Renewal input infinite buffer batch service queue with single exponential working vacation and accessibility to batches

P. Vijaya Laxmi*

Department of Applied Mathematics,
Andhra University,
Visakhapatnam, 530 003, Andhra Pradesh, India
E-mail: vijaya_iit2003@yahoo.co.in
*Corresponding author

Obsie Mussa Yesuf

Department of Mathematics,
Andhra University,
Visakhapatnam, 530 003, Andhra Pradesh, India
E-mail: yesufobsie@yahoo.com

Abstract: This paper considers an infinite buffer single server batch service queue with single exponential working vacation policy. The inter-arrival times are generally independent and identically distributed random variables and the service times are exponential. The server accesses new arrivals even after service has started on any batch of initial number a. This operation continues till the service time of the ongoing batch is completed or the maximum accessible limit d of the batch being served is attained whichever occurs first. The supplementary variable technique and the recursive method are used to develop the steady-state queue length distributions at pre-arrival and arbitrary epochs. Some performance measures and numerical results are discussed.

Keywords: accessible batch; non-accessible batch; supplementary variable; batch service; single vacation; working vacation.

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Biographical notes: P. Vijaya Laxmi is an Assistant Professor in the Department of Applied Mathematics, Andhra University, Visakhapatnam, India. She received her MSc and PhD from Indian Institute of Technology, Kharagpur, India, in 1995 and 2003, respectively. Her main areas of research interest are continuous and discrete-time queueing models and their applications. She has publications in various journals like *Operations Research Letters*, *Queueing Systems*, *Applied Mathematical Modelling*, etc.

Obsie Mussa Yesuf is a Lecturer in Jimma University, Ethiopia. He received his MSc in Mathematics from Addis Ababa University, Ethiopia, in 2000.

Currently, he is pursuing his PhD in the Department of Mathematics, Andhra University, India, since 2007. His research interests are stochastic modelling and queueing theory.

1 Introduction

Batch service queues have been studied extensively by many researchers owing to their practical applicability in modelling and analysis of complex communication networks, manufacturing and production systems, lift operations, cargo loading and unloading problems, etc. In many telecommunication systems, it is frequently observed that the server processes the packets in groups. For example, consider an ATM multiplexer with multiple input links where each input link may serve messages that consist of several packets. In semiconductor manufacturing processes, in service mechanism of a web server and computer operating systems, jobs are frequently processed in batches whose size usually varies depending on the total number of jobs accumulated. Neuts (1967) proposed the general batch service rule in which service begins only when a certain number of customers are available. Extensive studies on batch service queues are found in Chaudhry and Templeton (1983) and Medhi (1991).

There is another class of queueing systems known as server vacation models, which have been studied over the past few decades and are applied widely in several areas as mentioned earlier. The server stops service and goes for vacation once the system capacity drops to a specified level say zero. On return from vacation, if the server finds the system still at zero level, then

- it goes for another vacation and continues in this manner until it finds at least one waiting customer upon return from a vacation
- it remains idle till a customer arrives.

The former one is known as Multiple Vacation policy (MV) and the later one as Single Vacation policy (SV). The vacation period may be utilised to carry out additional works. For more details on this topic, one can refer to the comprehensive survey by Doshi (1986) and the monographs of Tian and Zhang (2006) and Takagi (1993) and the references therein.

Unlike the classical vacation models, in working vacation model, customers are served during the vacation period at a rate that is generally different and lower than that of the normal service rate. This concept of Working Vacation (WV) was introduced by Servi and Finn (2002) for $M/M/1/MWV$ model with a motivation to analyse a reconfigurable Wavelength-Division Multiplexing (WDM) optical access network.

In this paper, we focus on an infinite buffer single server queue with accessible and non-accessible batch services and with single exponential working vacations, i.e., $GI/M^{(a,d,b)}/1/\infty/SWV$ queue. The inter-arrival time of customers and service time of batches are respectively, arbitrarily and exponentially distributed. The supplementary variable technique is used to develop the steady-state equations treating the remaining inter-arrival time as the supplementary variable. A simple recursive method has been developed to obtain the steady-state distributions of the number in the system (queue) both at pre-arrival and arbitrary epochs. Performance measures of the system are presented in the form of tables and graphs. This queueing model has applications in the field of communication systems, manufacturing systems, computer networks, polling systems, cinema theatres and other such related areas.

The rest of this paper is organised as follows. Section 2 presents a brief literature review. In Section 3, we present the description of the model along with necessary notations. Formulation and solution of the model are given in Section 4. Performance measures and some special cases are demonstrated in Sections 5 and 6, respectively. Numerical results in the form of tables and graphs are illustrated in Section 7. Conclusions follow at the end.

2 Literature review

Considerable amount of research work is available on batch service queues. Gold and Tran-Gia (1993) have analysed a single server finite capacity queue with general batch service rule where customers arrive according to a Poisson process and service times of the batches are arbitrarily distributed. They have obtained the distribution of number of customers in the queue at departure and arbitrary epochs using embedded Markov chain technique. The relationships between departure and arrival epoch probabilities in a single server batch service queue have been studied by Hébuterne and Rosenberg (1999), where the arrivals and service times are generally distributed. The finite buffer continuous-time queues with general arrivals and batch service have been studied by Vijaya Laxmi and Gupta (1999) using both the supplementary variable and the embedded Markov chain techniques and queue length distribution at various epochs has been obtained.

Batch service may be with accessible batches (AB). If a batch being served does not employ its full capacity for service, late arrivals may join the ongoing service as long as the number in that service batch is less than a pre-defined threshold $d(a \leq d < b)$. The service time of the batch is not changed by inclusion of such arriving customers in course of ongoing service. Such batch is said to be accessible batch. However, if the number in the service batch exceeds d, the batch becomes non-accessible for the late arriving customers and such a batch is called non-accessible (NAB) batch. This has been considered by Gross (2008), Kleinrock (1975) and Medhi (1991). The infinite buffer queue with accessible and non-accessible batch service rule has been studied by Sivasamy (1990), where the arrival and service times are exponentially distributed. The finite and infinite buffer queues with accessible and non-accessible batch service rule in discrete-time systems have been studied by Goswami et al. (2006), by considering the arrivals and service times as geometrically distributed. Sivasamy and Pukazhenthi (2009) have analysed the discrete time batch service queue with accessible batch with the arrivals and service times as geometrically and negative binomially distributed, respectively. Recently, Goswami and Sikdar (2010) have presented the analysis of the discrete-time finite buffer batch service queue with accessible and non-accessible batches using the recursive method wherein arrivals occur according to a general process and service times of the batches are geometrically distributed.

Vacation queues have also gained notable attention owing to their wide applications in transportation, computer communication networks, etc. Tian (1993) studied the $GI/M/1$ queue with single exponential vacation. The infinite buffer single server batch service queue with multiple vacations has been analysed by Choi and Han (1994). Chae et al. (2006) have discussed stochastic decomposition in $GI/M/1$ queue by deriving the probability-generating function of the stationary queue length and the Laplace-Stieltjes Transform (LST) of the stationary sojourn time. Batch arrival batch service queue with single and multiple exponential vacations has been discussed by Sikdar and Gupta (2008).

Queues with working vacation are quite different from classical vacation models. During working vacation, customers are served by the server generally with a slower rate. Therefore, the working vacation models have more complicated modalities and their analysis is more difficult than classical vacation queues. Baba (2005) studied the $GI/M/1$ queue with multiple working vacations and obtained the stationary queue length distributions and waiting time. Wu and Takagi (2006) have considered the $M/G/1/MWV$ queue and generalised the work of Servi and Finn (2002). The $GI/M/1$ queue with exhaustive service discipline and multiple working vacations has been studied by Banik et al. (2007) using the supplementary variable and embedded Markov chain techniques to obtain system size distributions at pre-arrival and arbitrary epochs. The $M/M/1$ queue with single working vacation using the quasi birth-death process and matrix-geometric solution method has been discussed in Tian et al. (2008). Zhao et al. (2009) studied the $GI/M/1$ queue with set-up period and working vacation and vacation interruption using matrix-geometric solution method. The $GI/M/1$ and $GI/Geo/1$ queues both with single working vacation have been studied by Chae et al. (2009) and derived the steady-state distribution of the number of customers in the system.

Recently, Yesuf and Vijaya Laxmi (2009) have analysed the infinite buffer single server accessible and non-accessible batch service queue with multiple exponential working vacations. Using the supplementary variable technique and a simple recursive method, they have derived the steady-state distribution of number of customer in the queue at pre-arrival and arbitrary epochs. In this paper, the main purpose is to do both analytic and computational analysis of $GI/M^{(a,d,b)}/1/\infty/SWV$ queue with accessible and non-accessible batch service, which have importance both from theoretical and applied point of view.

3 Description of the model

Let us consider an infinite buffer single server accessible and non-accessible batch service queue with single working vacations. The inter-arrival times are independent and identically distributed random variables with probability distribution function $A(u)$, probability density function $a(u), u \geq 0$, LST $A^*(\theta)$ $(Re(\theta) \geq 0)$ and the mean inter-arrival time is $1/\lambda = -A^{*(1)}(0)$. The customers are served exponentially with parameter μ by a single server in batches whose minimum and maximum sizes are a and b, respectively. However, if the number of customers in the queue is less than the minimum threshold value a, the server takes single exponential working vacation with parameter ϕ.

