SIXTH-ORDER STABLE CENTRAL DIFFERENCE METHOD FOR SELF-ADJOINT SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Numerical Analysis)

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DECLARATION

I undersigned declare that this thesis entitled **"Sixth-Order Stable Central Difference Methods for Self-adjoint Singularly Perturbed Two-Point Boundary Value Problems**" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Abstract

In this thesis, sixth order stable central difference method has been presented for solving self-adjoint singularly perturbed two-point boundary value problems. First the given interval is discretized and the derivatives of the given differential equation are replaced by the central difference approximations. Then, the given differential equation is transformed to linear system of algebraic equations. Further, this algebraic system is transformed into the three term recurrence relation, which can easily be solved by using Thomas Algorithm. To validate the applicability of the proposed method, some model examples have been considered and solved for different values of perturbation parameter and mesh sizes. The stability and convergence of the method have been analyzed. As it can be observed from the numerical results in tables and graphs, the presented method approximates the exact solutions very well and provides better results than some existing numerical methods reported in the literature.

Chapter One
Introduction

1.1. Background of the Study

Numerical analysis is the branch of mathematics that deals with finding approximations to difficult problems such as finding the roots of non-linear equations, integration involving complex expressions and solving differential equations for which analytical solutions do not exist. Numerical analysis plays a significant role when we face difficulties in finding the exact solution of an equation using a direct method when it becomes very difficult to apply theoretical methods proposed earlier to find the exact solution. We can solve problems by higher order method or lower order method but a higher order method gives a more accurate numerical solution than the lower order method for a fixed step size and to obtain a numerical solution with an acceptable accuracy we have to use a very small step size.

Singularly perturbed differential equation is an ordinary differential equation in which the highest order derivative is multiplied by a small positive parameter $\varepsilon\,(0<\varepsilon<<1)$ and the behaviors of the solutions of these differential equations depends on the magnitude of the parameter. In general, any differential equation whose solution changes rapidly in some parts of the interval and changes slowly in some of the parts of the interval is known as singular perturbation problem and also known as boundary layer problem.

An adjoint differential equation is a linear differential equation usually derived from its primal equation using integration by part. It means that, a differential equation obtained from a given differential equation and having property that the solution of one equation is an integrating factor of the other and a differential equation that has the same solution as its adjoint equation is known as self-adjoint differential equation.

In general, self-adjoint singularly perturbed differential equation is a differential equation that has the same solution as its adjoint equation in which the highest order derivative is multiplied by a small positive parameter.

Details of self-adjoint Singular Perturbation Problems(SPPs) are discussed in the books of Delkhon and Delkhosh [4]and Miller et al. [14].Singular Perturbation Problems (SPPs) arise very frequently in diversified fields of applied mathematics and engineering, for instance fluid mechanics, elasticity, hydrodynamics, quantum mechanics, plasticity, chemical-reaction theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, oceanography, meteorology, modeling of semiconductor devices, diffraction theory and reaction-diffusion processes and many other allied area. Due to this, studying the singular perturbation problem is a very attractive issue in the contemporary mathematical circles. The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equation that has a multi-scale character .However, most of the problems cannot be completely solved analytically, as there is a thin layer where the solution varies rapidly, while away from the layer, the solution behaves regularly and varies slowly. Therefore, the usual numerical treatment of singularly perturbed problem gives major computational difficulties and in recent years a large number of special purpose methods have been developed to provide accurate numerical solutions. But the occurrence of sharp boundary layers as the coefficient of highest derivative approaches to zero creates difficulty for most standard numerical methods. Moreover there are a wide variety of asymptotic expansion methods available for solving the problem of the above type. However, there is a difficulty in applying these methods as finding the approximate asymptotic expansions in inner and outer region are not easy.

There are a lot of techniques for solving self-adjoint singular perturbation problem. For instance, discrete numerical methods have been suggested by various authors for self-adjoint singularly perturbed problems like, initial value technique method [15],quintic non-polynomial spline functions method [20,22],difference scheme using cubic spline [19],finite difference method with variable mesh [10], fitted mesh B-spline collocation method [6,14] etc . It is well-known that existing numerical methods produce good results only when we take $h < \varepsilon$ but, if we take h $\geq \varepsilon$ they produce oscillatory solutions (totally bad results) and also most of the classical methods are unstable and fail to give accurate results when the perturbation parameter is small relative to mesh length h$(h \geq \varepsilon)$ that is used for discretization of the difference equation. Farther, classical computational approaches to singularly perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory computed solutions and this is very costly and time consuming method [5]. Hence, one has to go for non-classical method Jain [12].Some non-classical methods are suggested by various authors, but few authors have developed numerical methods for self-adjoint singular perturbed boundary problems. Hence, the purpose of this research is to develop suitable numerical method, which is more efficient and simple to solve self-adjoint singularly perturbed two-point boundary value problems. In this study, we restrict ourselves to find the numerical solution for linear second order self-adjoint singularly perturbed two-point boundary value equation of the form:

$$-\varepsilon(p(x)y')' + q(x)y = f(x),\ 0 \le x \le 1 \qquad (1.1)$$

subjected to the boundary conditions:

$$y(0) = \alpha\ , \qquad y(1) = \beta \qquad (1.2)$$

where $\alpha,\ \beta \in \Re$ and ε is a small positive parameter and $p(x)$, $q(x)$ and $f(x)$ are smooth functions, such that $p(x) > 0$.

The main purpose of this study is to design/present sixth order stable central difference method for solving self-adjoint singularly perturbed two-point boundary value problems.

1.2. Statement of the Problem.

The increasing desire for the numerical solutions to mathematical problems, which are more difficult or impossible to solve explicitly, has become the present- day scientific research. Thus, this shows the importance and application of numerical methods to solve problems in real life. Among the methods used to find approximate solution of ordinary differential equations with boundary condition the finite difference method is one that approximates the solution of singularly perturbed two-point boundary value problems.

The numerical method used to find approximate solution of boundary value problem has an impressive importance but, solving singularly perturbed problems is unstable and fail to give accurate results when the parameter ε is very small. Classical computational approaches to singularly perturbed problems are known to be inadequate as they required extremely large number of mesh points to produce satisfactory computed solutions Farell et al [6].So, solving singularly perturbed problems have various difficulties to get accurate numerical solutions Kadalbajoo [10]. Thus, existing numerical methods produce good results only when we take step size $h < \varepsilon$ but, this is very costly and time consuming process. Therefore, this study aims to present numerical method which is stable, simple and more efficient for $h \ge \varepsilon$

As a result, this present study attempts to answer the following questions:

- How do we describe the sixth order stable central difference method for self-adjoint singularly perturbed differential equation?
- To what extent the method approximate the exact solution?
- To what extent the present method converges?

1.3. Objectives of the Study

1.3.1. General objective

The general objective of this study is to present numerical method which is simple, efficient and easily adaptable for computer used for solving self-adjoint singularly perturbed differential equation.

1.3.2. Specific objectives

The specific objectives of the study are:

- To describe sixth order stable central difference method for second order self-adjoint singularly perturbed two-point boundary value problem.
- To compare the solutions obtained with the exact solution.
- To establish the stability and convergence of the method.

1.4. Significance of the Study

The results of this study may contribute to research activities in this area. In addition to that it may be useful for students of the department to develop their research skills. Further, collaboration in this research project may be useful for the graduate program of the department and enhances the research skills and scientific communication of the researcher. As a result, the study might:

- serve as a reference material for anyone who works on this area.
- improve the application of numerical methods in different field of studies.

