Stability Analysis of Meshcherskii's Equation of Dynamic System with Variable Mass

A Research Report Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment of the Requirements for the Degree of Masters of Sciences in Mathematics

By

Adugna Gadisa

Supervisor: Dr. Chernet Tuge (Assistant Professor)

Co-Advisor: Dr. Yesuf Obsie (Assistant Professor)

June, 2016

Jimma, Ethiopia

DECLARATION

I, the undersigned declare that, the research entitled **“Stability Analysis of Meshcherskii’s Equation of Dynamic System, with Variable Mass”** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like.

Name: **Adugna Gadisa.**

Signature……………….. Date: ……………………

The work has been done with supervision and approval of advisors**:**

Principal advisor: **Dr. Chernet Tuge** (Ass. Professor)

Signature: ………………. Date: …………………….

Co-advisor: **Dr. Yesuf Obsie** (Ass. Professor)

Signature: ………………. Date: …………………….

ACKNOWLEDGEMENTS

First of all, I am indebted to my almighty GOD who gave me long life, strength and helped me to reach the precise time of my thesis report writing completion. Next my special cordial thanks goes to my Advisor Dr. Chernet Tuge and my co-advisor Dr. Yesuf Obsie for their constructive support and guidance throughout this research report preparation. Furthermore, I wish to thank my family for their continuous support while I am preparing this report from the beginning.

Table of Contents **page**

ABSTRACT

In this research report asymptotic stability of Meshcherskii equation of dynamic system with variable mass is investigated. The Lyapunov function method of stability analysis is employed. The model of the system is considered and stability is investigated for different given trajectories. A Lyapunov function is constructed and asymptotic stability using the function is proved. Moreover, the practical applicability of the result is demonstrated by simulation using MATLAB. The result of the simulation shows an excellent conformity with the theoretical proof made in ascertaining asymptotic stability.

Key words: Asymptotic stability, equilibrium point, Meshcherskii equation, Lyapunov stability, stability.

CHAPTER ONE

1. INTRODUCTION

1.1 Background of the Study

According to Newton's second law of motion, the acceleration $\boldsymbol{a}$ of a body of mass m is proportional to the force $\boldsymbol{F}$ acting on it [1, 2, 3, 4, 5]. That is

$$\boldsymbol{F} = m\boldsymbol{a}\,. \tag{1.1.}$$

Equation (1.1) can be used to develop the dynamics of particles or bodies with a fixed masses which is well known in the literature. But in general it is not possible to apply the dynamics of a fixed mass system to a variable mass dynamic system. For instance, a rocket is propelled by ejecting burnt fuel which causes the mass of the rocket to decrease substantially as the rocket accelerates. A rain-drop falling through the damp atmosphere coalesces with small droplets which increase its mass. In both of these illustrations the mass of the body is varying with time. The term "variable mass" is used in the sense that a mass is being removed and / or added to the original body under consideration [6, 7, 8, 9, 10].

Suppose the body is also subjected to an external force $\boldsymbol{F}$. Then Newton's 2nd law of motion is now interpreted in the form, force is equal to the change in linear momentum ($\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$). Hence the equation of motion assumes the form [11, 12, 13, 14]

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = m\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v}\frac{dm}{dt} \tag{1.2}$$

The study [16] of stability theory has begun in the works of Aristotle and Archimedes from the different point of view. Aristotle investigated the motion occurring after perturbation (small disturbance) and he determined the stability of unperturbed state from the course of the motion. Archimedes observed purely geometric situation occurring after perturbation of a system and used it to determine the stability of the unperturbed system. Therefore these two methods are called kinematic method and the geometric method which is connected with the names of Archimedes, Torricelli, Baldi and Lagrange dominated [16] the field of mechanics for extended period of time (from 17th to 19th century). The energy method, which is the work of Lagrange, represents the third method. Now days, Energy criteria are used to determine the stability of an equilibrium position. In this century the stability concept of Lyapunov has found wide acceptance [16, 17, 18].

Lyapunov stability criterion is general and use-full approach to analyze stability of nonlinear systems. Lyapunov stability concept includes two approaches: Lyapunov indirect method and Lyapunov direct method. The idea of Lyapunov indirect method is linearization of a system around a given point and discussing local stability with small stability regions on the other hand Lyapunov direct method is the most important tool for design and analysis of nonlinear systems. This method can be applied directly to a nonlinear system without the need of linearization and achieves global stability [17, 18].

1.2 Statement of the Problem

The stability of dynamic system with a constant mass is widely investigated using linearization techniques discussed above. Even though the use of Lyapunov direct method is more powerful than linearization, it is not widely applied in different researches since it is not easy to find a Lyapunov function. Moreover, as far as the knowledge of the researcher is concerned stability analysis of variable mass dynamic system is rarely found in the literature. However it is worth mentioning that as discussed in the background above, the famous Russian Mathematician Meshcherskii, I.V. [11, 14] constructed the dynamic equation of movement of variable mass system and opened a wide chapter of research in the area of modeling dynamics and stability investigation of such systems.

It is with this understanding that the study adapted a model developed by Meshcherskii I.V. [14], and planned to investigate its stability using different trajectories that has to be followed by the system given in equation (1.3). Hence, this study establish sufficient conditions for the stability of the motion of dynamic system with a variable mass along a given path in the XZ-plane. To simplify the investigation, we use the assumption that at each point, the tangential and the normal components of the velocity of the system have the form $X = k_1(t)\boldsymbol{v}$ and $Z = k_2(t)\boldsymbol{v}$, where $\boldsymbol{v}$-velocity of the mass, k_1, k_2 – are the coefficient of the velocity of the mass along the X and Z-components respectively. The dynamic model of such a system with a variable mass $m = m(t)$, is given by [11, 14].

$$\begin{cases} \ddot{x} = \dfrac{\dot{m}}{m}(\mu - 1)\dot{x} - \dfrac{k_1}{m}\dot{x} - \dfrac{k_2}{m}\dot{z}, \\ \ddot{z} = \dfrac{\dot{m}}{m}(\eta - 1)\dot{z} - \dfrac{k_2}{m}\dot{z} + \dfrac{k_2}{m}\dot{x} - g \end{cases} , \tag{1.3}$$

where $\mu = \mu(t), \eta = \eta(t)$ −the ratio of the projection of speed of change of mass and the mass on the x, z −coordinate respectively.