During any single working vacation period, the customers are served one by one exponentially at a rate η (say), which is generally different and lower than its normal service rate μ. The arrival times, vacation times and service times are mutually independent of each other. Furthermore, to start service in the working vacation period the server requires a minimum of a customers, otherwise it will wait until a minimum of a customers accumulate in the system and then start service one by one or wait till the single working vacation period ends whichever occurs first. On return from the single working vacation, if the server finds n $(0 \leq n \leq a-1)$ customers in the system, it enters into the idle period. On the other hand, if on return from the single working vacation there are $n(n \geq a)$ customers waiting in the queue, it begins to serve them with the normal service rate μ according to batch service rule. Moreover, the service interrupted at the end of the single working vacation restarts from the beginning. If b or more customers are present in the queue at service initiate or single working vacation completion epoch, then only b of them are taken into service and the rest will wait in the queue.

It is further assumed that the late entries can join a batch in course of ongoing service as long as the number of customers in that batch is less than $d < b$ (called maximum accessible limit).

At every departure epoch of service, the server may find the system in any one of the following three cases:

- i $0 \leq n \leq a-1$
- ii $a \leq n \leq d-1$
- iii $n \geq d$.

In case (i), it takes single working vacation. In case (ii), the server takes the entire queue for batch service and admits the subsequent arrivals in the batch while the service is on, till the accessible limit d is reached, and such a batch is called an accessible batch. In case (3), it takes min(n, b) customers for the service and does not allow further arrivals into the batch being served even if the current batch size is not b, i.e., when the batch size is greater than or equal to d, the batch becomes non-accessible for late arriving customers. If b or more customers are present in the queue at service initiate or single working vacation completion epoch, then only b of them are taken into service and the rest of the customers will wait in the queue. The traffic intensity is given by $\rho = \lambda / b\mu < 1$.

The state of the system at time t is described by the following random variables, namely

- $N_s(t) =$ number of customers present in the system including those in service
- $N_q(t) =$ number of customers present in the queue not counting those in service
- $U(t) =$ remaining inter-arrival time for the next arrival
- $\zeta(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is on working vacation,} \\ 2, & \text{if the server is busy with an accessible batch,} \\ 3, & \text{if the server is busy with a non-accessible batch.} \end{cases}$

Let us define the joint probabilities by

$$R_n(u,t)\mathrm{d}u = P(N_s(t) = n, \quad u < U(t) \leq u + \mathrm{d}u, \quad \zeta(t) = 0), \quad u \geq 0, \quad 0 \leq n \leq a-1,$$

$$P_{n,0}(u,t)\mathrm{d}u = P(N_s(t)=n, \quad u < U(t) \leq u+\mathrm{d}u, \quad \zeta(t)=1), \quad u \geq 0, \quad n \geq 0,$$
$$Q_{n,0}(u,t)\mathrm{d}u = P(N_s(t)=n, \quad u < U(t) \leq u+\mathrm{d}u, \quad \zeta(t)=2), \quad u \geq 0, \quad a \leq n \leq d-1,$$
$$Q_{n,1}(u,t)\mathrm{d}u = P(N_q(t)=n, \quad u < U(t) \leq u+\mathrm{d}u, \quad \zeta(t)=3), \quad u \geq 0, \quad n \geq 0.$$

These probabilities in steady state, i.e., as $t \to \infty$ are denoted by $R_n(u)$, $P_{n,0}(u)$, $Q_{n,j}(u)$ and their LSTs are $R_n^*(\theta)$, $P_{n,0}^*(\theta)$, $Q_{n,j}^*(\theta)$, $j = 0, 1$, respectively.

4 Formulation and solution of the model

In this section, we determine the distribution of number of customers in the system (queue) at various epochs using the supplementary variable technique and the recursive method. The former technique is used to develop the steady-state equations treating the remaining inter-arrival time as the supplementary variable while the latter one is used to obtain the steady-state distribution of the number in the system (queue) both at pre-arrival and arbitrary epochs.

Relating the states of the system at two consecutive time epochs t and $t + dt$, using definitions and probabilistic arguments, we have in steady state

$$-\frac{\mathrm{d}}{\mathrm{d}u}R_0(u) = \phi P_{0,0}(u), \tag{1}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}R_n(u) = \phi P_{n,0}(u) + a(u)R_{n-1}(0), \quad 1 \leq n \leq a-1, \tag{2}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{0,0}(u) = -\phi P_{0,0}(u) + \mu Q_{0,1}(u) + \mu\sum_{k=a}^{d-1} Q_{k,0}(u), \tag{3}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{n,0}(u) = -\phi P_{n,0}(u) + \mu Q_{n,1}(u) + a(u)P_{n-1,0}(0), \quad 1 \leq n \leq a-2, \tag{4}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{a-1,0}(u) = -\phi P_{a-1,0}(u) + \mu Q_{a-1,1}(u) + \eta P_{a,0}(u) + a(u)P_{a-2,0}(0), \tag{5}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}P_{n,0}(u) = -(\phi+\eta)P_{n,0}(u) + a(u)P_{n-1,0}(0) + \eta P_{n+1,0}(u), \quad n \geq a, \tag{6}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}Q_{a,0}(u) = -\mu Q_{a,0}(u) + \phi P_{a,0}(u) + \mu Q_{a,1}(u) + a(u)R_{a-1}(0), \tag{7}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}Q_{n,0}(u) = -\mu Q_{n,0}(u) + \phi P_{n,0}(u) + \mu Q_{n,1}(u) + a(u)Q_{n-1,0}(0), \quad a+1 \leq n \leq d-1, \tag{8}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}Q_{0,1}(u) = -\mu Q_{0,1}(u) + \phi\sum_{k=d}^{b} P_{k,0}(u) + \mu\sum_{k=d}^{b} Q_{k,1}(u) + a(u)Q_{d-1,0}(0), \tag{9}$$
$$-\frac{\mathrm{d}}{\mathrm{d}u}Q_{n,1}(u) = -\mu Q_{n,1}(u) + \mu Q_{n+b,1}(u) + a(u)Q_{n-1,1}(0) + \phi P_{n+b,0}(u), \quad n \geq 1, \tag{10}$$

where $R_n(0),\ 0 \leq n \leq a-1,\ P_{n,0}(0),\ n \geq 0,\ Q_{n,0}(0),\ a \leq n \leq d-1$ and $Q_{n,1}(0)$, $n \geq 0$ are the respective rates of arrivals.

Multiplying equations (1)–(10) by $e^{-\theta u}$ and integrating with respect to u from 0 to ∞ yields

$$-\theta R_0^*(\theta) = \phi P_{0,0}^*(\theta) - R_0(0), \tag{11}$$

$$-\theta R_n^*(\theta) = \phi P_{n,0}^*(\theta) + A^*(\theta)R_{n-1}(0) - R_n(0), \quad 1 \le n \le a-1, \tag{12}$$

$$(\phi-\theta)P_{0,0}^*(\theta) = \mu Q_{0,1}^*(\theta) + \mu \sum_{k=a}^{d-1} Q_{k,0}^*(\theta) - P_{0,0}(0), \tag{13}$$

$$(\phi-\theta)P_{n,0}^*(\theta) = \mu Q_{n,1}^*(\theta) + A^*(\theta)P_{n-1,0}(0) - P_{n,0}(0), \quad 1 \le n \le a-2, \tag{14}$$

$$(\phi-\theta)P_{a-1,0}^*(\theta) = \mu Q_{a-1,1}^*(\theta) + \eta P_{a,0}^*(\theta) + A^*(\theta)P_{a-2,0}(0) - P_{a-1,0}(0), \tag{15}$$

$$(\phi+\eta-\theta)P_{n,0}^*(\theta) = A^*(\theta)P_{n-1,0}(0) + \eta P_{n+1,0}^*(\theta) - P_{n,0}(0), \quad n \ge a, \tag{16}$$

$$(\mu-\theta)Q_{a,0}^*(\theta) = \phi P_{a,0}^*(\theta) + \mu Q_{a,1}^*(\theta) + A^*(\theta)R_{a-1}(0) - Q_{a,0}(0), \tag{17}$$

$$(\mu-\theta)Q_{n,0}^*(\theta) = \phi P_{n,0}^*(\theta) + \mu Q_{n,1}^*(\theta) + A^*(\theta)Q_{n-1,0}(0) - Q_{n,0}(0), \quad a+1 \le n \le d-1, \tag{18}$$

$$(\mu-\theta)Q_{0,1}^*(\theta) = \phi \sum_{k=d}^{b} P_{k,0}^*(\theta) + \mu \sum_{k=d}^{b} Q_{k,1}^*(\theta) + A^*(\theta)Q_{d-1,0}(0) - Q_{0,1}(0), \tag{19}$$

$$(\mu-\theta)Q_{n,1}^*(\theta) = \mu Q_{n+b,1}^*(\theta) + \phi P_{n+b,0}^*(\theta) + A^*(\theta)Q_{n-1,1}(0) - Q_{n,1}(0), \quad n \ge 1. \tag{20}$$

Adding equations (11)–(20), and taking limit as $\theta \to 0$ and using the normalisation condition

$$\sum_{n=0}^{a-1} R_n + \sum_{n=0}^{\infty} P_{n,0} + \sum_{n=a}^{d-1} Q_{n,0} + \sum_{n=0}^{\infty} Q_{n,1} = 1,$$

we get

$$\sum_{n=0}^{a-1} R_n(0) + \sum_{n=0}^{\infty} P_{n,0}(0) + \sum_{n=a}^{d-1} Q_{n,0}(0) + \sum_{n=0}^{\infty} Q_{n,1}(0) = \lambda. \tag{21}$$

The left-hand side of equation (21) denotes mean number of entrances into the system per unit time and is obviously equal to mean arrival rate λ.