1.5. Delimitation of the Study

This study delimited to solve self-adjoint singularly perturbed two-point boundary value problems by sixth order stable central difference method.

Chapter Two
Review of Related Literature

2.1. Singular Perturbation Problem

The problems in which the highest order derivative term is multiplied by a small parameter are known to be perturbed problems and the parameter is known as the perturbation parameter. A perturbed problem, whose solution can be approximated on the whole of its domain whether space or time by a single asymptotic expansion, is a regular perturbation.

Most often in applications, an acceptable approximation to a regularly perturbed problem is found by simply replacing the small parameter ε by zero everywhere in the problem statement. This leads to take only few terms of the expansion which have no coefficient perturbation parameter, yielding an approximation that converges slowly, to the true solution as ε decreases. The solution to a singularly perturbed problem cannot be approximated in this way, since taking the parameter to be zero changes the nature of the problem described by singular perturbation problem associated with various types of differential equations. These singular perturbed problems arise in modeling various modern complicated processes, such as a fluid flow at high Reynolds, chemical reactor theory, electro magnitude field problem in moving media, electro analytical chemistry etc

2.2 Self-adjoint Differential Equation

A differential equation that has the same solution as its adjoint equation is known as self-adjoint differential equation.

A second order linear differential equation is called self-adjoint if and only if it has the following form:

$(p(x)y')' + q(x)y = f(x)$; $a < x < b$ where $p(x) > 0$ on (a,b) and $q(x), p'(x)$ and $f(x)$ are continuous functions on $[a,b]$ Delkhon and Delkhosh [4]

2.3. Numerical Methods

It is well known fact that the solution of singular perturbed boundary value problem exhibits a multi-scale character. That is there is a thin layer, where the solution varies rapidly, while away from the layer the solutions behave regularly and varies slowly. Therefore, usual numerical treatment of SPP gives major computational difficulties and in recent years a large number of special purpose methods have been developed to provide accurate numerical solution. The occurrence of sharp boundary as the coefficient of highest derivative approaches zero creates difficulty for most standard numerical methods. There are a wide variety of perturbation methods such as asymptotic expansion method, WKB method and multiple scale method available for solving the problems of the above type. However, difficulties in applying these methods, such as finding the approximate asymptotic expansions in inner and outer regions are not easy. *So* many approximate methods have been developed and refined, including finite difference methods and spline method.

2.3.1 Spline Methods

In the last 29 years remarkable progress has been made in the theory, methods and applications for the singular perturbation in the mathematical circles and a lot of new results have appeared .To be more accessible for practicing engineers and applied mathematicians, there is a need for methods, which are easy and ready for computer implementation. The spline technique appears to be an ideal tool to obtain these goals. There have been considerable amount of work using various spline methods for solution of singular perturbation problems. For instance, Bawa [2] proposed a spline based computational technique suitable for parallel computing for singularly perturbed reaction problem. To solve the problem, the author first decomposed the domain in to three non-overlapping sub domains and sub problems corresponding to boundary layer regions are solved by using adaptive spline scheme.Khan et al. [9] describes a sixth order method based on sextic splines.The advantage of this method is higher accuracy with the same computational effort as Bawa. It is computationally efficient method and the algorithm can be easily implemented on a computer.

Surlaet.al.[21] Constructed a spline collocation method for singularly perturbed boundary value problem with two small parameters .The suitable choice of collocation points provides the inverse monotonicity enabling utilization of barrier function method in error analysis.

Rao and Kumar [16] presented a higher order cubic β-spline collocation method for the numerical solution of self-adjoint singularly perturbed boundary value problem that is much easier and more efficient for computing. The essential idea in this method is to divide the domain of the differential equation in to three non-overlapping sub domains and solve the regular problems obtained by transforming the differential equation with respective boundary conditions on these sub domains a higher order β-spline collocation method. The boundary conditions at the transition points are obtained using the zeros-order asymptotic approximation to the solution of the problem. The author used the two step spline approximate method, since a special type of tridiagonal system is obtained and proved the optimal order convergence of the method which is more efficient than classical finite difference scheme on piecewise uniform shishkin meshes as given by Farrell et al [5] and comparable with the quintic spline difference scheme of Bawa and Natesan[1]. Rashidinia et al. [18] developed the class of methods for the numerical solution of singularly perturbed two-point boundary value problems using spline in compression. The methods are second-order and fourth order accurate and applicable to both singular and non-singular perturbed problems. In the same year Rashidina et al [19] again developed a numerical technique for class of singularly perturbed two-point boundary value problems on a uniform mesh using polynomial cubic spline.

Tirmizi et al [22] used quinitic non-polynomial spline functions to develop a class of numerical methods for solving self-adjoint singularly perturbed problems. The methods are computationally efficient and the algorithm can easily be implemented on computer. Fourth and sixth order convergence is obtained. It has been shown that the relative errors in absolute value confirm the theoretical convergence.

Rao and Kumar [17] presented the exponential β-spline collocation method for numerical solution of self- ad joint singularly perturbed Dirichlet boundary value problem. It is relatively simple to collocate the boundary value problem at the nodal points the uniform mesh, to setup the collocation system and solve them. Examples show that this method is more efficient than the cubic β-spline collocation method on uniform mesh as well as the cubic β -spline collocation method on fitted mesh.

Mishra and Kumar [15] developed an initial value technique for self-adjoint singularly perturbed two-point boundary value problems by reducing the original problem to normal form and converted in to first order initial value problems. These initial value problems have been solved by the cubic spline method.

Rashidinia et al [20] developed non-polynomial quintic spline method for numerical solution of self-adjont singularly perturbed value problems, the relations have been derived using off-step points and the developed methods are fourth, sixth and eighth order accurate.

Recently Bisht and Khan [3] have applied the difference scheme using cubic spline for solving self-adjoint two-point boundary value problem. Their scheme leads to tridiagonal linear system. The convergence analysis is given, which shows the method is second and fourth order convergent depending upon the choice of the parameters

2.3.2 Finite Difference Methods

The finite difference methods are always a convenient choice for solving boundary value problems because of their simplicity. Finite difference methods are one of the most widely used numerical schemes to solve differential equations. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of equations to be solved by Thomas Algorithm in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward difference method, backward difference methods, central difference method, etc.The challenge in analyzing finite difference methods for new classes of problems is often to find an appropriate definition of stability that allow one to prove convergence and to estimate the error in approximation.

There are some finite-difference methods which have been suggested by various authors for self-adjoint singularly perturbed problems. For instance; Kadalbajoo and Kumar [10] developed a numerical method based on finite difference method with variable mesh for self adjoint singularly perturbed two-point boundary value problem by reducing the original problem to normal form and solved the reduce problem by finite difference method taking variable mesh. Lubuma and Patidar [13] designed non-standard finite difference scheme for self-adjoint singularly perturbed two-point boundary value problems by using appropriate renormalization of the denominator of the discrete derivative. In addition to this Kadalbajoo and Sharma [7] present a numerical scheme for a second order singularly perturbed boundary value problem, which works nicely in both cases, i.e. when the delay argument is the bigger one as well as the smaller one. To handle the delay argument, they constructed a special type of mesh so that the term containing delay lies on nodal points after discretization. Yadaw [8] presented B-spline collocation method for solving a class of two-parameter singularly perturbed boundary value problems and stabilized second order uniform convergence.