In particular this research will focus on the following problems.

i. Constructions of a Lyapunov function for the dynamic system given by equation (1.3) in order to ascertain asymptotic stability (Investigation of stability analysis).
ii. Setting different stability conditions depending on a given trajectory of the system (1.3)
iii. Demonstration of the accuracy of the results using MATLAB based on a given numerical data.

1.3 Objective

1.3.1. General Objective

The general objective of this research is to establish sufficient conditions for the stability of a dynamic system with a variable mass given by equation (1.3) in the sense of Lyapunov.

1.4 Significance of the Study

The theory of stability is a core part of any designed dynamical model (control system). For instance, any control system needs to have the stability condition investigated and conditions be set before further applications or before using the system for any designed purpose. Otherwise it may be dangerous to use the system as it may not be manageable. Hence, the result of this research can be used:

- for practical purpose such as in control design to the dynamic system of the model under discussion which can be adapted to formulate the dynamic equation of many variable mass systems such as Rockets.
- as a model in transforming mathematical concepts to other applied sciences such as stability of dynamics of Rockets and spin stabilization of modern space crafts.
- as a stepping stone for other researchers on investigation of stability of dynamical system with variable mass.

1.5 Delimitation of the Study

This study is delimited to discussing the stability conditions of the equation of variable mass dynamic system given by equation (1.3). This research is conducted in Jimma university department of mathematics.

CHAPTER TWO

2. LITERATURE REVIEW

2.1. Stability

Stability theory plays a central role in systems theory and engineering. In a dynamic system we have by definition an equilibrium means $\dot{\mathbf{X}}(\mathrm{t}) = \mathbf{0}$ for all t. In other words: The equilibrium points are those points which satisfies $\mathbf{f}(\mathbf{X}(\mathbf{t})) = \mathbf{0}$ for autonomous (time invariant) systems and for non-autonomous systems, of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathrm{t})$ equilibrium points x_0 are defined by $\mathbf{f}(\mathrm{x}_0, \mathrm{t}) = 0 \quad \forall \mathrm{t} > \mathrm{t}_0$. In this case equilibrium refers to certain points in phase space
(a set whose elements called “points” of the system at any moment of time) which have the property that if the system is located for such a point, it sits there forever, because the derivative of all the coordinates are zero [19, 20, 21].
An equilibrium is also encounter in the earth motion around the Sun which is different from the other. In this case we do not have an equilibrium situation like the one we just described. Still this system is also in some kind of equilibrium in the sense that it repeats its motion periodically. We have a closed path which after a year repeats itself. It is meaningful to ask the question about stability in both these cases, however, these two phenomenon are so different that they require different concepts of stability in order to catch the important properties of each system [19].

The other possibility of discussing about stability is one can imagine a motion that is a solution curve for $\dot{\boldsymbol{X}} = f(\boldsymbol{X})$ which starts in x_0. What will happen if we start a motion close by x_0? Are we then going to find a motion which always will be close to the motion that started in x_0. Lyapunov has given the name to the stability definition which deals with this problem [19].

Other concept of stability is due to Laplace. All motion which is limited, that is $|\boldsymbol{X}(t)| < \infty \; \forall t$ is stable [19].

In general, stability theory [1, 2, 19, 20] addresses the stability of solutions of differential equations and trajectories of dynamical systems under small disturbance or perturbations of initial conditions. In dynamical systems, an orbit is called Lyapunov stable if the forward orbit of any point is in a small enough neighborhoods or it stays in a small neighborhood. Various criteria have been developed to prove stability or instability of an orbit. Under favorable circumstances, the question may be reduced to a well-studied problem involving eigenvalues of matrices. A more general method involves Lyapunov functions. In practice, any one of a number of different stability criteria is applied.

Stability of equilibrium points is usually characterized in the sense of Lyapunov, a Russian mathematician and engineer who laid the foundation of the theory. Lyapunov states that an equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. Also he showed that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization of the system about that point has an exponentially stable equilibrium at the origin. Lyapunov stability analysis can be used to show boundedness of the solution, even when the system has no equilibrium points [19, 20].

Lyapunov stability criterion is a general and use-full approach to analyze stability of nonlinear systems. Lyapunov stability concept includes two approaches: Lyapunov indirect method and Lyapunov direct method. The idea of Lyapunov indirect method is linearization of a system around a given point and discussing local stability with small stability regions on the other hand Lyapunov direct method is the most important tool for design and analysis of nonlinear systems. This method can be applied directly to a nonlinear system without the need of linearization and achieves global stability [20].

Even though Lyapunov function method is more general and powerful, it has some drawbacks in that it is not easy to find the Lyapunov function V and the theorems are only sufficient conditions. That is, the fact that Lyapunov function doesn’t exist or difficult to find doesn’t mean the dynamic system is not stable. Moreover, failure of Lyapunov function candidate to satisfy asymptotic stability condition doesn’t mean that the system is not asymptotically stable. These two are series drawbacks of Lyapunov stability analysis application in practice [20, 21].

In Lyapunov stability theory we think of mainly three kinds of stability theorems: neutral stability, asymptotic stability and exponential stability theorems. Exponential stability is stronger than asymptotic stability in that asymptotic stability tells us the stability and convergence, exponential stability indicates in addition the rate of convergence. Asymptotic stability is stronger than neutral stability. In many practical application, especially in engineering, asymptotic stability is more desired as it finally indicates the system converging to the equilibrium point [20, 21].

CHAPTER THREE

3. METHODOLOGIES

3.1 Study Site and Period

This study was conducted in Jimma University in the department of mathematics from October 2015 to June 2016.

