THE SCHAUDER FIXED POINT THEOREM ANDANALYSIS METHOD FOR SOLVING POSITIVE SOLUTIONS FOR SECOND ORDER M-POINT B0UNDARY VALUE PROBLEMS

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Declaration

I, the undersigned, declare that this research work, "The schauder fixed point theorem and analysis method for solving positive solution of second order m-point boundary value problem without super and sub linear cases " has been composed exclusively by me, and no part of this thesis formed the basis for the award of any degree, diploma or other similar titles.

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Advisor's comment

This research has been done by Kebede Obsi in the Department of Mathematics, Jimma University. The work has been done under the supervision of;

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ABSTRACT

This thesis is designed to develop a scheme for finding positive solution of second order m-point boundary value problem by using schauder fixed point theorem and analysis method without the use of super linear and sub linear cases. The mathematical operation which were performed by schauder fixed point theorem and analysis method were deduced from krasoleskii [9] fixed point theorem and some definition of open sets. The discussion of the study were made by considering the well known fixed point theorem due to Krasnoselskii and some definition of open subsets as a preliminary concept. This study asserts the fact that positive solution of second order m-point B.V.P can be obtained without super linear and sub linear cases ant it gives at least two positive solutions to the model.

CHAPTER ONE

1. INTRODUCTION

1.1. Background of the Study

Mathematics is the body of knowledge centered on the science that draws necessary conditions. A point of central importance in the study of nonlinear singular boundary value problems in the second order ordinary differential equation is to avoid the singularity and then to solve its solution and check the existence and uniqueness of solution in the neighborhood of singular point for the given boundary condition. As a result singular B.V.Ps of nonlinear second order ordinary differential equations is applicable in many areas of science and technology, like nuclear physics, chemical reactions, etc. Singular boundary value problems of non linear second order ordinary differential equations can be solved by different mathematical approaches such as analytical and numerical method. The numerical treatment of singular boundary value problem has always been difficult and challenging task due to the singular behavior that occurs at a point. The schauder fixed point theorem and analysis method is one of a very effective analytical way of solving positive solution of second order 3-point boundary value problems [10] and m-point B.V.Ps with the consideration of super and sublinear cases.

Fixed point theorem has many important applications in differential equations. Such as existence of positive solution of nonlinear three point boundary value problems [16]. Among different fixed point theorems, Brower's is particularly well known due to its use across numerous fields of mathematics and Henri Poincare (1886) proved results equivalent to Brower fixed point theorem. The simplest forms of Brower's theorem are: for continuous function f from closed disc to itself or from closed interval I in the real numbers to itself have at least one fixed point. In this original field, this result is one of the key theorems characterizing the topology of Euclidean space .In economics, Brouwer's fixed point theorem and its extension, play a central role in the proof of general equilibrium in market. Economics as developed in the 1950 s by economics Nobel Prize winners Kenneth Arrow and Gerard Debrew.

The theorem was first studied in view of work on differential equations by the French mathematicians around Poincare. And Picard proving results requires the use of topological method.

Brower's fixed point theorem is a fixed point theorem in topology named after Luitzen Brouwer's. The study of multi point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseer [11] since then nonlinear multi point boundary value problems have been studied by several authors. The boundary value problems arise in several branches of physics; Problems involving the wave equation such as the determination of normal modes are often stated as boundary value problems of Sturm Liouville problems [2]. To be useful in applications, a boundary value problem should be well posed, this means that given the input to the problem there exists a unique solution which depends continuously on the input. Much theoretical work in the field of partial differential equations is devoted to proving that, boundary value problems arising from scientific and engineering applications are in fact well posed.

A fixed point theorem is a theorem that asserts every function that satisfies some given property will have a fixed point .If you have an equation and want to prove that it has a solution, and if it is hard to find that solution explicitly, then consider to re- write the equation in the form $f(x) = x$ and apply a fixed point theorem. This method can be applied not just to numerical equations but also to analytical? That often used to prove the existence of positive solutions of second order three point boundary value problems [10]. Boundary value problems (BVPs) for non-linear second ordinary differential equations arise in a variety of areas of applied mathematics, physics, and different problems of control theory. Singular two-point boundary value problems (BVPs) of non-linear second order ordinary differential equations (odes) arise very frequently in many applications like gas dynamics, nuclear physics, chemical reactions, atomic structure, atomic calculation, and study of positive solutions of nonlinear elliptic equations. The problems of nonlinear BVPs in ordinary differential equation have been treated in different mathematical approaches. In this study, positive solution of second order M-point boundary value problems were found by using theorem and Krasnoselskii [9] without considering super linear and sublinear cases.