2.4 Numerical versus Analytical Methods

The techniques used for calculating the exact solution are known as analytic methods because we used the analysis to solve it. Analytical solution is continuous. The exact solution is also referred to as a closed form solution or analytical solution. But this tends to work only for simple differential equations with simple coefficients, for higher order or non-linear differential equations with complex coefficients; it becomes very difficult to find exact solution. Therefore, we need numerical methods for solving these equations. Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically, but numerical methods can give an approximate solution to any equation.

Chapter Three

Methodology

This chapter consists of the following methods and materials that have been used to carry out the study. These are; study design, study site and period, source of information, study procedure, and ethical considerations

3.1. Study Site and Period

The study was conducted at Jimma University, which is Ethiopia's first innovative community oriented education institution of higher learning, department of Mathematics from September 2014 to September 2015.

3.2. Study Design

This study was employed mixed design:

- Documentary review design
- Experimental design

3.3. Study Area

Conceptually this study has been focused on sixth order stable central difference method for self-adjoint singularly perturbed two point boundary value problems of differential equation with Dirichlet boundary conditions.

3.4. Source of Information

This study mostly depends on documentary materials and the numerical solutions which will be obtained by the help of MATLAB software. So, the sources of information for the study are books, journals and different related studies from internet services and numerical data obtained by writing MATLAB code for the present numerical method. The proposedmethod is programmed using MATLAB.

3.5. Study Procedures

The study is an experimental as it involves entirely laboratory work with the help of computer and MATLAB software. Further, important materials for the study were collected by the researcher using documentary analysis. The required numerical data was collected by coding and running using MATLAB software to get the numerical results and the graphs of some examples to check the validity and efficiency of the method.

In order to achieve the above mentioned objectives, the study follows the following steps:

1. Defining the Problem/formulation of the problem.
2. Discretizing the given interval.
3. Replacing the differential equation by the central difference approximations.
4. The given differential equation was transformed to algebraic equation then to tri-diagonal system.
5. The tri-diagonal system (TDS) obtained in step 4 is easily solved by Discrete Invariant Imbedding Algorithm (Tomas Algorithm).
6. Coding program in MATLAB software for the obtained tri-diagonal system.
7. Validatation of the present scheme by implementing it on some numerical examples.

3.6. Ethical Issues

To be legal for collecting all the information and materials for study, it is important to have a permission letter. So, the researcher got a letter of permission from ethical committee of the college and then the researcher explained the aim of collecting of those materials to concern body.

Chapter Four

Description of the Method, Results and Discussion

4.1. Description of the Method

To describe the method, considering Eqs(1.1) and (1.2), denoting $p(x_i) = p_i$, $q(x_i) = q_i$, $f(x_i) = f_i$, $y(x_i) = y_i$ for simplicity and rearranging, we get:

$$-y_i'' + a_i y_i' + b_i y_i = c_i \tag{4.1}$$

$$y(0) = \alpha \ , \quad y(1) = \beta \tag{4.2}$$

where $a_i = -\dfrac{p_i'}{p_i}, \quad b_i = \dfrac{q_i}{\varepsilon p_i} \ \ and \ \ c_i = \dfrac{f_i}{\varepsilon p_i}$

we divide the interval $[0, 1]$ into N equal subintervals with uniform step length h.

Let $0 = x_0 \le x_1 \le x_2 \le \ . \ . \ . \le x_N = 1$ be the mesh points. Then we have $x_i = x_0 + ih$ $i = 1, 2, ..., N-1$. Assume that $y(x)$ has continuous sixth order derivatives on $[0, 1]$.

By using Taylor series expansion, we obtain:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{(4)} + \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} + \frac{h^7}{7!} y_i^{(7)} + \frac{h^8}{8!} y_i^{(8)} + O(h^9) \tag{4.3}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{(4)} - \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} - \frac{h^7}{7!} y_i^{(7)} + \frac{h^8}{8!} y_i^{(8)} + O(h^9) \tag{4.4}$$

Then, by subtracting Eq. (4.4) from Eq. (4.3), we obtain:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' - \frac{h^4 y_i^{(5)}}{120} - \frac{h^6 y_i^{(7)}}{5040} + O(h^8)$$

Thus, the six- order central difference approximation for the first derivative of y_i is given by:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' - \frac{h^4 y_i^{(5)}}{120} + T_1 \tag{4.5}$$

where $T_1 = -\frac{h^6 y_i^{(7)}}{5040} + O(h^8)$

Similarly, adding Eqs. (4.3) and (4.4), we obtain the sixth order central difference approximation for the second derivative of y_i as:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} - \frac{h^4}{360} y_i^{(6)} + T_2 \tag{4.6}$$

where $T_2 = -\frac{h^6}{20160} y_i^{(8)} + O(h^8)$

Substituting Eqs (4.5) and (4.6) into Eq. (4.1), we get:

$$\left(-\frac{1}{h^2} - \frac{a_i}{2h}\right) y_{i-1} + \left(\frac{2}{h^2} + b_i\right) y_i + \left(\frac{a_i}{2h} - \frac{1}{h^2}\right) y_{i+1} - \frac{a_i h^2}{6} y_i''' + \frac{h^2}{12} y_i^{(4)} - \frac{a_i h^4}{120} y_i^{(5)} + \frac{h^4}{360} y_i^{(6)} = c_i + T_3 \tag{4.7}$$

where $T_3 = T_2 - a_i T_1$

By differentiating both sides of Eq. (4.1) successively and rearranging it, we obtain:

$$y_i''' = a_i y_i'' + (a_i' + b_i) y_i' + b_i' y_i - c_i' \tag{4.8}$$

$$y_i^{(4)} = (a_i^2 + 3a_i' + b_i) y_i'' + (a_i a_i' + a_i b_i + a_i'' + 2b_i') y_i' + (a_i b_i' + b_i'') y_i - a_i c_i' - c_i'' \tag{4.9}$$

$$\begin{aligned} y_i^{(5)} = {} & \left(a_i^2 a_i' + a_i^2 b_i + a_i a_i'' + 2a_i b_i' + 3a_i'^2 + 4a_i' b_i + b_i^2 + a_i''' + 3b''\right) y_i' + \\ & \left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b_i'\right) y'' + \left(a_i^2 b_i' + a_i b_i'' + 3a_i' b_i' + b_i b_i' + b_i'''\right) y_i \\ & - \left(a_i^2 + 3a_i' + b_i\right) c_i' - a_i c_i'' - c_i''' \end{aligned} \tag{4.10}$$

$$y_i^{(6)} = \left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b''\right) y_i'' +$$
$$\left(a_i^3 a_i' + a_i^3 b_i + a_i^2 a_i'' + 2a_i^2 b_i' \ + 7a_i a_i'^2 + 9a_i b_i a_i' + 2a_i b_i^2 + a_i a_i''' + 3a_i b_i'' + 10a_i' a_i'' +\right.$$
$$\left. 12a_i' b_i' + 7b_i a'' + 6b_i b_i' + a_i^{(4)} + 4b_i'''\right) y_i' +$$
$$\left(a_i^3 b_i' + a_i^2 b_i'' + 7a_i a_i' b_i' + 2a_i b_i b_i' + a_i b_i''' + 4a_i' b_i'' + b_i b_i'' + 6b_i' a_i'' + 4b_i'^2 + b_i^{(4)}\right) y_i$$
$$-\left(a_i^3 + 7a_i a_i' + 2a_i b_i + 6a_i'' + 4b_i' \right) c_i' - \left(a_i^2 + 4a_i' + b_i\right) c_i'' - a_i c_i''' - c_i^{(4)} \tag{4.11}$$

