A POINT OF COINCIDENCE AND COMMON FIXED POINT THEOREM FOR EXPANSIVE TYPE MAPPINGS IN b-METRIC SPACES

A Thesis Submitted to The Department of Mathematics, College of Natural Sciences, Jimma University in Partial Fulfillment of the Requirements for The Award of The Degree of Master of Science in Mathematics (MSc.)

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ABSTRACT

The main purpose of this research paper is to establish sufficient condition for the existence of point of coincidence and common fixed point for a pair of self-maps satisfying some expansive type conditions in a b-metric space.

In this research undertaking, we followed analytical design. Secondary sources of data such as journal, internet and books were used for this study. The analysis techniques which we adopted for the successful completion of this study were that of Mohanta (2016).

This study was conducted from September 2016 to June 2017.

Keywords: b-metric space, point of coincidence, common fixed point, expansive mappings.

CHAPTER ONE: INTRODUCTION

1.1 Background of the study

Let (X, d) be a metric space. A point z in X is said to be a fixed point of a self-map $T: X \to X$ if $T(z)$ = z. We denote the set of fixed points of T by $F(T)$.

A self-map $T: X \to X$ is said to be a contraction, if there exists a constant $\alpha \epsilon [0, 1)$ such that for all $x, y \epsilon X$,

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1.1}$$

Banach (1922) stated his celebrated theorem on the existence and uniqueness of fixed point of a contraction map defined on a complete metric space for the first time. This theorem states that, if (X, d) is a complete metric space and $T: X \to X$ is a contraction map, then T has a unique fixed point. Since its first appearance, the Banach contraction mapping principle has become the main tool to study contractions as they appear abundantly in a wide array of quantitative sciences. This theorem provides a technique for solving a variety of applied problems in Mathematical sciences and engineering. Its most well-known application is in ordinary differential equations, particularly, in the proof of the Picard - Lindelöf theorem which guarantees the existence and uniqueness of solution of first order initial value problems (Smith, 2014).

It is worth emphasizing that its remarkable strength of the Banach principle originates from the constructive processes it provides to identify the fixed points. This notable strength further attracted the attention of not only many prominent Mathematicians related to non-linear analysis, but also many researchers who are interested in iterative methods to examine the quantitative problems involving certain mapping and space structures required in their work in various areas such as social science, biology, economics and computer science.

Inequality (1.1.1) implies continuity of the self-map T. A natural question is whether we can find contractive conditions which will imply existence of a fixed point of a self-map in a complete metric space but do not imply the continuity the map.

The Banach contraction principle has been generalized in many ways over the years. In some generalizations, the contractive nature of the map is weakened; see (Branciari, 2002; Kannan, 1969; Petric, 2010; Kikkawa and Suzuki, 2008; Kirk *et al.*, 2003; Rhoades, 1977; Sessa, 1982 and others). In other generalizations, the topology (ambient space) is weakened; for example rectangular metric (Branciari, 2000), *b*-metric space (Czerwik, 1993), cone metric space (Huang and Zhang, 2007), cone rectangular metric space (Azam *et al*. 2009), and cone *b*-metric space (Jovanovi´c *et al*., 2010) are some of the generalized metric spaces introduced by different authors in the recent past. See also (Aage *et al*., 2008; Abbas *et al*., 2011; Chaipunya *et al*., 2012; Panthi *et al.,* 2015; Zeyada *et al*., 2005; Zoto et al. 2012) and others.

A metrical common fixed point theorem is broadly comprised of conditions on commutativity, continuity, completeness and contraction besides suitable containment of range of one map into the range of the other. For proving new results, the researchers of this domain are required to improve one or more of these conditions.

Let (X, d) be a metric space and S, T two self-mappings on (X, d). A point $z \in X$ is said to be a common fixed point of S and T if $Sz = Tz = z$.

Jungck proved a common fixed point theorem for commuting maps by generalizing the Banach's fixed point theorem (Jungck, 1976). With a view to accommodate a wider class of mappings in the context of common fixed point theorems, Sessa (1982) introduced the notion of weakly commuting mappings which was further generalized by Jungck (1986) by defining compatible mappings. After this, there came a host of such definitions which are scattered throughout the recent literature whose survey and illustration (up to 2001) is available in Murthy (2001). A minimal condition merely requiring the commutativity at the set of coincidence points of the pair called weak compatibility was introduced by Jungck and Rhoades (1998). This new notion was extensively utilized to prove new results.

Bakhtin introduced b-metric space as a generalization of metric space and proved the contraction mapping principle in b-metric space that generalized the famous Banach contraction principle in metric space (Bakhtin, 1989). Since then, many researchers including Czerwik (1993), Akkouchi (2011), Aydi *et al*. (2012), Boriceanu (2009), Bota *et al.* (2011), Kir and Kiziltunc (2013) and Pacurar (2010) studied the extension of the existing fixed point theorems in *b*-metric spaces for singlevalued and multivalued functions.

The study of expansive mappings is a very interesting research area in fixed point theory. A mapping satisfying the condition $\mathrm{d(Tx,Ty)} \geq \beta\mathrm{d(x,y)}$ for all $\mathrm{x,y} \in \mathrm{X}$, where $\beta > 1$, is called expansive mapping. Wang *et al*. (1984) introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. Rhoades (1985) and Taniguchi (1989) generalized the results of Wang (1984) for pair of mappings. Later, Khan *et al*. (1986) generalized the result of Wang (1984) by making use of the functions. Kang (1993) generalized these results of Khan *et al.* (1986), Rhoades (1985) and Taniguchi (1989) for expanding mappings. Ahmed (2009) established a common fixed point theorem for expansive mappings by using the concept of compatibility of type (A) in 2-metric spaces. The theorem proved by Ahmed (2009) was the generalization of the result of Kang *et al.* (1993) for expansive mappings. Şahin and Telci (2010) also presented a common fixed point theorem for expansion type mappings in complete cone metric spaces which generalizes and extends the theorem of Wang *et al.* (1984) for a pair of mappings to cone metric spaces.

