THE REDUCED DIFFERENTIAL TRANSFORM METHOD FOR ONE DIMENSIONAL TIME FRACTIONAL CAHN- HILLIARD EQUATION

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[**Differential Equation**]

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Declaration

This is to certify that this thesis entitled "**Reduced Differential Transform Method For One dimensional Time Fractional Cahn-Hilliard Equation**" submitted in partial fulfillment of the requirement for the award of the Degree of Masters of Science (MSc) in Differential Equation to the school of Graduate studies, Jimma University, through the Department of Mathematics studies conducted by Mr. Bikila Biru .

To best of my knowledge and beliefs, the substances included under this thesis work has not been submitted for the fulfillment of any award qualification in earlier time.

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ACRONYMS

ADM - Adomain Decomposition Method

DTM - Differential Transform Method

FDE - Fractional Differential Equation

FPDEs - Fractional Partial Differential Equations

HPM - Homotopy Perturbation Method

IVPs - Initial Value Problems

PDEs - Partial Differential Equations

RDTM - Reduced Differential Transform Method

VIM - Variation Iteration Method

Abstract

The main purpose of this study is to develop a scheme to find analytic approximate solutions of initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equation by reduced differential transform method. The reduced differential transform method procedures for solving one dimensional homogeneous time fractional Cahn-Hilliard equation subjected to the initial condition are newly developed and introduced. The reduced and inverse reduced differential transformed functions in one dimension for solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equation are defined. Some theorems and Corollaries used in one dimension for solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equation are defined and proved. The time fractional Cahn-Hilliard equation is obtained from the standard Cahn-Hilliard equation by replacing the α integer order time derivative by a fractional derivative. The fractional derivative involved here is in sense of Caputo fractional derivatives, for its advantage that the initial conditions for fractional differential equations take the traditional form as for integer order differential equations. In order to show the reliability of the solutions, examples are constructed and 3D figures for some of the solutions are sketched.

Key Words: Reduced Differential Transform Method, One Dimensional Homogeneous Time Fractional Cahn-Hilliard Equation.

CHAPTER ONE

1. INTRODUCTION

1.1. Background of the study

Nonlinear partial differential equations are widely used to describe many important phenomena and dynamic processes in Physics, Mechanics, Chemistry, Biology, etc. The study of non- linear partial differential equations plays an important role in Physical Sciences and Engineering fields. The investigation of exact solutions of non- linear partial differential equations plays an important role in the study of non -linear physical phenomena. Many methods, exact, approximate, and purely numerical are available in the literature for the solution non-linear partial differentials [17].

Fractional calculus deals with fractional derivatives and integrals of any order. That is a generalization of ordinary (standard) differentiation and integration to arbitrary (non-integer) order [20]. Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number, or even number , powers of the differential operator

D = $\frac{d}{dx}$ and the integration operator [20].

Fractional derivatives provide an excellent instrument for the descriptive and hereditary properties of various materials and processes. So solving FPDEs is completely important in the circumstance of Applied Mathematics, Theoretical Physics and Engineering Sciences [26]. For further understanding of their practical application refer to [26].

Fractional order partial differential equations, as generalizations of classical integer order PDEs, have been used to model problems in fluid flow and other areas of differential equations, [25]. For example, in order to formulate certain electrochemical problems, half order derivatives and integrals are more useful than the classical models [25].

There are well known definitions of a fractional derivative and integrals of order, $\alpha > 0$ ($\alpha \varepsilon R$) such as Riemann- fractional Liouville, Grunwald- used Letnikow, Caputo and generalized functions approach from calculus . The most commonly used definitions are

Riemmann and Caputo. The Riemann- Liouville fractional derivative is mostly used by mathematicians but it is not suitable for physical problems of the real world since it requires the definition of fractional order initial conditions which have no physically meaningful explanation yet [25]. Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem. So, Caputo fractional derivative is the base for FDE with integer order initial conditions such as TFPDEs with integer order initial conditions [25],

Time fractional partial differential equations (TFPDEs) are differential equations which can be obtained from the standard partial differential equations by replacing the integer order time derivative by a fractional derivative. Some of these are time fractional heat equations, time fractional wave equations, time fractional telegraphic equations and so on and are represented by linear and nonlinear PDEs [11].

Several real phenomena emerging in engineering and science fields can be demonstrated successfully by developing model using the fractional calculus theory. The fractional differential theory has gained much more attention as the fractional order system response ultimately converges to the integer order equation. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential equations [29].

In recent past, the glorious developments have been envisaged in the field of fractional calculus and fractional differential equations .Differential equation involving fractional order derivatives are used to model a variety of systems of real world physical electrolyte polarization, heat conduction electromagnetic waves, diffusion equation, etc [1].

Mathematical approaches to partial differential equations are divided in to two methods called Analytical methods which strive to find exact formulae for the dependent variable as a function of independent variables and numerical methods which result in approximate values of dependent variable at prescribed and discrete location within a finite domain of the independent variables [23].

But, there are mathematical approaches which can be neither of the two methods. These are semi analytical and semi numerical methods. For example, reduced differential transform method

(RDTM) is semi analytical method and used to find exact solutions or closed approximate solution of a differential equation [27]. It is an iterative procedure for obtaining Taylor series solution of differential equation [26].

The RDTM was first introduced by a Turkish Mathematician Keskin.Y [16]. This method based on the use of the traditional DTM techniques. Usually, a few numbers of iteration needed of the series solution for numerical purposes to get high accuracy. The solution procedure of the RDTM is simpler than that of traditional DTM, and the amount of computation required in RDTM is much less than that in traditional DTM. The solution obtained by the RDTM is an infinite power series for initial value problems, which can be in turn, expressed in a closed form, the exact solution [16].

As in [18], RDTM can be successfully applied to solve telegraph and Cahn-Hilliard equations. But, nothing will discussed about how to solve IVPs of one dimensional homogeneous time fractional Cahn- Hilliard equations by applying reduced differential transform method in the existing literature. This motivated the researcher to choose this topic and fill the gap of the work of Mahmoud and Nazek [18].

Therefore, the main purpose of this study is to develop a scheme to find analytic approximate solutions of one dimensional homogeneous time fractional Cahn- Hilliard equations of form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2} u(x,t)}{\partial x^{2}} - u(x,t) + u^{3}(x,t) = 0 \quad , 0 < \alpha \le 1, \; t > 0.$$

Subjected to the initial condition, $u(x,0) = g(x)$, $x \in \Re$ *and* where α is a parameter that describe the order of time derivatives by fractional derivatives in sense of Caputo fractional derivatives.

1.2. Statement of the problem

Cahn-Hilliard equation can be used in a wide variety of engineering and mathematical physics applications [18], solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equations by applying reduced differential transform method is not presumably presented in the existing literature. As a result the objective of this paper is to fill the gap and mainly to answer the following questions.

1. How can we define the reduced and inverse reduced differential transformed function in 1D for solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equation by reduced differential transform method?
2. How can we apply reduced differential transform method (RDTM) to obtain analytic approximate solutions of initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard?
3. How can we construct supportive examples for solving IVPs of one dimensional homogeneous time fractional Cahn-Hilliard equation by reduced differential transform method ?

1.3. Objectives of the study

1.3.1. General objective

The general objective of the study is to develop a scheme to find analytic approximate Solutions of one dimensional homogeneous time fractional Cahn- Hilliard equation subject to the initial condition by reduced differential transform method (RDTM).

1.3.2. Specific objectives

The specific objectives of the study are:

- ✓ To define the reduced and inverse reduced differential transformed function in 1D for solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equation by reduced differential transform method.
- ✓ To apply reduced differential transform method to obtain analytic approximate solutions for IVPs of one dimensional homogeneous time fractional Cahn-Hilliard equation.
- ✓ To construct supportive examples for solving IVPs of one dimensional homogeneous time fractional Cahn-Hilliard equation by reduced differential transform method.

1.4 The significance of the Study

This research is considered of vital importance for the following reasons.

- ✓ It develops the researcher skill on mathematical (applied) research.
- ✓ It provides techniques of solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equations by using RDT method.
- ✓ It familiarize researcher with the scientific communication in mathematics.
- ✓ It was used as a reference material for anyone who will work on this area.

1.5 Delimitation of the study

The study is delimited to initial value problems of one dimensional homogeneous time fractional Cahn- Hilliard equation and focus only on developing a scheme to find analytic approximate solutions of one dimensional homogeneous time fractional Cahn-Hilliard equation subjected to the initial conditions by the RDT method in sense of Caputo fractional derivatives.