By substituting Eqs (4.8)- (4.11) into Eq. (4.7) and rearranging, we get:

$$\left(-\frac{1}{h^2} - \frac{a_i}{2h} \right) y_{i-1} + \left(\frac{2}{h^2} + b_i - \frac{a_i b_i' h^2}{6} + \frac{h^2}{12}\left(a_i b_i' + b_i''\right) - \frac{a_i h^4}{120}\left(a_i^2 b_i' + a_i b_i'' + 3a_i' b_i' + b_i b_i' + b_i'''\right)\right.$$
$$\left. + \frac{h^4}{360}\left(a_i^3 b_i' + a_i^2 b_i'' + 7a_i a_i' b_i' + 2a_i b_i b_i' + a_i b_i''' + 4a_i' b_i'' + b_i b_i'' + 6b_i' a_i'' + 4b_i'^2 + b_i^{(4)}\right)\right) y_i + \left(\frac{a_i}{2h} - \right.$$
$$\left. \frac{1}{h^2} \right) y_{i+1} + \left(\left(-\frac{a_i h^2}{6}\left(a_i' + b_i\right) + \frac{h^2}{12}\left(a_i a_i' + a_i b_i + a_i'' + 2b_i'\right) - \frac{a_i h^4}{120}\left(a_i^2 a_i' + a_i^2 b_i + a_i a_i'' + 2a_i b_i' \right.\right.\right.$$
$$\left. + 3a_i'^2 + 4a_i' b_i + b_i^2 + a_i''' + 3b_i'' \right) + \frac{h^4}{360}\left(a_i^3 a_i' + a_i^3 b_i + a_i^2 a_i'' + 2a_i^2 b_i' + 7a_i a_i'^2 + 9a_i b_i a_i' + 2a_i b_i^2 + \right.$$
$$\left.\left. a_i a_i''' + 3a_i b'' + 10a_i' a_i'' + 12a_i' b_i' + 7b_i a_i'' + 6b_i b_i' + a_i^{(4)} + 4b_i''' \right)\right) y_i' + \left(-\frac{a_i^2 h^2}{6} + \frac{h^2}{12}\left(a_i^2 + 2a_i' + b_i\right)\right.$$
$$- \frac{a_i h^4}{120}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b_i' \right) + \frac{h^4}{360}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + \right.$$
$$\left.\left. 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i'' \right) \right) y_i''$$
$$= \quad c_i \ + \left(-\frac{a_i h^2}{12} - \frac{a_i h^4}{120}\left(a_i^2 + 3a_i' + b_i\right) + \frac{h^4}{360}\left(a_i^3 + 7a_i a_i' + 2a_i b_i + 6a_i'' + 4b_i'\right)\right) c_i'$$
$$+ \left(\frac{h^2}{12} - \frac{a_i^2 h^4}{120} + \frac{h^4}{360}\left(a_i^2 + 4a_i' + b_i\right)\right) c_i'' + \left(-\frac{a_i h^4}{120} + \frac{a_i h^4}{360} \right) c_i''' + \frac{h^4}{360} c_i^{(4)} + T_3 \tag{4.12}$$

Further, from Taylor series the second order approximation of first and second derivatives of y_i are given as: $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$ (4.13)

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{4.14}$$

Substituting Eqs (4.13) and (4.14) into Eq. (4.12),we obtain:

$$\Big(-\frac{1}{h^2}-\frac{a_i}{2h}+\frac{a_ih}{12}\left(a_i'+b_i\right)-\frac{h}{24}\left(a_ia_i'+a_ib_i+a_i''+2b_i'\right)+\frac{a_ih^3}{240}\big(a_i^2a_i'+a_i^2b_i+a_ia_i''+2a_ib_i'+3a_i^2+$$
$$4a_i'b_i+b_i^2+a_i'''+3b_i''\big)-\frac{h^3}{720}\big(a_i^3a_i'+a_i^3b_i+a_i^2a_i''+2a_i^2b_i'+7a_ia_i'^2+9a_ib_ia_i'+2a_ib_i^2+a_ia_i'''+3a_ib_i''$$
$$+10a_i'a_i''+12a_i'b'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b_i'''\big)-\frac{a_i^2}{6}+\frac{1}{12}\left(a_i^2+2a_i'+b_i\right)-\frac{a_ih^2}{120}\big(a_i^3+5a_ia_i'+2a_ib$$
$$+3a_i''+3b'\big)\frac{h^2}{360}\big(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''\big)\Big)y_{i-1}+$$
$$\Big(b_i+\frac{2}{h^2}-\frac{a_ib_i'h^2}{12}+\frac{b_i''h^2}{12}-\frac{a_ih^4}{120}\left(a_i^2b_i'+3a_i'b_i'+a_ib_i''+b_ib_i'+b_i'''\right)+\frac{h^4}{360}\big(a_i^3b_i'+a_i^2b_i''+7a_ia_i'b_i'+$$
$$2a_ib_ib_i'+a_i\,b_i'''+4a_i'b_i''+b_ib_i''+6b_i'a_i''+4b_i'^2+b_i^{(4)}\big)+\frac{a_i^2}{3}-\frac{1}{6}\left(a_i^2+2a_i'+b_i\right)+\frac{a_ih^2}{60}\big(a_i^3+5a_ia_i'+$$
$$2a_ib_i+3a_i''+3b'\big)-\frac{h^2}{180}\left(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''\right)\Big)y_i+$$
$$\Big(-\frac{1}{h^2}+\frac{a_i}{2h}-\frac{a_ih}{12}\left(a_i'+b_i\right)+\frac{h}{24}\left(a_ia_i'+a_ib_i+a_i''+2b_i'\right)-\frac{a_ih^3}{240}\big(a_i^2a_i'+a_i^2b_i+a_ia_i''+2a_ib_i'+3a_i^2$$
$$+4a_i'b_i+b_i^2+a_i'''+3b_i''\big)+\frac{h^3}{720}\big(a_i^3a_i'+a_i^3b_i+a_i^2a_i''+2a_i^2b_i'+7a_ia_i'^2+9a_ib_ia_i'+2a_ib_i^2+a_ia_i'''+$$
$$3a_ib_i''+10a_i'a_i''+12a_i'b_i'+7b_ia_i''+6b_ib_i'+a_i^{(4)}+4b_i'''\big)-\frac{a_i^2}{6}+\frac{1}{12}\left(a_i^2+2a_i'+b_i\right)-\frac{a_ih^2}{120}\big(a_i^3+5a_ia_i'+$$
$$2a_ib+3a_i''+3b_i'\big)+\frac{h^2}{360}\big(a_i^4+9a_i^2a_i'+3a_i^2b_i+9a_ia_i''+7a_ib_i'+8a_i'^2+6a_i'b_i+b_i^2+4a_i'''+6b_i''\big)\Big)y_{i+1}$$
$$=c_i+\Big(-\frac{a_ih^2}{12}-\frac{a_ih^4}{120}\left(a_i^2+3a_i'+b_i\right)+\frac{h^4}{360}\left(a_i^3+7a_ia_i'+2a_ib_i+6a_i''+4b_i'\right)\Big)c_i'+$$
$$\Big(\frac{h^2}{12}-\frac{a_i^2h^4}{120}+\frac{h^4}{360}\left(a_i^2+4a_i'+b_i\right)\Big)c_i''+\Big(-\frac{a_ih^4}{120}+\frac{a_ih^4}{360}\Big)c_i'''+\frac{h^4}{360}c_i^{(4)}+T_3 \tag{4.15}$$