Recently, Mohanta (2016) established sufficient conditions for existence of point of coincidence and common fixed points for a pair of self-maps satisfying the following expansive type conditions in b-metric spaces:

Theorem 1.1 (Mohanta, 2016) Let (X, d) be a b-metric space with coefficient $s \geq 1$. Suppose the mappings $g, f: X \to X$ satisfy the conditions

$$d(fx, fy) \geq \alpha_1 d(gx, gy) + \alpha_2 d(fx, gx) + \alpha_3 d(fy, gy) \qquad (1.1.2)$$

for all $x, y \in X$, where $\alpha_i \geq 0$ for each $i = 1,2,3$ with $\alpha_1 + \alpha_2 + \alpha_3 > s$.

Assume the following hypotheses:

(i) $\alpha_2 < 1$ and $\alpha_1 \neq 0$, (ii) $g(X) \subseteq f(X)$, and (iii) $f(X)$ or $g(X)$ is complete. Then f and g have a point of coincidence in X. Moreover, if $\alpha_1 > 1$, then the point of coincidence is unique. If f and g are weakly compatible and $\alpha_1 > 1$, then f and g have a unique common fixed point in X.

Motivated and inspired by the work of Mohanta (2016), in this research work the researchers studied the sufficient conditions for the existence of points of coincidence and common fixed points for a pair of self-maps satisfying some expansive type conditions in b-metric spaces by replacing inequality (1.1.2) with a more general expansive type condition. We have also supported our main result with examples.

1.2 Statement of the Problems

This study focuses on the sufficient conditions for existence of points of coincidence and common fixed points for a pair of self-maps satisfying some expansive type conditions in b-metric spaces by replacing inequality (1.1.2) with a more general expansive type condition of the form

$$d(fx,fy)+\frac{\beta}{s}[d(gx,fy)\ +\ d(gy,fx)]\geq\alpha_1 d(gx,gy)\ +\ \alpha_2\, d(fx,gx)\ +\ \alpha_3\, d(fy,gy) \qquad (1.2.1)$$

for all $x,y\in X$, where $\alpha_i\geq 0$ for each $i=1,2,3$, $s\geq 1$ and $\beta\geq 0$.

Thus, this study answers the following questions:

1. How can we prove the existence of a point of coincidence of the maps f and g satisfying the expansive type condition under investigation in b-metric space?
2. How can we get a unique point of coincidence and hence the unique common fixed point of the maps f and g satisfying the expansive type condition under investigation in b-metric space?
3. How can we validate our main results by supporting with an applicable example?

1.3 Objective of the Study

1.3.1 General objective

The main objective of this study was to establish sufficient conditions for existence of point of coincidence and common fixed points for a pair of self-maps satisfying an expansive type condition (1.2.1) in b-metric spaces.

1.3.2 Specific objectives

1. To prove the existence of point of coincidences of the maps f and g satisfying the expansive type condition under investigation in b-metric space.

2. To prove the uniqueness of point of coincidences and hence prove existence unique common fixed point of the maps f and g satisfying the expansive type condition under investigation in b-metric space.
3. To validate the main results of this study using applicable example.

1.4 Significance of the Study

Fixed point theory has been a subject of growing interest of many researchers for various types of well-known Contraction principle in this space. The researchers hope that the results obtained in this study contribute to further research activities in this area. Furthermore, collaboration in this research is useful for the graduate program of the department. The researchers also benefit from this study since it helps to develop scientific research writing skill and scientific communication in mathematics.

1.5 Delimitation of the Study

This study was conducted under the stream of functional analysis and delimited to the study of existence of points of coincidence and unique common fixed point for a pair of self-maps satisfying expansive type condition in b-metric space.

CHAPTER TWO: REVIEW OF RELATED LITERATURE

The theory of fixed point is one of the most powerful tools of modern mathematics. Not only it is used on a daily basis in pure and applied mathematics but also serves as a bridge between analysis and topology and provides a very fruitful area of interaction between the two.

Let X be a nonempty set and $T: X \longrightarrow X$ be a self-map on X. An element p $\in X$ is called a periodic point for T if there exists a positive integer k such that $T^k p = p$. If $k = 1$, then p is called a fixed point of T.

Theorem 2.1 (Banach, 1922) Let (X, d) be a complete metric space and $T: X \longrightarrow X$ a strict contraction, i.e., a map satisfying

$$d(Tx, Ty) \leq \alpha d(x, y), for\ x, y \epsilon X, \quad (2.1.1)$$

where $0 \leq \alpha < 1$ is constant. Then

(P1) T has a unique fixed point p in X (i.e.$Tp = p$)

(P2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, 3, \cdots$ converges to p for $x_0 \in X$.

Kannan (1968) has been the first one to consider discontinuous self-mapping T, by considering, instead of (2.1.1), the following alternative and independent contractive condition.

Theorem 2.2 (Kannan, 1968). Let (X, d) be a complete metric space, T is a self-map of X.

Assume that there exists $\beta \epsilon [0, \frac{1}{2})$ such that for all x, y ϵX

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \quad (2.1.2)$$

Then T has a unique fixed point in X.

Mappings satisfying the inequality (2.1.2) are called Kannan type mapping.