CHAPTER TWO

2 LITERATURE REVIEW

Partial differential equations (PDEs) have numerous essentials applications in various fields of science and engineering such as fluid mechanics, thermodynamic, heat transfer, Physics[15].The classical Taylor series method has been one of the earlier methods for solving the differential equations with an advent of high speed computers there has been an increasing trend towards exploring new ideas out of traditional techniques for the last couple of decades.In1986 an up dated version of Taylor series method, called the differential transform method (DTM) was introduced by [32] and then applied DTM in order to solve electric circuit.

In past several decades many authors mainly had paid attention to study the solution of fractional differential equations by using various developed method such as RDTM, VIM, DTM, ADM, Tanh-Coth method, and Sine-Cosine method. Among of these VIM, DTM, ADM, Tanh-coth method and sine –cosine method [18] used to solve non- linear partial differential equations (PDEs). Recently, researchers have applied the reduced differential transform method (RDTM) successfully to obtain analytic solution. For example:

Mahmoud Rawashden [17] used the RDTM, to find exact and approximate solution for Garden equation, Variant Non-linear Water Wave equation and the Fifth-order korteweg-de Vries(FKdv) equation. Bayram and Ibis [11] used the RDTM, to find approximate solution for the (KdvB) equation, Drinefel'd- sokolov- Wilson equations, Coupled Burgers equations and modified Boussinesq equation. Keskin .Y and Oturanc.G [16] used the RDTM, to solve linear and non-linear wave equations and they showed the effectiveness and accuracy of the proposed method. Saravanan and Magesha [24] used the RDTM and ADM, to solve analytic solution for linear and non-linear Newell-White head- Segel equation. Murat Gubes [23] used the RDTM, to obtain analytic solution for non-linear time-dependent Foam Drainage equations. Vinet Srivastava [20] used the RDTM, to obtain analytic solution of telegraph equation.

In 1998 the first analytical method the variation iteration method (VIM), was proposed by [21], to solve fractional differential equations and after it also used to solve more complex fractional differential equations such as linear and non-linear viscoelastic models with fractional derivatives, non-linear equations of fractional order, linear fractional partial differential equations arising in fluid mechanics and the fractional heat and wave like equations with variable coefficients.

In 2007, the Homotopy perturbation method (HPM) was applied to both non-linear and linear fractional differential equations and it was showed that HPM is an alternative analytical method for fractional differential equations. HPM also used to solve the fractional heat and wave like equations with variable coefficients [21].

In 2009 another improved approach for solving initial-value problem for partial differential equation, known as reduced differential transform method (RDTM) has recently been used by [14] and developed the reduced differential transform method for the fractional differential equations and showed that reduced differential transform method is the easily useable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations.

Some examples of Analytical methods are the Adomain decomposition method, Viration iteration method, Differential transform method, Homotopy perturbation method, Homotopy analysis method, Sine-Cosine method, Inverse scattering method, Balance method and Hirota's bilinear method[15].

The Cahn- Hilliard equation can be found in a wide variety of engineering and scientific applications. In recent years, numerous works have focused on the development of more advanced and efficient method for Cahn- Hilliard equations such as Differential transform method, extended fractional Ricatti Expansion method and Fractional sub-equation method. As reference [18], reduced differential transform method (RDTM) can be successfully applied to solve Cahn- Hilliard equations. However, how to solve initial value problems of one dimensional time fractional Cahn- Hilliard equations by applying RDTM is not presumably discussed in [18] and in other existing literature.

Therefore, this study is aimed to develop a scheme to find analytic approximate solutions of one dimensional time fractional Cahn- Hilliard equation subjected to the initial conditions by reduced differential transform method (RDTM).

CHAPTER THREE

3. METHODOLOGY

3.1 Study Site, Area and period

This study is conducted in Jimma University, under Department of Mathematics (Differential Equation Stream) from September,2014- September, 2015.

3.2 Study Design

The study design is Analytic design.

3.3 Source of information (data)

The information that is used to conduct this study is collected from secondary sources such as reference books, internets, published and unpublished research articles (Journals).

3.4 Procedures of the study

In order to achieve the objectives of the study, the following procedures are undertaken:

Step (I): Apply the reduced differential transform to the initial conditions.

Step (II): Apply the reduced differential transform to the one dimensional homogeneous time fractional Cahn-Hilliard equation to obtain a recursion system for the unknown function u_1(x), u_2 (x), u_3 (x)……

Step (III): Use the transformed initial conditions and solve the recursion system for the unknown functions u_1(x), u_2 (x), u_3 (x)…...

Step (IV): Use the differential inverse transform formula to obtain the analytic approximate solutions for the one dimensional homogeneous time fractional Cahn-Hilliard equation in infinite power series. That is, $u(x, t) = \sum_{k=0}^{\infty} u_k(x) t^{k\alpha} = u_0(x) + u_1(x)\, t^{\alpha} + u_2(x)\, t^{2\alpha} + \ldots$

3.5 Ethical Issues

For this study it needs books, Journals and other related materials, but there may be a problem for collecting all above listed materials without any permitted letters. So, the researcher needs to take a letter of permission from Mathematics department before going to collect data and have good approaches during data collection period.

CHAPTER FOUR

4. RESULTS AND DISCUSSION

4.1. Preliminaries

4.1.1. The Gamma Function

Definition 4.1.1. The gamma function, Γ (z) is defined [8] as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt, \quad z > 0 \qquad (1)$$

Properties of Gamma Function for, Z∈ℕ

(i) Γ (z+1)=z Γ(z)

(ii) Γ (z+1) = z !

(iii) Γ(1)=1, where z = 1

4.1.2. Fractional Calculus

In this section, some definitions and properties of the fractional calculus are given.

Definition 4.1.2.1 . Real function f (t), t > 0 is said to be in the space Cμ, μ∈ℝ , if there exists a real number p (>μ), such that f (t) = $t^{p} f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n , if and only if $f^{(n)}(f^{(n)} = \frac{d^n}{dx^n} f(t)) \in$ Cμ, n∈ℕ [5].

Definition 4. 1. 2. 2. The Riemann- Liouville fractional integral operator (J^{α}) of order α ≥ 0, of a function f ∈ c_μ, μ ≥ -1, is defined [9] as:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ \alpha > 0 \qquad (2)$$

Properties of the operator (J^{α}), for f ∈ c_μ, μ ≥ -1, α, β ≥ 0 and γ ≥ 0 were:

$$(i) \quad J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t) \qquad (3)$$

$$(ii) \quad J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t) \qquad (4)$$

$$(iii) \quad J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma} \qquad (5)$$

Definition 4.1.2.3.

For n to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0, in$ [31] defined as:

$$D^{\alpha}u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{0}^{t}[(t-s)^{n-\alpha-1}\frac{\partial^n}{\partial x^n}u(x,s)]ds\,, & n-1 < \alpha < n \\ \dfrac{\partial^{n}}{\partial t^{n}}\,u(x,t)\,, \alpha = n \in N \end{cases} \tag{7}$$

Properties of the operators α and β such that α, β > 0, n-1 < α < n and $\gamma > -1$.

(*i*) $J^{\alpha}J^{\beta}$f(t)= $J^{\alpha+\beta}$f(x) (8)

(*ii*) $J^{\alpha}(t-s)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-s)^{\alpha+\gamma}$ (9)

(*iii*) $(J^{\alpha}D^{\alpha}f)(t) = J^{n}D^{n}f(t) = f(t) - \sum_{k=0}^{n-1} J\ f^{(k)}(s)\frac{(t-s)}{k!}$, t > s (10)

4.1.3. Generalized Taylor Formula

In this section, we define generalized Taylor formula and before we see that mean value theorem.

Theorem 4.1.3.1. [Mean value theorem]

Suppose $f(x) \in c([a,b])$ and $D_{a}^{\alpha}\ f(x) \in c([a,b]),\ for\, 0 < \alpha \leq 1$, in [19] we have

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)}\left(D_{a}^{\alpha} f(\xi)\right)(x-a)^{\alpha} \tag{11}$$

with $0 \leq \xi \leq x, \forall x \in [a,b]\, and\ \ D^{\alpha}$ is the Caputo fractional derivative of order $\alpha > 0$.