Eq. (4.15) can be written as the three- term recurrence relation of the form:

$$-E_iy_{i-1}+F_iy_i-G_iy_{i+1}=H_i,\qquad i=1,2,3,\;.\;.\;.\,N-1 \tag{4.16}$$

where:

$$E_i = \frac{1}{h^2} + \frac{a_i}{2h} - \frac{a_i h}{12}\left(a_i' + b_i\right) + \frac{h}{24}\left(a_i a_i' + a_i b_i + a_i'' + 2b_i'\right) - \frac{a_i h^3}{240}\left(a_i^2 a_i' + a_i^2 b_i + a_i a_i'' + 2a_i b_i' + 3a_i^2 + 4a_i' b_i + b_i^2 + a_i''' + 3b_i''\right) + \frac{h^3}{720}\left(a_i^3 a_i' + a_i^3 b_i + a_i^2 a_i'' + 2a_i^2 b_i' + 7a_i a_i'^2 + 9a_i b_i a_i' + 2a_i b_i^2 + a_i a_i''' + 3a_i b'' + 10a_i' a_i'' + 12a_i' b_i' + 7b_i a_i'' + 6b_i b_i' + a_i^{(4)} + 4b_i'''\right) + \frac{a_i^2}{6} - \frac{1}{12}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{120}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b_i'\right) - \frac{h^2}{360}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i''\right)$$

$$F_i = b_i + \frac{2}{h^2} - \frac{a_i b_i' h^2}{12} + \frac{b_i'' h^2}{12} - \frac{a_i h^4}{120}\left(a_i^2 b_i' + 3a_i' b_i' + a_i b_i'' + b_i b_i' + b_i'''\right) + \frac{a_i^2}{3} - \frac{1}{6}\left(a_i^2 + 2a_i' + b_i\right) + \frac{h^4}{360}\left(a_i^3 b_i' + a_i^2 b_i'' + 7a_i a_i' b_i' + 2a_i b_i b_i' + a_i b_i''' + 4a_i' b_i'' + b_i b_i'' + 6b_i' a_i'' + 4b_i'^2 + b_i^{(4)}\right) + \frac{a_i h^2}{60}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b_i'\right) - \frac{h^2}{180}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i''\right)$$

$$G_i = \frac{1}{h^2} - \frac{a_i}{2h} + \frac{a_i h}{12}\left(a_i' + b_i\right) - \frac{h}{24}\left(a_i a_i' + a_i b_i + a_i'' + 2b_i'\right) + \frac{a_i h^3}{240}\left(a_i^2 a_i' + a_i^2 b_i + a_i a_i'' + 2a_i b' + 3a_i^2 + 4a_i' b_i + b_i^2 + a_i''' + 3b''\right) - \frac{h^3}{720}\left(a_i^3 a_i + a_i^3 b_i + a_i^2 a_i'' + 2a_i^2 b_i' + 7a_i a_i'^2 + 9a_i b_i a_i' + 2a_i b_i^2 + a_i a_i''' + 3a_i b_i'' + 10a_i' a_i'' + 12a_i' b_i' + 7b_i a_i'' + 6b_i b_i' + a_i^{(4)} + 4b'''\right) + \frac{a_i^2}{6} - \frac{1}{12}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{120}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i'' + 3b_i'\right) - \frac{h^2}{360}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i''\right)$$

$$H_i = c_i + \left(-\frac{a_i h^2}{12} - \frac{a_i h^4}{120}\left(a_i^2 + 3a_i' + b_i\right) + \frac{h^4}{360}\left(a_i^3 + 7a_i a_i' + 2a_i b_i + 6a_i'' + 4b_i'\right)\right) c_i' + \left(\frac{h^2}{12} - \frac{a_i^2 h^4}{120} + \frac{h^4}{360}\left(a_i^2 + 4a_i' + b_i\right)\right) c_i'' + \left(-\frac{a_i h^4}{120} + \frac{a_i h^4}{360}\right) c_i''' + \frac{h^4}{360} c_i^{(4)} + T_3$$

Eq. (4.16) gives us the tri-diagonal system which can easily be solved by Thomas Algorithm.

4.2. Stability and Convergence Analysis

Definition (Keller [11]**):** The linear difference operator L_h is stable if, for sufficiently small h, there exists a constant k, independent of h, such that

$\left|v_j\right| \le k\{ \max\left(|v_0|,|v_N| \right) + \max_{1\le i\le N-1}|L_h v_i| \}, \quad j = 0,1,2,...,N$ for any mesh function $\{ v_j \}_{j=0}^{N}$

Theorem 4.1:under the assumption $b(x_i) \equiv \theta > 0$ for positive constant θ,

$\left(-a_i^2 + 2a_i' + b_i + 4a_i^2 a_i' + a_i^2 b_i + 6a_i' a_i'' + 4a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i''\right) > 0$ and

$$h < \min\left\{ \frac{2\left(-a_i^2 + 2a_i' + b_i + 4a_i^2 a_i' + a_i^2 b_i + 6a_i' a_i'' + 4a_i b_i' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' + 6b_i''\right)}{\left|\begin{array}{l} -a_i a_i' - a_i b_i + a_i'' + 2b_i' + 4a_i a_i'^2 + 5a_i a_i' b_i + a_i b_i^2 + 10a_i a_i'' + 12a_i' b_i' + 7b_i a_i'' + 6b_i b_i' + a_i^{(4)} \\ + 4b''' \end{array}\right|} \right\}$$

The linear difference operator of Eq. (4.16) is stable for $k = \max\left\{1, \dfrac{1}{\theta}\right\}$

Proof:

Let $L_h(.)$ denoted the difference operator on left side of Eq. (4.16) and w_i be any mesh function satisfying:

$$L_h(w_i) = H_i \tag{4.17}$$

If the $\max|w_i|$ occurs for $i = 0$ or $i = N$ then definition holds trivially. Since $k \ge 1$ so assume that $\max|w_i|$ occurs for $i = 1,2,...N-1$ under the given assumptions

$$E_i > 0, \; G_i > 0, \; F_i > E_i + G_i \quad \text{and} \; |E_i| \le |G_i|$$

This implies the tri-diagonal system in Eq. (4.16) is diagonally dominant and its solution exists and is unique. Then by rearranging the difference Eq. (4.16) and using the non negativity of the coefficients, we have:

$$F_i|w_i| \le E_i|w_{i-1}| + G_i|w_{i+1}| + |H_i|$$

$$\Rightarrow \quad F_i|\,w_i| \le E_i|\,w_{i-1}| + G_i|\,w_{i+1}| + |L_h\, w_i| \tag{4.18}$$

Since $b(x_i) \equiv \theta$ is a constant and by assumption $b'(x_i) = 0$.