3.2 Study Design

In this research both analytical and approaches ware employed. Analytically constructing Lyapunov function and proving if the function satisfies the required Lyapunov stability criteria was made and experimentally the well-known software MATLAB was used to demonstrate the applicability of the results using given particular numerical data.

3.3 Source of Information

The source of information used in this research was books related to stability theory and control system, articles on stability of a system, related journals and lecture notes.

3.4 Procedure of the Study

- Define Lyapunov function candidate.
- Check if the defined function satisfies the Lyapunov function conditions.
- Set stability conditions for different given trajectories.
- Demonstrate the result for some given numerical data using MATLAB.

3.5 Ethical Consideration

Ethical clearance was obtained from Mathematics department, College of Natural Sciences, Jimma University and any concerned body was informed about the purpose of the study.

CHAPTER FOUR

4. RESULT AND DISCUSSION

4.1. Preliminaries

Definition 4.1: [21] Autonomous system

A system of ordinary differential equations is said to be autonomous (time invariant) if it does not explicitly contain the independent time (t).

Let us consider system of differential equation

$$\dot{x}(t) = X(x(t)),$$

where,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix},$$

otherwise it non autonomous (time variant).

Definition 4.2: [21, 22] (Equilibrium point)

Equilibrium point is a point $x_0 \in \mathbb{R}^n$ such that $\dot{x}(x_0) = 0$ (for all future time).

Definition 4.3: [19, 20, 21, 22] (Lyapunov equilibrium stability)

The equilibrium point $x_0 = 0$ is called

i. Stable if for each $\varepsilon > 0$, there exist a $\delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon.$$

ii. Asymptotically stable if it is stable and δ can be choosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

iii. Exponentially stable if there exist two strictly positive numbers $\boldsymbol{a}$ and λ such that

$$\left|\left|x(t)\right|\right| \leq a\left|\left|x(0)\right|\right|e^{-\lambda t}$$

in some ball B_r around the origin.

iv. Unstable if it not stable.

Definition 4.4: [19] (Lyapunov motion stability)

i. A motion (path) $x(t)$ that starts in $x^*(t_0) = x^*_0$, and look simultaneously at another motion (path) which starts in $x(t_0) = x_0$. If for arbitrary $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0)$, such that $||x_0 - x^*_0|| < \delta \Rightarrow ||x(t) - x^*(t)|| < \varepsilon$ for $t > t_0$, then the motion $x(t)$is stable.

ii. Asymptotic stability

If in addition to (i) the following is satisfied

$$\lim_{t\to\infty} ||x(t) - x^*(t)|| = 0,$$

then the system is said to be asymptotically stable.

iii. Unstable if it not stable

4.1.1 The Second (Direct) Method of Lyapunov

The main qualitative method for investigating stability properties of unperturbed motion is the direct method of Lyapunov also known as the second method of Lyapunov. The main aim of the method is to reduce the system stability analysis to the analysis of the properties of some special Lyapunov function V as described below [21]

Definition:4.5:[21,22]

1. A function $V: D \to \mathbb{R}$ is said to be positive definite in domain D if it satisfies the following conditions:

$(i)\ 0 \in D\ and\ V(0) = 0.$

$(ii) V(x) > 0$, for all $x\ in\ D/\{0\}$

2. A function $V: D \rightarrow \mathbb{R}$ is said to be negative definite in D if

(i) $0 \in D$ *and* $V(0) = 0$.

(ii) $V(x) < 0$, for all x *in* $D/\{0\}$.

3. A function $V: D \rightarrow \mathbb{R}$ is said to be positive semi definite in D if it satisfies the following conditions: (i) $0 \in D$ *and* $V(0) = 0$.

(ii) $V(x) \geq 0, for\ all\ x\ in\ D/\{0\}$.

4. A function $V: D \rightarrow \mathbb{R}$ is said to be negative semi definite in D if

(i) $0 \in D$ *and* $V(0) = 0$.

$(ii) V(x) \leq 0, for\ all\ x\ in\ D/\{0\}$.

5. Unless and other wise $V: D \rightarrow \mathbb{R}$ is indefinite function.

Theorem 4.1: [21, 22, 24, 25]. Lyapunov theorem of stability

Let $x = 0$ be an equilibrium point of $X: D \rightarrow \mathbb{R}^n$, X is locally Lipchitz map and $D \subset \mathbb{R}^n$ be any domain containing $x = 0$, let $V: D \rightarrow \mathbb{R}$ be continuously differentiable function such that $V(0) = 0$ and $V(x) > 0$ in $D/\{0\}$ and

$\dot{V}(x) \leq 0$ in D, then the equilibrium point $x = 0$ is stable, further if $\dot{V}(x) < 0\ in$

$D/\{0\}\ then$ $x = 0$ is asymptotically stable.

Proof:

Given $\varepsilon > 0$. We need to construct $\delta > 0$ such that any trajectory starting in $B(0, \delta)$ doesn't leave $B(0, \varepsilon)$. Let construct $\Omega_\beta \subset B(0, \varepsilon)$ B_δ: is an open ball with radius ε around the origin

since $\varepsilon > 0$ is given $B_\varepsilon \subset D$

∂B_ε: Boundary of the ball $B_\varepsilon = \{x \in \mathbb{R}^n / \|x\| = \varepsilon\}$ is. Then, value of the Lyapunov function V on the boundary ∂B_ε is strictly positive. Since $V(x) > 0$ for all x except $x = 0$ all points

In ∂B_ε are at distance ε away from 0.

Hence, $V(x) > 0 \; \forall x \in \; \partial B_\varepsilon$

$$\text{Let } \alpha \coloneqq \min_{\|x\|} V(x) \text{ then } \alpha > 0$$

Take any $\beta \in (0, \alpha)$ and define

$$\Omega_\beta \coloneqq \{x \in \beta_\varepsilon | V(x) \leq \beta\}$$

<u>Claim 1</u>: Ω_β is interior of B_ε.

Suppose: Ω_β was not in the interior of B_ε

Then there would be a point $p \in \partial B_\varepsilon \cap \Omega_\beta$. $p \in \partial B_\varepsilon \Longrightarrow V(p) \geq \alpha$

$p \in \Omega_\beta \Longrightarrow V(p) \leq \beta (< \alpha)$.