Proof: By (2), we have;

$$J_{a}^{\alpha}D_{a}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-s)^{\alpha-1} D_{a}^{\alpha} f(s)ds, \quad \alpha > 0. \tag{12}$$

Using, integral mean value theorem, $\int_a^b f(x)dx = f(\xi)(b-a)$, we have

$$J_a^{\alpha} D_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} D_a^{\alpha} f(\xi) \int_a^x (x-s)^{\alpha-1} ds,$$

$$= \frac{1}{\Gamma(\alpha)} D_a^{\alpha} f(\xi)(x-a)^{\alpha} \text{, for } 0 \leq \xi \leq 0 \quad (13)$$

Also, from (10), we have:

$$\left(J_a^{\alpha} D_a^{\alpha} f(x)\right) = f(x) - f(a) \quad (14)$$

Lastly, from (13) and (14), we have:

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D_a^{\alpha} f(\xi)(x-a)^{\alpha}$$

Theorem 4.1.3.2. Suppose that $\left(D_a^{\alpha}\right)^n f(x), \left(D_a^{\alpha}\right)^{n+1} f(x) \in c(a,b]$, for $0 < \propto \leq 1$, [19] *we have*

$$\left(J_a^{n\alpha} \left(D_a^{\alpha}\right)^n f\right)(x) - \left(J_a^{(n+1)\alpha} \left(D_a^{\alpha}\right)^{n+1} f\right)(x) = \frac{(x-a)^{na}}{\Gamma(n\alpha+1)} \left(\left(D_a^{\alpha}\right)^n f\right)(a) \quad (15)$$

, where $\left(D_a^{\alpha}\right)^n = D_a^{\alpha}.D_{a.}^{\alpha} ....D_a^{\alpha}\left(n - times D_a^{\alpha}\right)$

Proof: From (15), we have:

$$\left(J_a^{n\alpha} \left(D_a^{\alpha}\right)^n f\right)(x) - \left(J_a^{(n+1)\alpha} \left(D_a^{\alpha}\right)^{n+1} f\right)(x) = \left(J_a^{n\alpha} \left(D_a^{\alpha}\right)^n f\right)(x) - \left(J_a^{n\alpha} J_a^{\alpha} D_a^{n\alpha} D_a^{\alpha} f\right)(x)$$

$$= J_a^{n\alpha} \left(D_a^{\alpha}\right)^n \left[f(x) - \left(J_a^{\alpha} D_a^{\alpha} f\right)(x)\right]$$

$$= J_a^{n\alpha} \left(D_a^{\alpha}\right)^n \left[f(x) - \left(J_a^{\alpha} D_a^{\alpha}\right) f(x)\right]$$

By (14), $\left(J_a^{\alpha} D_a^{\alpha} f\right)(x) = f(x) - f(a)$

$$= J_a^{n\alpha}\left(D_a^{\alpha}\right)^n \left[f(x) - \left(f(x) - f(a)\right)\right]$$

$$= J_a^{n\alpha}\left(D_a^{\alpha}\right)^n f(a)$$

Using (9), $J_a^{n\alpha}\left(D_a^{\alpha}\right)^n f(a) = \dfrac{\Gamma(0+1)}{\Gamma(n\alpha+0+1)}(t-a)^{0+n\alpha}\left(\left(D_a^{\alpha}\right)^n f\right)(a)$

$$= \frac{1}{\Gamma(n\alpha+1)}(t-a)^{n\alpha}\left(\left(D_a^{\alpha}\right)^n f\right)(a)$$

$$= \frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)}\left(\left(D_a^{\alpha}\right)^n f\right)(a)$$

Hence, $\left(J_a^{n\alpha}\left(D_a^{\alpha}\right)^n f\right)(x) - \left(J_a^{(n+1)\alpha}\left(D_a^{\alpha}\right)^{n+1} f\right)(x) = \dfrac{(x-a)^{na}}{\Gamma(n\alpha+1)}\left(\left(D_a^{\alpha}\right)^n f\right)(a)$

Theorem 4.1.3.3. [Generalized Taylor's Formula]

Suppose $\left(D_a^{\alpha}\right)^k f(x) \in C(a,b],\ for k = 0,1,2,\ldots,n+1,\ for,\ 0 < x \leq 1$, then

$$f(x) = \sum_{i=1}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^i f\right)(a) + \frac{\left(D_a^{\alpha}\right)^{n+1} f(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \quad (16)$$

with a$\leq \xi \leq x, \forall x \in (a,b]$[19].

Proof: From (15), we have:

$$\left(J_a^{i\alpha}\left(D_a^{\alpha}\right)^i f\right)(x) - \left(J_a^{(i+1)\alpha}\left(D_a^{\alpha}\right)^{i+1} f\right)(x) = \frac{(x-a)^{ia}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^i f\right)(a)$$

$$\sum_{i=1}^{n}\left(J_a^{i\alpha}\left(D_a^{\alpha}\right)^i f\right)(x) - \left(J_a^{(i+1)\alpha}\left(D_a^{\alpha}\right)^{i+1} f\right)(x) = \sum_{i=1}^{n}\frac{(x-a)^{ia}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^i f\right)(a)$$

$$f(x) - \left(J_a^{(n+1)\alpha}\left(D_a^{\alpha}\right)^{n+1} f\right)(x) = \sum_{i=1}^{n}\frac{(x-a)^{ia}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^i f\right)(a) \quad (17)$$

By (2), $\left(J_a^{(n+1)\alpha}\left(D_a^{\alpha}\right)^{n+1} f\right)(x)=\frac{1}{\Gamma((n+1)\alpha+1)}\int_a^x (x-t)^{(n+1)\alpha}\left(\left(D_a^{\alpha}\right)^{n+1} f\right)(t)dt$ and also by integral mean value theorem, $\left(J_a^{(n+1)\alpha}\left(D_a^{\alpha}\right)^{n+1} f\right)(x)=\frac{1}{\Gamma((n+1)\alpha+1)}\left(D_a^{\alpha}\right)^{n+1} f(\xi)\int_a^x (x-t)^{(n+1)\alpha} dt$

$$\left(J_a^{(n+1)\alpha}\left(D_a^{\alpha}\right)^{n+1} f\right)(x)=\frac{1}{\Gamma((n+1)\alpha+1)}\left(D_a^{\alpha}\right)^{n+1} f(\xi)(x-a)^{(n+1)\alpha} \tag{18}$$

Substituting (18) in to (17), we have:

$$f(x)-\frac{1}{\Gamma((n+1)\alpha+1)}\left(D_a^{\alpha}\right)^{n+1} f(\xi)(x-a)^{(n+1)\alpha}=\sum_{i=1}^{n}\frac{(x-a)^{ia}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^{i} f\right)(a)$$

Hence, $f(x)=\sum_{i=1}^{n}\frac{(x-a)^{ia}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^{i} f\right)(a)+\frac{1}{\Gamma((n+1)\alpha+1)}\left(D_a^{\alpha}\right)^{n+1} f(\xi)(x-a)^{(n+1)\alpha}$

In particularly, if $\alpha = 1$, the generalized Taylor's formula reduces to a classical Taylor's formula, which is:

$$f(x)=\sum_{i=1}^{n}\frac{(x-a)^{i}}{\Gamma(i+1)}\left(\left(D_a\right)^{i} f\right)(a)+\frac{1}{\Gamma((n+1)+1)}\left(D_a\right)^{n+1} f(\xi)(x-a)^{(n+1)}$$

$$f(x)=\sum_{i=1}^{n}\frac{(x-a)^{i}}{i!}\left(\left(D_a\right)^{i} f\right)(a)+\frac{1}{(n+1)!}\left(D_a\right)^{n+1} f(\xi)(x-a)^{(n+1)} \tag{19}$$

The radius of convergence, R for generalized Taylor's series, $f(x)=\sum_{i=1}^{n}\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(\left(D_a^{\alpha}\right)^{i} f\right)(a)$, is given by R= $|t-a|^{\alpha}\lim_{n\to\infty}\left|\frac{\Gamma(n\alpha+1)}{\Gamma((n\alpha+1)\alpha+1)}\frac{\left(\left(D_a^{\alpha}\right)^{n+1} f\right)(a)}{\left(\left(D_a^{\alpha}\right)^{n} f\right)(a)}\right|$ (20)

4.1.4.The Reduced Differential Transform Method (RDTM)

Consider a function of two variables f(x, t) and suppose that it can be represented as a product of two single-variable functions, i.e., f(x, t) = g(x) h(x). Based on the properties of differential transform function f(x, t) can be represented as f(x, t) = $\sum_{i=0}^{\infty} G(i)x^i \sum_{j=o}^{\infty} H(j)t^j = \sum_{k=0}^{\infty} f_k(x)t^k$ and

f_k (x) is called t-dimensional spectrum function of f (x, t) [16].

The basic definition of the reduced differential transform and inverse reduced differential transform in [13,14,15,16,24,27,28,29] are discussed below.