In this study, we consider also the result in [10] to a multi point boundary value problem of the form

$$u''(t) + a\ (t)f(u) = 0 \qquad t \in (0,1)$$

$u(0) = 0\ \ , u(1) - \sum_{i=1}^{m-2} k_i\ \ x_i\ \ = b\ ,$ ……………………………………………(1.1

Under the following assumptions'

A$_1$) $b_i, k_i > 0 \quad (i = 1,2,3, \dots \dots \dots \dots \dots m-2)\ , 0 < x_1\ < x_2 \quad \dots \dots < x_{m-2}\ < 1$

and $0 < \sum_{i=1}^{m-2} k_i < 1$

A$_{2)}$ $f \in C([0,\infty),[0,\infty))$; a $\in c[0,1], (0,\infty)$ is continuous and $a(t) \neq 0$ on any sub interval of $[0,1]$

Then the boundary value problem (1.1) has at least one positive solution in one of the following cases

i) $f_0 = 0$ and $f_\infty = +\infty$ (Super linear case)
ii) $f_0 = +\infty$ and $f_\infty = 0$ (sublinear case)

This is proved by schauder fixed point theorem and analysis method. Then the researcher went to prove that equation (1.1) without the superliner or sub linear condition, which gives at least two positive solutions to the question stated above. (M.A Krasnoselskii) [9].

Therefore, Schauder fixed point theorem is an extension of Brouwer's fixed point to the topological space, it may have an infinite dimension. However, positive solution of second order m-point boundary value problems by applying schauder fixed point theorem and analysis method will be develop after this study. As a result, this motivated the researcher to conduct the study. Therefore, the main purpose of this study is to develop a method that can be used to find positive solution of second order m- point boundary value problems, by using fixed point theorem and analysis method without sublinear and super linear cases.

1.2 Statement of the problem

As the extensive applications of boundary value problems for differential equations in physics, biology and engineering sciences, the solvability of the problem has received great attention from many authors [9].Then this study will try to develop method of solving positive solution of second order m-point boundary value problems by considering some facts from fixed point theorem and analysis method M.AKrasnoselskii[9]

The existence of positive solutions of boundary value problem (1.1) by using Krasnoselskii fixed point theorem obtained $f_0 = 0$ and $f_\infty = \infty$ for super linear cases, and $f_0 = \infty$ and $f_\infty = 0$ for sublinearcases, then by Lerayschauder fixed point theorem proved that the boundary value problem has at least one positive solution. Ma and Castaneda [11] established existence results for positive solution of (1.1) under the assumptions $f_0 = 0, f_\infty = +\infty$, or $f_0 = +\infty, f_\infty = 0$. Then the study consider without sublinear and super linear cases $f_0 = f_\infty = 0$ or $f_0 = f_\infty = \infty$ the boundary value problem (1.1) has at least two positive solutions with $f_0, f_\infty \epsilon [0, \infty)$

As, a result the study is intended to answer the following questions.

1 How can we develop techniques for solving positive solution of second order m- point boundary value problems without super linear and sub linear cases?

2 What theorem is essential to develop a method for solving positive solution of second order m-point boundary value problem without super linear and sublinear cases?

3. How can we justify method of solving positive solution of second order m- point boundary value problems, by using fixed point theorem [9] and analysis method?

1.3 Objectives of the study

1.3.1. General objectives

The general objective of this study is to develop a scheme for solving positive solution of second order m-point boundary value problems without super linear and sublinear cases.

1.3.2 Specific objectives of the study

The following are the specific objectives of this study;

- To develop techniques for finding positive solution of second order m-point Boundary value problems, by using fixed point theorem and analysis method without super linear and sublinear cases.

1.4. Significance of the Study

This study is considered for vital importance for the following reasons

- It will develop techniques of solving positive solution of second order m-point boundary Value problems, by using fixed point theorem and analysis method without super linear and sublinear cases.
- It will be used as reference material for anyone who will work on similar area.

1.5 Delimitation of the study

Even though there are different types of boundary value problems of differential equation, which can be solved by different analytical and numerical method, such as homogenous and non homogenous, linear and nonlinear, etc. This study is delimited to positive solution of second order m-point boundary value problems, by the use of Schauder fixed point theorem and analysis method without sublinear and super linear cases.

