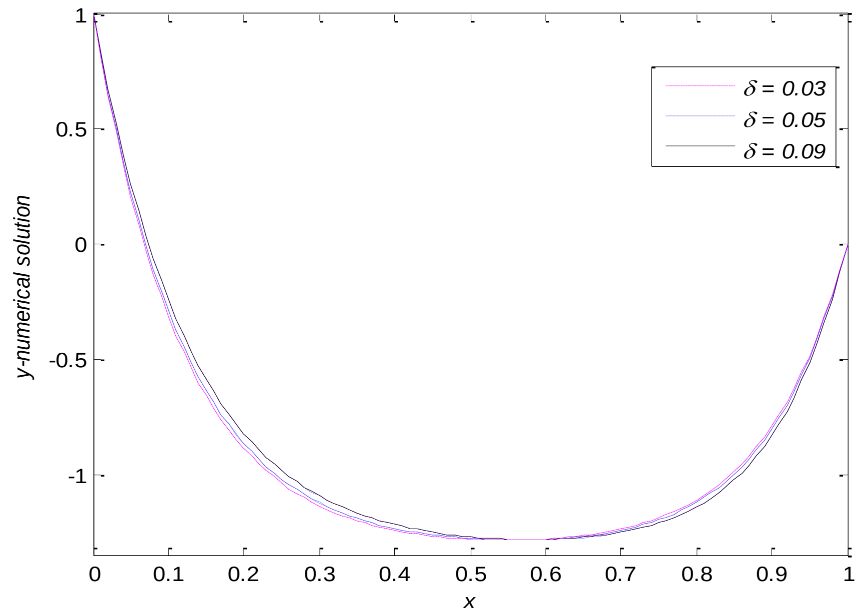

Fig. 4. 1: The numerical solution of **Example 4.1** with $\varepsilon = 0.01$ and $N = 100$.

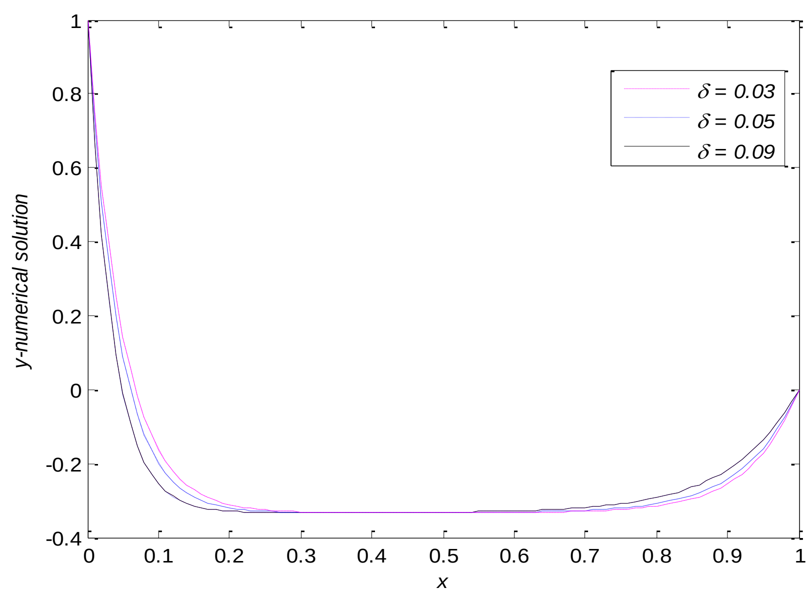

Fig. 4. 2: The numerical solution of **Example 4.2** with $\varepsilon = 0.01$ and $N = 100$.