Definition 4.1.4.1. If f(x, t) is analytic and continuously differentiable with respect to the space variable x and time variable t in the domain of interest, then the spectrum function (reduced transform function) was defined in [13,14, 15,16,24,27,28,29] as :

$$R_D[f(x,t)] = f_k(\text{x}) = \frac{1}{\Gamma(\text{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} \quad (21)$$

Where, R_D - is the reduced differential transform operator.

f_k(x)- is the transformed function (reduced transformed function).

Note: In the above definition, particular if $\alpha = 1$, we have:

$$f_k(\text{x}) = \frac{1}{\Gamma(\text{k}+1)}\left[\frac{\partial^{k}}{\partial t^{k}} f(x,t)\right]_{t=t_o} \quad (22)$$

and by gamma function property (ii), $\Gamma(\text{k} + 1)$ = k !

Hence, $f_k(\text{x}) = \frac{1}{\text{k!}}\left[\frac{\partial^{k}}{\partial t^{k}} f(x,t)\right]_{t=t_o}$ (23)

Définition 4.1.4.2. If f (x, t) is analytic and continuously differentiable with respect to the space variable x and time variable t in the domain of interest, then the inverse reduced differential transformed function was defined in [13,14, 15,16,24,27,28,29] as :

$$R_D^{-1}[f_k(\mathrm{x})] = f(x,t) = \sum_{k=0}^{\infty} f_k(\mathrm{x})\, t^{k\alpha} \tag{24}$$

Where, $R_D^{-1} -$ Donates the inverse reduced differential transform operator.

f(x, t) - is the inverse reduced differential transformed function.

$\alpha -$ is the order of time derivatives by fractional derivatives in sense of Caputo fractional derivatives.

Substituting equation (21) into equation (24), we have:

$$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t) \right]_{t=t_o} t^{k\alpha} \tag{25}$$

Note: In the above definition, particularly if $\alpha = 1$, (25) becomes:

$$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} \left[\frac{\partial^{k}}{\partial t^{k}} f(x,t) \right]_{t=t_0} t^{k}$$ and by gamma function property(ii), $\Gamma(k+1) = k!$

Hence, $$f(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^{k}}{\partial t^{k}} f(x,t) \right]_{t=t_0} t^{k} \tag{26}$$

Definitions 4.1.4.1 and 4.1.4.2 are stated in [13,14,24,26] for solving fractional linear heat equations, linear and Non-linear Newell-white head-segel equations and time fractional non-linear evolution equations having time fractional derivative of order, α such that $0< \alpha \leq 1$ respectively . These definitions are also stated in [16,27,29] for solving linear and Non-linear wave equation and analytical approximations of two and three dimensional time fractional telegraphic equation respectively such that $0< \alpha \leq 2$ .But, the definitions of the reduced differential transformed function (t-dimensional spectrum function) and the inverse reduced differential transformed functions are not defined in work of Mahmoud S. and Nazek A.

Some of the fundamental theorems in one dimensional performed by reduced differential transformed method [18] are discussed below.

Let $f(x,t), u(x,t)$ and $v(x,t)$ be analytical and k-times continuously differentiable functions with respect to the space variable x and time variable t ,then the following theorems holds.

Theorem 4.1.4.1. If $f(x,t) = u(x,t)$, then f_k(x)= u_k(x)

Proof: Suppose f_k(x) and u_k(x) are the t-dimensional spectrum functions (transformed functions) of f(x,t) and u (x, t) respectively.

Aim: we want to show, f_k (x) = u_k(x)

Applying the reduced differential transform operator RDT, on both sides of $f(x,t) = u(x,t)$, we have: RDT $[f(x,t)]$ = RDT $[u(x,t)]$ (27)

By definition 4.1.4.1, RDT $[f(x,t)] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o}$ and

RDT $[u(x,t)] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o}$ (28)

Now substituting (28) into (27), we have:

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} \quad (29)$$

Also by definition 4.1.4.1,

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = f_k(x) \text{ and } \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} = u_k(x) \quad (30)$$

substituting(30) *into* (29), we have

$$f_k(x) = u_k(x).$$

Theorem 4.1.4.2. If $f(x,t)$ = u(x, t) ± v(x, t), then f (x)= u_k(x) ± v_k(x)

Proof: suppose f_k(x), u_k(x), and v_k(x) are the t-dimensional spectrum functions (transformed functions) of f(x, t), u (x, t) and v(x,t) respectively.

Aim: we want to show, f_k(x) = u_k(x) ± v_k(x)

Applying reduced differential transform operator RDT, on both side of f_k(x,t)= u_k(x,t) ± v_k(x,t), we have:

$$\text{RDT}\left[f(x,t)\right] = \text{RDT}\left[f(x,t)\right] \pm \text{RDT}\left[v(x,t)\right] \tag{31}$$

By definition 4.1.4.1, RDT $\left[f(x,t)\right] = \frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o}$

RDT $\left[u(x,t)\right] = \frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o}$ and

$$\text{RDT}\left[v(x,t)\right] = \frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} v(x,t)\right]_{t=t_o} \tag{32}$$

Now, substituting (32) into (31), we have:

$$\frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = \frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} \pm \frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} v(x,t)\right]_{t=t_o} \tag{33}$$

$also\ by\ definition$ 4.1.4.1

$\frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = f_k(\mathrm{x})$, $\frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} = u_k(\mathrm{x})$ and

$$\frac{1}{\Gamma(\mathrm{k}\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} v(x,t)\right]_{t=t_o} = v_k(\mathrm{x}) \tag{34}$$

Substituting (34) into (33), we have: f_k(x)= u_k(x) ± v_k(x).

Theorem 4.1.4.3. If $f(x,t)= \alpha\ u(x,t)$, where α is constant, then f_k(x)= α u_k(x)

Proof: suppose f_k(x) and u_k (x) are the t-dimensional spectrum functions (transformed functions) of f(x, t) and u (x, t) respectively and α be a constant.

Aim: we want to show, $f_k(x) = \alpha\ u_k(x)$

Applying reduced differential transform operator, on both side $f(x,t) = \alpha u(x,t)$, we have

$$\text{RDT } [f(x,t)] = \text{RDT } [\alpha u(x,t)] = \alpha \text{ RDT } [u(x,t)] \tag{35}$$

By definition 4.1.4.1, we get, RDT $[f(x,t)] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o}$ and

$$\alpha \text{ RDT } [u(x,t)] = \frac{\alpha}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} \tag{36}$$

Now, Substituting (36) into (35), we have:

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = \frac{\alpha}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} \tag{37}$$

By also definition 4.1.4.1 *, we have*:

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = f_k(\text{x, t}),\quad \frac{\alpha}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=t_o} = \alpha\ u_k(\text{x}) \tag{38}$$

Substituting (38) into (37), we have:

$$f_k(\text{x}) = \alpha\ u_k(\text{x}).$$

Theorem 4.1.4.4. If $f(x,t) = \frac{\partial^n}{\partial x^n} u(x,t)$, then $f_k(\mathrm{x}) = \frac{\partial^n}{\partial x^n} u_k(\mathrm{x})$

Proof: suppose $f_k(\mathrm{x})$ and $u_k(\mathrm{x})$ are the t-dimensional spectrum functions (transformed function) of $f(x,t)$ and $u(x,t)$ respectively.

Aim: we want to show, $f_k(\mathrm{x}) = \frac{\partial^n}{\partial x^n} u_k(\mathrm{x})$

Applying reduced differential transform operator RDT on $f(x,t) = \frac{\partial^n}{\partial x^n} u$ (x,t), we have

$$\text{RDT } [f(x,t)] = \text{RDT} \left[\frac{\partial^n}{\partial x^n} u(x,t) \right] \tag{39}$$

By definition 4.1.4.1, we have, RDT $[f(x,t)] = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t) \right]_{t=t_o}$ and

$$\text{RDT}\left[\frac{\partial^n}{\partial x^n} u(x,t) \right] = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^n}{\partial x^n} \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_o}$$

$$= \frac{\partial^n}{\partial x^n} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_o} \tag{40}$$

Now substituting (40) into (39), we have:

$$\frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t) \right]_{t=t_o} = \frac{\partial^n}{\partial x^n} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_o} \tag{41}$$

$$\frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t) \right]_{t=t_o} = f_k \text{ (x)} \;\; and \;\; \frac{\partial^n}{\partial x^n} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t) \right]_{t=t_o} = \frac{\partial^n}{\partial x^n} u_k \text{ (x)} \tag{42}$$

Substituting (42) into (41), we have: $f_k(\mathrm{x}) = \frac{\partial^n}{\partial x^n} u_k(\mathrm{x})$.

Corollary 4.1.4.4.