CHAPTER TWO

2 LITERTURE REVIEW

Many phenomena in engineering , physics, chemistry, and other sciences can be described very successfully by models using mathematical tools from calculus. The non local boundary value problem for ordinary differential equations arise in variety of applied mathematics and physics, and describe many phenomena in applied mathematical sciences .For example vibrations of a guy wire of uniform cross –section and composed on N parts of different densities can be set up as a multi point boundary value problems, many problems in the theory of elastic stability can be handled by the method of multi point problems. The existence of positive solutions for second order non linear three point integral boundary value problems by using the Leray –schauder fixed point theorem, some sufficient conditions for the existence of positive solutions are obtained which improve the results of literature Tariboon and Sitthiwiratthan by using Leray –schauder fixed point [6].

Several problems arising in science and engineering are modeled by differential equations that involves homogeneous and non homogeneous, linear and non linear, singular and non singular point boundary value problems O. Regan[12] .there are mathematically modeled observations certainly give enlargement to a class of singular point second order ordinary differential equations along coherent boundary or initial conditions . Some existence, non existence and multiplicity results of positive solutions are established for four point boundary value problem. Then Gupta[4] studied three point boundary value problems for non linear ordinary differential equations .Since then more general non –linear multi –point boundary value problems have been studied by many authors by using the Leray-Schauder continuation theorem Gupta (3) for some recent results .

Multi point boundary value problem that arise from different areas of applied mathematics and physics have received a lot of attention in the literature in the last decades (see for example, 2, 3, 5, 10, 12 and references there.

The study of multi point boundary value problems for linear second order ordinary differential equations was initiated by Il´in and moiseer [12] for additional back grounds and results.

Since more general non liner singular two point boundary value problems have been broadly studied but there are so many effective ways of solving class at linear as well as non linear singular two point boundary value problems.

In most cases the specified ordinary differential equation do not only need analytical approach as result it can be treated through various approximate and numerical approaches.

R.Ma[10,13] obtained the existence of one positive solution for more general three point boundary value problems under the assumption that f is super linear where $f \in c[0,\infty],[0,\infty]$,there main tool is a fixed point theorem for mapping defined on banach spaces with cone [3].for back ground information and applications of such a fixed point theorem ,one can refer to[17].

The existence of at least three positive solutions for three point boundary value problems is Leggett-williams fixed point theorem [11]. most of literature about multi point B.V.Ps did not argue positive solution ,mean while in many situations ,only positive solutions is meaningful [10] .recently Ma show the existence and multiplicity of positive solutions for some three point non linear ordinary differential equation of boundary value problems.[4] .The fact that such type of class of difficulties are related to specific area of the field of differential equation that has been the problem of enormous research and powerful interest to researchers in recent past . A sub class of second order ordinary differential equation of the type followed by imposed initial or boundary condition problems come in to existence.

Existence, uniqueness and positive solution of a system of second order boundary value problems are discussed by banach and schauder fixed point theorem. Then applications of schauder fixed point theorem on the existence of positive periodic solutions to differential equation generalize results contained in multiplicity of positive periodic solution. [5]

Singular two point boundary value problems of non linear second order ordinary differential equations arise very frequently in many applications like gas dynamics ,nuclear physics, chemical reactions, atomic calculation and the study of positive solutions of non linear elliptic equations .The problems of non linear boundary value problems in ordinary differential equations have been treated in different mathematical approaches and by using fixed point theorem, we present sufficient conditions which ensure the existence of positive solution of second order m-point boundary value problems .

Guo discussed also the existence and uniqueness of solutions of a two point boundary value problem for second order non linear differential equations using fixed point theorem, by using fixed point theorem the authors study the existence of at least one positive solution for second order m-point boundary value problem.

Therefore this study was targeted to develop scheme to find positive solutions of second order m-point boundary value problems by applying schauder fixed point theorem and analysis method.[R.MA.N castaned] without the use of cases of super linear and sublinear

A point of central importance in the study of non-linear singular point initial value problems (IVPs) and boundary value problems (BVPs) in second order ordinary differential equations (odes) is to avoid the singularity and then to solve its solution and check the existence and uniqueness of solution in the neighborhood of singular point for the given initial and boundary condition. During the last two decades, the theory of singular two-point boundary value problem (BVP) has been comprehensively developed by R. P. Agarwal and D. O'ReganAgarwal [14]

As a result singular point initial value problems (IVPs) and boundary value problems (BVPs) of nonlinear second order ordinary differential equations (odes)are applicable in many areas of science and Several problems arising in science and engineering are modeled by ordinary differential equations that involves homogeneous and non-homogeneous, linear and nonlinear, singular and nonsingular point boundary value problems (BVPs)O'Regan**[**15**]**.