This now implies $\alpha \leq V(p) \leq \beta < \alpha$ which is contradiction thus there cannot be a point

$p \in \partial B_\varepsilon \cap \Omega_\beta$ or $\partial B_\varepsilon \cap \Omega_\beta = \emptyset$.

Therefore Ω_β is contained in the interior of B_ε.

Also Ω_β is closed set. ($V(x) \leq \beta$: Boundary of Ω_β is inside Ω_β) it is bounded: $\Omega_\beta < B_\varepsilon$

Hence Ω_β is compact.

<u>Claim 2:</u> The set Ω_β satisfies any trajectory in Ω_β at $t = 0$ stays in Ω_β at $t \geq 0$

$$\dot{V}\big(x(t)\big) \leq 0 \Longrightarrow \int_0^t \dot{V}\left(x(t)\right) \leq 0$$

$$\Longrightarrow \big[V\big(x(t)\big)\big]_0^t \leq 0$$

$$\Longrightarrow V\big(x(t)\big) \leq V\big(x(0)\big) \leq \beta. \text{ For all } t \geq 0$$

This proves Ω_β is positively invariant.

Since Ω_β is compact set $\dot{X} = f(X)$ has unique solution defined for all $t \geq 0$ for each $x(0) \in \Omega_\beta$.

Now to find a $\delta > 0$ such that $B_\delta \subset \Omega_\beta$ as $V(x)$ is continuous and $V(0) = 0, V(x)$ is closed to zero for all x in some B_δ also (V is continuos at $x = 0$ if and only if for every $\beta > 0$ there exists a $\delta > 0$ such that $x \in B_\delta \Longrightarrow |v(x) - V(0)| < \beta$.

Using $V(0) = 0$ and $V(x) > 0$ for $x \in D$

$$\Longrightarrow x \in B_\delta \Longrightarrow V(x) < \beta$$

Thus there exist a ball B_δ contained in side Ω_β for some $\delta > 0$ we have shown:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\beta_\delta \subset \Omega_\beta \subset \beta_\varepsilon \text{ and}$$

$$x(0) \in B_\delta \Longrightarrow x(0) \in \Omega_\beta$$

$$\Longrightarrow \text{for all } t \geq 0 \text{ we have } x(t) \in \Omega_\beta \text{ and hence } x(t) \in \beta_\varepsilon$$

Hence the point of equilibrium $x = 0$ is stable.

To show asymptotically stable

If $\dot{V}(x) < 0$ in $D/\{0\}$ also holds.

We want to show $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$

Since $V(x) = 0 \Leftrightarrow x = 0$ we can instead show $V(x) \longrightarrow 0$

$\dot{V}(x) < 0$. This implies, V is monotonically decreasing with time.

Hence a limit does exist.

As $t \longrightarrow 0, V(x(t)) \longrightarrow c$

To show that $c = 0$

Suppose $c > 0$ by continuity of (x) , there is $d > 0$ such that $B_d \subset \Omega_c$

The limit $V\big(x(t)\big) \geq c$ for all $t \geq 0$

Let $-\gamma \coloneqq \max\limits_{dc\|x\|\leq\varepsilon} (\dot{V}(x))$

Remark: over compact set, a continuous function achieves its maximum and minimum

The compact set : $d < \|x\| \leq \gamma$, the continuous function on this set : $\dot{V}(x)V\big(x(t)\big) = V\big(x(0)\big) + \int_0^t \dot{V}\left(x(\tau)\right)d\tau \leq V\big(x(0)\big) - \gamma t$ for $x(0) \in B_\varepsilon$

Since $-\gamma \coloneqq max x\dot{V}(x)$ for x satisfaying $d \leq \|x\| \leq \varepsilon)$

Then $\gamma > 0$

Hence RHS (right hand side) eventually become negative

Hence the set $d \leq \|x\| \leq \varepsilon)$ can't be invariant, and our assumption about $c > 0$ is causes this contradiction.

Thus $V\big(x(t)\big) \longrightarrow 0$ as $t \longrightarrow \infty$ and hence $x(t) \longrightarrow 0$ also this prove asymptotically stable

Definition: 4.6: [21, 26]

A continuously differentiable function V satisfying $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$ with $\dot{V}(x) \leq 0$ in D and used to prove the stability of equilibrium point is called Lyapunov function.

Theorem 4.2: [21, 22] Let $x = 0$ be an equilibrium point $\dot{x} = f(x)$. Let $V: R^n \to \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \quad , \quad \forall x \neq 0$$

$$\|x\| \to \infty \Rightarrow V(x) \to 0.$$

$$\dot{V}(x) \neq 0, \forall x \neq 0$$

Then, $x = 0$ is globally asymptotically stable equilibrium point

Theorem 4.3: [21, 22, 26] Let $x = 0$ is an equilibrium point of $\dot{x} = f(x)$

Let $V: D \to \mathbb{R}$ be continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrary small $\|x_0\|$ define a set $\cup$ as

$$\cup = \{x \in B_r | v(x) > 0\},$$

and suppose that $\dot{V}(x) > 0$ in $\cup$ then, $x = 0$ is unstable

4.1.2 Lyapunov's Indirect Method

[21] Consider a nonlinear autonomous system

$$\dot{x} = f(x) \tag{4.1}$$

$f:\ D \rightarrow \mathbb{R}^n$, Continuously differentiable $D \subset \mathbb{R}^n$ in to $\mathbb{R}^n$

$x = 0\ \epsilon D$ and $f(0) = 0$

- $\Rightarrow\ \frac{f_i(x)-f_i(0)}{x-0} = \frac{\partial f_i}{\partial x}(z_i)$
- $\Rightarrow\ f_i(x) - f_i(0) = \frac{\partial f_i}{\partial x}(z)x$ (mean value theorem)