4.1 Steady-state distribution at pre-arrival epoch

Let R_n^- be the probability that n $(0 \le n \le a-1)$ customers waiting in the system at pre-arrival epoch and the server is idle and $P_{n,0}^-$ be the probability that n $(n \ge 0)$ customers are in the system at pre-arrival epoch and the server is on working vacation. Furthermore, let $Q_{n,0}^-$ denote the probability that the server is busy with an accessible batch of size $n(a \le n \le d-1)$, and $Q_{n,1}^-$ denote the probability that the server is busy with non-accessible batch with $n(n \ge 0)$ customers waiting in the queue at pre-arrival epoch. These are given by

$$R_n^- = \frac{1}{\lambda}R_n(0), P_{n,0}^- = \frac{1}{\lambda}P_{n,0}(0), Q_{n,0}^- = \frac{1}{\lambda}Q_{n,0}(0),\ Q_{n,1}^- = \frac{1}{\lambda}Q_{n,1}(0), \tag{22}$$

where λ is given by equation (21).

To obtain R_n^-, $P_{n,0}^-$ and $Q_{n,i}^-$, $i = 0, 1$, we need to evaluate $R_n(0)$, $P_{n,0}(0)$ and $Q_{n,i}(0)$, $i = 0, 1$ which is done here.

Evaluation of rate probabilities $R_n(0), P_{n,0}(0)$ and $Q_{n,i}(0)$

Using the displacement operator D defined by $Dx_n = x_{n+1}$ for all n on equation (16) we obtain,

$$(\phi + \eta - \theta - \eta D)P_{n,0}^*(\theta) = (A^*(\theta) - D)P_{n-1,0}(0), \quad n \geq a. \tag{23}$$

Setting $\theta = \phi + \eta - \eta D$ in equation (23) we get

$$[A^*(\phi + \eta - \eta D) - D]P_{n-1,0}(0) = 0, \tag{24}$$

whose solution is given by

$$P_{n,0}(0) = C\nu^n, \quad n \geq a - 1, \tag{25}$$

where C is an arbitrary constant and ν is the unique root of the equation $A^*(\phi + \eta - \eta D) - D = 0$ inside $(0, 1)$ for $\rho = \frac{\lambda}{b\mu} < 1$.

Substituting equation (25) in equation (23), we get the general solution as

$$P_{n,0}^*(\theta) = \frac{C(A^*(\theta) - \nu)\nu^{n-1}}{(\phi_1 - \theta)}, \quad n \geq a, \tag{26}$$

where $\phi_1 = \phi + \eta(1 - \nu)$.

Applying the displacement operator D, we can write equation (20) as

$$(\mu - \mu D^b - \theta)Q_{n,1}^*(\theta) = (A^*(\theta) - D)Q_{n-1,1}(0) + \phi P_{n+b,0}^*(\theta), \quad n \geq 1. \tag{27}$$

Setting $\theta = \mu - \mu D^b$ in equation (27) and using equation (26), we obtain

$$(A^*(\mu - \mu D^b) - D)Q_{n,1}(0) = -\frac{C\phi(A^*(\mu - \mu D^b) - \nu)\nu^{n+b}}{\phi_1 - \mu(1 - D^b)}, \quad n \geq 0. \tag{28}$$

Thus, the general solution of equation (28) is given by

$$Q_{n,1}(0) = Kr^n - \frac{C\phi\nu^{n+b}}{\phi_2}, \quad n \geq 0, \tag{29}$$

where K is an arbitrary constant and r is the unique root of the equation $A^*(\mu - \mu D^b) - D = 0$ in $(0, 1)$ and $\phi_2 = \phi_1 - \mu(1 - \nu^b)$.

Let $z_j(\theta)$, $1 \leq j \leq b$, be the b roots of $\mu - \mu D^b - \theta = 0$ for a fixed θ with $Re(\theta) \geq 0$. Then, the complementary solution of the homogeneous difference equation $(\mu - \mu D^b - \theta)Q_{n,1}^*(\theta) = 0$ of equation (27) is given by

$$Q_{n,1}^{*(c)}(\theta) = \sum_{j=1}^{b} d_j z_j(\theta),$$

where d_js are arbitrary constants. Since $\sum_{n=1}^{\infty} Q^*_{n,1}(0) \leq 1$, so we must have all $d_j = 0$.

Thus, the general solution of equation (27) using equations (26) and (29) is

$$Q^*_{n,1}(\theta) = \frac{Kr^{n-1}(A^*(\theta) - r)}{\mu - \mu r^b - \theta} - \frac{C\phi(A^*(\theta) - \nu)\nu^{n+b-1}}{\phi_2(\phi_1 - \theta)}, \quad n \geq 1. \tag{30}$$

Let $\alpha = Q_{a,0}(0)$ and $\omega = A^*(\mu)$.

Setting $\theta = \mu$ in equation (18), using equations (26) and (30) and after recursive substitution, we obtain

$$Q_{n,0}(0) = \frac{C\phi(\omega^{n-a} - \nu^{n-a}\nu^n)}{\phi_2} - Kr^{a-b}(\omega^{n-a} - r^{n-a}) + \alpha\omega^{n-a}, \quad a \leq n \leq d-1. \tag{31}$$

Hence, the solution of equations (17) and (18) is

$$\begin{aligned} Q^*_{n,0}(\theta) = \frac{1}{\mu - \theta}\Bigg[&\frac{C\phi}{\phi_2}\left\{\frac{\nu^{n-1}(A^*(\theta) - \nu)(\phi_1 - \mu)}{(\phi_1 - \theta)} + \nu^a H(\theta, n, \nu)\right\} \\ &+K\left\{\frac{\mu r^{n-1}(A^*(\theta) - r)}{(\mu - \mu r^b - \theta)} - r^{a-b}H(\theta, n, r)\right\} \\ &+\alpha(A^*(\theta) - \omega)\omega^{n-a-1}\Bigg], a \leq n \leq d-1, \end{aligned} \tag{32}$$

where $H(\theta, n, x) = \omega^{n-a-1}(A^*(\theta) - \omega) - x^{n-a-1}(A^*(\theta) - x)$. Using equations (26) and (30) in equation (17) after substituting $\theta = \mu$, we get

$$R_{a-1}(0) = \frac{1}{\omega}\left[\alpha - \frac{C\phi(\omega - \nu)\nu^{a-1}}{\phi_2} + Kr^{a-b-1}(\omega - r)\right]. \tag{33}$$

From equation (19), we get

$$\begin{aligned} Q^*_{0,1}(\theta) = \frac{1}{\mu - \theta}\Bigg[&\frac{C\phi}{\phi_2}\left\{\frac{(\nu^{d-1} - \nu^b)(A^*(\theta) - \nu)(\phi_1 - \mu)}{(\phi_1 - \theta)(1 - \nu)} + \nu^b G(\theta, \nu)\right\} \\ &+ K\left\{\frac{\mu(r^{d-1} - r^b)(A^*(\theta) - r)}{(\mu - \mu r^b - \theta)(1 - r)} - G(\theta, r)\right\} + \alpha A^*(\theta)\omega^{d-a-1}\Bigg], \end{aligned} \tag{34}$$

where $G(\theta, x) = A^*(\theta)(\omega^{d-a-1} - x^{d-a-1})x^{a-b} + 1$.

Let $\beta = A^*(\phi)$. Setting $\theta = \phi$ in equations (15) and (14), we get after recursive substitution,

$$\begin{aligned} P_{n,0}(0) = \beta^{n-a+1}\Bigg[&\frac{C\nu^{a-1}(1 - \beta)}{1 - \nu} + \frac{C\phi\mu(\beta^{a-n-1} - \nu^{a-n-1})\nu^{n+b}}{\eta(1 - \nu)\phi_2} \\ &- \frac{K\mu r^n(\beta^{a-n-1} - r^{a-n-1})}{\mu(1 - r^b) - \phi}\Bigg], \quad 0 \leq n \leq a-2. \end{aligned} \tag{35}$$

This completes the evaluation of the rate probabilities and their Laplace transform counterparts, but their expressions involve still unknown constants C, K and α, which are evaluated by deriving three independent equations as discussed here.

Evaluation of the constants $C,\ K$ and α:

Using equations (26), (29), (30) and (31) in equation (19), substituting $\theta=\mu$ and simplifying, we get

$$\frac{C\phi}{\phi_2}T(\nu)-KT(r)+\alpha\omega^{d-a}=0 \tag{36}$$

where $T(x)=\frac{(\omega-x)(x^{d-b-1}-1)}{1-x}+x^{a-b}\omega(\omega^{d-a-1}-x^{d-a-1})+1$.