(i).If $f(x,t) = \frac{\partial}{\partial_X} u(x, t)$, then $f_k(x) = \frac{\partial}{\partial_X} u_k(x)$ (43)

(ii).If $f(x,t) = \frac{\partial^2}{\partial_{x^2}} u(x, t)$, then $f_k(x) = \frac{\partial^2}{\partial_{x^2}} u_k(x)$ and … (44)

To prove the above corollaries 4.1.4.4 (i) and 4.1.4.4 (ii), we follow that the prove of theorem 4.1.4.4

Theorem 4.1.4.5. If $f(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)$, then $f_k(x) = \frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} u_{k+N}(x)$

Proof: suppose $f_k(x)$ is the t-dimensional spectrum function (transformed function) of $f(x,t)$.

Aim: we want to show, $f_k(x) = \frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} u_{k+N}(x)$

Applying reduced differential transform operator RDT on $f(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)$, we get

$$\text{RDT}\left[f(x,t)\right] = \text{RDT}\left[\frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)\right] \tag{45}$$

By definition 4.1.4.1, we have

$$\text{RDT}\left[f(x,t)\right] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right] \text{ and}$$

$$\text{RDT}\left[\frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)\right] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{K\alpha}}{\partial t^{K\alpha}}\left(\frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x,t)\right)\right] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{K\alpha+N\alpha}}{\partial t^{K\alpha+N\alpha}} u(x,t)\right]_{t=to} \tag{46}$$

Multiplying on right side of (46) by $\frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(K\alpha + N\alpha + 1)}$, we have:

$$\text{RDT}\left[\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)\right] = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(K\alpha+N\alpha+1)}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{K\alpha+N\alpha}}{\partial t^{K\alpha+N\alpha}}u(x,t)\right]_{t=to}$$

$$= \frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\left[\frac{1}{\Gamma(K\alpha+N\alpha+1)}\frac{\partial^{K\alpha+N\alpha}}{\partial t^{K\alpha+N\alpha}}u(x,t)\right]_{t=to}$$

$$= \frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\left[\frac{1}{\Gamma(K\alpha+N\alpha+1)}\frac{\partial^{(K+N)\alpha}}{\partial t^{(K+N)\alpha}}u(x,t)\right]_{t=to}$$

$$\text{RDT}\left[\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)\right] = \frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\left[\frac{1}{\Gamma(K\alpha+N\alpha+1)}\frac{\partial^{(K+N)\alpha}}{\partial t^{(K+N)\alpha}}u(x,t)\right]_{t=to} \quad (47)$$

By definition 4.1.4.1, equations (46) and (47) becomes, $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}f(x,t)\right]_{t=t_o} = f_k(x)$

and $\frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\left[\frac{1}{\Gamma((K+N)\alpha+1)}\frac{\partial^{(K+N)\alpha}}{\partial t^{(K+N)\alpha}}u(x,t)\right]_{t=to} = \frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}u_{k+N}(\mathrm{x})$ (48)

where $u_{k+N}(\mathrm{x}) = \frac{1}{\Gamma((N+k)\alpha+1)}\left[\frac{\partial^{(K+N)\alpha}}{\partial t^{(K+N)\alpha}}u(x,t)\right]_{t=to}$

Lastly, substituting (48) into (45), we have

$$f_k(\mathrm{x}) = \frac{\Gamma(N\alpha+k\alpha+1)}{\Gamma(k\alpha+1)}u_{k+N}(\mathrm{x})$$

Corollary 4.1.4.5

(i).If $f(x,t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}}u(\mathrm{x,t})$, then $f_k(\mathrm{x}) = \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(\mathrm{x})$, where, N=1 (49)

(ii).If $f(x,t) = \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(\mathrm{x,t})$, *then* $f_k(\mathrm{x}) = \frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}u_{k+2}(\mathrm{x})$, where, N=2 (50)

To prove the above corollaries 4.1.4.5 (i) and 4.1.4.5 (ii), we follow the above theorem 4.1.4.5

Theorem 4.1.4.6. If $f(x,t) = u(x,t)v(x,t)$,then f_k(x) = $\sum_{i=0}^{k} u_i(x)V_{k-i}(x)$

Proof: suppose f_k(x), u_k(x) and v_k(x) are the t-dimensional spectrum functions (transformed functions) of $f(x,t)$, $u(x,t)$ and $v(x,t)$respectively.

Aim : we want to show f_k(x) = $\sum_{i=0}^{k} u_i(x)V_{k-i}(x)$

Applying the reduced differential transform operator RDT on $f(x,t) = u(x,t)v(x,t)$, we have:

$$RDT[f(x,t)] = RDT[u(x,t)v(x,t)] \tag{51}$$

By definition 4.1.4.1, we have $RDT[f(x,t)] = \frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_0}$

$$\text{and } RDT[u(x,t)v(x,t)] = \frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{K\alpha}}{\partial t^{k\alpha}} u(x,t)v(x,t)\right]_{t=t_0} \tag{52}$$

Substituting (52) in to (51), we have:

$$\frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_0} = \frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{K\alpha}}{\partial t^{k\alpha}} u(x,t)v(x,t)\right]_{t=t_0} \tag{53}$$

Also by definition 4.1.4.1, we have:

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x,t)\right]_{t=t_o} = f_k(x) \text{ and } \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)v(x,t)\right]_{t=t_o} = \sum_{r=0}^{K} u_r(x)v_{K-r}(x) \tag{54}$$

Lastly, substituting (54) in to (53), we have: $f_K(x) = \sum_{r=0}^{K} u_r(x)v_{K-r}$ (55)

Corollary 4.1.4.6. If $f(x,t) = u^3(x,t)$, then $f_k(x) = \sum_{i=0}^{k}\sum_{j=0}^{i} u_{i-j}(x)u_j(x)u_{k-i}(x)$ (56)

To prove corollary 4.1.4.6 , we follow the above theorem 4.1.4.6

4.2 Main Results

To the best of my knowledge no other researcher have developed a scheme to find exact analytic approximate solutions of one dimensional homogenous time fractional Cahn-Hilliard equation subjected to the initial condition by reduced differential transform method (RDTM). Due to this, the gap of work of Mahmoud S. Rawahden and Nazek A, Obeidat [18], were filled by the researcher to develop a scheme to find exact analytic approximate solutions of one dimensional homogenous time fractional Cahn-Hilliard equation subject to the initial condition by reduced differential transform method (RDTM).

Based on the above definitions and theorems on this paper the main result of the researcher is presented below.

4.2.1 Reduced differential transform method procedures for solving analytic approximate solutions of 1D homogeneous time fractional Cahn –Hilliard equations

Under this section, the reduced differential transform method procedures for solving one dimensional homogeneous time fractional Cahn –Hilliard equation of the following form is newly introduced and developed .

Consider one dimensional homogeneous time fractional Cahn –Hilliard equation of form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2} u(x,t)}{\partial x^{2}} - u(x,t) + u^{3}(x,t) = 0,\ 0 < \alpha \leq 1,\ t > 0 \qquad (57)$$

Subject to the initial condition, $u(x,0) = g(x),\ x \in \mathbb{R}$ (58)

Where α is a parameter that describes the order of time derivatives by fractional derivatives in sense of Caputo fractional derivatives .

Step (i): Applying the reduced differential transform operator RDT on both sides of the problem equation [57] and [58], we have:

$$\text{RDT}\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)\right]=\text{RDT}\left[\frac{\partial^{2}}{\partial x^{2}}u(x,t)\right]+\text{RDT}[\,u(x,t)]-RDT[u^{3}(x,t)] \tag{59}$$

And RDT [$u(x,o)$= g (x)] (60)

Step (ii): By corollary 4.1.4.5 (i): RDT $[\frac{\partial^{\alpha}}{\partial x^{\alpha}}\mathrm{u}(x,t)] = \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x)$, by corollary 4.1.4.4 (ii): RDT $[\frac{\partial^{2}U(x,t)}{\partial x^{2}}] = \frac{\partial^{2}}{\partial x^{2}}u_{k}(x)$, by theorem 4.1.4.1: RDT $[u(x,t)] = u_k(x)$ and by corollary 4.1.4.6:RDT $[U^{3}(x,t)] = \sum_{i=o}^{k}\sum_{j=o}^{i}u_{i-j}(x)u_{j}(x)u_{k-i}(x) = F_k(x)$ in [3], where $F_k(x)$ is the transformed values of $U^{3}(x,t)$ and we get the following iteration formulae.