Thus from Eq. (4.16) we have:

$$F_i = \frac{2}{h^2} + \frac{a_i^2}{3} - \frac{1}{6}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{60}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i''\right)$$
$$- \frac{h^2}{180}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i'''\right) + \theta$$

Now, using the fact that,

$$E_i + G_i = \frac{2}{h^2} + \frac{a_i^2}{3} - \frac{1}{6}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{60}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i''\right)$$
$$- \frac{h^2}{180}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 7a + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i''' \right)$$

and from Eq. (4.18), we get:

$$\Big[\ \frac{2}{h^2} + \frac{a_i^2}{3} - \frac{1}{6}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{60}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i''\right)$$
$$- \frac{h^2}{180}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i'''\right) + \theta\ \Big]\,|w_i|$$
$$\le E_i|w_{i-1}| + G_i|w_{i+1}| + |L_h w_i|$$
$$\le \left(E_i + G_i\right) \max_{1\le k\le N-1}|w_k| + \max_{1\le k\le N-1}|L_{hK}| \tag{4.19}$$

Since the inequality in Eq. (4.19) holds for every i, it follows that:

$$\Big[\ \frac{2}{h^2} + \frac{a_i^2}{3} - \frac{1}{6}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{60}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i''\right)$$
$$- \frac{h^2}{180}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i'''\right) + \theta\ \Big] \max_{1\le k\le N-1}|w_i|$$
$$\le \Big[\ \frac{2}{h^2} + \frac{a_i^2}{3} - \frac{1}{6}\left(a_i^2 + 2a_i' + b_i\right) + \frac{a_i h^2}{60}\left(a_i^3 + 5a_i a_i' + 2a_i b_i + 3a_i''\right)$$
$$- \frac{h^2}{180}\left(a_i^4 + 9a_i^2 a_i' + 3a_i^2 b_i + 9a_i a_i'' + 8a_i'^2 + 6a_i' b_i + b_i^2 + 4a_i'''\right)\Big] \max_{1\le k\le N-1}|\,w_k| + \max_{1\le k\le N-1}|\,L_h w_k|$$

This implies $\theta \max_{1\le i\le N-1} |w_i| \le \max_{1\le k\le N-1} |L_h w_K|$

Hence, $\max_{1\le k\le N-1} |w_i| \le \frac{1}{\theta} \max_{1\le k\le N-1} |L_h w_k| \le \frac{1}{\theta}\left\{\max(|w_0|,|w_N|) + \max_{1\le k\le N-1} |L_h w_k|\right\}$

Therefore, $|w_i| \le k\left\{\max(|w_0|,|w_N|) + \max_{1\le k\le N-1} |L_h w_k|\right\}$ where $k = \frac{1}{\theta}$

Hence, L_h is stable and this implies that the solution of the system of the difference Eq. (4.16) are uniformly bounded, independent of mesh size h and the parameter ε. Hence the scheme is stable for all step sizes.

Corollary 4.1: Under the conditions for theorem 4.1, the error $e_i = y(x_i) - y_i$ between the solutions of $y(x)$ of the continuous problem and y_i of the discrete problem, with boundary conditions satisfies the estimate

$$|e_i| \le k \max_{1\le i\le N-1} |T_i| \tag{4.20}$$

where $T_i = \frac{h^6}{20160}\left|y_i^{(8)}\right| + \frac{a_i h^6}{5040}\left|y_i^{(7)}\right|$ is the truncation error and

$$T_i \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{h^6}{20160}\left|y_i^{(8)}\right| \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{a_i h^6}{5040}\left|y_i^{(7)}\right| \right\}$$

Proof: under the given conditions it is clear that the error e_i satisfies

$$L_h(e_i) = L_h(y(x_i) - y_i) = T_i,\ i = 1,2,\ldots N-1 \quad \textit{and}\ e_0 = e_N = 0$$

Then theorem 4.1, the stability of L_h implies that

$$|y(x_i) - y_i| = |e_i| \le k \max_{1\le i\le N-1} |T_i| \tag{4.21}$$

Hence the estimate in Eq. (4.20) establishes the convergence of the scheme for the fixed value of the perturbation parameter ε.

4.3. Thomas Algorithm

The tri-diagonal matrix algorithm, also known as Thomas Algorithm, is a specified form of Gauss elimination that can be used to solve tri-diagonal system of equations as stated above.

The tri-diagonal matrix algorithm is based on the Gauss elimination procedure and consists of two parts; a forward elimination phase and backward substitution phase.

A description for solving the tri-diagonal system using Discrete Invariant Imbebedding Algorithm called Thomas Algorithm is given as follows.

Consider Eq.(4.19) with the boundary conditions:

$$\begin{aligned} y_0 &= y(0) = \alpha \\ y_N &= y(1) = \beta \end{aligned} \tag{4.22}$$

Assume that the solution of Eq. (1.1) can be written:

$$y_i = w_i y_{i+1} + T_i,\ i = 1,2 \ldots N-1 \tag{4.23}$$

where $w_i = w(x_i)$ and $T_i = T(x_i)$ to be determined.

Evaluating Eq. (4.23) at $x_i = x_{i-1}$, we have:

$$y_{i-1} = w_{i-1} y_i + T_{i-1} \tag{4.24}$$

Now, substituting Eq. (4.24) into Eq.(4.16) gives:

$$y_i = \frac{G_i}{F_i - E_i w_{i-1}} y_{i+1} + \frac{H_i + E_i T_{i-1}}{F_i - E_i w_{i-1}} \tag{4.25}$$

Comparing Eq. (4.23) with Eq.(4.25), we get the recurrence relation:

$$w_i = \frac{G_i}{F_i - E_i w_{i-1}} \tag{4.26}$$

$$T_i = \frac{H_i + E_i T_{i-1}}{F_i - E_i w_{i-1}} \tag{4.27}$$

To solve these recurrence relations for $i = ,1, 2, 3\ .\ .\ .\ N-1$ we need the initial conditions for w_0 and T_0.For, this we take $y_0 = y(0) = w_0 y_1 + T_0$ .Choose $w_0 = 0$ then the value of $T_0 = y(0) = \alpha$, with these initial values ,we compute w_i and T_i for $i = 1, 2, 3\ .\ .\ .N-1$ from Eqs.(4.26)and(4.27) in forward process, and then obtained y_i in the backward process from Eqs.(4.22)and(4.23).

4.4. Numerical Examples

To demonstrate the applicability of the method, three self-adjoint singular perturbation problems have been considered. From these examples two of them have exact solution and one is without exact solution. These examples have been chosen because they have been widely discussed in the literature their exact solutions were available for comparison.

Example 4.1.: consider the following self-adjont singular perturbation problem:

$$-\varepsilon y'' + y = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x), \qquad 0 \le x \le 1$$

with boundary conditions $y(0) = 0 = y(1)$

The exact solution is given by:

$$y(x) = \frac{\exp\left(-(1-x)/\sqrt{\varepsilon}\right) + \exp\left(-x/\sqrt{\varepsilon}\right)}{1 + \exp\left(-1/\sqrt{\varepsilon}\right)} - \cos^2(\pi x)$$

The numerical solutions are given in Tables 4.1.and comparison of the exact and numerical solutions for $h \ge \varepsilon$ is given in figure 4.1.

Example 4.2.: consider the following self-adjoint singular perturbation problem: $-\varepsilon y'' + \frac{4}{(x+1)^4}\left(1 + \sqrt{\varepsilon}(x+1)\right)y = f(x), \quad 0 \le x \le 1$ with boundary conditions $y(0) = 2$, $y(1) = -1$ and $f(x)$ is chosen, such that the exact solution is given by:

$$y(x) = -\cos\left(\frac{4\pi x}{x+1}\right) + \frac{3\left(\exp\left(\frac{-2x}{\sqrt{\varepsilon}(x+1)}\right) - \exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)\right)}{1 - \exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)}$$

The numerical solutions in terms of maximum absolute errors are given in Tables 4.2.for different values of the perturbation parameters ε and N and comparison of the exact and numerical solutions for $h \geq \varepsilon$ is given in figure 4.2.