Setting $\theta=0$ in equations (11) and (12), we get

$$R_n(0)=\phi\sum_{j=0}^{n}P_{j,0},\quad 0\le n\le a-1, \tag{37}$$

where $P_{j,0}, 0\le j\le a-1$, are obtained by setting $\theta=0$ in equations (13)–(15) as:

$$\begin{aligned}P_{0,0}=\frac{1}{\phi}\Bigg[&\frac{C\phi}{\phi_2}\Bigg\{\frac{(\phi_1-\mu)(\nu^{a-1}-\nu^b)}{\phi_1}-\frac{(\omega-\nu)\nu^{a-1}}{\omega}+\nu^b\\&-\frac{\mu\beta^{1-a}\nu^b(\beta^{a-1}-\nu^{a-1})}{\eta(1-\nu)}\Bigg\}-\frac{C\beta^{1-a}\nu^{a-1}(1-\beta)}{1-\nu}\\&+K\Bigg\{\frac{r^{a-1}-1}{1-r^b}+\frac{(\omega-r)r^{a-b-1}}{\omega}+\frac{\mu\beta^{1-a}(\beta^{a-1}-r^{a-1})}{\mu(1-r^b)-\phi}\Bigg\}+\frac{\alpha}{\omega}\Bigg].\end{aligned} \tag{38}$$

$$\begin{aligned}P_{n,0}=\frac{1}{\phi}\Bigg[&\frac{C\phi\mu}{\phi_2}\Bigg\{\frac{\nu^b[(1-\nu)\nu^{n-1}-(1-\beta)\nu^{a-1}\beta^{n-a}]}{\eta(1-\nu)}\\&-\frac{(1-\nu)\nu^{n+b-1}}{\phi_1}\Bigg\}+\frac{C(1-\beta)^2\nu^{a-1}\beta^{n-a}}{1-\nu}\\&+K\Bigg\{\frac{(1-r)r^{n-1}}{1-r^b}-\frac{\mu[(1-r)r^{n-1}-(1-\beta)r^{a-1}\beta^{n-a}]}{\mu(1-r^b)-\phi}\Bigg\}\Bigg],\\&1\le n\le a-2.\end{aligned} \tag{39}$$

$$\begin{aligned}P_{a-1,0}=\frac{1}{\phi}\Bigg[&\frac{C}{\beta(1-\nu)\phi_1}\Bigg\{\frac{\phi\mu[(\beta-\nu)\phi_1-\eta\beta(1-\nu)^2]\nu^{a+b-2}}{\eta\phi_2}\\&+[(1-\beta)\phi_1-\beta\phi(1-\nu)]\nu^{a-1}\Bigg\}\\&+K\Bigg\{\frac{(1-r)r^{a-2}}{1-r^b}-\frac{\mu r^{a-2}(\beta-r)}{\beta(\mu(1-r^b)-\phi)}\Bigg\}\Bigg].\end{aligned} \tag{40}$$

Setting $\theta=\phi$ in equation (13) and using equations (32), (34) and (35), we obtain another equation as

$$\begin{aligned}&\frac{C\phi}{\phi_2}\Bigg\{\frac{M(\nu)(\phi_1-\mu)-\nu^bM_1(\nu)}{\eta(1-\nu)}+\nu^bF(\nu)-\frac{\beta^{1-a}\nu^{a-1}(1-\beta)(\mu-\phi)\phi_2}{\mu\phi(1-\nu)}\Bigg\}\\&+K\Bigg\{\frac{\mu M(r)+M_1(r)}{\mu(1-r^b)-\phi}-F(r)\Bigg\}+\alpha\Bigg\{\frac{\omega^{d-a+1}(1-\beta)-(\omega-\beta)}{\omega(1-\omega)}\Bigg\}=0,\end{aligned} \tag{41}$$

where

$$M(x) = \frac{(\beta - x)(x^{a-1} - x^b)}{1 - x},$$
$$F(x) = x^{a-b}\Big(\frac{\omega^{d-a+1}(1-\beta) - (\omega - \beta)}{\omega(1-\omega)} - \frac{x^{d-a+1}(1-\beta) - (x-\beta)}{x(1-x)}\Big) + 1 \text{ and}$$
$$M_1(x) = \beta^{1-a}(\beta^{a-1} - x^{a-1})(\mu - \phi).$$

Elimination of α from equations (36) and (41) gives

$$C = \frac{T_2 K}{T_1}, \tag{42}$$

where

$$T_1 = \frac{\phi}{\phi_2}\bigg\{\frac{M(\nu)(\phi_1 - \mu) - \nu^b M_1(\nu)}{\eta(1-\nu)} - \frac{\beta^{1-a}\nu^{a-1}(1-\beta)(\mu-\phi)\phi_2}{\mu\phi(1-\nu)} + \nu^b(F(\nu) - J(\nu))\bigg\},$$
$$T_2 = F(r) - \frac{\mu M(r) + M_1(r)}{\mu(1-r^b) - \phi} - J(r),$$

where

$$J(x) = \bigg(\frac{x^{a-b}\omega^{d-a}(1-x) + (1-\omega)(1-x^{d-b})}{1-x}\bigg)\bigg(\frac{1-\beta-(\omega-\beta)\omega^{a-d-1}}{1-\omega}\bigg).$$

Now using equations (25), (29), (31), (35) and (37)–(40) in equation (21), we obtain

$$CT_3 + KT_4 = \lambda, \tag{43}$$

where

$$T_3 = \frac{\phi}{\phi_2}\bigg[\frac{\mu\nu^b}{\eta(1-\nu)}\bigg\{\frac{1}{\beta\phi_1}(\nu^{a-2}[(\beta-\nu)\phi_1 - \eta\beta(1-\nu)^2] - \beta\phi M_2(\nu)) + M_3(\nu) + S(\nu)\bigg\} + \frac{a(\nu^{a-1}-\nu^b)(\phi_1-\mu)}{\phi_1} + \frac{[a(1-\nu)-1]\nu^b}{1-\nu} - \nu^a E(\nu) - \nu^b\omega^{a-d}T(\nu)\bigg(\frac{a}{\omega} + \frac{1-\omega^{d-a}}{1-\omega}\bigg)\bigg] + \frac{\eta(1-\nu)\nu^{a-1}}{\phi_1} + \frac{1}{1-\nu}\bigg(\beta^{1-a}\nu^{a-1}[1-(1-\beta)(a+M_2(\beta))] + \frac{\nu^{a-1}(1-2\beta+\beta\nu)}{\beta}\bigg),$$
$$T_4 = \frac{1}{\mu(1-r^b)-\phi}\bigg\{\frac{\phi M_2(r)}{1-r^b} - \mu S(r) - \frac{\mu(\beta-r)r^{a-2}}{\beta} - \mu M_3(r)\bigg\} + \omega^{a-d}T(r)\bigg(\frac{a}{\omega} + \frac{1-\omega^{d-a}}{1-\omega}\bigg) + \frac{a(r^{a-1}-1)+(1-r)r^{a-2}}{1-r^b} + \frac{1}{1-r} + r^{a-b}E(r),$$

where

$$
\begin{aligned}
E(x) &= \frac{a(\omega - x)}{\omega x} - \left(\frac{1-\omega^{d-a}}{1-\omega} - \frac{1-x^{d-a}}{1-x}\right),\\
M_2(x) &= \frac{(1-a) + x^{a-2}(1-2x) + ax}{1-x},\\
M_3(x) &= \frac{1-x^{a-1}}{1-x} + \frac{x^{a-1}(1-\beta^{1-a})}{1-\beta} \quad \text{and}\\
S(x) &= x^{a-1}\beta^{1-a}M_2(\beta) - a\beta^{1-a}(\beta^{a-1} - x^{a-1}).
\end{aligned}
$$

Solving the linear equations (42) and (43), we obtain C and K after some simplification as:

$$
C = \lambda T_2(T_2T_3 + T_1T_4)^{-1}, \tag{44}
$$

$$
K = \lambda T_1(T_2T_3 + T_1T_4)^{-1}. \tag{45}
$$

Using equations (44) and (45) in equation (36), we get

$$
\alpha = \lambda\omega^{a-d}\Big(T(r)T_1 - \frac{\phi\nu^b}{\phi_2}T(\nu)T_2\Big)(T_2T_3 + T_1T_4)^{-1}. \tag{46}
$$

We summarise the above-mentioned results for pre-arrival epoch probabilities in the following theorem.