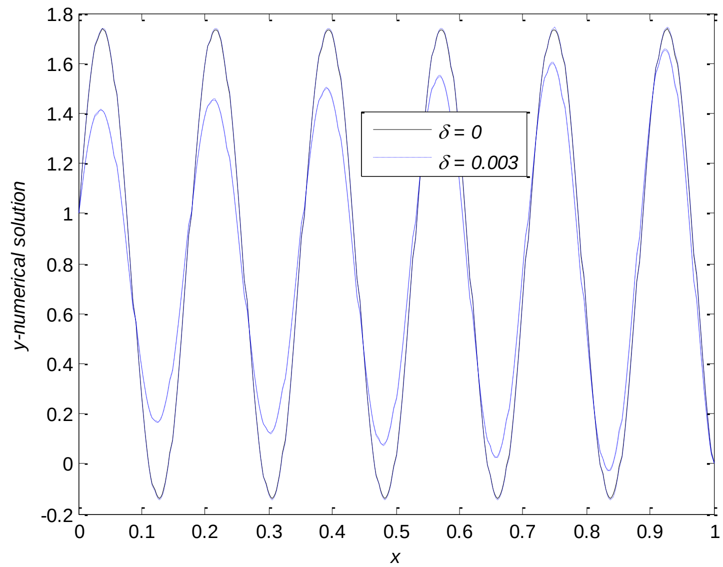

Fig. 4. 3: The numerical solution of **Example 4.3** with $\varepsilon = 0.001$ and $N = 300$.

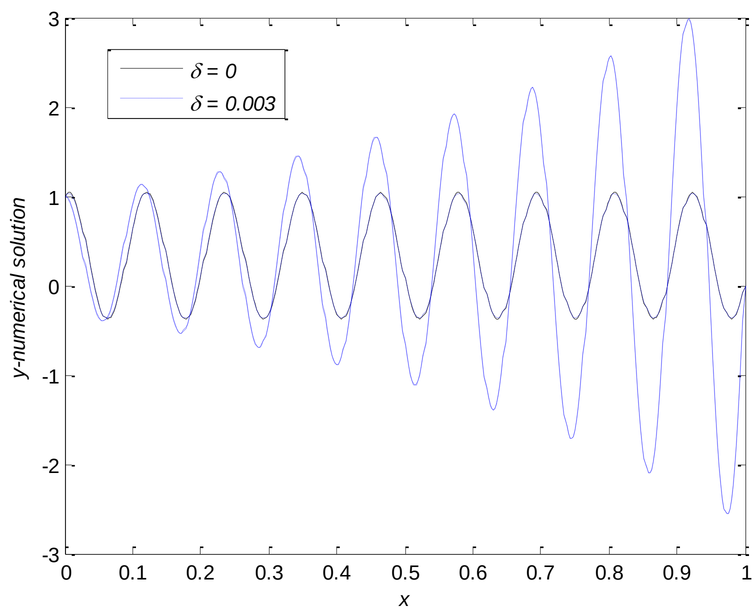

Fig. 4. 4: The numerical solution of **Example 4.4** with $\varepsilon = 0.001$ and $N = 300$.

4.3.2. Illustration of the Effect of Delay for Method II

The following graphs (Figs. (4.5) – (4.8)) show the numerical solutions obtained by the present method (*i.e.*, fourth order) for different values of delay parameter δ.

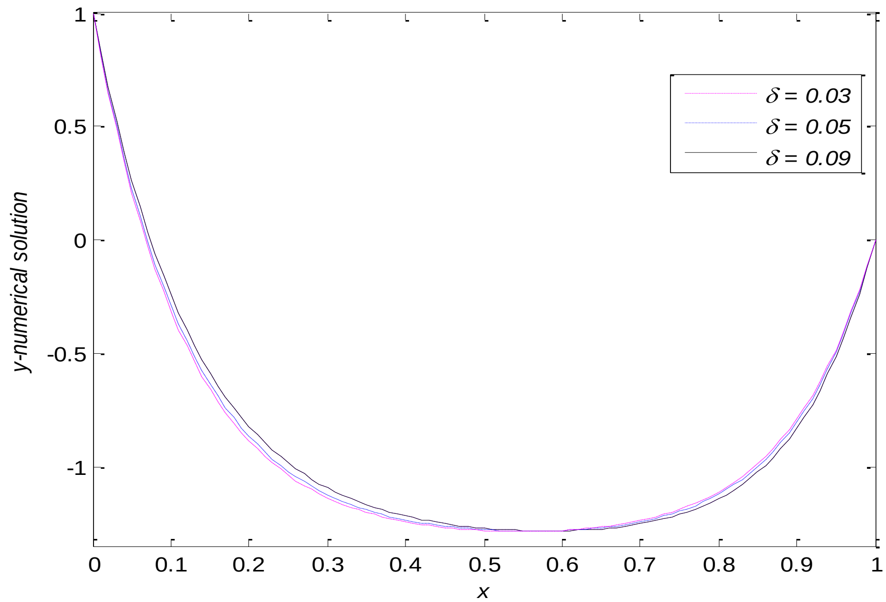

Fig. 4. 5: The numerical solution of **Example 4.1** with $\varepsilon = 0.01$ and $N = 100$.