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x) = \frac{\partial^{2}}{\partial x^{2}}u_{k}(x) + u_{k}(x) - F_{k}(x),\ 0<\alpha\leq 1,\ t>0 \tag{61}$$

And $u_o(x) = g(x)$, x∈ ℝ (62)

Step (iii): Substituting (62) into (61) by direct forward, we get the $u_k(x)$ values, $\forall_k = 1,2,3....$

$$For\ K=0,\frac{\Gamma(\alpha+1)}{\Gamma(1)}u_{1}(x)=\frac{\partial^{2}}{\partial x^{2}}u_{0}(x)+u_{0}(x)\_F_{0}(x),and\ F_{0}(x)=u_{0}^{3}(x)$$

$$=\frac{\partial^{2}}{\partial x^{2}}u_{0}(x)+u_{0}(x)-u_{0}^{3}(x)$$

$$u_{1}(x) = \frac{\Gamma(1)}{\Gamma(\alpha+1)}\left[\frac{\partial^{2}}{\partial x^{2}}u_{0}(x)+u_{0}(x)-u_{0}^{3}(x)\right]$$

$$For\ K=1,\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}u_{2}(x)=\frac{\partial^{2}}{\partial x^{2}}u_{1}(x)+u_{1}(x)-F_{1}(x),where,F_{1}(x)=3u_{0}^{2}(x)u_{1}(x)$$

$$=\frac{\partial^{2}}{\partial x^{2}}u_{1}(x)+u_{1}(x)-3u_{0}^{2}(x)u_{1}(x)$$

$$=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_1(x)+u_1(x)-3u_0^{\,2}(x)u_1(x)\right]$$

$$u_2(x)=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_1(x)+u_1(x)-3u_0^{\,2}(x)u_1(x)\right]$$

$$For\ \ K=2,\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}u_3(x)=\frac{\partial^2}{\partial x^2}u_2(x)+u_2(x)-F_2(x), where, F_2(x)=3u_0(x)u_1^{\,2}(x)+3u_0^{\,2}(x)u_2(x)$$

$$=\frac{\partial^2}{\partial x^2}u_2(x)+u_2(x)-\left(3u_0(x)u_1^{\,2}(x)+3u_0^{\,2}(x)u_2(x)\right)$$

$$=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_2(x)+u_2(x)-\left(3u_0(x)u_1^{\,2}(x)+3u_0^{\,2}(x)u_2(x)\right)\right]$$

$$u_3(x)=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_2(x)+u_2(x)-\left(3u_0(x)u_1^{\,2}(x)+3u_0^{\,2}(x)u_2(x)\right)\right],\ldots$$

Step (iv): By definition 4.1.4.2, we have;

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x)t^{k\alpha}$$

$$=u_0(x)+u_1(x)t^{\alpha}+u_2(x)t^{2\alpha}+.....$$

$$=u_o(x)+\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left[\frac{\partial^2}{\partial x^2}u_0(x)+u_0(x)-u_0^{\,3}(x)\right]t^{\alpha}+\ldots \quad (63)$$

Particularly for, $\alpha=1$ (63), becomes;

$$u(x,t)=u_o(x)+\left[\frac{\partial^2}{\partial x^2}u_0(x)+u_0(x)-u_0^{\,3}(x)\right]t^{\alpha}+\ldots \quad (64)$$

4.3 Supportive Examples

In this section, the reduced differential transform method (RDTM) to find the analytic approximate solutions of one dimensional homogeneous time fractional Cahn-Hilliard equation subject to the initial conditions is applied.

Example 4.3.1: Consider 1D homogeneous non-linear time fractional Cahn-Hilliard equation,

$$\frac{\partial^\alpha}{\partial t^\alpha} \text{u(x, t)} - \frac{\partial^2}{\partial x^2} \text{u(x, t)} - u(x,t) + u^3(x,t) = 0, \ \text{o} < \alpha \leq 1 , t > 0 \tag{65}$$

$$\textit{subjected to the initial condition}: u(x,t) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}} \tag{66}$$

Solution: Applying the reduced differential transform operator RDT on (65) and (66), we have:

$$RDT\left[\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2}(u(x,t)) + u(x,t) - u^3(x,t)\right] \tag{67}$$

And,

$$RDT\left[u(x,t) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\right] \tag{68}$$

By corollary 4.1.4.5 (i): we have $RDT\left[\frac{\partial^\alpha}{\partial t^\alpha} u(x,t)\right] = \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} u_{k+1}(x)$, by Corollary 4.1.4.4(ii): we have $RDT\left[\frac{\partial^2}{\partial t^2} u(x,t)\right] = \frac{\partial^2}{\partial x^2} u_k(x)$, also by theorem 4.1.4.1:

$$\text{RDT [u(x, t)]} = u_K(x) \textit{ and by corollary } 4.1.4.6: RDT\left[u^3(x,t)\right] = F_k(x) \tag{69}$$

$$\text{And} \quad u_0(x) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}} \tag{70}$$

Substituting (69) into (67), we have:

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}u_{k+1}(x)=\frac{\partial^2}{\partial x^2}u_k(x)+\ u_k\,(x)-\ F_k(x) \tag{71}$$

Also, substituting (70) into (71), we get $u_k(x), \forall_k = 1,2,3,.....$

$$For K = 0, \frac{\Gamma(\alpha+1)}{\Gamma(1)}u_1(x)=\frac{\partial^2}{\partial x^2}u_0(x)+\ u_0\,(x)-\ F_0(x)$$

Where $F_0(x) = u_0{}^3(x) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}$ *and* $u_0(x) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}$

$$\frac{\Gamma(\alpha+1)}{\Gamma(1)}u_1(x)=\frac{\partial^2}{\partial x^2}\left(\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\right)+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}$$

$$=\frac{e^{\frac{2x}{\sqrt{2}}}-e^{\frac{x}{\sqrt{2}}}+2+4e^{\frac{x}{\sqrt{2}}}+2e^{\frac{2x}{\sqrt{2}}}-2}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}$$

$$=\frac{3e^{\frac{2x}{\sqrt{2}}}+3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}=\frac{3e^{\frac{x}{\sqrt{2}}}\left(1+e^{\frac{x}{\sqrt{2}}}\right)}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}$$

$$=\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}$$

$$\Gamma(\alpha+1)u_1(x)=\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}$$

$$u_1(x)=\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2\Gamma(\alpha+1)}$$

$$For K=1, \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}u_2(x)=\frac{\partial^2}{\partial x^2}u_1(x)+u_1(x)-F_1(x)=\frac{\partial^2}{\partial x^2}u_1(x)+u_1(x)-3u_0{}^2(x)u_1(x)$$

$$=\frac{\partial^2}{\partial x^2}\left(\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2\Gamma(\alpha+1)}\right)+\frac{3e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2\Gamma(\alpha+1)}-\frac{9e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4\Gamma(\alpha+1)}$$

$$\Gamma(2\alpha+1)u_2(x)=\frac{9e^{\frac{x}{\sqrt{2}}}\left(e^{\frac{2x}{\sqrt{2}}}-1\right)}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}$$

$$u_2(x)=\frac{9e^{\frac{x}{\sqrt{2}}}\left(e^{\frac{2x}{\sqrt{2}}}-1\right)}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4\Gamma(2\alpha+1)}$$

.

.

.

Applying inverse differential transform formula and by definition 4.1.4.2, we have:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x)t^{k\alpha}=u_0(x)+u_1(x)t^{\alpha}+u_2(x)t^{2\alpha}+.......$$

$$u(x,t)=\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)}+\frac{3\left(e^{\frac{x}{\sqrt{2}}}\right)t^{\alpha}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{2}\Gamma(\alpha+1)}+\frac{9e^{\frac{x}{\sqrt{2}}}\left(e^{\frac{2x}{\sqrt{2}}}-1\right)t^{2\alpha}}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{4}\Gamma(2\alpha+1)}+\ldots\ldots\ldots\ldots \tag{72}$$

which is the general analytic approximate solution of the problems (65) and (66) in infinite power series.