In order to acknowledge and study the essential properties and systematic behavior associated to a class of singularity occurring on various fronts relating to astrophysics, electro hydrodynamics and human physiology in multidisciplinary sciences, mathematical modeling is must and that has proved an efficient way to mathematicians and large now.

There are mathematically modeled observations certainly give enlargement to a class of singular point second order ordinary differential equations along with coherent boundary / or initial conditions Wang, J, Gao, W, Zhang, Z **[**18**]**. In 1986, the study of singular point boundary value problems (BVPs) for non-linear differential equations (odes) was initiated by Kinguradze and Lomtatidze**[7]**.In the study of nonlinear ordinary differential equations manifest in physics, engineering and other sciences.

Many mathematical representations, associated the singular two-point boundary value problems (BVPs) with nonlinear second order ordinary differential equations and such ordinary differential equations (odes)transpire in different areas of applied mathematics, astrophysics and control theory problems Castro, A,Kurepa[1**]** and their problems can be solved by different mathematical approaches such as analytical and numerical methods.

One of the common things that for solving singular boundary value problems are that the original differential equation is represented differently at singular point and at other points, that is, non-singular points in the given interval in its original form. In most cases, this problem does not always treated analytically at initial and boundary conditions

The techniques of solving positive solution of second order m-point boundary value problems of non-linear second order ordinary differential equations without sublinear and super linear cases have not been discussed so far. Hence developing such techniques the most successive way in order to avoid the singularity and solve the singular m-point boundary value problems of nonlinear second order ordinary differential equations in the neighborhood of the singular points for the given boundary conditions by using the concept of open subsets.

CHAPTER THREE

METHODOLOGY

3.1 Study site, area and period

The study will be conducted to develop a scheme of finding analytical positive solutions of second order m-point boundary value problems by using Schauder fixed point theorem and analysis method under mathematics department in Jimma University from September 2014 to September 2015.

3.2 Study design

The study was conducted analytically.

3.3 Source of information

The sources of data for this study was secondary data and which can be collected through reference books ,internet, reading on lines books and different published research articles (or journals) that heads to find positive solution of second order m-Point boundary value problem of ordinary differential equations.

3.4 Procedure of the study

In order to achieve the objectives of this study, the researcher will use the procedures which are almost the same to the standard techniques which were used in R.Ma (10) and R.Ma (12).

3.5 Ethical issues

Books, journals, internet and other related materials will be needed for this study to collect related information's. But there may be a problem to get and use all these materials without any cooperation request letter .so the cooperation request letter was taken by the researcher to the institutes where these materials are available to get consent from them. Moreover, rules, regulations of the university from which information was collected and get materials were kept by the researcher.

CHAPTER FOUR

4 DISCUSSIONS AND MAIN RESULT

4.1 Preliminary result

Definition 1 Open set:- It is an abstract concept generalizing the idea of an open interval in the

Real line

Definition 2 Open subset:-Given a metric space X and let $S \subset Y \subset X$.S is an open subset of Y, if

for all $p \in S$ there exists $u > 0$ such that $d\,(p, q) < u, q \in X$implies $q \in S$

: S to be an open sub set of Y if it is open and subset of Y

: S is open in Y if and only $S = Y \cap T$ for some open subset $T \subseteq X$

Definition 3 Banach spaces

. Areal or a complex normedspace is a real or complex vector space X together with

a map: X→ ℝ is the norm ‖.‖ such that:-

i) $\|x\| \geq 0$for all x∈ X and $\|x\| = 0$ if and only if $x = 0$
ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and α ∈ ℂorℝ
iii) $\|x + y\| \leq \|x\| + \|y\| For\ x, y \in X$

Definition 4 A metric space d on X satisfies $d\,(p, q) = \|x - y\|$ and a sequence $\{x_n\}$ in metric

Space(x, d) is Cauchy sequence if $\lim_{p,q\to\infty} d(x_p\,, x_q\,) = 0$

Definition 5 A metric space(x, d) is banach space (complete metric space) if every Cauchy

Sequence in metric space converges [Roden]

♥ Consider the following boundary value problems

$$u''\,(t) + a(t)f(u) = 0, \qquad 0 < t < 1$$

$$u'(0) = \sum_{i=1}^{m-2} b_i u'(x_i), u(1) = \sum_{i=1}^{m-2} k_i u\,(x_i)$$

Under the following assumptions'

(A_1) $x_i \in (0,1)$ with $0 < x_1 < x_2 < \cdots < x_{m-2} < 1.\, k_i$, $b_i \in (0,\infty), (i = 1,2,3, \dots m-2)$ satisfying$0 < \sum_{i=1}^{m-2} k_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$

(A_2) $f \in C([0,\infty),[0,\infty))$; $a \in C([0,1],[0,\infty))$and $a(t) \neq 0$on$[0,1]$ .Then the boundary value problem (1.1) has at least one positive solution in one of the following cases.