Where $z_i\ \epsilon(x,0)$ or $(0,x)$

Since $f_i(0) = 0$

- $\Rightarrow\ f_i(x) = \frac{\partial f_i}{\partial x}(z_i)x \Rightarrow \frac{\partial f_i}{\partial x}(0) + \left[\frac{\partial f_i(z_i)}{\partial x} - \frac{\partial f_i}{\partial x}(0)\right]$
- $\Rightarrow\ \frac{\partial f_i}{\partial x}(z_i)x = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right]x$
- $\Rightarrow\ f(x) = Ax + g(x)$

Where $A = \frac{\partial f_i}{\partial x}(0)$ $\qquad g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right]x$

$|g_i(x)| \leq \left\|\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right\| \|x\|$

Since $\frac{\partial f}{\partial x}$ continuous we have

$\|g(x)\| \leq \left\|\frac{\partial f}{\partial x}(z) - \frac{\partial f}{\partial x}(0)\right\| \|x\|$

- $\Rightarrow\ \frac{\|g(x)\|}{\|x\|} \leq \left\|\frac{\partial f}{\partial x}(z) - \frac{\partial f}{\partial x}(0)\right\| < \epsilon$
- $\Rightarrow\ \frac{\|g(x)\|}{\|x\|} < \epsilon\ \Rightarrow \lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} = 0$

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system (4.1) by its linearization about the origin

$$\dot{x} = Ax \text{ Where } A = \frac{\partial f_i}{\partial x}(0). \tag{4.2}$$

Theorem4.4: [21, 27, 28]

Consider a general non-homogenous non-autonomous linear system of the form

$$\dot{x} = A(t)x + f(t), \tag{4.3}$$

where $f(t)$ is a column vector.

Suppose $x^*(t)$is a solution of the equation (4.3). To be able to investigate the stability of the solution $x^*(t)$,define

$$\xi(t) = x(t) - x^*(t).$$

where

$x(t)$ is any other solution. Then we obtain the following homogenous equation

$$\dot{\xi}(t) = A(t)\xi\ . \tag{4.4}$$

Then all solutions of the linear system (4.3) have the same stability properties with the zero solution of (4.3) [19, 27, 28].

Theorem 4.5:.[27, 29] The zero solution of the system (4.3) is stable if and only if every solution is

bounded as $t \to \infty$. If A is a constant matrix and every solution is bounded, then the solutions are uniformly stable

Theorem 4.6: [21, 30] Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(x)$,

where $f: D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin.

Let $A = \frac{\partial f}{\partial x}(x)\Big|_{x=0}$ then,

1. The origin is asymptotically stable if real part of all eigenvalues A are less than zero.

2. The origin is unstable if real part of eigenvalues A is greater than zero for one or more eigenvalues of A.

This method allows us to determine the stability of the nonlinear system about the equilibrium point on the basis of the linearized system [21, 26, 31, 32].

i. If the Eigenvalues of a matrix A in the linearized system have negative real parts, the nonlinear system is stable about the equilibrium point.

ii. If at least one Eigenvalue of a matrix A in the linearized system has positive real part, the nonlinear system is unstable about the equilibrium point.

iii. If at least one Eigenvalue of a matrix in the linearized system has zero real part, the test is inconclusive. The linear approximation is in sufficient to determine stability. However, methods exist to include higher order terms.

Example 1:

$$\dot{x} = x(1 - x - 2y) = f(x, y)$$

$$\dot{y} = y(1 - 2x - y) = g(x, y)$$

The system is non linear

- ✓ To show both $f \& g$ has continuous partial derivative up to 2nd order

$f_x = 1 - 2x - 2y$ $\quad f_y = -2x$

$f_{xx} = -2$ $\quad f_{yy} = 0$ $\quad f_{xy} = -2$

$g_x = -2y$ $\quad g_y = 1 - 2x - 2y$

$g_{xx} = 0$ $\quad g_{yy} = -2$ $\quad g_{xy} = -2$

Hence f & g has continuous

s partial derivative up to 2nd order

- ✓ Their equilibrium point is

$$(0,0), (0,1), (1,0), (\tfrac{1}{3}, \tfrac{1}{3})$$

- ✓ The Jacobian is $\quad J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1 - 2x - 2y & -2x \\ -2y & 1 - 2x - 2y \end{pmatrix}$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 = \lambda_1 = \lambda_2 = 1$$

Hence the nonlinear system is unstable at $(0,0)$.

$$J(0,1) = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ -2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -1$$

Hence the nonlinear system is stable at $(0,1)$.

$$J(1,0) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{vmatrix} -1-\lambda & -2 \\ 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -1$$

Hence the nonlinear system is stable at$(1,0)$.

4.2 Main Results

Consider the system given by:

$$\begin{cases} \ddot{x} = \frac{\dot{m}}{m}(\mu - 1)\dot{x} - \frac{k_1}{m}\dot{x} - \frac{k_2}{m}\dot{z} \\ \ddot{z} = \frac{\dot{m}}{m}(\eta - 1)\dot{z} - \frac{k_2}{m}\dot{z} + \frac{k_2}{m}\dot{x} - g \end{cases} \tag{4.5}$$

Let $x = \varphi(t)$ and $z = \psi(t)$ are trajectory

Then solve for μ and η from (4.5)

$$\begin{cases} \ddot{\varphi} = \frac{\dot{m}}{m}(\mu - 1)\dot{\varphi} - \frac{k_1}{m}\dot{\varphi} - \frac{k_2}{m}\dot{\psi} \\ \ddot{\psi} = \frac{\dot{m}}{m}(\eta - 1)\dot{\psi} - \frac{k_2}{m}\dot{\psi} + \frac{k_2}{m}\dot{\varphi} - g \end{cases} \tag{4.6}$$

$$\begin{cases} \mu = 1 + \frac{m}{\dot{m}}\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_1\dot{\varphi} + k_2\dot{\psi}}{m\dot{\varphi}}\right) \\ \eta = 1 + \frac{m}{\dot{m}}\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} + \frac{k_2\dot{\psi} - k_2\dot{\varphi}}{m\dot{\psi}}\right) \end{cases} \tag{4.7}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_2 = \frac{\dot{m}}{m}(\mu - 1)x_2 - \frac{k_1}{m}x_2 - \frac{k_2}{m}x_4 \\ \dot{x}_4 = \frac{\dot{m}}{m}(\eta - 1)x_4 - \frac{k_2}{m}x_4 + \frac{k_2}{m}x_2 - g \end{cases} \tag{4.8}$$

Substituting (4.7) in (4.8) we have

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_2 = \left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2}{m}x_4 \\ \dot{x}_4 = \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 + \frac{k_2}{m}x_2 - g \end{cases} \tag{4.9}$$

Where x_2 and x_4 are velocities and x_1 and $x_3$are coordinates.