Theorem 4.1: *The pre-arrival epoch queue length distributions R_n^- that an arrival sees n customers in the system and the server is idle, $P_{n,0}^-$ that an arrival sees n customers in the system and the server is on working vacation, $Q_{n,j}^-, j = 0, 1$ that the server is busy with an accessible/non-accessible batch are given by*

$$
\begin{aligned}
R_n^- &= \frac{\phi}{\lambda}\sum_{j=0}^{n} P_{j,0}, \quad 0 \le n \le a-1,\\
P_{n,0}^- &= \frac{\beta^{n-a+1}}{\lambda}\left[\frac{C\nu^{a-1}(1-\beta)}{1-\nu} + \frac{C\phi\mu(\beta^{a-n-1} - \nu^{a-n-1})\nu^{n+b}}{\eta(1-\nu)\phi_2}\right.\\
&\quad \left. - \frac{K\mu r^n(\beta^{a-n-1} - r^{a-n-1})}{\mu(1-r^b) - \phi}\right], \quad 0 \le n \le a-2,\\
P_{n,0}^- &= \frac{C}{\lambda}\nu^n, \quad n \ge a-1,\\
Q_{n,0}^- &= \frac{1}{\lambda}\left[\frac{C\phi(\omega^{n-a} - \nu^{n-a})\nu^a}{\phi_2} - Kr^{a-b}(\omega^{n-a} - r^{n-a}) + \alpha\omega^{n-a}\right],\\
&\qquad a \le n \le d-1,\\
Q_{n,1}^- &= \frac{1}{\lambda}\left[Kr^n - \frac{C\phi\nu^{n+b}}{\phi_2}\right], \quad n \ge 0.
\end{aligned}
$$

Proof: To get the desired results, we use equation (22) in equations (37), (35), (25), (31) and (29), respectively, and using equations (38)–(40).

4.2 Steady-state distribution at arbitrary epoch

The arbitrary epoch queue length distributions R_n that an arrival sees n customers in the system and the server is idle, $P_{n,0}$ that an arrival sees n customers in the system and the server is on working vacation, $Q_{n,j}, j = 0, 1$ that server is busy with an accessible/non-accessible batch are summarised in the following theorem.

Theorem 4.2: *The arbitrary epoch probabilities are given by*

$$P_{0,0} = \frac{1}{\phi}\left[\frac{C\phi}{\phi_2}\left\{\frac{(\phi_1+\mu)(\nu^{a-1}-\nu^b)}{\phi_1} - \frac{(\omega-\nu)\nu^{a-1}}{\omega} + \nu^b - \frac{\mu\beta^{1-a}\nu^b(\beta^{a-1}-\nu^{a-1})}{\eta(1-\nu)}\right\} - \frac{C\beta^{1-a}\nu^{a-1}(1-\beta)}{1-\nu} + K\left\{\frac{r^{a-1}-1}{1-r^b} + \frac{(\omega-r)r^{a-b-1}}{\omega} + \frac{\mu\beta^{1-a}(\beta^{a-1}-r^{a-1})}{\mu(1-r^b)-\phi}\right\} + \frac{\alpha}{\omega}\right],$$

$$P_{n,0} = \frac{1}{\phi}\left[\frac{C\phi\mu}{\phi_2}\left\{\frac{\nu^b[(1-\nu)\nu^{n-1}-(1-\beta)\nu^{a-1}\beta^{n-a}]}{\eta(1-\nu)} - \frac{(1-\nu)\nu^{n+b-1}}{\phi_1}\right\} + \frac{C(1-\beta)^2\nu^{a-1}\beta^{n-a}}{1-\nu} + K\left\{\frac{(1-r)r^{n-1}}{1-r^b} - \frac{\mu[(1-r)r^{n-1}-(1-\beta)r^{a-1}\beta^{n-a}]}{\mu(1-r^b)-\phi}\right\}\right], \quad 1 \le n \le a-2,$$

$$P_{a-1,0} = \frac{1}{\phi}\left[\frac{C}{\beta(1-\nu)\phi_1}\left\{\frac{\phi\mu[(\beta-\nu)\phi_1-\eta\beta(1-\nu)^2]\nu^{a+b-2}}{\eta\phi_2} + [(1-\beta)\phi_1-\beta\phi(1-\nu)]\nu^{a-1}\right\} + K\left\{\frac{(1-r)r^{a-2}}{1-r^b} - \frac{\mu r^{a-2}(\beta-r)}{\beta(\mu(1-r^b)-\phi)}\right\}\right],$$

$$P_{n,0} = \frac{C(1-\nu)\nu^{n-1}}{\phi_1}, \quad n \ge a,$$

$$Q_{n,0} = \frac{1}{\mu}\left[\frac{C\phi}{\phi_2}\left\{\frac{(1-\nu)\nu^{n-1}(\phi_1-\mu)}{\phi_1} + \nu^a((1-\omega)\omega^{n-a-1}-(1-\nu)\nu^{n-a-1})\right\} + K\left\{\frac{(1-r)r^{n-1}}{1-r^b} - r^{a-b}((1-\omega)\omega^{n-a-1}-(1-r)r^{n-a-1})\right\} + \alpha(1-\omega)\omega^{n-a-1}\right], \quad a \le n \le d-1,$$

$$Q_{0,1} = \frac{1}{\mu}\left[\frac{C\phi}{\phi_2}\left\{\frac{(\nu^{d-1}-\nu^b)(\phi_1-\mu)}{\phi_1} + \nu^a(\omega^{d-a-1}-\nu^{d-a-1}) + \nu^b\right\} + K\left\{\frac{r^{d-1}-1}{1-r^b} - r^{a-b}(\omega^{d-a-1}-r^{d-a-1})\right\} + \alpha\omega^{d-a-1}\right],$$

$$Q_{n,1} = \frac{K(1-r)r^{n-1}}{\mu(1-r^b)} - \frac{C\phi(1-\nu)\nu^{n+b-1}}{\phi_1\phi_2}, \quad n \ge 1.$$

Proof: The results of $P_{n,0}, 0 \leq n \leq a-1$ are obtained from equations (38)–(40). The other results of the theorem are obtained by setting $\theta = 0$ in equations (26), (32), (34) and (30), respectively.

Remark: Equations (35), (38)–(43) and their corresponding equations in Theorems 4.1 and 4.2 become indeterminate when $\eta = 0$. The problem may be overcome by using L'Hopital's rule.

Theorem 4.3: *The arbitrary epoch probabilities* $\{R_n\}_0^{a-1}$ *are given by*

$$R_n = R^-_{n-1} - \phi P^{*(1)}_{n,0}(0), \quad 1 \leq n \leq a-1.$$

Finally, the only unknown quantity R_0 *is obtained by using the normalisation condition* $R_0 = 1 - \left(\sum_{n=1}^{a-1} R_n + \sum_{n=0}^{\infty} P_{n,0} + \sum_{n=a}^{d-1} Q_{n,0} + \sum_{n=0}^{\infty} Q_{n,1}\right)$.

Proof: Differentiating equation (12) with respect to θ, and setting $\theta = 0$, we obtain

$$R_n = -\phi P^{*(1)}_{n,0}(0) + R^-_{n-1}, \quad 1 \leq n \leq a-1,$$

where $P^{*(1)}_{n,0}(0), 1 \leq n \leq a-2$ and $P^{*(1)}_{a-1,0}(0)$ can be obtained by differentiating equations (14) and (15) with respect to θ and setting $\theta = 0$, we obtain

$$P^{*(1)}_{n,0}(0) = \frac{1}{\phi}\big(P_{n,0} - P^-_{n-1,0} + \mu Q^{*(1)}_{n,1}(0)\big), \quad 1 \leq n \leq a-2,$$
$$P^{*(1)}_{a-1,0}(0) = \frac{1}{\phi}\big(P_{a-1,0} - P^-_{a-2,0} + \mu Q^{*(1)}_{a-1,1}(0) + \eta P^{*(1)}_{a,0}(0)\big).$$

Finally, to know the quantities $Q^{*(1)}_{n,1}(0), 1 \leq n \leq a-1$, and $P^{*(1)}_{a,0}(0)$, we differentiate equations (30) and (26) with respect to θ and setting $\theta = 0$, we obtain

$$Q^{*1)}_{n,1}(0) = Kr^{n-1}\left\{\frac{\lambda(1-r) - \mu(1-r^b)}{\lambda\mu^2(1-r^b)^2}\right\}$$
$$-C\phi\nu^{n+b-1}\left\{\frac{(\lambda-\eta)(1-\nu)-\phi}{\lambda\phi_1^2\phi_2}\right\}, \quad n \geq 1,$$
$$P^{*(1)}_{n,0}(0) = C\nu^{n-1}\left\{\frac{(\lambda-\eta)(1-\nu)-\phi}{\lambda\phi_1^2}\right\}, \quad n \geq a,$$

and setting $n = a$ in $P^{*(1)}_{n,0}(0)$, we get

$$P^{*(1)}_{a,0}(0) = C\nu^{a-1}\left\{\frac{(\lambda-\eta)(1-\nu)-\phi}{\lambda\phi_1^2}\right\}.$$

4.3 Algorithm for computing state probabilities

To demonstrate the working schemes of the recursive method, we describe an algorithm for calculating the steady-state probabilities. Given the values of $\lambda, \mu, \phi, \eta, a, d$ and b, first obtain ν and r, which are the roots of the equations $A^*(\phi + \eta - \eta\nu) - \nu = 0$ and $A^*(\mu - \mu r^b) - r = 0$, respectively (see equations (25) and (29)). Mathematica 6.0 software is used to compute them as their analytical expression is difficult to derive though not impossible for different arrival distributions. The remaining steps of the solution algorithm are stated as follows:

Step 1: Compute $P_{n,0}(0)$ for $n \geq a - 1$ using equation (25).

Step 2: Compute $Q_{n,1}(0)$ for $n \geq 0$ using equation (29).

Step 3: Compute $Q_{n,0}(0)$ for $a \leq n \leq d - 1$ using equation (31).