(i) $f_0 = 0$ *and* $f_\infty = +\infty$ (Super linear case),

(ii) $f_0 = +\infty$ *and* $f_\infty = 0$ (Sublinear case),

Where, $f_0 = lim_{u \to 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}$

This is proved by Schauder fixed point theorem and analysis method. Then the aim of this study is the above B.V.P without the super linear condition or sub linear condition, which gives at least two positive solutions to the question stated by (1.1). The key tool in finding our main results is the well known fixed point theorem due to Krasnoselskii [9].

Theorem 1.1 Let E be a Banach space and K be a cone in E. Assume Ω_1 and $\Omega_2$are open subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $A: K \cap (\Omega_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that

Either $\|Au\| \leq \|u\|$for $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$

Or

$\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$

A has a fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$

Suppose let $E = C[0,1]$ the space of all continuous functions $u: [0,1] \to \mathbb{R}$.This is banach space when it is capable with the usual sup norm

$$\|u\| = Sup|u(t)|, t \in [0,1]$$

For the sake of convenience we set

$$\mu = \frac{\sum_{i=1}^{m-2} k_i(1-x_i)}{1-\sum_{i=1}^{m-2} k_i x_i}\mu_1 = \frac{1-\sum_{i=1}^{m-2} k_i}{\sum_{i=1}^{m-2} k_i(1-x_i)\int_0^1 a(s)ds}$$

$$\mu_2 = \frac{1-\sum_{i=1}^{m-2} k_i}{\int_0^1 (1-s)a(s)ds + \frac{\sum_{i=1}^{m-2} b_i \int_0^{x_i} a(S)ds\,(1-\sum_{i=1}^{m-2} k_i x_i)}{1-\sum_{i=1}^{m-2} b_i}}$$

Set $k = \{u \in E: u \geq 0, inf\ u(t) \geq \mu\|u\|\}$ is a cone in E where$t \in [0,1]$

Lemma 1 *suppose* $\sum_{i=1}^{m-2} k_i \neq 1, \sum_{i=1}^{m-2} b_i \neq 1$ *If*$h(t) \in C(0,1)$ *and* $h(t) \geq 0$. *Then the B.V.P*

$$u"(t) + h(t) = 0 \qquad\qquad 0 < t < 1$$

$$u'(0) = \sum_{i=1}^{m-2} b_i\, u'(x_i) \quad , \qquad u(1) - \sum_{i=1}^{m-2} k_i u(x_i) = 0$$

has a unique solution $u(t) = -\int_0^t (t-s)h(s)ds - \alpha t + \beta$

Where $\boldsymbol{\alpha} = \frac{\sum_{i=1}^{m-2} b_i \int_0^{x_i} h(s)ds}{1-\sum_{i=1}^{m-2} b_i}$

$$\beta = \frac{1}{1-\sum_{i=1}^{m-2} k_i}\left(\int_0^1 (1-s)h(s)ds\right) - \frac{\sum_{i=1}^{m-2} k_i \int_0^{x_i}(x_i-s)h(s)ds}{1-\sum_{i=1}^{m-2} k_i} + \frac{1}{1-\sum_{i=1}^{m-2} k_i}(\alpha\left(1-\sum_{i=1}^{m-2} k_i x_i\right))$$

Proof From (2.1), we have $u"(t) = -h(t)$

For $t \in [0, 1)$ integration from 0 to t yield

$$\Rightarrow u'(t) = -\int_0^t h(t)ds + u'(0)$$

For $t \in [0,1]$, integration from 0 to t yields

$$\Rightarrow u(t) = -\int_0^t (\int_0^x h(s)ds))dx + u'(0)t$$

i.e. $u(t) - \int_0^t (t-s)h(s)ds + u'(0)t$, Then we have: $u(1) = -\int_0^1 (1-s)h(s)ds + u'(0)$ and $u(x_i) = -\int_0^{x_i}(x_i - s)h(s)ds + u'(0)x_i$