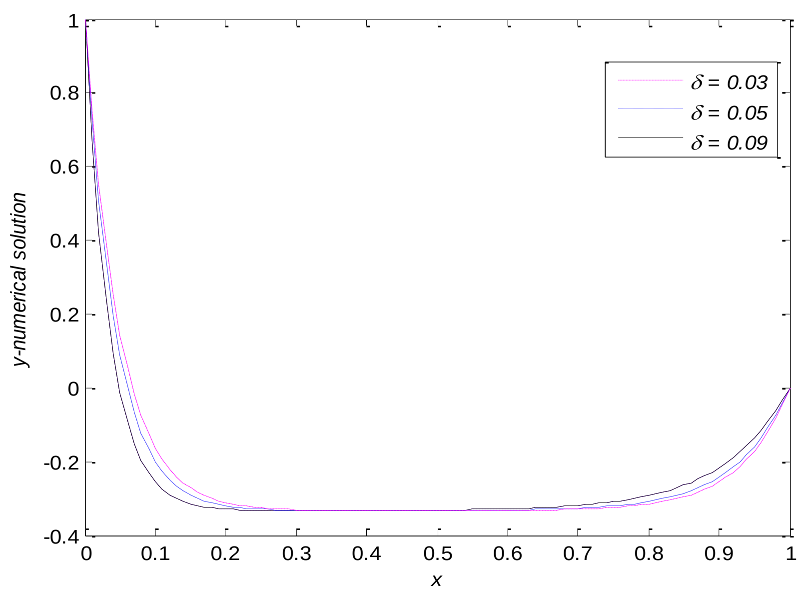

Fig. 4. 6: The numerical solution of **Example 4.2** with $\varepsilon = 0.01$ and $N = 100$.

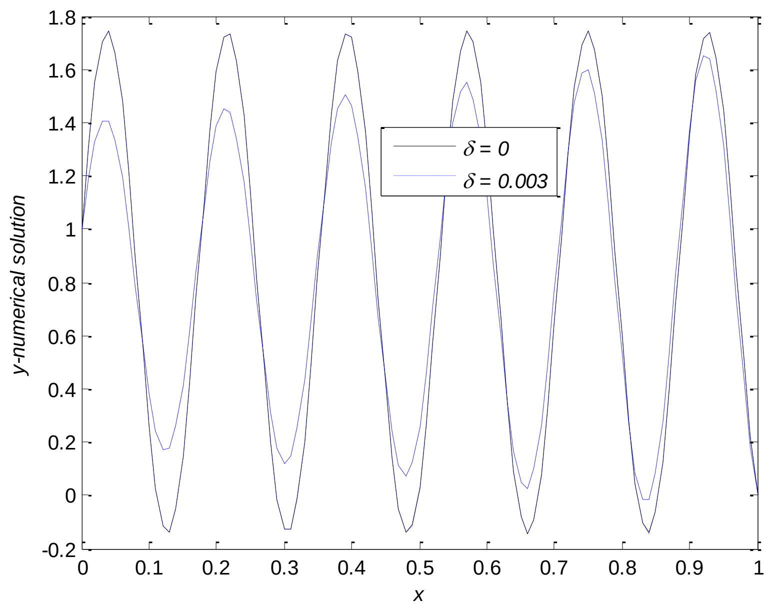

Fig. 4. 7: The numerical solution of **Example 4.3** with $\varepsilon = 0.001$ and $N = 100$.