Based on the above equation (4.2) and theorem 4.4 we can consider the system

$$\dot{x} = Ax\ ,$$

Where

$$A = \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_2 = \left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2}{m}x_4 \\ \dot{x}_4 = \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 + \frac{k_2}{m}x_2 \end{cases},$$

To investigate the stability of the above system let us consider a Lyapunov function candidate given by:

$$V = \alpha(t){x_2}^2 + \beta(t){x_4}^2,$$

where $\alpha(t), \beta(t)$ are nonnegative functions and both of them not equal to zero.

Then.

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \left[\frac{\partial V}{\partial x_2}\frac{\partial V}{\partial x_4}\right]\begin{bmatrix}\dot{x}_2 \\ \dot{x}_4\end{bmatrix}$$

Let us consider different cases to establish stability conditions.

<u>Case I</u>

Suppose $\alpha(t) = \beta(t)$ is constant and positive.

Then

$\frac{dV}{dt} = \left[\frac{\partial V}{\partial x_2}\frac{\partial V}{\partial x_4}\right]\begin{bmatrix}\dot{x}_2\\ \dot{x}_4\end{bmatrix}$

$=[2\alpha x_2 \quad 2\beta x_4]\begin{bmatrix}\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2}{m}x_4\\ \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 + \frac{k_2}{m}x_2\end{bmatrix}.$

The first derivative of the Lyapunov function candidate V becomes

$$\dot{V} = 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2{}^2 - 2\alpha\frac{k_2}{m}x_2x_4 + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\varphi}}\right)x_4{}^2 + 2\beta\frac{k_2}{m}x_2x_4$$

$$= 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2{}^2 + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\varphi}}\right)x_4{}^2 + \left(-2\alpha\frac{k_2}{m} + 2\beta\frac{k_2}{m}\right)x_2x_4 \qquad (4.10)$$

To guarantee asymptotical stability we need to have $\dot{V} < 0$ which can be satisfied if and only if

$2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right) < 0,\quad 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right) < 0$ and $\frac{2k_2}{m}(\alpha - \beta) = 0.$

This leads to the stability condition given by:

$$\frac{\ddot{\varphi}}{\dot{\varphi}} < -\frac{k_2\dot{\psi}}{m\dot{\varphi}},\quad \frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}} < 0 \qquad (4.11)$$

Case II

$\alpha(t)$ and $\beta(t)$ are positive functions.

$V = \alpha(t)x_2{}^2 + \beta(t)x_4{}^2$

The first derivative of the Lyapunov function candidate V becomes

$\dot{V} = \dot{\alpha}x_2{}^2 + 2\alpha x_2\dot{x}_2 + \dot{\beta}x_4{}^2 + 2\beta x_4\dot{x}_4$

$= \dot{\alpha}x_2{}^2 + \dot{\beta}x_4{}^2 + 2\alpha x_2\left\{\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2x_4}{m}\right\} + 2\beta x_4\left\{\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 - \frac{k_2}{m}x_2\right\}$

$$= \dot{\alpha}x_2{}^2 + \dot{\beta}x_4{}^2 + 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2{}^2 + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4{}^2 + \left(\frac{2\beta k_2}{m} - \frac{2\alpha k_2}{m}\right)x_2x_4 \qquad (4.12)$$

To simplify the function let as take $\alpha = \frac{1}{\dot{\varphi}^2}$ and $\beta = \frac{1}{\dot{\psi}^2}$

Then $\dot{\alpha} = \frac{-2\ddot{\varphi}}{\dot{\varphi}^3}$ and $\dot{\beta} = \frac{-2\ddot{\psi}}{\dot{\psi}^3}$

$$\dot{V} = \left\{\dot{\alpha} + 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)\right\}{x_2}^2 + \left\{\dot{\beta} + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)\right\}{x_4}^2 + \left(\frac{2\beta k_2}{m} - \frac{2\alpha k_2}{m}\right)x_2x_4$$

$$= \left\{\frac{-2\ddot{\varphi}}{\dot{\varphi}^3} + \frac{2}{\dot{\varphi}^2}\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)\right\}{x_2}^2 + \left\{\frac{-2\ddot{\psi}}{\dot{\psi}^3} + \frac{2}{\dot{\varphi}^2}\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)\right\}{x_4}^2 + \frac{2k_2}{m}\left(\frac{1}{\dot{\varphi}^2} - \frac{1}{\dot{\varphi}^2}\right)x_2x_4 \qquad (4.13)$$

To guarantee asymptotical stability we need to have $\dot{V} < 0$ which can be satisfied if and only if

$$\frac{-2\ddot{\varphi}}{\dot{\varphi}^3} + \frac{2}{\dot{\varphi}^2}\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right) < 0 \qquad \frac{-2\ddot{\psi}}{\dot{\psi}^3} + \frac{2}{\dot{\varphi}^2}\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right) < 0 \quad \frac{2k_2}{m}\left(\frac{1}{\dot{\varphi}^2} - \frac{1}{\dot{\varphi}^2}\right) = 0$$

$$\frac{2k_2\dot{\psi}}{m\dot{\varphi}^3} < 0 \qquad \frac{2g}{\dot{\psi}\dot{\varphi}^2} - \frac{2k_2\dot{\varphi}}{m\dot{\psi}\dot{\varphi}^2} < 0 \qquad \dot{\varphi}^2 = \dot{\psi}^2$$

$$\frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}} < 0$$

This leads to the stability condition given by:

$$\frac{2k_2\dot{\psi}}{m\dot{\varphi}^3} < 0\,, \qquad \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}} < 0 \text{ and } \qquad \dot{\varphi}^2 = \dot{\psi}^2. \qquad (4.14)$$

Case III

$\alpha(t)\ and\ \beta(t)$ are functions given by

$$\alpha = e^m = \beta.$$

Then,

$$\dot{\alpha} = \dot{m}e^m = \dot{\beta},$$

where m is function of t.