$u(1) - \sum_{i=1}^{m-2} k_i u(x_i) = 0$This implies that

$$-\int_0^1 (1-s)h(s)ds + u'(0) - \sum_{i=1}^{m-2} k_i(-\int_0^{x_i}(x_i - s)h(s)ds + u'(0)x_i) = 0$$

$$u'(0) - \sum_{i=1}^{m-2} k_i\, u'(0)x_i = \int_0^1 (1-s)h(s)ds - \sum_{i=1}^{m-2} k_i \int_0^{x_i}(x_i - s)h(s)ds$$

$$u'(0)\ (1 - \sum_{i=1}^{m-2} k_i\, x_i) = \int_0^1 (1-s)h(s)ds - \sum_{i=1}^{m-2} k_i \int_0^{x_i}(x_i - s)h(s)ds$$

$u'(0) = \frac{1}{1-\sum_{i=1}^{m-2} k_i x_i}\int_0^1 (1-s)h(s)ds - \frac{1}{1-\sum_{i=1}^{m-2} k_i x_i}\sum_{i=1}^{m-2} k_i \int_0^{x_i}(x_i - s)h(s)ds.$

$$u(t) = -\int_0^t (t-s)h(s)ds + \frac{t}{1-\sum_{i=1}^{m-2} k_i\, x_i}\int_0^1 (1-s)h(s)ds - \frac{t}{1-\sum_{i=1}^{m-2} k_i\, x_i}\sum_{i=1}^{m-2} k_i \int_0^{x_i}(x_i - s)h(s)ds$$

From this equation (2.1) is satisfied .Therefore it has a unique solution.

$$u(t) = -\int_0^t (\int_0^x h(s)d(s))dx - \alpha t + \beta$$

$\Rightarrow u(t) = -\int_0^t (t-s)h(s)ds - \alpha t + \beta$. The proof is completed.

Lemma 2.Suppose the condition A_1 is satisfied If$h\,(t) \in C[0,1]$and $h(t) \geq 0$ on $[0,1]$then the unique solution $u\,(t)$ of (2.1) satisfies $u\,(t) \geq 0$for $t\epsilon[0,1]$

Lemma 3 Suppose the condition $A_1$is satisfied. If $h\,(t) \in [0,1]$ and $h\,(t) \geq 0$ on $[0,1]$ then unique solution $u\,(t)$ of (2.1) satisfies $inf u\,(t) \geq \mu\|u\|\ \ t \in [0,1]$

We define the operator $A: K \to E$

By$Au\,(t) = -\int_0^t (t-s)a(s)f(u(s)ds - \alpha t + \beta$……………………………………….(2.2)

Where αandβ are as in lemma (2.1) with $h(s) = a(s)\,f\,(u(s))$.

Lemma.4 If the conditions A_1 and A_2 are satisfied then the operator defined in (2.2) is

Completely continuous and satisfies $A\,(k) \in k$

4.2 Proof of the main result

The literature [6] studied the existence of positive solution of B.V.P(1.1) by using Krasnoselskii fixed point theorem obtained super linear and sublinear cases ,then B.V.P has at least one positive solution.

Now by using theorem 1.1 and Krasnoselskii [9] positive solutions of operator equations, we are in a position to present and prove main results. In what follows, we always assume that the condition A_1 and A_2 are satisfied.

Theorem 4.1The boundary value problem (1.1) has at least two positive solutions $u_1(t)$ and u_2 (t) such that$0 < \|u_1\| < p < \|u_2\|$ If the following assumptions are satisfied

(H_1)$f_0 = f_\infty = +\infty$

(H_2) There exist constant $p > 0$ and $m \in (0, \mu_2)$ such that $f(u) \leq mp,\ u \in [0, p]$

Proof : u(t) is the solution of the boundary value problem(1.1) if and only if u(t) solve the operator equation Au(t)= u(t) where the operator A is defined in (2.2) thus we only need to verify that the operator A has two positive fixed point in k

At first, in view of $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u} = +\infty$ then for a constant $m_1 \in \left[\frac{\mu_1}{\mu}, \infty\right]$, there exists apositive constant $p_1 \in (0, p)$ such that $f(u) \geq m_l u$ for

$u \in [0, p_1]$……………………………………………………………………..4.1

Define the first open set of E by

$\Omega_1 = \{u/u \in E\ \ \|u\| < p1\}$……………………………………………………4.2

Since $Au(0) = \frac{1}{1-\sum_{i=1}^{m-2} k_i}[\int_0^1 (1-s)a(s)f(u(s))ds - \sum_{i=1}^{m-2} k_i \int_0^{x_i}(x_i - s)a(s)f(u(s))ds + \alpha\,(1 - \sum_{i=1}^{m-2} k_i x_i\ )]$

$$\geq \frac{1}{1-\sum_{i=1}^{m-2} k_i}\left(\sum_{i=1}^{m-2} k_i \int_0^1 (1-s)a(s)h(s)ds - \sum_{i=1}^{m-2} k_i \int_0^{x_i} (x_i - s)a(s)h(s)ds\right)$$

$$\geq \frac{\sum_{i=1}^{m-2} k_i\ (1-x_i) \int_0^1 a(s)h(s)ds}{1-\sum_{i=1}^{m-2} k_i}$$