Theorem 2.3 (Chatterjea, 1972**)** Let (X, d) be a complete metric space, T be a self-map of X. Assume that there exists $\gamma\epsilon[0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)] \text{ For all x, y} \in X \qquad (2.1.3)$$

Then T has a unique fixed point in X

The inequality (2.1.1), (2.1.2), (2.1.3) are independent of one another (Rhoades, 1977).

Definition 2.4 Let (X, d) be a metric space. A mapping $T: X \longrightarrow X$ is called Zamfirescu operator if there exist real numbers α, β and γ satisfying $0 \leq \alpha < 1, 0 \leq \beta < \frac{1}{2} and\ 0 \leq \gamma < \frac{1}{2}$ such that for each x, y $\in X$ at least one of the following is true:

(Z1) $d(Tx, Ty) \leq \alpha d(x, y)$;

(Z2) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$; and

(Z3) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

Zamfirscu (1979) established the following theorem which is a generalization of Banach contraction principle (Banach, 1922), Kannan's theorem (Kannan, 1968) and Chatterjea's theorem (Chatterjea, 1972).

Theorem 2.5 (Zamfirescu, 1979) Let (X, d) be a complete metric space and T be a self-map on X. If T is a Zamfirscu operator, then T has fixed point in X.

Further generalizations of Banach contraction principle are done by several authors, such as (Bailey, 1966; Branciare, 2002; Ciric, 1971; Das, 1980; Dutta and Choudhury, 2008; Edelstein, 1962; Hardy and Rogers, 1973; Jaggi, 1980; Kannan, 1968; Kannan, 1969 and Kannan, 1984; Meir and Keeler, 1969; Reich, 1971).

Jungck (1976) established a common fixed point theorem for commuting mapping, which generalized the well-known Banach contraction principle.

Jungck'*s* theorem was generalized and extended for commuting mapping in various ways with several contraction type by many authors, such as, (Bianchini, 1972; Das and Naik, 1979; Ding, 1983; Fisher, 1978; Jungck, 1986). Sessa (1982) introduced a concept that generalizes commuting maps, namely, weakly commuting maps and proved a common fixed point theorem for such maps. Jungck (1986) initiated the concept of compatible pair of maps, as a generalization of weakly commuting maps, in order to obtain common fixed points of a pair of self-maps.

Jungck *et al.* (1993) introduced an independent notion of compatible maps, namely compatible map of type (A) and established common fixed point theorem. Jungck and Rhodes (1998) introduced the notation of weakly compatible maps which is found to be very helpful in obtaining coincidence point and common fixed point of various classes of mapping on metric space, by such researchers as, (Abbas and Rhoades, 2007; Abbas and Jungck, 2008; Ahmad, 2003; Bari and Vetro, 2008; Beg and Abas, 2006; Fisher, 1978; Jha, 2007; Khan and Domlo, 2006 and Pant, 1994).

CHAPTER THREE: METHODOLOGY OF THE STUDY

3.1 Study site and period

This study focuses on an interesting topic from Functional Analysis especially, fixed point theory. The goal of this research is to establish sufficient conditions for existence of point of coincidence and common fixed points for a pair of self-maps satisfying an expansive type condition (1.2.1) in b-metric spaces. This study is conducted from September 2016 - June 2017 G.C. in Jimma University.

3.2 Study design

The study design we followed to achieve the objective of this study is analytical method.

3.3 Source of information

The study depends on various sources of information such as; related books in functional analysis in particular, fixed point theory, journals, different related unpublished / published research works and the internet.

3.4 Mathematical Procedure of the Study

Relevant materials and data for the study were collected by means of documentary review. Hence, in this study we followed the standard procedure used in the published work of Mohanta (2016) in achieving the proposed goal of this research work. That is, we constructed a sequence $\{x_n\}$ in X such that $fx_n = gx_{n-1}$, for all $n \geq 1$, where f and g are expansive self-mappings on X, and then showed that the sequence $\{gx_{n-1}\}$ and hence $\{fx_n\}$ is Cauchy in either $g(X)$ or $f(X)$, since at least one of them is complete. Further, we showed that the limit point of the Cauchy sequence is a point of coincidence of the pair of maps f and g.

3.5 Ethical consideration

Ethical considerations have been taken care of at in all stages of the research process. So, to make the study legal, permission has been obtained from Ethical committee of College of Natural Sciences of Jimma University. Moreover, the researchers have kept the rules and regulations of the university and have acknowledged all sources of information.

CHAPTER FOUR: RESULT AND DISCUSION

4.1 PRELIMINARIES

In this section we need to recall some basic notations, definitions and necessary results from existing literature. The definition of metric space, complete metric space, *b*- metric space, the Cauchy sequence, complete *b*-metric space, the notion of convergence and other results that we need in the sequel.

Definition 4.1.1. Let X be a non-empty set and let $d : X \times X \rightarrow \Re^+$ be a function satisfying the conditions,

$d_1)$ $d(x,y) = 0 \Leftrightarrow x = y$;

$d_2)$ $d(x,y) = d(y,x)$;

$d_3)$ $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. Then d is called metric on X, and the pair (X, d) is called a metric space.

Definition 4.1.2 (Czerwik, 1993). Let X be a non-empty set and $k \geq 1$ be a real number. A mapping $d: X\, x\, X \rightarrow \Re^+$ is said to be a *b*-metric if for all $x, y, z \in X$, the following conditions are satisfied:

$d_1)$ $d(x,y) = 0 \Leftrightarrow x = y$

$d_2)$ $d(x,y) = d(y,x)$

$d_3)$ $d(x,y) \leq k[\, d(x,z) + d(z,y)]$

The pair (X, d) is called a *b*-metric space. From the above definition it is evident that the *b*-metric space extends the metric space. Here, for *k* = 1 it reduces into standard metric space. But the converse is not true as is clear from the following examples.