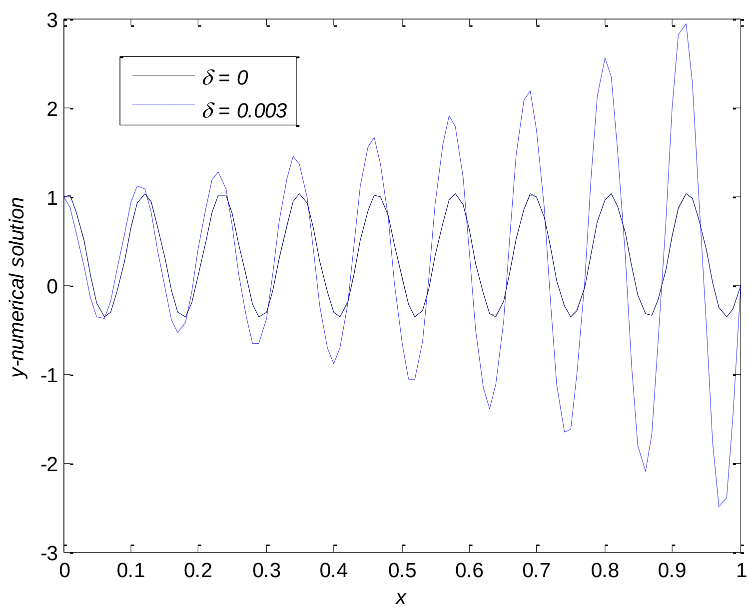

Fig. 4. 8: The numerical solution of **Example 4.4** with $\varepsilon = 0.001$ and $N = 100$.

4.3.3. The Rate of Convergence ρ for the Present Methods

In the same way in Eq. (4.46) one can define $Z_{h/2}$ by replacing h by $h/2$ and $N-1$ by $2N-1$, that is:

$$Z_{h/2} = \max_i \left| y_i^{h/2} - y_i^{h/4} \right| \text{, for } i = 1, 2, \ldots, 2N-1.$$

The computational rate of convergence ρ is also obtained by using the double mesh principle defined as, Doolan et al [7]:

$$\rho = \frac{\left(\log(Z_h) - \log\left(Z_{h/2}\right)\right)}{\log 2}.$$

The following Tables (*i.e.*, Tables (4.7) – (4.10)) shows the rate of convergence ρ of the present methods for different values of the mesh size h.

Table 4. 7: Rate of Convergence ρ for **Example 4.1** ($\varepsilon = 0.1$ and $\delta = 0.05$).

Method	h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	ρ
Second Order						
	1/100	1/200	2.4393e-05	1/400	6.0986e-06	1.9999
	1/200	1/400	6.0986e-06	1/800	1.5247e-06	2.0000
	1/300	1/600	2.7105e-06	1/1200	6.7764e-07	2.0000
Fourth Order						
	1/100	1/200	1.2135e-09	1/400	7.5855e-11	3.9998
	1/200	1/400	7.5860e-11	1/800	4.9848e-12	3.9277
	1/300	1/600	1.4980e-11	1/1200	9.4408e-13	3.9880

Table 4. 8: Rate of Convergence ρ for **Example 4.2** ($\varepsilon = 0.1$ and $\delta = 0.05$).

Method	h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	ρ
Second Order						
	1/100	1/200	6.1292e-05	1/400	1.5325e-05	1.9998
	1/200	1/400	1.5325e-05	1/800	3.8314e-06	2.0000
	1/300	1/600	6.8113e-06	1/1200	1.7028e-06	2.0000
Fourth Order						
	1/100	1/200	3.3028e-09	1/400	2.0653e-10	3.9993
	1/200	1/400	2.0657e-10	1/800	1.2909e-11	4.0002
	1/300	1/600	4.0807e-11	1/1200	2.5474e-12	4.0017

Table 4. 9: Rate of Convergence ρ for **Example 4.3** ($\varepsilon = 0.1$ and $\delta = 0.03$).

Method	h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	ρ
Second Order						
	1/100	1/200	5.2227e-04	1/400	1.3061e-04	1.9996
	1/200	1/400	1.3061e-04	1/800	3.2655e-05	1.9999
	1/300	1/600	5.8052e-05	1/1200	1.4513e-05	2.0000
Fourth Order						
	1/100	1/200	3.9856e-08	1/400	2.4913e-09	3.9998
	1/200	1/400	2.4916e-09	1/800	1.5603e-10	3.9971
	1/300	1/600	4.9143e-10	1/1200	3.0611e-11	4.0049

Table 4. 10: Rate of Convergence ρ for **Example 4.4** ($\varepsilon = 0.1$ and $\delta = 0.03$).