Thus from (4.1) and (4.2) and noting that $Inf\ u\ (t) \geq \mu\left|\left|u\right|\right|,\ u \in K \cap \partial\Omega_1 \in k$ we get for any $u \in k\ \cap \partial\Omega_1$

$$Au\ (0) \geq \frac{\sum_{i=1}^{m-2} k_i (1-x_i) \int_0^1 a(s) m_1 u(s) ds}{1-\sum_{i=1}^{m-2} k_i} \geq \frac{\mu m_1 \|u\| \sum_{i=1}^{M-2} k_i (1-x_i) \int_0^1 a(s) ds}{1-\sum_{i=1}^{m-2} k_i} \geq ||u||$$

Which implies $\|Au(t)\| \geq ||u||, u \in\ k \cap \partial\Omega_1$……………………………………….4.3

On the other hand since $\lim_{u\to+\infty} \frac{f(u)}{u} = +\infty$ .Then for a constant $m_2 \in [\frac{\mu_1}{\mu}, +\infty]$, there exists a positive constants $p_2 > p$ such that $f(u) \geq m_2 u$ for $u > \mu p_2$

Define the second open subset of E by:

$\Omega_2 = \{u: u \in E\ \|u\| < p_2\}$in the same way as above we have

$\|Au(t)\| \geq \|u\|, u \in k \cap \partial\Omega_2$…………………………………………………………….4.4

Finally define the open subset of E by:

$\Omega = \{u: u \in E: \left||u|\right| < p\}$ From the expression of $Au(t)$ equation (2.2) we can see

$$Au\ (t) \leq \frac{1}{1-\sum_{i=1}^{m-2} k_i} [\int_0^1 (1-s) a(s) f\big(u(s)\big) ds]. + \frac{\sum_{i=1}^{m-2} b_i \int_0^{x_i} a(s) f\big(u(s)\big) ds}{1-\sum_{i=1}^{m-2} b_i} \times (1 - \sum_{i=1}^{m-2} k_i x_i)$$

Thus for any $u \in K \cap \partial\Omega$ from condition (H_2) we obtain

$$Au\ (t) \leq \frac{m\left||u|\right|}{1-\sum_{i=1}^{m-2} k_i} [\int_0^1 (1-s) a(s) ds\ + \frac{\sum_{i=1}^{m-2} b_i \int_0^{x_i} a(s) ds}{1-\sum_{i=1}^{m-2} b_i} (1 - \sum_{i=1}^{m-2} k_i\, x_i)]$$

$$= \frac{m||u||}{\mu} \leq ||u||$$

Which implies $\|Au(t)\| \leq\ ||u||\ , u \in k \cap \partial\Omega$……………………………………………….4.5

Therefore it follows from (4.3), (4.4), (4.5) and theorem 1.1 that A has a fixed in $\cap\ (\bar{\Omega}/_{\Omega_1})$ and a fixed point in $k\ \cap\ (\ \bar{\Omega}_2/_{\Omega})$. We finish the proof of theorem (3.1)

Theorem 3.2 The boundary value problem (1.1) has at least two positive solution $u_1(t)$ and $u_2(t)$ such that $0 < \left||u_1|\right| < q < ||u_2||$ if the following assumptions are satisfied

H_3. $f_0 = f_\infty = 0$

H_4. There exists $q > 0$ and $n \in (\frac{\mu_1}{\mu}, \infty)$ such that $f(u) \geq nq, u \in [\mu q, q]$

Proof: since $f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = 0$ then for constant $\varepsilon \in (0, \mu]$ there exists positive constant $q1 < q$ such that $f(u) \leq \varepsilon u$ for $u \in [0, q_1]$……………………………………………4.6

Assume γ_1 and γ_2 are open subsets of E with $0 \in \gamma_1 \subset \bar{\gamma_1} \subset \gamma_2$ and let $A: k \cap (\bar{\gamma}_2/\gamma_1) \to K$ be a completely continuous operator satisfying the properties in theorem (1.1).