Example 4.1.3 (Berinde,1993). The space $X = l_p$ with $0 < p < 1$ where

$$l_p(\Re) = \left\{ x = \{x_n\} \subset \Re \left| \sum_{n=1}^{\infty} |x_n|^p < \infty \right. \right\},$$

together with the function d: $lp \times lp \rightarrow \Re^+$ defined by

$$d(x,y) = (\textstyle\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p}$$

where $x = \{x_n\},\ y = \{y_n\} \in l_p$ is a b-metric space.

Here since conditions (d_1) and (d_2) of Definition 4.1.2 hold clearly, we only check condition (d_3).

Indeed, since $0 < p < 1$, the mapping $x \mapsto x^p$ is concave on $\mathfrak{R}^+$and also it is subadditive. Thus we get the pointwise bound of $|f(x) + g(x)|^p \le |f(x)|^p + |g(x)|^p$. Consequently, we have

$$||f + g||_p^p \le ||f||_p^p + ||g||_p^p.$$

$$= 2\left(\frac{1}{2}||f||_p^p + \frac{1}{2}||g||_p^p\right)$$

$$\le 2\left(\frac{1}{2}||f||_p + \frac{1}{2}||g||_p\right)^p$$

$$= 2^{1-p}\left(||f||_p + ||g||_p\right)^p.$$

This implies

$$||f + g||_p \le 2^{1/p}\left(||f||_p + ||g||_p\right).$$

Now putting $f = \{u_n\}$ and $g = \{v_n\}$, where let $u_n = x_n - z_n$, $v_n = z_n - y_n$ for each $n = 1,2,\cdots$, we then get $x_n - y_n = u_n + v_n$ for each $n = 1,2,\cdots$.

So, we obtain

$$d(x,y) = \left(\sum_{n=1}^{\infty}|x_n - y_n|^p\right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty}|u_n + v_n|^p\right)^{1/p}$$

$$= \|f + g\|_p$$

$$\le 2^{1/p}\left(||f||_p + ||g||_p\right)$$

$$\le 2^{1/p}\left(\left(\sum_{n=1}^{\infty}|x_n - z_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty}|z_n - y_n|^p\right)^{1/p}\right)$$

$$\le 2^{1/p}\left(d(x,z) + d(z,y)\right) \text{ for all } x = \{x_n\},\ y = \{y_n\}, z = \{z_n\} \in X$$

Therefore, d is a b-metric with constant $s = 2^{1/p}$.

Similarly, we have

Example 4.1.4 (Berinde, 1993) Let $X \coloneqq Lp\,[0,1]$ be the space of all real-valued functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, with $0 < p < 1$. Define $d: X \times X \to \mathfrak{R}^+$as:

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}$$

for each $x, y \in X$. Then d is a b-metric with coefficient $s = 2^{1/p}$.

Remark 1: When $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces are effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true.

Example 4.1.5 (Akkouchi, 2011). Let $X = \{0, 1, 2\}$. Define $d: X \times X \to \mathfrak{R}^+$ as follows

$d(2,0) = d(0,2) = m,\ m \geq 2,$
$d(0,1) = d(1,0) = d(1,2) = d(2,1) = 1$, and
$d(0,0) = d(1,1) = d(2,2) = 0.$

Then, $d(x,y) \leq \frac{m}{2}[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

(X,d) is a b-metric space with constant $s = \frac{m}{2}$. However, if $m > 2$ the ordinary triangle inequality does not hold. Thus (X,d) is not a metric space. If we take $0, 1, 2 \in X$, then we get

$$d(2,0) = m \geq d(2,1) + d(1,0) \Rightarrow m \geq 2.$$

Hence it does not satisfy ordinary triangle inequality.
Therefore the function defined above is a b-metric space but not a metric for $m > 2$.

Example 4.1.6 (Roshan *et al*., 2014). Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$. However, (X,ρ) is not necessarily a metric space. $\rho(x,y) = (d(x,y))^p > 0$ for $x \neq y$, for all $x, y \in X$.

This implies $\rho(x, y) > 0$ if $x \neq y$.

$$\begin{aligned}\rho(x, y) = 0 &\Leftrightarrow (d(x, y))^p = 0\\ &\Leftrightarrow d(x, y) = 0\\ &\Leftrightarrow x = y.\end{aligned}$$

Now $\rho(x, y) = (d(x, y))^p = (d(y, x))^p = \rho(y, x)$.

This implies $\rho(x, y) = \rho(y, x)$.

Hence conditions (d_1) and (d_2) of Definition 4.1.2 are satisfied. If $1 < p < \infty$, then the function $g(x) = x^p (x > 0)$ is strictly convex, and hence $\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}(a^p + b^p)$. This in turn implies that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$.

So for all $x, y, z \in X$, we have

$$\begin{aligned}\rho(x, y) = (d(x, y))^p &\leq \big(d(x, z) + d(z, y)\big)^p\\ &\leq 2^{p-1}\Big(\big(d(x, z)\big)^p + \big(d(z, y)\big)^p\Big)\\ &= 2^{p-1}(\rho(x, z) + \rho(z, y)).\end{aligned}$$

So condition (d_3) of Definition 4.1.2 holds and hence ρ is a b-metric with constant $s = 2^{p-1}$. If $X = \Re$, the set of real numbers and $\rho(x, y) = |x - y|^2$, then ρ is a b-metric on $\Re$ with $s = 2$, but not a metric on $\Re$ since the ordinary triangle inequality for a metric does not hold.