Method	h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	ρ
Second Order						
	1/100	1/200	8.3415e-04	1/400	2.0830e-04	2.0016
	1/200	1/400	2.0833e-04	1/800	5.2068e-05	2.0004
	1/300	1/600	9.2578e-05	1/1200	2.3141e-05	2.0002
Fourth Order						
	1/100	1/200	1.5497e-07	1/400	9.6846e-09	4.0001
	1/200	1/400	9.6846e-09	1/800	6.0394e-10	4.0032
	1/300	1/600	1.9131e-09	1/1200	1.1975e-10	3.9978

4.4. Discussion

In this thesis, parameter uniform second and fourth order numerical methods are presented for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour. First, the given singularly perturbed delay reaction-diffusion equation is converted into an asymptotically equivalent singularly perturbed boundary value problem by using the Taylor series expansion for the delay term as the delay parameter is sufficiently small. Then, given interval is discretized and the derivative of the given differential equation is replaced by the finite difference approximations. And, the given differential equation is transformed into a three-term recurrence relation, which can easily be solved by using Thomas Algorithm. The stability and convergence of the methods have been investigated. Further, the present methods are ε-uniformly convergent methods for which have not been sufficiently developed for a wide class of singularly perturbed delay differential equations, Pratima and Sharma [34]. The numerical results have been presented in Tables (4.1) – (4.6) for different values of the perturbation parameter ε and delay parameter δ and number of mesh points N. The results obtained by the present method are compared with Swamy et al [45].

It can be observed from the tables that the present methods give better results than the method by Swamy et al [45]. Further, it can be observed from the tables that, the accuracy of the problem increased by increasing the resolution of the grid, Kadalbajoo and Ramesh [20]. *i.e.* it is significant that all of the maximum absolute errors decrease rapidly as N increases.

The graphs of the considered examples for different values of delay parameter and for fixed step size h and ε are plotted in Figs. (4.1) – (4.8), to examine the effect of delay on the twin boundary layer and oscillatory behavior of the solution. Tables (4.7) – (4.10) depicts that the present methods have the rate of convergence which are in agreement with the theoretical proofs.

Moreover, to demonstrate the effect of delay on the twin boundary layer, we consider examples (4.1) and (4.2). From Fig. (4.1) or (4.5), we observed that when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour. From Fig. (4.2) or (4.6), we observed that when the delay is $O(\varepsilon)$, the solution maintains layer behaviour although the coefficient of the delay term in the equation is of $O(1)$ and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases. To demonstrate the effect of delay on the oscillatory behaviour, we consider the examples (4.3) and (4.4). Accordingly, one can conclude that the solution oscillates throughout the domain for different values of shift parameter δ (Figs. (4.3), (4.4), (4.7) and (4.8)).

CHAPTER FIVE

CONCLUSION AND SCOPE FOR FUTURE WORK

5.1. Conclusion

This study is implemented on four linear examples without exact solutions by taking different small values for the perturbation parameter ε and delay parameter δ and the computational results are presented in the tables. Generally for both methods, we conclude that; the results observed from the tables demonstrate that the present methods approximate the solution very well. A numerical result presented in this thesis shows the superiority of the proposed methods over some existing methods reported in the literature. Furthermore, fourth order method is more accurate than second order method for the same examples. The stability and ε-uniform convergence of the methods are established well. The results presented confirmed that computational rate of convergence as well as theoretical estimates indicate that second order method is a second order convergent and fourth order method is a fourth order convergent. The graphs show the effect of delay on the twin boundary layer depending on the order of the coefficient of the delay term and also for oscillatory behavior, the solution is oscillated throughout the domain for different values of shift parameter δ.

5.2. Scope for Future Work

The schemes proposed in this thesis can also be extended to sixth and higher order numerical methods for singularly perturbed delay reaction-diffusion equation. And also, this thesis considered the ε-uniform convergent. So, one can establish the numerical methods of fourth and higher order whose convergence of the difference scheme is for fixed values of the parameter ε. *i.e.*, convergence is dependent on ε.

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