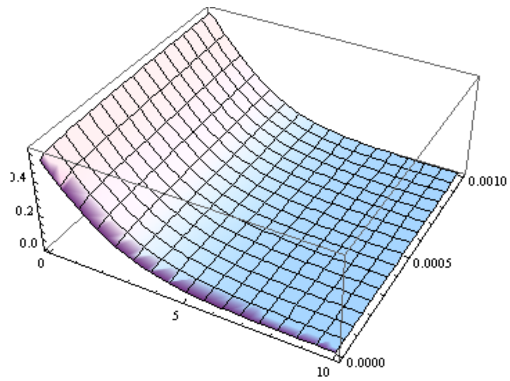
In particularly if, $\alpha=1$ from (72) we have:

$$u(x,t)=\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)}+\frac{3\left(e^{\frac{x}{\sqrt{2}}}\right)t}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{2}\Gamma(1+1)}+\frac{9e^{\frac{x}{\sqrt{2}}}\left(e^{\frac{2x}{\sqrt{2}}}-1\right)t^{2}}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{4}\Gamma(2+1)}+\ldots\ldots\ldots\ldots$$

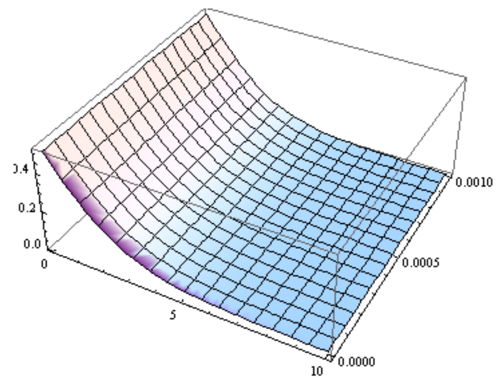
$$u(x,t)=\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)}+\frac{3\left(e^{\frac{x}{\sqrt{2}}}\right)t}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{2}1!}+\frac{9e^{\frac{x}{\sqrt{2}}}\left(e^{\frac{2x}{\sqrt{2}}}-1\right)t^{2}}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{4}2!}+\ldots\ldots\ldots\ldots \tag{73}$$

which is also the analytic approximate solutions of the problems (65) and (66) in infinite power series.

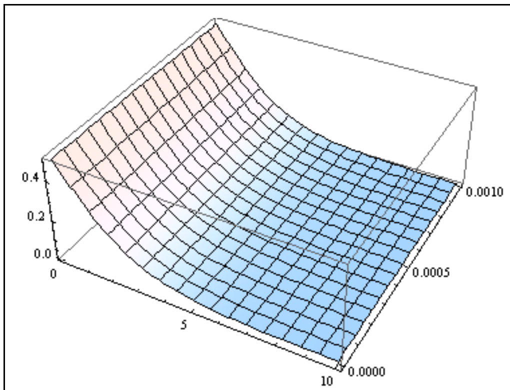
The 3D plot of solution $for\ \alpha = 0.25, \alpha = 0.5,\ \ \alpha = 0.75\ , and\ \alpha = 1$ are shown as follow.



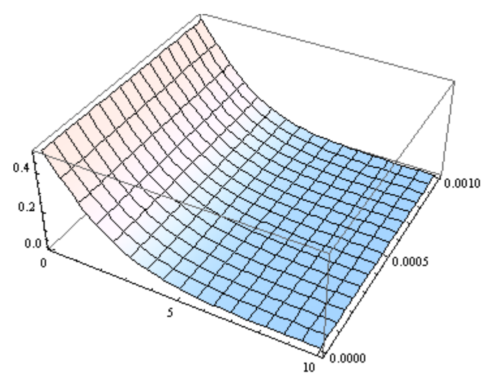
For $\alpha = 0.25$



For $\alpha = 0.5$



For $\alpha = 0.75$



For $\alpha = 1$

Fig 1: The 3D plot solution for example 4.3.1 when $0 < x < 10$ and $0.0000 < t < 0.0010$

Example 4.3.2: Consider one dimensional homogeneous time fractional Cahn-Hilliard equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{2} u(x,t)}{\partial x^{2}} + u(x,t) - u^{3}(x,t), \quad \text{o} < \alpha \leq 1 \, , t > 0 \tag{74}$$

Subject to the initial condition, $u(x,0) = e^{x}$ (75)

Solution: Applying the reduced differential transform operator RDT on both sides of problems (74) and (75), we have: $RDT\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \frac{\partial^{2}}{\partial x^{2}}(u(x,t)) + u(x,t) - u^{3}(x,t)\right]$ (76)

And,

$$RDT\left[u(x,t) = e^{x}\right] \tag{77}$$

By Corollary 4.1.4.5 (i): RDT$\left[\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right] = \frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} u_{k+1}(x)$, by Corollary 4.1.4.4 (ii):

RDT$\left[\frac{\partial^{2} u(x,t)}{\partial x^{2}}\right] = \frac{\partial^{2}}{\partial x^{2}} u_{k}(x)$, by theorem 4.1.4.1: RDT$[u(x,t)] = u_{k}(x)$ and by Corollary 4.1.4.6:

RDT$\left[u^{3}(x,t)\right] = F_{k}(x)$ (78)

And,

RDT[u(x, t) $= e^{x}$] $\Rightarrow$ $u_0(x) = e^{\chi}$ (79)

Substituting (78) into (76), we get: $\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} u_{k+1}(x) = \frac{\partial^{2}}{\partial x^{2}} u_{k}(x) + u_{k}(x) - F_{k}(x)$ (80)

Substituting (79) into (80), we get $u_{k}(x), \forall_{k=1,2,3......}$

For K=0, $\frac{\Gamma(\alpha + 1)}{\Gamma(1)} u_{1}(x) = \frac{\partial^{2}}{\partial x^{2}} u_{0}(x) + u_{0}(x) - F_{0}(x)$, where $u_{0}(x,0) = e^{x}$ and $F_{0}(x) = e^{3x}$

$$\frac{\Gamma(\alpha + 1)}{\Gamma(1)} u_{1}(x) = \frac{\partial^{2}}{\partial x^{2}}(e^{x}) + e^{x} - e^{3x} = 2e^{x} - e^{3x}$$

And by gamma function property (iii), $\Gamma(1)=1!$

$$\frac{\Gamma(\alpha+1)}{1!}u_1(x) \quad = 2e^x - e^{3x}$$

$$u_1(x) = \frac{2e^x - e^{3x}}{\Gamma(\alpha+1)}$$

For k=1, $\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}u_2(x) = \frac{\partial^2}{\partial x^2}u_1(x) + u_1(x) - F_1(x)$, where $F_1(x) = 3u_0{}^2(x)u_1(x)$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{2e^x - e^{3x}}{\Gamma(\alpha+1)}\right) + \frac{2e^x - e^{3x}}{\Gamma(\alpha+1)} - 3e^{2x}\left(\frac{2e^x - e^{3x}}{\Gamma(\alpha+1)} \right)$$

$$= \frac{2e^x - 9e^{3x}}{\Gamma(\alpha+1)} + \frac{2e^x - e^{3x}}{\Gamma(\alpha+1)} - \frac{3e^{2x}\left(2e^x - e^{3x}\right)}{\Gamma(\alpha+1)}$$

$$= \frac{2e^x - 9e^{3x} + 2e^x - e^{3x} - 6e^{3x} + 3e^{5x}}{\Gamma(\alpha+1)}$$

$$\Gamma(2\alpha+1)u_2(x) \quad = 4e^x - 16e^{3x} + 3e^{5x}$$

$$u_2(x) = \frac{4e^x - 16e^{3x} + 3e^{5x}}{\Gamma(2\alpha+1)}$$

.

.

Applying inverse transform formulae and by definition 4.1.4.2, we have;

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x)t^{k\alpha} = u_0(x) + u_1(x)t^{\alpha} + u_2(x)t^{2\alpha} + \ldots\ldots$$

$$= e^x + \left(\frac{2e^x - e^{3x}}{\Gamma(\alpha+1)} \right)t^{\alpha} + \left(\frac{4e^x - 16e^{3x} + 3e^{5x}}{\Gamma(2\alpha+1)} \right)t^{2\alpha} + \ldots. \qquad (81)$$

In particularly if $\alpha = 1$, (91) becomes:

$$u(x,t) = e^x + \left(\frac{2e^x - e^{3x}}{\Gamma(1+1)} \right) t + \left(\frac{4e^x - 16e^{3x} + 3e^{5x}}{\Gamma(2+1)} \right) t^2 + \ldots$$

And, by gamma function property (ii), we have: $\Gamma(1+1) = 1!, \Gamma(2+1) = 2!, \ldots\ldots$

Hence, $$u(x,t) = e^x + \left(\frac{2e^x - e^{3x}}{1!} \right) t + \left(\frac{4e^x - 16e^{3x} + 3e^{5x}}{2!} \right) t^2 + \ldots \qquad (82)$$

which is also the analytic approximate solutions of the problems (74) and (75) in infinite power series.