Definition 4.1.7 (Boriceanu, 2009) Let (X, d) be a b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} \mathrm{d}(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$

(ii) $\{x_n\}$ is Cauchy if and only if $\lim_{n.m\to\infty} \mathrm{d}(x_n, x_m) = 0$.

(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 2 (Boriceanu, 2009) In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

The following example shows that a b-metric need not be continuous.

Example 4.1.8 (Hussein *et al.*, 2012). Let $X = \mathbb{N} \cup \{\infty\}$ and let $d: X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0, \text{if } m = n, \\ \left|\dfrac{1}{m} - \dfrac{1}{n}\right|, \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{or } \infty, \\ 2, \text{otherwise.} \end{cases}$$

Then considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m,p) \leq \tfrac{5}{2}\big(d(m,n) + d(n,p)\big).$$

Then, (X, d) is a b-metric space (with $s = \frac{5}{2}$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \tfrac{1}{2n} \to 0 \text{ as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \nrightarrow 5 = d(\infty, 1)$ as $n \to \infty$

Theorem 4.1.9. (Aghajani, *et al.*, 2014). Let (X, d) be a b-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ converge to $x, y \in X$, respectively. Then, we have

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to\infty}(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s\, d(x,z).$$

Proof: Using the triangle inequality in a b-metric space it is easy to see that

$$d(x,y) \leq s\, d(x, x_n) + s^2 d(x_n, y_n) + s^2\, d(y_n, y)$$

and

$$d(x_n, y_n) \leq s\, d(x_n, x) + s^2 d(x,y) + s^2\, d(y, y_n\,).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the first desired result. Similarly, using again the triangle inequality the last assertion follows.

Definition 4.1.10. (Mohanta, 2016). Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence $\{x_n\}$ in X, such that $x_n \to x_0$ as $n \to \infty$ then $Tx_n \to Tx_0$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.

Definition 4.1.11. (Mohanta, 2016). Let (X, d) be a b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Tx, Ty) \geq k\, d(x, y) \qquad \text{for all } x, y \in X.$$

Definition 4.1.12. (Roshan *et al*., 2014) Two selfmaps f and T of a b-metric space (X, d) are said to be compatible if and only if $\lim_{n\to\infty} d(fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Thus $d(fTx, Tfx) \to 0$ as $d(fx, Tx) \to 0$. This implies the pair of maps f and T are compatible. So, if f and T commute, then they are compatible.

Definition 4.1.13. (Roshan *et al*., 2014) Let X be a set and $f, T: X \to X$ be self-maps of X. A point x in X is called a coincidence point of f and T if $fx = Tx$. We shall call $w = fx = Tx$ a point of coincidence of f and T. The set of coincidence points of T and f is denoted by $C(T, f)$.

Example 4.1.14. Take $X = [0,1], Sx = x^2$, $Tx = \frac{x}{2}$. It is clear that $\{0, \frac{1}{2}\}$ is the set of coincidence point of S and T and 0 is the unique common fixed point.

Definition 4.1.15. (Roshan *et al*., 2014) Two self-maps f and T of a nonempty set X are said to be weakly compatible if they commute at their coincidence point (i.e., $fTx = Tfx$ for all $x \in C(f, T)$).

Example 4.1.16. Let $X = [1, \infty]$ with the usual metric. Define $f, T: X \to X$ by $f(x) = 4x - 3$ and $T(x) = x^2$ for all $x \in X$. Then 1 and 3 are the only coincidence points of f and T. Also, $fT(1) = 1 = Tf(1)$. But, $fT(3) = 33 \neq 81 = Tf(3)$. Hence f and T are occasionally weakly compatible but not weakly compatible.

Proposition 4.1.17. (Abbas *et al*., 2008) Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T .

Proof : Let v the unique point of coincidence of S and T. Then $v = Su = Tu$ for some $u \in X$. By weakly compatibility of the pair of maps (S, T), we have $Sv = STu = TSu = Tv$, which implies that $Sv = Tv = w$ (say). Thus, w is also a point of coincidence of S and T. Therefore, by the uniqueness of the point of coincidence of the selfmaps S and T, we have $v = w$.

Thus, v is the unique common fixed point of S and T.

4.2 MAIN RESULT

In this section, we prove some point of coincidence and common fixed point results in b-metric spaces.

Theorem 4.2.1. Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f, g : X \to X$ satisfy the condition

$$d(fx, fy) + \frac{\beta}{s}[d(gx, fy) + d(gy, fx)]$$
$$\geq \alpha_1 d(gx, gy) + \alpha_2 d(fx, gx) + \alpha_3 d(fy, gy), \qquad (4.2.1)$$

for all $x, y \in X, x \neq y$ where α_i is nonnegative real numbers for each $i = 1,2,3$ and $\beta \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 > (1 + 2\beta)s$ and $\beta < \frac{\alpha_1}{s} + \alpha_3$. Assume the following hypotheses:

(i) $\alpha_2 < 1$ and $\alpha_1 \neq 0$, (ii) $g(X) \subseteq f(X)$, and (iii) $f(X)$ or $g(X)$ is complete. Then f and g have a point of coincidence in X. Moreover, if $\alpha_1 > 1 + \frac{2\beta}{s}$, then the point of coincidence is unique. If f and g are weakly compatible and $\alpha_1 > 1 + \frac{2\beta}{s}$, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ such that $gx_0 = fx_1$. This is possible since $g(X) \subseteq f(X)$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $fx_n = gx_{n-1}$, for all $n \geq 1$.

If $fx_{n_0+1} = gx_{n_0}$ for some $n_0 \in \mathbb{N}$, then $fx_{n_0} = gx_{n_0}$.

Thus, x_{n_0} is a coincidence point of f and g.