Step 4: Compute $P_{n,0}(0)$ for $0 \leq n \leq a - 2$ using equation (35).

Step 5: Compute $P_{0,0}$, $P_{n,0}, 0 \leq n \leq a - 2$ and $P_{a-1,0}$ using equations (38)–(40), respectively.

Step 6: Compute $R_n(0)$ for $0 \leq n \leq a - 1$ using equation (37).

Step 7: Compute $P^-_{n,0}, 0 \leq n \leq a - 2$, $P^-_{n,0}, n \geq a - 1$, $Q^-_{n,1}, n \geq 0$, $Q^-_{n,0}$, $a \leq n \leq d - 1$ and R^-_n, $0 \leq n \leq a - 1$ using equation (22).

Step 8: Compute $P_{n,0}$ for $n \geq a$ using equation (26).

Step 9: Compute $Q_{n,0}$ for $a \leq n \leq d - 1$ using equation (32).

Step 10: Compute $Q_{0,1}$ using equation (34).

Step 11: Compute $Q_{n,1}$ for $n \geq 1$ using equation (30).

Step 12: Compute R_n for $0 \leq n \leq a - 1$ from Theorem 4.3.

This completes the evaluation of arbitrary and pre-arrival epoch probabilities from Theorems 4.1–4.3. In the following sections, we discuss some performance measures, special cases and numerical results.

5 Performance measures

Performance measures are important features of queueing systems as they reflect the efficiency of the queueing system under consideration. Once the state probabilities at pre-arrival and arbitrary epochs are known, we can evaluate various performance measures. The average queue length when server is idle (L_{q0}), the average queue length when server is in single working vacation (L_{q1}), the average queue length when the server is busy with non-accessible batch (L_{q2}) and the average number of customers in the queue at an arbitrary epoch (L_q) are given, respectively, by

$$L_{q0} = \sum_{n=0}^{a-1} nR_n, L_{q1} = \sum_{n=0}^{\infty} nP_{n,0}, L_{q2} = \sum_{n=0}^{\infty} nQ_{n,1}$$
$$\text{and } L_q = \sum_{n=0}^{a-1} nR_n + \sum_{n=0}^{\infty} n(P_{n,0} + Q_{n,1}).$$

The average waiting time in the queue (W_q) of a customer using Little's rule is given by $W_q = L_q/\lambda$.

Waiting time analysis

It is known from the literature that the waiting time analysis of the $GI/M^{(a,b)}/1/\infty$ model is not available though it is discussed for $M/M^{(a,b)}/1/\infty$ queue, see Medhi (1991). However, we present the waiting time distribution in the queue of an admitted customer for $GI/M^{(a,d,b)}/1/\infty/SWV$ queueing model for the special case $a = d = 1$.

Let W_A be the actual waiting time distribution in the queue of an admitted customer and let $W_A^*(\theta)$ be its LST. Because of the memoryless property of the service times during the normal service period, service time during working vacation and the length of working vacation time distributions, an arrival may find the system in one of the following two situations:

Case I: A busy server with $n(0 \le n \le \infty)$ customers in the queue during the normal service period. In this case, the arriving customer will wait in the queue till the service completion of $[\frac{n}{b}] + 1$ batches, where $[x]$ is the greatest integer contained in x.

Case II: A busy server with n $(0 \le n \le \infty)$ customers in the queue during the working vacation period. Here, the customer has to wait in the queue till

- the service completion of n customers during working vacation
- the service completion of $k(0 \le k \le n)$ customers during working vacation, the server returns from a working vacation and the service completion of $[\frac{n-k}{b}]$ batches of customers during the normal service period.

Hence, combining all the above-mentioned cases, we have

$$W_A^*(\theta) = \sum_{n=0}^{\infty} Q_{n,1}^{-} \left(\frac{\mu}{\theta+\mu}\right)^{[\frac{n}{b}]+1} + \sum_{n=0}^{\infty} P_{n,0}^{-} \left(\frac{\eta}{\theta+\phi+\eta}\right)^{n+1}$$
$$+ \sum_{n=0}^{\infty} P_{n,0}^{-} \sum_{k=0}^{n} \left(\frac{\phi}{\theta+\phi+\eta}\right) \left(\frac{\eta}{\theta+\phi+\eta}\right)^{k} \left(\frac{\mu}{\theta+\mu}\right)^{[\frac{n-k}{b}]}.$$

From this expression, we can easily obtain mean waiting time in the queue (W_A), which is given by:

$$
\begin{aligned}
W_A &= -W_A^{*(1)}(0) \\
&= \frac{1}{\mu}\sum_{n=0}^{\infty} Q_{n,1}^{-}\left(\left[\frac{n}{b}\right]+1\right) + \sum_{n=0}^{\infty} P_{n,0}^{-}\frac{(n+1)\eta^{n+1}}{(\phi+\eta)^{n+2}} \\
&\quad + \sum_{n=0}^{\infty} P_{n,0}^{-}\sum_{k=0}^{n}\frac{\phi\eta^k}{(\phi+\eta)^{k+1}}\left(\frac{k+1}{\phi+\eta}+\frac{1}{\mu}\left[\frac{n-k}{b}\right]\right).
\end{aligned}
$$

It may be noted here that the numerical value of the average waiting time in the queue obtained through waiting time analysis matches exactly with the one obtained earlier using Little's rule, as it should be.

6 Special cases

The following special cases are deduced from our model by taking specific values of the parameters a, d, b, ϕ and η.

Case 1: $a = d$, i.e., the server is not accessible for late arrivals. In this case, the model reduces to $GI/M^{(a,b)}/1/\infty/SWV$ queue.

Case 2: $a = d = b = 1$, i.e., the batch size one. In this case, the model reduces to $GI/M/1/\infty/SWV$ queue and our results match analytically with the one obtained by Chae et al. (2009).

Case 3: $\eta \to 0$, i.e., the server with single vacation. In this case, the model reduces to $GI/M^{(a,d,b)}/1/\infty/SV$ queue.

Case 4: $a = d$ and $\eta \to 0$, i.e., the server with single vacation is not accessible for late arrivals. In this case, the model reduces to $GI/M^{(a,b)}/1/\infty/SV$ queue.

Case 5: $a = d = b = 1$ and $\eta \to 0$, i.e., the batch size one with single vacation. In this case, the model reduces to $GI/M/1/\infty/SV$ queue and our results match with the one discussed by Tian (1993).

7 Numerical results

To demonstrate the applicability of the results obtained in previous sections, we have presented numerical results in self-explanatory tables and graphs by considering different inter-arrival time distributions such as exponential (M), deterministic (D), Erlang (E_k) and hyperexponential (HE_2, with parameters $\sigma_1, \sigma_2, \lambda_1, \lambda_2$). It can be seen from the theorems in Section 4 that to obtain the pre-arrival and arbitrary epoch probabilities for various inter-arrival time distributions we need first to obtain the roots ν and r. But, because of the analytical complexity it is difficult to give the explicit expressions of ν and r though numerically it is easier to evaluate them using the softwares like Mathematica 6.0.

In Table 1, comparison of performance measures has been done for E_2 inter-arrival time distribution by taking the parameters $\lambda = 2.5, \rho = 0.5, a = 3,$

$d = 6, b = 10$, $\eta = 0.2$ and 0.8 and ϕ varying from 0.2 to 1.2. One can observe that all the performance measures except L_{q0} decrease as ϕ increases. This is because, for fixed a and λ as ϕ increases the server attends the system more frequently and may find the system with less than a number of customer, thereby the queue length during idle period L_{q0} increases. On the other hand, when the system has minimum a number of customers, i.e., ready for service, the other performance measures L_{q1}, L_{q2}, L_q, W_q naturally decrease with increasing value of ϕ. Furthermore, except L_{q0} all performance measures corresponding to $\eta = 0.2$ are larger than those corresponding to $\eta = 0.8$, which may be explained in a similar way. This observation may help us in suitably choosing ϕ and η so that the server reach its optimal serving capacity to enhance the system performance.

Table 1 Comparison of performance measures for $E_2/M^{(3,6,10)}/1/\infty/SWV$ queue

ϕ	L_q	W_q	L_{q0}	L_{q1}	L_{q2}
			$\eta = 0.2$		
0.2	11.239600	4.495850	0.010635	5.020450	6.208560
0.4	6.423580	2.569430	0.031894	2.091270	4.300410
0.6	5.147310	2.058930	0.054713	1.180080	3.912520
0.8	4.631170	1.852470	0.075811	0.764525	3.790830
1.0	4.370960	1.748380	0.094322	0.538466	3.738170
1.2	4.219990	1.688000	0.110244	0.401552	3.708190
			$\eta = 0.8$		
0.2	9.077620	3.631050	0.012404	4.336270	4.728950
0.4	5.666810	2.266720	0.035506	1.836070	3.795230
0.6	4.780180	1.912070	0.059123	1.053600	3.667460
0.8	4.418780	1.767510	0.080353	0.693507	3.644920
1.0	4.233050	1.693220	0.098672	0.495323	3.639050
1.2	4.123250	1.649300	0.114264	0.373784	3.635200

In Tables 2 and 3, few results of queue length distributions at pre-arrival and arbitrary epochs along with some performance measures are presented for different inter-arrival time distributions. For this case, we have taken $\lambda = 1.25$, $\mu = 0.3125$, $\phi = 0.6$, $\eta = 1.0$, $a = 3$, $d = 6, b = 10$ and $\rho = 0.4$. As can be seen from Table 2, the queue length distributions at pre-arrival and arbitrary epochs are same for exponential inter-arrival times due to its memoryless property.