Example 4.3.: consider the following self-adjoint singular perturbation problem:

$$-\varepsilon y'' + (1+x)^2 y = (12x^2 - 13x + 5)(1+x)^2, \qquad 0 \leq x \leq 1$$

with boundary conditions

$$y(0) = 0 = y(1)$$

The exact solution of the problem is not known. The numerical results in terms of maximum absolute errors are given in Table 4.3 and its graph is given in figure 4.3.

4.5 Numerical Results

Table 4.1: Numerical solution of Example 4.1 for $\varepsilon = 10^{-4}$, $h = 10^{-3}$

x	Exact solution	Mishra et.al.[15]		Our Method	
		$y(x)$	Absolute error	$y(x)$	Absolute error
0.000	0.0000000	0.0000000	0.0000000	0.0000000000000	0.0000000000000
0.001	-0.0951527	-0.0953418	0.0001891	-0.0951527438021	0.0000000314100
0.010	-0.6311339	-0.6323828	0.0012489	-0.6311340507449	0.0000001277022
0.020	-0.8607221	-0.8624169	0.0016948	-0.8607221613765	0.0000000939559
0.030	-0.9413565	-0.9431993	0.0018428	-0.9413566088407	0.0000000518442
0.040	-0.9659759	-0.9678527	0.0018768	-0.9659759671028	0.0000000254272
0.050	-0.9687903	-0.9706548	0.0018645	-0.9687903228384	0.0000000116899
0.100	-0.9044631	-0.9060599	0.0015968	-0.9044630974109	0.0000000001532
0.300	-0.3454915	-0.3448815	0.0006100	-0.3454915028141	0.0000000000017
0.500	-0.0000000	0.0019739	0.0019739	-0.0000000000054	0.0000000000054
0.700	-0.3454914	-0.3448814	0.0006100	-0.3454915028141	0.0000000000017
0.900	-0.9044631	-0.9060598	0.0015967	-0.9044630974109	0.0000000001532
1.000	0.0000000	0.0000000	0.0000000	0.0000000000000	0.0000000000000

Table 4.2: Maximum Absolute Errors for Example 4.2

N	$\varepsilon = (1/N)^{0.25}$	$\varepsilon = (1/N)^{0.5}$	$\varepsilon = (1/N)^{0.75}$	$\varepsilon = (1/N)^{1.0}$
Our Method				
16	2.9718E-04	4.9658E-04	8.9268E-04	1.7181E-03
32	2.0905E-05	4.1607E-05	9.0798E-05	2.3653E-04
64	1.4884E-06	3.4999E-06	9.8228E-06	3.9036E-05
128	1.0650E-07	3.0026E-07	1.1659E-06	7.4775E-06
256	7.6403E-09	2.6424E-08	1.5321E-07	1.5612E-06
Kadalbajoo et.al.[10]				
16	2.0E-02	1.7E-02	1.5E-02	1.4E-02
32	4.7E-03	4.0E-03	3.4E-03	4.1E-03
64	1.1E-03	9.1E-04	9.3E-04	1.1E-03
128	2.6E-04	2.0E-04	2.4E-04	3.2E-04
256	6.1E-05	5.0E-05	6.4E-05	9.6E-05
Lubuma [13] as reported in Kadalbajoo et. al.[10]				
16	3.8E-02	2.5E-02	1.6E-02	1.4E-02
32	9.6E-03	6.3E-03	4.3E-03	7.9E-03
64	2.4E-03	1.6E-03	1.1E-03	2.4E-03
128	6.0E-04	3.9E-04	2.7E-04	6.2E-04
256	1.5E-04	9.8E-05	6.9E-05	1.6E-05

Table 4.3: Maximum Absolute Errors for Example 4.3.

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-4}	4.9633E-04	3.1259E-05	1.9577E-06	1.2244E-07	7.6565E-09	4.7870E-10	3.0028E-11
2^{-5}	1.6268E-03	1.0705E-04	6.7125E-06	4.2143E-07	2.6349E-08	1.6469E-09	1.0297E-10
2^{-6}	6.0702E-03	3.8490E-04	2.4186E-05	1.5140E-06	9.4658E-08	5.9180E-09	3.6995E-10
2^{-7}	2.0883E-02	1.3555E-03	8.9366E-05	5.6001E-06	3.5125E-07	2.1957E-08	1.3725E-09
2^{-8}	6.3612E-02	5.3582E-03	3.3894E-04	2.1282E-05	1.3318E-06	8.3267E-08	5.2046E-09
2^{-9}	1.6774E-01	1.9166E-02	1.2357E-03	8.1656E-05	5.1156E-06	3.2064E-07	2.0043E-08
2^{-10}	3.8994E-01	5.9855E-02	5.0279E-03	3.1782E-04	1.9951E-05	1.2484E-06	7.8048E-08
2^{-11}	8.7295E-01	1.6013E-01	1.8343E-02	1.1787E-03	7.8013E-05	4.8869E-06	3.0617E-07

The computational rate of convergence can also be obtained by using the double mesh principle defined below. Let

$Z_h = \max\left|y_i^h - y_i^{h/2}\right|,\ i=1,2,...,N-1$ where y_i^h is the numerical solution on the mesh $\{x_i\}_1^{N-1}$ at nodal point x_i, where $x_i = x_0 + ih, \quad i=1,2,...,N-1$ and where $y_i^{h/2}$ is the numerical solution at the nodal point x_i on the mesh $\{x_i\}_1^{2N-1}$ where $x_i = x_0 + ih/2, \quad i=1,2,...2N-1$

In the same way one can define $Z_{h/2}$ by replacing h by $h/2$ and $N-1$ by $2N-1$. That is,

$Z_{h/2} = \max\left|y_i^{h/2} - y_i^{h/4}\right|, \quad i=1,2,...,2N-1$. The computed rate of convergence is defined as:

$$Rate = \frac{\log Z_h - \log Z_{h/2}}{\log 2}$$

The following tables show the rate of convergence of the two Examples 4.2 and 4.3 for different values of the mesh size h.

Table 4.4: Rate of Convergence for Example 4. 2, $\varepsilon = \left(\frac{1}{16}\right)^{0.25}$

h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	Rate
$1/16$	$1/32$	2.7628E-04	$1/64$	1.9417E-05	3.8307
$1/32$	$1/64$	1.9417E-05	$1/128$	1.3819E-06	3.8126
$1/64$	$1/128$	1.3819E-06	$1/256$	9.8860E-08	3.8051

Table 4.5: Rate of Convergence for Example 4.3, $\varepsilon = 2^{-4}$

h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	Rate
$1/16$	$1/32$	4.6507E-04	$1/64$	2.9301E-05	3.9884
$1/32$	$1/64$	2.9301E-05	$1/128$	1.8353E-06	3.9969
$1/64$	$1/128$	1.8353E-06	$1/256$	1.1478E-07	3.9991
$1/128$	$1/256$	1.1478E-07	$1/512$	7.1778E-09	3.9992
$1/256$	$1/512$	7.1778E-09	$1/1024$	4.4867E-10	3.9998