So, we assume that $fx_n \neq fx_{n+1}$ for all $n \in \mathbb{N}$.

By (4.2.1), we have

$$d(fx_n, fx_{n+1}) + \frac{\beta}{s}[d(gx_n, fx_{n+1}) + d(gx_{n+1}, fx_n)]$$
$$\geq \alpha_1 d(gx_n, gx_{n+1}) + \alpha_2 d(fx_n, gx_n) + \alpha_3 d(fx_{n+1}, gx_{n+1})$$
$$\Rightarrow d(gx_{n-1}, gx_n) + \frac{\beta}{s}[d(gx_n, gx_n) + d(gx_{n+1}, gx_{n-1})]$$
$$\geq \alpha_1 d(gx_n, gx_{n+1}) + \alpha_2 d(gx_{n-1}, gx_n) + \alpha_3 d(gx_n, gx_{n+1})$$
$$\Rightarrow d(gx_{n-1}, gx_n) + \frac{\beta}{s} d(gx_{n+1}, gx_{n-1})$$
$$\geq (\alpha_1 + \alpha_3) d(gx_n, gx_{n+1}) + \alpha_2 d(gx_{n-1}, gx_n)$$
$$\Rightarrow d(gx_{n-1}, gx_n) + \frac{\beta}{s} s[d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]$$
$$\geq (\alpha_1 + \alpha_3) d(gx_n, gx_{n+1}) + \alpha_2 d(gx_{n-1}, gx_n)$$

$$\Rightarrow (1+\beta-\alpha_2)d(gx_{n-1},gx_n) \geq (\alpha_1+\alpha_3-\beta)d(gx_n,gx_{n+1})$$

$$\Rightarrow d(gx_n,gx_{n+1}) \leq \frac{1+\beta-\alpha_2}{\alpha_1+\alpha_3-\beta}d(gx_{n-1},gx_n)$$

$$\Rightarrow d(gx_n,gx_{n+1}) \leq \lambda d(gx_{n-1},gx_n) \text{ for all } n \geq 0,$$

where $\lambda = \frac{1+\beta-\alpha_2}{\alpha_1+\alpha_3-\beta} \in \left(0,\frac{1}{s}\right)$.

By induction, we get that

$$d(gx_n,gx_{n+1}) \leq \lambda^n d(gx_0,gx_1), \tag{4.2.2}$$

for all $n \geq 0$.

For $m,n \in \mathbb{N}$ with $m > n$, by repeated use of (4.2.2) we have

$$\begin{aligned} d(gx_n,gx_m) &\leq s\,[d(gx_n,gx_{n+1}) + d(gx_{n+1},gx_m)] \\ &\leq sd(gx_n,gx_{n+1}) + s^2 d(gx_{n+1},gx_{n+2}) + \cdots \\ &\qquad + s^{m-n-1}[d(gx_{m-2},gx_{m-1}) + d(gx_{m-1},gx_m)] \\ &\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}]\,d(gx_0,gx_1) \\ &\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}]\,d(gx_0,gx_1) \\ &\leq s\lambda^n[1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}]\,d(gx_0,gx_1) \\ &\leq \frac{s\lambda^n}{1-s\lambda}d(gx_0,gx_1). \end{aligned}$$

This implies that $d(gx_n,gx_m) \to 0$ (as $m,n \to \infty$), since $\frac{s\lambda^n}{1-s\lambda}d(gx_0,gx_1) \to 0$ as $n \to \infty$.

So, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Suppose that $g(X)$ is a complete. Then there exists $y \in g(X) \subseteq f(X)$ such that $gx_n \to y$ and also$fx_n \to y$. In case, $f(X)$ is complete, this holds also with $y \in f(X)$. Let $u \in X$ be such that $fu = y$.

By (4.2.1), we have

$$\begin{aligned} d(fx_n,fu) + \frac{\beta}{s}[d(gx_n,fu) + d(gu,fx_n)] \\ \geq \alpha_1 d(gx_n,gu) + \alpha_2 d(gx_n,fx_n) + \alpha_3 d(gu,fu) \\ \Rightarrow d(fx_n,y) + \frac{\beta}{s}[d(gx_n,y) + d(gu,fx_n)] \\ \geq \alpha_1 d(gx_n,gu) + \alpha_2 d(gx_n,fx_n) + \alpha_3 d(gu,y). \end{aligned}$$

If $\alpha_1 \neq 0$, then

$$\begin{aligned} d(gx_n,gu) \leq \frac{1}{\alpha_1} d(gx_{n-1},y) + \frac{\beta}{\alpha_1 s}[d(gx_n,y) + d(gu,fx_n)] \\ -\frac{\alpha_2}{\alpha_1}d(gx_n,fx_n) - \frac{\alpha_3}{\alpha_1}d(gu,y). \end{aligned}$$

Therefore,

$$d(y, gu) \leq s[d(y, gx_n) + d(gx_n, gu)]$$
$$\leq s[d(y, gx_n) + \tfrac{1}{\alpha_1} d(gx_{n-1}, y) + \tfrac{\beta}{\alpha_1 s}[d(gx_n, y) + d(gu, fx_n)]$$
$$-\frac{\alpha_2}{\alpha_1} d(gx_n, fx_n) - \frac{\alpha_3}{\alpha_1} d(gu, y)]$$
$$\leq s[d(y, gx_n) + \tfrac{1}{\alpha_1} d(fx_n, y) + \tfrac{\beta}{\alpha_1 s} d(gx_n, y) + \tfrac{\beta}{\alpha_1} d(gu, y) + \tfrac{\beta}{\alpha_1} d(y, fx_n)$$
$$-\frac{\alpha_2}{\alpha_1} d(gx_n, fx_n) - \frac{\alpha_3}{\alpha_1} d(gu, y)].$$

Taking limit as $n \to \infty$, using Theorem 4.1.9 we obtain

$$d(y, gu) \leq s\left[\tfrac{\beta}{\alpha_1} d(gu, y) - \tfrac{\alpha_3}{\alpha_1} d(gu, y)\right] = \tfrac{s}{\alpha_1}(\beta - \alpha_3) d(gu, y).$$
$$\Rightarrow \left(1 + \tfrac{s}{\alpha_1}(\alpha_3 - \beta)\right) d(y, gu) \leq 0.$$
$$\Rightarrow d(y, gu) = 0. \qquad \text{(Since } \beta < \tfrac{\alpha_1}{s} + \alpha_3\text{)}$$

Hence, $gu = y$ and hence $fu = gu = y$.