Example 4.3.3: Consider one dimensional homogeneous time fractional Cahn-Hilliard equation

$$\sin^2 x \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - u^3(x,t), \quad 0 < \alpha \leq 1, t > 0 \qquad (83)$$

Subject to the initial condition, $u(x,0) = \sin x$ (84)

Solution: Applying the reduced differential transform operator RDT on both sides of problems (83) and (84), we have: $RDT\left[\sin^2 x \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^2}{\partial x^2}(u(x,t)) + u(x,t) - u^3(x,t) \right]$ (85)

And,

$$RDT[u(x,t) = \sin x] \qquad (86)$$

By Corollary 4.1.4.5(i): RDT $\left[\sin^2 x \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \right] = \sin^2 x \frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} u_{k+1}(x)$, by Corollary 4.1.4.4 (ii): RDT $\left[\frac{\partial^2 u(x,t)}{\partial x^2} \right] = \frac{\partial^2}{\partial x^2} u_k(x)$, by theorem 4.1.4.1: RDT $[u(x,t)] = u_k(x)$ and by Corollary 4.1.4.6: RDT$[u^3(x,t)] = F_k(x)$ (87)

And,

RDT[u(x, t) =sinx] $\Rightarrow \quad u_0(x) = \sin x$ (88)

Substituting (87) into (85), we get:

$$\sin^2 x \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} u_{k+1}(x) = \frac{\partial^2}{\partial x^2} u_k(x) + u_k(x) - F_k(x) \qquad (89)$$

Substituting (88) into (89), we get $u_k(x), \forall_{k=1,2,3......}$

For K=0, $\sin^2 x \frac{\Gamma(\alpha+1)}{\Gamma(1)} u_1(x) = \frac{\partial^2}{\partial x^2} u_0(x) + u_0(x) - F_0(x)$, where $u_0(x,0) = \sin x$ and $F_0(x) = \sin^3 x$

$$\sin^2 x \frac{\Gamma(\alpha+1)}{\Gamma(1)} u_1(x) = \frac{\partial^2}{\partial x^2}(\sin x) + \sin x - \sin^3 x = -\sin^3 x$$

And by gamma function property (iii), $\Gamma(1) = 1!$

$$\sin^2 x \frac{\Gamma(\alpha+1)}{1!} u_1(x) \quad = -\sin^3 x$$

$$u_1(x) = \frac{-\sin x}{\Gamma(\alpha+1)}$$

For k=1, $\sin^2 x \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} u_2(x) = \frac{\partial^2}{\partial x^2} u_1(x) + u_1(x) - F_1(x)$, where $F_1(x) = 3u_0{}^2(x) u_1(x)$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{-\sin x}{\Gamma(\alpha+1)}\right) - \frac{\sin x}{\Gamma(\alpha+1)} - 3\sin^2 x\left(\frac{-\sin x}{\Gamma(\alpha+1)}\right)$$

$$= \frac{\sin x}{\Gamma(\alpha+1)} - \frac{\sin x}{\Gamma(\alpha+1)} + \frac{3\sin^3 x}{\Gamma(\alpha+1)}$$

$$= \frac{3\sin^3 x}{\Gamma(\alpha+1)}$$

$$\sin^2 x \Gamma(2\alpha+1) u_2(x) \quad = 3\sin^3 x$$

$$u_2(x) = \frac{3\sin x}{\Gamma(2\alpha+1)}$$

.

.

Applying inverse transform formulae and by definition 4.1.4.2, we have ;

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x) t^{k\alpha} = u_0(x) + u_1(x) t^{\alpha} + u_2(x) t^{2\alpha} + \ldots\ldots$$

$$= \sin x - \left(\frac{\sin x}{\Gamma(\alpha+1)} \right) t^{\alpha} + \left(\frac{3\sin x}{\Gamma(2\alpha+1)} \right) t^{2\alpha} + \ldots. \tag{90}$$

which is the general analytic approximate solutions of the problems (83) and (84) in infinite power series.

In particularly if $\alpha = 1$, (90) becomes:

$$u(x,t) = \sin x - \left(\frac{\sin x}{\Gamma(1+1)} \right) t + \left(\frac{3\sin x}{\Gamma(2+1)} \right) t^2 + \ldots.$$

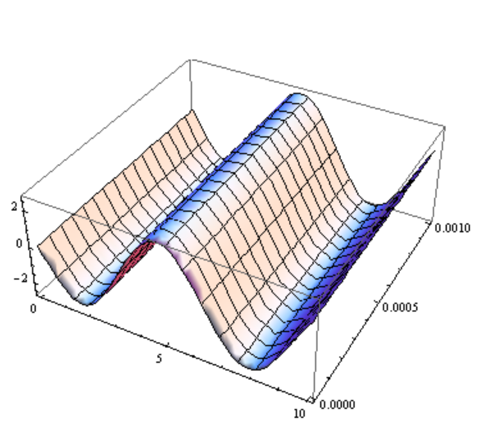
And, gamma function property (ii), we have: $\Gamma(1+1) = 1!, \Gamma(2+1) = 2!, \ldots\ldots..$

Hence, $$u(x,t) = \sin x - \left(\frac{\sin x}{1!} \right) t + \left(\frac{3\sin x}{2!} \right) t^2 + \ldots. \tag{91}$$

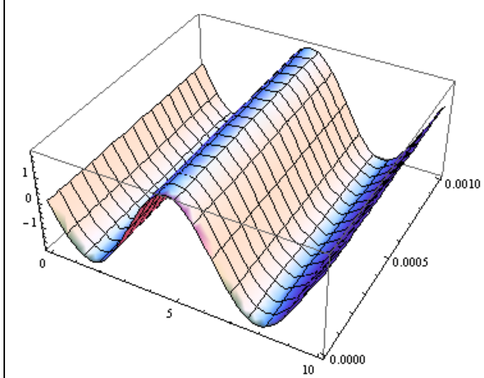
$$= \sin x \left[1 - \frac{t}{1!} + \frac{3t^2}{2!} - \ldots\ldots\ldots \right]$$

which is the analytic approximate solutions of the problems (83) and (84) in infinite power series.

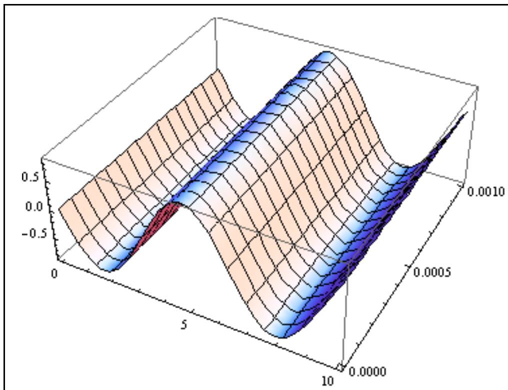
The 3D plot of the solution for example 4.3.3 in domain $x \in \Re$ for $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 1$ are shown below.



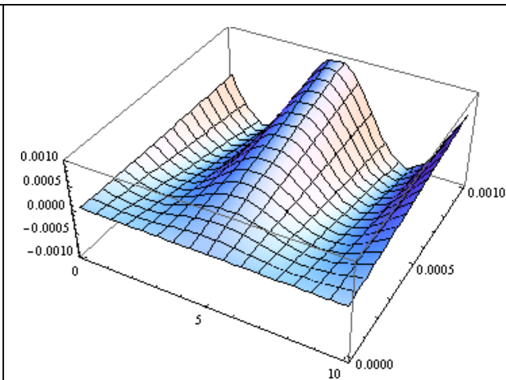
For $\alpha = 0.25$



For $\alpha = 0.5$



For $\alpha = 0.75$



For $\alpha = 1$

Fig 2: The 3D plot solution for example 4.3.3 when $0 < x < 10$ and $0.0000 < t < 0.0010$

CHAPTER FIVE

5 CONCLUSION AND FUTURE SCOPE

In this study, the reduced differential transform method (RDTM) is proposed to solve the exact analytic approximate solution of non- linear initial value problems of one dimensional homogeneous time fractional Cahn Hilliard equation in sense of Caputo fractional derivatives. The reduced and inverse reduced differential transformed function in one dimension for solving initial value problems of one dimensional homogeneous time fractional Cahn-Hilliard equation are defined. Six mathematical operations (theorems) with some corollaries are used for solving exact analytic approximate solutions of one dimensional homogeneous time fractional Cahn-Hilliard equation subject to the initial condition by using definitions of reduced and inverse reduced differential transformed function are given and proved. The procedures of solving exact analytic approximate solutions of one dimensional homogeneous time fractional Cahn-Hilliard equation subject to the initial condition by reduced differential transform method is newly developed and introduced. The solution are obtained in infinite power series. Thus, we conclude that the proposed method is very effective, simple and can be applied to other non -linear partial differential equations models in area of Physics and Engineering. My future proposed is applying the method used to solve another non-linear time fractional partial differential equation .

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