First derivative of Lyapunov function candidate is,

$$\dot{V} = \left\{\dot{\alpha} + 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)\right\}{x_2}^2 + \left\{\dot{\beta} + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)\right\}{x_4}^2 + \frac{2k_2}{m}(\beta - \alpha)x_2x_4 \qquad (4.15)$$

To guarantee asymptotical stability we need to have $\dot{V} < 0$ which can be satisfied if and only if

$$\dot{m}e^m + 2e^m\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right) < 0 \qquad \dot{m}e^m + 2e^m\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right) < 0$$

$$e^m\left\{\dot{m} + \frac{2\ddot{\varphi}}{\dot{\varphi}} + \frac{2k_2\dot{\psi}}{m\dot{\varphi}}\right\} < 0 \qquad e^m\left\{\dot{m} + 2\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)\right\} < 0$$

$\dot{m} + \frac{2\ddot{\varphi}}{\dot{\varphi}} + \frac{2k_2\dot{\psi}}{m\dot{\varphi}} < 0$

$\frac{2\ddot{\varphi}}{\dot{\varphi}} < -\dot{m} - \frac{2k_2\dot{\psi}}{m\dot{\varphi}}$

$\frac{\ddot{\varphi}}{\dot{\varphi}} < \frac{-1}{2}\dot{m} - \frac{k_2\dot{\psi}}{m\dot{\varphi}}$

$\dot{m} + 2\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right) < 0$

$\frac{\ddot{\psi}}{\dot{\psi}} < -\frac{1}{2}\dot{m} - \frac{g}{\dot{\psi}} + \frac{k_2\dot{\varphi}}{m\dot{\psi}}$

Hence we have a stability condition given by, $\frac{\ddot{\varphi}}{\dot{\varphi}} < \frac{-1}{2}\dot{m} - \frac{k_2\dot{\psi}}{m\dot{\varphi}}$ and $\frac{\ddot{\psi}}{\dot{\psi}} < -\frac{1}{2}\dot{m} - \frac{g}{\dot{\psi}} + \frac{k_2\dot{\varphi}}{m\dot{\psi}}$. (4.16)

Example 1:

Let $x = \varphi = e^{2t}$ $\quad \dot{x} = \dot{\varphi} = 2e^{2t}$ $\quad \ddot{x} = \ddot{\varphi} = 4e^{2t}$

$Z = \psi = -e^{2t}$ $\quad \dot{Z} = \dot{\psi} = -2e^{2t}$ $\quad \ddot{Z} = \ddot{\psi} = -4e^{2t}$

Where $k_2(t) = e^{-3t}$ and mass $m = e^{-t}$

The criteria of stability is given by

$$\frac{2k_2\dot{\psi}}{m\dot{\varphi}^2} < 0 \quad \text{and} \quad \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}} < 0$$

$$-\frac{4e^{-3t}e^{2t}}{e^{-t}e^{4t}} = -4e^{-4t} < 0 \qquad \frac{10}{-2e^{2t}} - \frac{e^{-3t}(2e^{2t})}{e^{-t}(-2e^{2t})} = -5e^{-2t} + e^{-2t} = 4e^{-2t} < 0$$

Since the criteria is satisfied

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_2 = \left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2}{m}x_4 \\ \dot{x}_4 = \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 + \frac{k_2}{m}x_2 \end{cases}$$

Is asymptotically stable system by theorem (4.1) for Lyapunov function given by,

$V = \alpha(t){x_2}^2 + 4\beta(t){x_4}^2$ Where $\alpha(t) = \frac{1}{4}e^{-4t} = \beta(t)\ \forall t$.

Proof:

$$V = \alpha(t){x_2}^2 + \beta(t){x_4}^2$$

$$\dot{V} = \dot{\alpha}{x_2}^2 + 2\alpha x_2\dot{x}_2 + \dot{\beta}{x_4}^2 + 2\beta x_4\dot{x}_4$$

$$\dot{V} = \dot{\alpha}{x_2}^2 + \dot{\beta}{x_4}^2 + 2\alpha x_2\left\{\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2x_4}{m}\right\} + 2\beta x_4\left\{\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 - \frac{k_2}{m}x_2\right\}$$

$$\dot{V} = \dot{\alpha}{x_2}^2 + \dot{\beta}{x_4}^2 + 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right){x_2}^2 + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right){x_4}^2 + \left(\frac{2\beta k_2}{m} - \frac{2\alpha k_2}{m}\right)x_2x_4$$

Then substitute values of $\alpha, \beta, m, \varphi\ and\ \psi$

$$\dot{V} = -e^{-4t}{x_2}^2 - e^{-4t}{x_4}^2 + \frac{1}{2}e^{-4t}\left(\frac{4e^{2t}}{2e^{2t}} + \frac{e^{-3t}(-2e^{2t})}{e^{-t}(2e^{2t})}\right){x_2}^2 +$$

$$\frac{1}{2}e^{-4t}\left(\frac{-4e^{2t}}{-2e^{2t}} + \frac{g}{-2e^{2t}} - \frac{e^{-3t}(2e^{2t})}{e^{-t}(-2e^{2t})}\right){x_4}^2 + \frac{2(e^{-3t})}{e^{-t}}\left(\frac{1}{4}e^{-4t} - \frac{1}{4}e^{-4t}\right)x_2x_4$$

$$\dot{V} = -\frac{1}{2}e^{-6t}{x_2}^2 + (-\frac{g}{4}e^{-6t} + e^{-6t}){x_4}^2$$

$$\dot{V} = -\frac{e^{-6t}}{2}[{x_2}^2 + 4{x_4}^2] < 0,$$

where gravitational acceleration $g = 10$ and for any $x_2$and $x_4 \in D$, $\dot{V} < 0$ for $D/\{0\}$.