Figure 1 shows the effect of rate of vacation (ϕ) on the average waiting time in the queue (W_q) when inter-arrival time distribution is E_3 with $\lambda = 2.5, \rho = 0.4, \eta = 1$, $a = 3, b = 15$. We have used different accessibility limit $d = 5, 9, 13$ and ϕ varying from 0.1 to 2. It can be observed that as ϕ increases, W_q drastically decreases initially and then behaves as a constant. With increasing ϕ, the server is frequently available in the system clearing the backlog customers thereby decreasing the waiting time. Furthermore, for fixed ϕ, as the accessibility limit d increases W_q decreases. This is how accessible batch service is important in providing better service and improving the performance of the system.

Table 2 Queue length distributions at pre-arrival and arbitrary epochs

	$M/M^{(3,6,10)}/1/\infty$ $\lambda = 1.25,\ \phi = 0.6,$ $\rho = 0.4, \mu = 0.3125$ $\eta = 1.0$		$E_4/M^{(3,6,10)}/1/\infty$ $\lambda = 1.25,\ \phi = 0.6,$ $\rho = 0.4, \mu = 0.3125$ $\eta = 1.0$	
	Pre-arrival	*Arbitrary*	*Pre-arrival*	*Arbitrary*
R_0	0.023909	0.023909	0.019152	0.009844
R_1	0.045093	0.045093	0.045234	0.035803
R_2	0.071192	0.071192	0.075549	0.063786
$P_{0,0}$	0.049811	0.049811	0.060297	0.039900
$P_{1,0}$	0.044134	0.044134	0.050873	0.054337
$P_{2,0}$	0.054373	0.054373	0.066195	0.063156
$P_{3,0}$	0.029441	0.029441	0.028580	0.040248
$P_{4,0}$	0.015943	0.015943	0.012339	0.017377
$P_{5,0}$	0.008632	0.008632	0.005328	0.007502
$P_{10,0}$	0.000402	0.000402	0.000080	0.000113
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$Q_{3,0}$	0.076689	0.076689	0.084959	0.082995
$Q_{4,0}$	0.074419	0.074419	0.081237	0.083227
$Q_{5,0}$	0.068574	0.068574	0.072847	0.076175
$Q_{0,1}$	0.075196	0.075196	0.076061	0.075400
$Q_{1,1}$	0.062028	0.062028	0.061367	0.066632
$Q_{2,1}$	0.051141	0.051141	0.049505	0.053755
$Q_{3,1}$	0.042150	0.042150	0.039932	0.043361
$Q_{4,1}$	0.034733	0.034733	0.032208	0.034975
$Q_{5,1}$	0.028617	0.028617	0.025978	0.028210
$Q_{10,1}$	0.010855	0.010855	0.008867	0.009629
$Q_{50,1}$	0.000004	0.000004	0.000001	0.000001
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sum	1.000000	1.000000	1.000000	1.000000
	$L_q = 2.608780, W_q = 2.087030,$ $L_{q0} = 0.187477, L_{q1} = 0.421325,$ $L_{q2} = 1.999980$		$L_q = 2.391570, W_q = 1.913250,$ $L_{q0} = 0.163375, L_{q1} = 0.446945,$ $L_{q2} = 1.781250$	

The effect of service rate during vacation (η) on the average queue length (L_q) for HE_2 inter-arrival time distribution is depicted in Figure 2. For this, we have taken the same $\lambda,\ \rho,\ a, b$ and accessibility limits d as in Figure 1, $\phi = 0.2,\ \sigma_1 = 0.4,\ \sigma_2 = 0.6,$ $\lambda_1 = 2,\ \lambda_2 = 3$ and ηs varying from 0 to 1.6. It can be seen from this figure that L_q decreases as η increases and all the graphs are almost linear. The server is busy all the time except the idle period so that the accumulation of customers in the queue decreases. Furthermore, L_q decreases faster for larger d values.

Figure 3 presents the effect of service rate during vacation (η) on the average queue lengths (L_{q0}, L_{q1}, L_{q2}, L_q) when inter-arrival time distribution is E_4 with parameters $\lambda = 2.5,\ \ \phi = 0.2,\ \ \rho = 0.4,\ \ a = 3,\ \ d = 7,\ \ b = 11$ and η varying from 0 to 1.6.

Table 3 Queue length distributions at pre-arrival and arbitrary epochs

	$D/M^{(3,6,10)}/1/\infty$ $\lambda = 1.25, \phi = 0.6,$ $\rho = 0.4, \mu = 0.3125$ $\eta = 1.0$		$HE_2/M^{(3,6,10)}/1/\infty$ $\lambda_1 = 2, \lambda_2 = 1, \sigma_1 = 0.4$ $\phi = 0.6, \lambda = 1.25, \rho = 0.4$ $\mu = 0.3125, \eta = 1.0$	
	Pre-arrival	*Arbitrary*	*Pre-arrival*	*Arbitrary*
R_0	0.016861	0.005834	0.024521	0.027989
R_1	0.045254	0.031512	0.044867	0.046589
R_2	0.077244	0.060555	0.070111	0.072456
$P_{0,0}$	0.064976	0.035127	0.047968	0.051086
$P_{1,0}$	0.053604	0.059151	0.042881	0.042387
$P_{2,0}$	0.071903	0.066647	0.052650	0.052592
$P_{3,0}$	0.026996	0.045841	0.029298	0.027973
$P_{4,0}$	0.010135	0.017211	0.016303	0.015566
$P_{5,0}$	0.003805	0.006462	0.009072	0.008662
$P_{10,0}$	0.000028	0.000048	0.000484	0.000462
⋮	⋮	⋮	⋮	⋮
$Q_{3,0}$	0.088832	0.085241	0.075224	0.075093
$Q_{4,0}$	0.084021	0.087149	0.073179	0.072691
$Q_{5,0}$	0.074266	0.079318	0.067727	0.067090
$Q_{0,1}$	0.075956	0.075642	0.074908	0.075084
$Q_{1,1}$	0.060771	0.068082	0.062069	0.061033
$Q_{2,1}$	0.048619	0.054470	0.051401	0.050541
$Q_{3,1}$	0.038896	0.043577	0.042551	0.041838
$Q_{4,1}$	0.031117	0.034862	0.035215	0.034625
$Q_{5,1}$	0.024893	0.027889	0.029139	0.028650
$Q_{10,1}$	0.008157	0.009138	0.011291	0.011101
$Q_{50,1}$	0.000001	0.000001	0.000005	0.000005
⋮	⋮	⋮	⋮	⋮
Sum	1.000000	1.000000	1.000000	1.000000
	$L_q = 2.311460, W_q = 1.849170,$ $L_{q0} = 0.152623, L_{q1} = 0.456760,$ $L_{q2} = 1.702080$		$L_q = 2.656610, W_q = 2.125290,$ $L_{q0} = 0.191500, L_{q1} = 0.415894,$ $L_{q2} = 2.049220$	

From this figure, it can be observed that all the average queue lengths except L_{q0} decrease as η increases. Moreover, L_{q0} is too small and it approaches asymptotically to zero for all values of η.

The effect of accessible limit (d) on the average queue lengths (L_{q0}, L_{q1}, L_{q2}, L_q) for deterministic inter-arrival time distribution is depicted in Figure 4. The parameters considered in this case are $\lambda = 2.5,\ \phi = 0.2,\ \rho = 0.4,\ \eta = 0.5,\ a = 3,\ b = 15$ and d varies from 3 to 15. As expected, the queue lengths decrease as d increases. Furthermore, L_{q0} is not influenced by d and it is zero asymptotically.