Therefore, y is a point of coincidence of f and g.

Now we suppose that $\alpha_1 > 1 + \frac{2\beta}{s}$. Let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$d(fx, fu) + \tfrac{\beta}{s}[d(gx, fu) + d(gu, fx)] \geq \alpha_1 d(gx, gu) + \alpha_2 d(fx, gx) + \alpha_3 d(fu, gu)$$
$$\Rightarrow d(v, y) + \tfrac{\beta}{s}[d(v, y) + d(y, v)] \geq \alpha_1 d(v, y),$$
$$\Rightarrow \left(\alpha_1 - 1 - \tfrac{2\beta}{s}\right) d(v, y) \leq 0.$$
$$\Rightarrow d(v, y) = 0. \qquad \text{(Since } \alpha_1 > 1 + \tfrac{2\beta}{s}\text{)}$$

Hence, $v = y$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 4.1.17, f and g have a unique common fixed point in X.

Remark (i) If we take $\beta = 0$ in Theorem 4.2.1, we get Theorem 1.2 as a corollary to Theorem 4.2.1.

(ii) If we take $\beta = \alpha_2 = \alpha_3 = 0$ in Theorem 4.2.1, we get the following as a corollary.

Corollary 4.2.2. Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f, g : X \rightarrow X$ satisfy the condition

$$d(fx, fy) \geq \alpha_1 d(gx, gy)$$

for all $x, y \in X$, where $\alpha_1 > s$ is a constant. If $g(X) \subseteq f(X)$ and $f(X)$ or $g(X)$ is complete, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

The following Corollary is the b-metric version of Banach's contraction principle.

Corollary 4.2.3. Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $g : X \rightarrow X$ satis es the contractive condition

$$d(gx, gy) \leq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{s})$ is a constant. Then g has a unique fixed point in X. Furthermore, the iterative sequence $\{g^n x\}$ converges to the fixed point.

Proof. It follows by taking $\beta = \alpha_2 = \alpha_3 = 0$ and $f = I$, the identity mapping on X, in Theorem 4.2.1.

Corollary 4.2.4. Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \rightarrow X$ is onto and satisfies

$$d(fx, fy) \geq \alpha_1 d(x, y)$$

for all $x, y \in X$, where $\alpha_1 > s$ is a constant. Then f has a unique fixed point in X.

Proof. Taking $g = I$ and $\beta = \alpha_2 = \alpha_3 = 0$ in Theorem 4.2.1, we obtain the desired result.

Corollary 4.2.5. Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ is onto and satisfies the condition

$$d(fx, fy) \geq \alpha_1 d(x, y) + \alpha_2 \beta d(fx, x) + \alpha_3 \gamma d(fy, y)$$

for all $x, y \in X$, where α_i is nonnegative real numbers for each $i = 1,2,3$ with $\alpha_1 \neq 0, \alpha_2 < 1, \alpha_1 + \alpha_2 + \alpha_3 > s$. Then f has a fixed point in X. Moreover, if $\alpha_1 > 1$, then the fixed point of f is unique.

Proof. It follows by taking $\beta = 0$ and $g = I$ in Theorem 4.2.1.

Now we give an example in support of our main result.

Example 4.2.6: Let $\mathrm{X} = [0,1) \cup (1,4]$ and define $\mathrm{d}: \mathrm{X} \times \mathrm{X} \to \mathfrak{R}$ by $\mathrm{d(x, y)} = |\mathrm{x} - \mathrm{y}|^2$. Then (X, d) is a b-metric space with $\mathrm{s} = 2$. Define mappings $\mathrm{f}, \mathrm{g}: \mathrm{X} \to \mathrm{X}$ by

$$f(x) = \begin{cases} 4x, & x \in \left[0, \frac{1}{4}\right) \\ 4, & x \in [\frac{1}{4}, 1) \cup (1,4] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x, & x \in \left[0, \frac{1}{4}\right] \\ 0, & x \in \left(\frac{1}{4}, 1\right) \cup (1,4] \end{cases}.$$

Then $(X) = [0,1) \cup \{4\}$, and $g(X) = [0, \frac{1}{4}]$, so that $g(X) \subseteq f(X)$ and $g(X)$ is complete.

Also, f and g satisfy the expansive type condition (4.2.1) with $\alpha_1 = 6, \alpha_2 = \frac{1}{9} = \alpha_3$ and $\beta = 1$, as it is explained in the following five cases.

Case (i): $x, y \in [0, \frac{1}{4})$, $x \neq y$. In this case we assume $x > y$. Then

$d(fx, fy) = 16(x - y)^2$, $d(gx, gy) = (x - y)^2$, $d(fx, gx) = 9x^2$, $d(fy, gy) = 9y^2$, $d(fx, gy) = (4x - y)^2$, and $d(fy, gx) = (4y - x)^2$.