$\gamma_1 = \{u: u \in E: ||u|| < q_1\}$………………………………………………… ………4.7

The same derivative can be used with m replaced by ε in equation (4.5)we get

$Au(t) \leq \varepsilon \frac{||u||}{\mu_1} \leq ||u||, u \in k \cap \partial\gamma_1$……………………………………………………..4.8

Next, in view of $f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u} = 0$ then for constant $\varepsilon \in (0, \mu_1]$ there exists a positive constant $q_2 > q$ such that

$f(u) \leq \varepsilon u, u \in [\mu q_2, \infty)$…………………………………………………….......4.9

Now define the second open subset of E by $\gamma_2 = \{u: u \in E\ \|u\| < q_2\}$…………………4.10
In the same way as above we have:

$\|Au(t)\| \leq \|u <\| < q_2\}$……………………………………………………………………. 4.11

Finally, define the open subset of E by $\gamma = \{u: u \in E, ||u|| < q\}$………………………………4.12

In virtue of $\inf_{t \in [0,1]} u(t) \geq \mu||u||$ for $u \in K \cap \partial\gamma \in K$ and the condition (H_4), the same derivation can be used with m_1 replaced by m in equation (4.3)

We obtain for any $u \in k \cap \partial\gamma, Au(0) \geq \mu n||u|| \sum_{i=1}^{m-2} k_i(1 - x_i) \int_0^1 a(s)ds > ||u||$ which implies $||A u(t)|| \geq ||u||, u \in K \cap \partial\gamma$…………………………………………………………4.13

Therefore it follows from (4.8), (4.11), (4.13) and theorem (1.1) that A has a fixed point in

$K \cap \left(\bar{\gamma}/\gamma_1\right)$ and a fixed point in $k \cap (\bar{\gamma_2}/\gamma)$, the proof of (3.2) is completed.

From the proof of theorem (3.1) and theorem(3.2) above ,we justify that there exists at least two positive solutions to the boundary value problem stated above by(1.1) .

Example 1. Let $f\ (u) = \frac{u^2}{u^2+1}$

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to +\infty} \frac{f(u)}{u} = \lim_{u\to 0^+} \frac{u}{u^2+1} = \lim_{u\to +\infty} \frac{u}{u^2+1} = 0$$

Since$f_0 = 0 = f_\infty$ without the use of super linear & sublinear cases, the B.V.P (1.1) has at least two positive solutions

Example 2. Let $f(u) = u^p$

- If $p > 1$, $\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to 0^+} u^{p-1} = 0$ and $\lim_{u\to +\infty} \frac{f(u)}{u} = \lim_{u\to +\infty} u^{p-1} = \infty$
- If $p \in C\ (0,1), lim_{u\to 0^+} \frac{f(u)}{u} = lim_{u\to 0^+} u^{p-1} = \infty$ and $\lim_{\to +\infty} \frac{f(u)}{u} = \lim_{u\to +\infty} u^{p-1} = 0$

Then B.V.P (1.1) has at least two positive solutions.

Example 3. Let $f(u) = \frac{u \sin(u)}{e^u}$ then$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to \infty} \frac{f(u)}{u} = 0$

This example illustrate that there exists at least two positive solutions to the equation (1.1) without considering super linear and sub linear cases.

Example 4. let $f\ (u) = u|lnu|$, where $u > 0$ then$\frac{f(u)}{u} = \frac{u|\ln u|}{u} = |\ln u|$

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to +\infty} \frac{f(u)}{u} = \lim_{u\to 0^+} |inu| = \lim_{u\to +\infty} |inu| == +\infty$$

Since$f_0 = f_\infty = +\infty$ the boundary value problem stated in (1.1) above has at least two positive solutions

CHAPTER-FIVE

5. CONCLUSION AND FUTURE SCOPES

In this study, new application of schauder fixed point theorem and analysis method without super linear and sub linear cases of solving positive solution of second order m-point boundary value problems, were introduced to handle multi point boundary value problems of ordinary differential equation .The mathematical operation which were performed by schauder fixed point theorem and analysis method, were deduced from the krasoleskii [9] fixed point theorem and some definition of open subsets.

Multi point B.V.Ps for non linear differential equations have been studied by many authors such as [6], [19] and soon. To see the effectiveness and applicability of schauder fixed point theorem and analysis method of solving positive solution of second order m-point boundary value problems without super linear and sublinear cases, for the test some examples were presented. This leads to the method used by the researcher was successfully implemented to obtain positive solution of second order m-point boundary value problems.

Therefore, the subsequent researcher can extend the techniques that in this work to solve positive solution of m-point boundary value problems of third and more than third ordinary differential equations.

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