Figure 5 compares the effect of ρ on the average queue length (L_q) for the SWV model having $\eta = 1.2$ and the SV model having $\eta \to 0$ when the inter-arrival time distribution is deterministic with the following parameters : $\lambda = 2.5,\ \phi = 0.2,$ $a = 3,\ d = 7,\ b = 11$ and ρ varying from 0.05 to 0.9. As an effect of ρ on L_q, we

Figure 1 Effect of ϕ on W_q

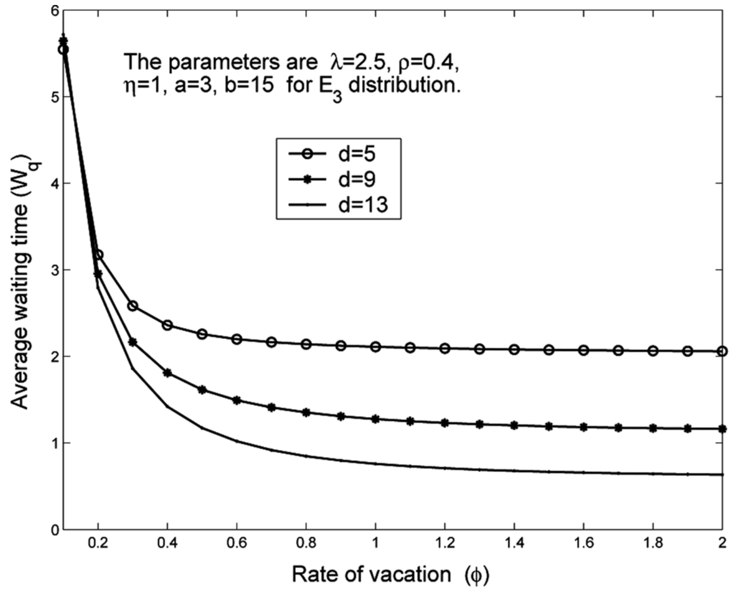
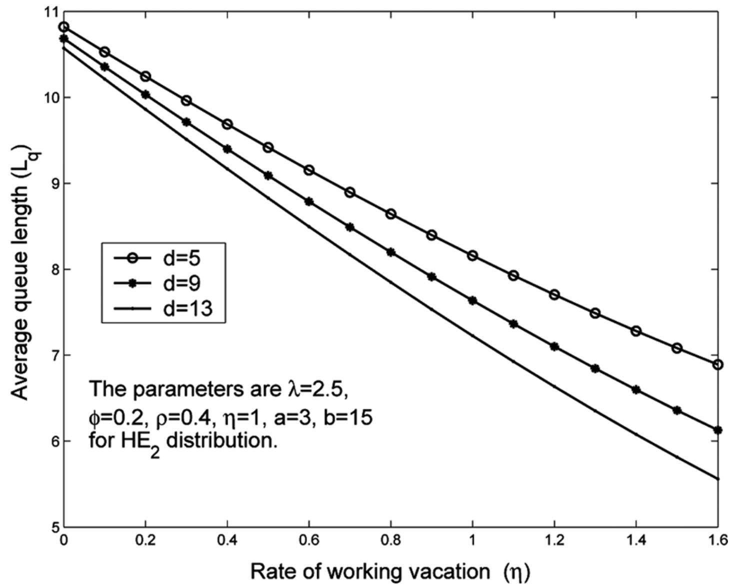


Figure 2 Effect of η on L_q



observe that the average queue lengths are almost constant for $\rho \leq 0.4$ and they increase drastically for $\rho \geq 0.8$. Furthermore, the SWV outperforms the SV model as ρ increases.

Figure 6 shows the impact of service rate (μ) on the average waiting time in the queue (W_q) for the inter-arrival time distributions M, E_4, D and HE_2 with parameters $\lambda = 2.5, \phi = 0.2, \eta = 1, a = 3, d = 7, b = 11$ and μ varying from 0.3 to 1.0. It can be

Figure 3 Effect of η on average queue lengths

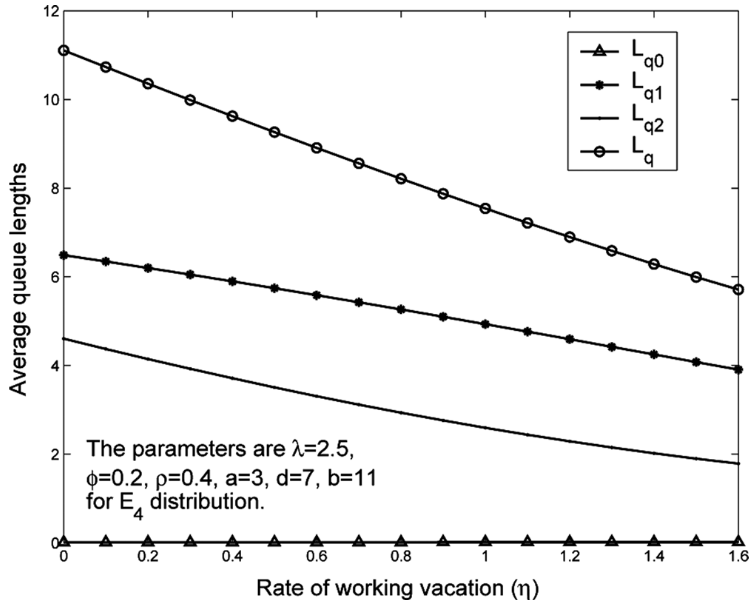
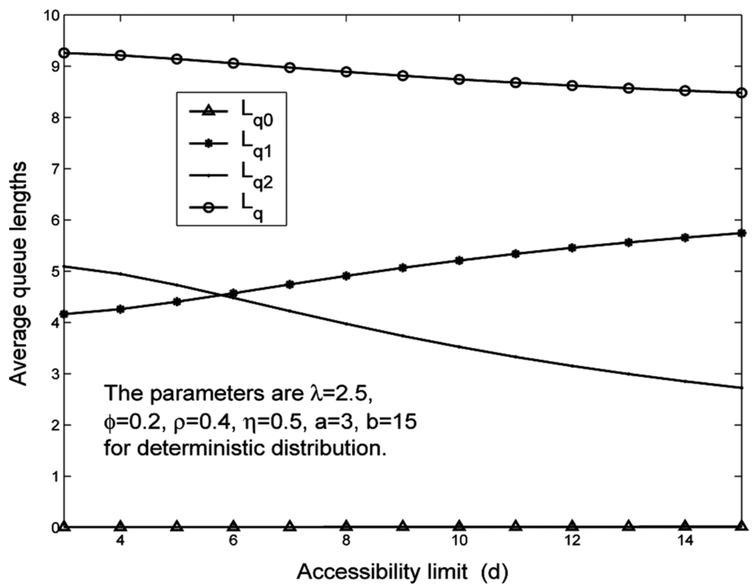


Figure 4 Effect of d on average queue lengths



seen that for any inter-arrival time distribution, W_q drastically decreases initially and becomes constant as μ increases. This is because when service rate increases the server utilisation is more, which in turn decreases the waiting time of a customer. Furthermore, for $\mu \leq 0.7$, the deterministic distribution has the lowest average waiting time whereas the hyperexponential distribution has the highest.

Figure 5 Effect of ρ on L_q

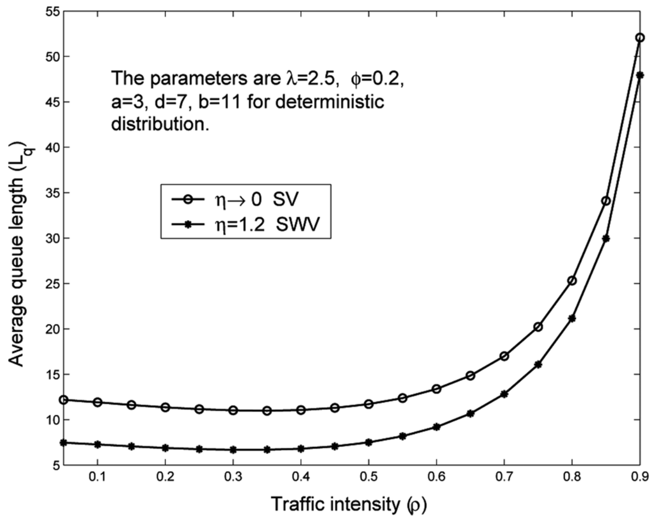
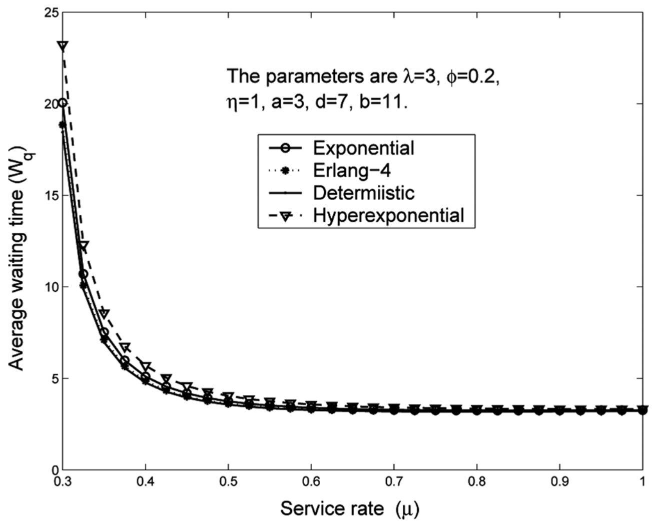


Figure 6 Effect of μ on W_q



8 Conclusions

In this paper, we have analysed an infinite buffer single server accessible and non-accessible batch service queue with general independent arrivals, exponential services and single exponential working vacation. Using the supplementary variable technique and the recursive method, the steady-state system (queue) length distribution of number of customers at pre-arrival and arbitrary epochs are obtained. The recursive method used in this paper is simple and easy to apply. The performance measures tables and graphs clearly show that the single working vacation model performs better than the model with single vacation. This queueing model have potential applications in the areas of communication systems, manufacturing systems, computer networks, etc. The limitations of this research are that though we are able to find out the approximate numerical results for $GI/M^{(a,d,b)}/1/SV$ queue by taking $\eta \to 0$, the analytical expressions are lengthy and cumbersome to derive. The techniques used in this paper can be applied to analyse more complex queueing models such as $GI/G^{(a,d,b)}/1/\infty/SWV$, and $MAP/G^{(a,d,b)}/1/\infty/SWV$ which are left for future investigations.

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