Now we get

$$16(x - y)^2 + \frac{1}{2}[(4x - y)^2 + (4y - x)^2] \geq 6(x - y)^2 + \frac{1}{9}(9x^2) + \frac{1}{9}(9y^2),$$

since the above inequality reduces to

$$29(x - y)^2 + 6x^2 + 6y^2 + 2xy \geq 0.$$

Case (ii): $x \in [0, \frac{1}{4})$ and $y \in \left(\frac{1}{4}, 1\right) \cup (1,4]$.

Here also we have $d(fx, fy) = 16(x-1)^2$, $d(gx, gy) = x^2$, $d(fx, gx) = 9x^2$, $d(fy, gy) = 16$, $d(fx, gy) = 16x^2$, and $d(fy, gx) = (4-x)^2$.

Now again we get

$$16(x-1)^2 + \frac{1}{2}[16x^2 + (4-x)^2] \geq 6x^2 + x^2 + \frac{16}{9},$$

since the above inequality reduces to

$$16(x-1)^2 + \frac{3}{2}x^2 + \frac{56}{9} - 4x \geq 0.$$

Case (iii): $x \in [0, \frac{1}{4})$ and $y = \frac{1}{4}$.

Here also we have $d(fx, fy) = 16(x-1)^2$, $d(gx, gy) = \left(x - \frac{1}{4}\right)^2$, $d(fx, gx) = 9x^2$, $d(fy, gy) = \frac{225}{16}$, $d(fx, gy) = \left(4x - \frac{1}{4}\right)^2$, and $d(fy, gx) = (4-x)^2$.

Here also

$$16(x-1)^2 + \frac{1}{2}\left[\left(4x - \frac{1}{4}\right)^2 + (4-x)^2\right] \geq 6\left(x - \frac{1}{4}\right)^2 + x^2 + \frac{225}{144},$$

which implies

$$16(x-1)^2 + \frac{1}{2}\left(4x - \frac{1}{4}\right)^2 + \frac{1}{2}\left[(4-x)^2 - 12\left(x - \frac{1}{4}\right)^2 - 2x^2 - \frac{225}{72}\right] \geq 0,$$

since

$$(4-x)^2 - 12\left(x - \frac{1}{4}\right)^2 - 2x^2 - \frac{225}{72} > 0 \text{ for } x \in [0, \tfrac{1}{4}).$$

Case (iv): $x, y \in (1,4]$, with $x \neq y$.

Here also we have $d(fx, fy) = 0$, $d(gx, gy) = 0$, $d(fx, gx) = 16$, $d(fy, gy) = 16$, $d(fx, gy) = 16$, and $d(fy, gx) = 16$.

Now again we get

$$0 + \frac{1}{2}[16 + 16] \geq 6(0) + \frac{1}{9}(16) + \frac{1}{9}(16),$$

since $16 \geq \frac{32}{9}$.

Case (v): $x \in \left(\frac{1}{4}, 1\right) \cup (1,4]$, and $y = \frac{1}{4}$.

Here also we have $d(fx, fy) = 0$, $d(gx, gy) = \frac{1}{16}$, $d(fx, gx) = 16$, $d(fy, gy) = \frac{225}{16}$, $d(fx, gy) = \frac{225}{16}$, and $d(fy, gx) = 16$.

Now again we get

$$0 + \frac{1}{2}\left[\frac{225}{16} + 16\right] \geq 6\left(\frac{1}{16}\right) + \frac{1}{9}\left(\frac{225}{16}\right) + \frac{1}{9}(16),$$

since $\frac{441}{32} \geq \frac{405}{144}$.

Thus in all the five cases for each $x \neq y$ the expansive condition (4.2.1) holds with $\alpha_1 = 6, \alpha_2 = \frac{1}{9} = \alpha_3$ and $\beta = 1$.

Here we observe the pair (f, g) is weakly compatible at $x = 0$, which is the only point of coincidence of f and g. Hence $x = 0$ is the unique common fixed point of f and g.

We note from case (v) that the expansive condition (1.2.1) in Theorem 1.2 fails to hold for any $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 > s$ for $s = 2$ when $x \in \left(\frac{1}{4}, 1\right) \cup (1,4]$ and $y = \frac{1}{4}$,

since $d(fx, fy) = 0, d(gx, gy) = \frac{1}{16}$, $d(fx, gx) = 16$ and $d(fy, gy) = \frac{225}{16}$ and hence the inequality (1.2.1) becomes $0 \geq \frac{1}{16}\alpha_1 + 16\alpha_2 + \frac{225}{16}\alpha_3$ which is absurd.

CHAPTER FIVE: CONCLUSION AND FUTURE SCOPE

5.1 CONCLUSION

In 2016, Mohanta established the sufficient conditions for existence of point of coincidence and common fixed point for pair of self-maps satisfying expansive type condition (1.2.1).

In this thesis, we established the sufficient condition for existence of point of coincidence and common fixed point, namely Theorem 4.2.1 for pair of self-maps satisfying the expansive type condition (1.2.2) in b-metric spaces.

Also, we have supported our main result of this research work by an example. Example 4.2.6 shows our work is more general than the main result of Mohanta (2016).

5.2 FUTURE SCOPE

Fixed point theory is one of active and vigorous areas of research in Mathematics and other sciences. There are several published results related to the existence of point of coincidence and common fixed point theorems for a pair of self-maps satisfying some expansive type condition in b-metric spaces.

The researcher believes that the search for the existence of point of coincidence and common fixed point for a pair of self-maps satisfying some expansive type condition in b-metric space is an active area of study. So, we recommend to the forthcoming post graduate students or any other interested researchers of the department of mathematics can exploit this opportunity and conduct their research work in this area.

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