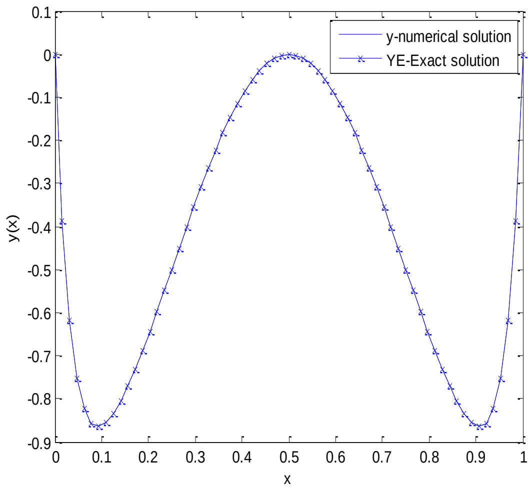


Fig 4.1 (a): Numerical Solution of Example 4.1,

for $\varepsilon = 10^{-3}$ *and* $N = 64$

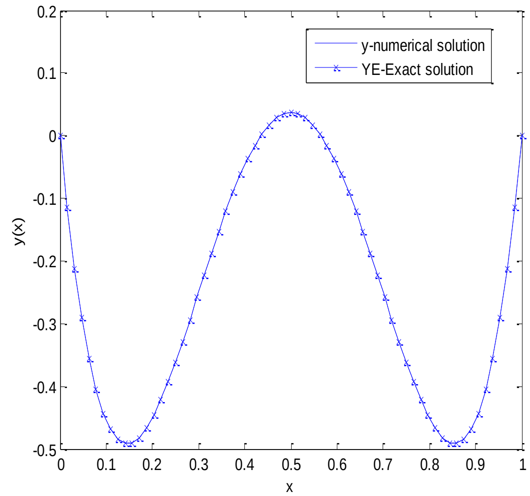


Fig4.1 (b): Numerical Solution of Example 4.1,

for $\varepsilon = \dfrac{1}{64}$ *and* $N = 64$

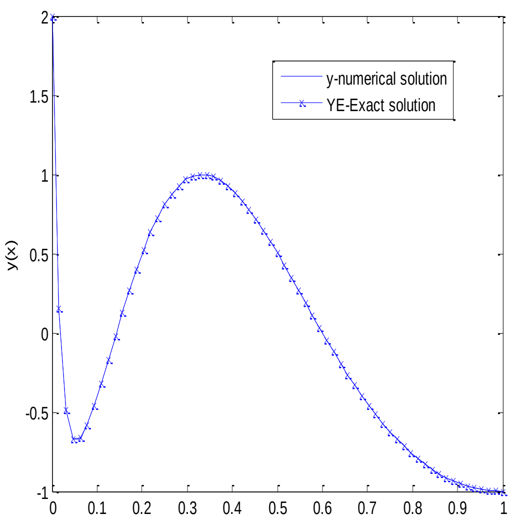


Fig 4.2 (a) Numerical Solution of Example 4.2

$\varepsilon = 10^{-3}$ *and* $N = 64$

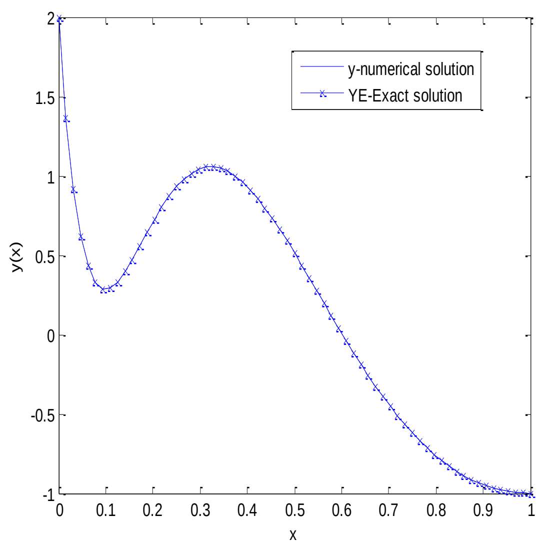
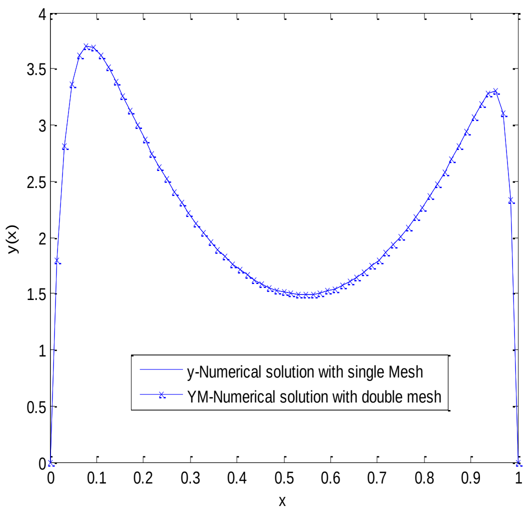


Fig 4.2 (b): Numerical Solution of Example 4.2 for

for $\varepsilon = \dfrac{1}{64}$ *and* $N = 64$



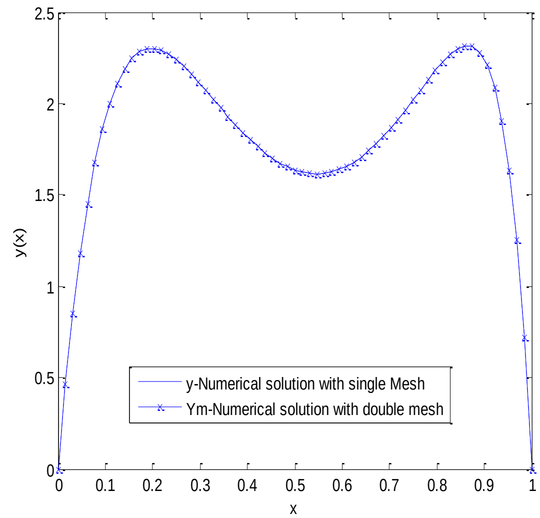


Fig 4.3 (a): Numerical Solution of Example 4.3 for $\varepsilon = 10^{-3}$ and $N = 64$

Fig 4.3 (b): Numerical Solution of Example 4.3 for $\varepsilon = \frac{1}{64}$ and $N = 64$

4.6. Discussion

In this thesis, sixth-order stable central difference method has been presented for solving self-adjoint singularly perturbed two point boundary value problems. The numerical results have been presented in tables 4.1-4.3 for different values of the perturbation parameter ε and different number of mesh points N .The results obtained by the present method have been compared with numerical results obtained by Mishra et.al [15] and Kadalbajoo et.al [10] and the results are summarized in the tables 4.1-4.3.

As it can be observed from the tables, the present method approximates the exact solution better than the methods proposed by Mishra et.al [15] and Kadalbajoo [10].

Further, as it can be observe from the graphs (Fig.4.1 -4.3), the present method approximates the exact solution very well for $h \geq \varepsilon$ for which most of classical numerical methods fail to give good result.

Chapter Five
Conclusion and Scope of Future Work

5.1. Conclusion

In this thesis the sixth order stable central difference method has been presented for solving self-adjoint singularly perturbed boundary value problem. It has been implemented on three examples by taking different values for perturbation parameter ε and the computational results are presented in tables and graphs. The results show that the present method approximates the exact solution very well and it shows that the betterment of the present method over some existing methods reported in the literature.

The results presented confirmed that computational rate of convergence and theoretical estimates indicate that sixth order method is stable and convergent. In addition to this the present method is conceptually simple, easy to use and readily adaptable for computer implementation for solving self-adjoint singularly perturbed boundary value problems.

5.2. Scope of Future Work

In the present thesis, the numerical method based on sixth order stable central difference scheme was constructed for solving self-adjoint singularly perturbed boundary problems by using three points (three-term recurrence relation). Hence, the method presented in this thesis can also be extended to five points and higher order than sixth order stable central difference methods for solving self-adjoint singularly perturbed boundary problems.

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