Hence by theorem (4.1) the system is asymptotically stable.

This implies that the system given by (1.3)

$$\begin{cases} \ddot{x} = \dfrac{\dot{m}}{m}(\mu - 1)\dot{x} - \dfrac{k_1}{m}\dot{x} - \dfrac{k_2}{m}\dot{z} \\ \ddot{z} = \dfrac{\dot{m}}{m}(\eta - 1)\dot{z} - \dfrac{k_2}{m}\dot{z} + \dfrac{k_2}{m}\dot{x} - g \end{cases},$$

is asymptotically stable by theorem 4.4.

Example 2: (Vertical Motion)

Let $x = \varphi = t \qquad \dot{x} = \dot{\varphi} = 1 \qquad \ddot{x} = \ddot{\varphi} = 0$

$Z = \psi = -t \quad \dot{Z} = \dot{\psi} = -1 \quad \ddot{Z} = \ddot{\psi} = 0$

Where $k_2(t) = e^{-3t}$ and mass $m = e^{-t}$

Based on the result in case II the stability condition for this system is given by

$$\frac{2k_2\dot{\psi}}{m\dot{\varphi}^2} < 0 \text{ and } \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}} < 0$$

Which leads to

$$-\frac{2e^{-3t}}{e^{-t}} = -2e^{-2t} < 0 \qquad \frac{10}{-1} - \frac{e^{-3t}}{e^{-t}(-1)} = -5 + e^{-2t} < 0$$

Since the criteria is satisfied

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_2 = \left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2}{m}x_4 \\ \dot{x}_4 = \left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 + \frac{k_2}{m}x_2 \end{cases}$$

Let us claim that is the system stable for Lyapunov function given by

$V = \alpha(t){x_2}^2 + \beta(t){x_4}^2$

Where $\alpha(t) = 1 = \beta(t) \ \forall t.$

Indeed,

$V = \alpha(t){x_2}^2 + \beta(t){x_4}^2$

$\dot{V} = \dot{\alpha}{x_2}^2 + 2\alpha x_2\dot{x}_2 + \dot{\beta}{x_4}^2 + 2\beta x_4\dot{x}_4$

$\dot{V} = \dot{\alpha}{x_2}^2 + \dot{\beta}{x_4}^2 + 2\alpha x_2\left\{\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right)x_2 - \frac{k_2x_4}{m}\right\} + 2\beta x_4\left\{\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right)x_4 - \frac{k_2}{m}x_2\right\}$

$\dot{V} = \dot{\alpha}{x_2}^2 + \dot{\beta}{x_4}^2 + 2\alpha\left(\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{k_2\dot{\psi}}{m\dot{\varphi}}\right){x_2}^2 + 2\beta\left(\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}}\right){x_4}^2 + \left(\frac{2\beta k_2}{m} - \frac{2\alpha k_2}{m}\right)x_2x_4$

Then substitute values of $\alpha, \beta, m, \varphi$ and ψ

$$\dot{V} = 2\left(0 + \frac{4e^{-t}(-1)}{e^{-t}(1)}\right){x_2}^2 + 2\left(0 + \frac{g}{-1} - \frac{4e^{-t}(1)}{e^{-t}(-1)}\right){x_4}^2 + \frac{2(4e^{-t})}{e^{-t}}(1-1)x_2x_4$$

$$\dot{V} = -4{x_2}^2 - 12{x_4}^2,$$

where gravitational acceleration $g = 10$, for any x_2 and $x_4 \in D$, $\dot{V} < 0$ for $D/\{0\}$.

Hence by theorem (4.1) the system is asymptotically stable.

Which implies that the system given by (1.3)

$$\begin{cases} \ddot{x} = \frac{\dot{m}}{m}(\mu - 1)\dot{x} - \frac{k_1}{m}\dot{x} - \frac{k_2}{m}\dot{z} \\ \ddot{z} = \frac{\dot{m}}{m}(\eta - 1)\dot{z} - \frac{k_2}{m}\dot{z} + \frac{k_2}{m}\dot{x} - g \end{cases},$$

is asymptotically stable system.

Let us see the simulation result using MATLAB 2008B based on the data in example 2 of vertical motion.

Accordingly the coefficient matrix, A, is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -9 \end{pmatrix}$$

Eigenvalues

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1.127 & 0 \\ 0 & 0 & 0 & -8.8730 \end{pmatrix}$$

Eigenvector

$$p = \begin{pmatrix} 1 & 0 & -0.6584 & -0.0141 \\ 0 & 0 & 0.7420 & 0.1252 \\ 0 & 1 & -0.0836 & -0.1111 \\ 0 & 0 & 0.0943 & 0.9858 \end{pmatrix}$$

In this case the general solution of the system is given by

$$c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+c_3\begin{pmatrix}-0.6584\\0.7420\\-0.0836\\-0.0943\end{pmatrix}\mathrm{e}^{-1.127\mathrm{t}}+c_4\begin{pmatrix}-0.0141\\0.1252\\-0.1111\\0.9858\end{pmatrix}\mathrm{e}^{-8.8730\mathrm{t}}$$

It turns out that any point in the plane generated by the null-space of AX, that is

$$\aleph(A)=\left\{c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\0\\1\\0\end{pmatrix}: c_1, c_2\ \epsilon\mathbb{R}\right\},$$

Consists of the equilibrium points and the phase trajectories are rays parallel to the plane generated by the other two eigenvectors. Since the nonzero eigenvectors are negative the motion as $t\to\infty$ converges to a point in the null space (asymptotic stability is then achieved). The simulation result, including the code (see appendix 1A), is shown in the figure below. In this particular simulation graph the same initial point is used for both the coordinates and the velocities of the system.

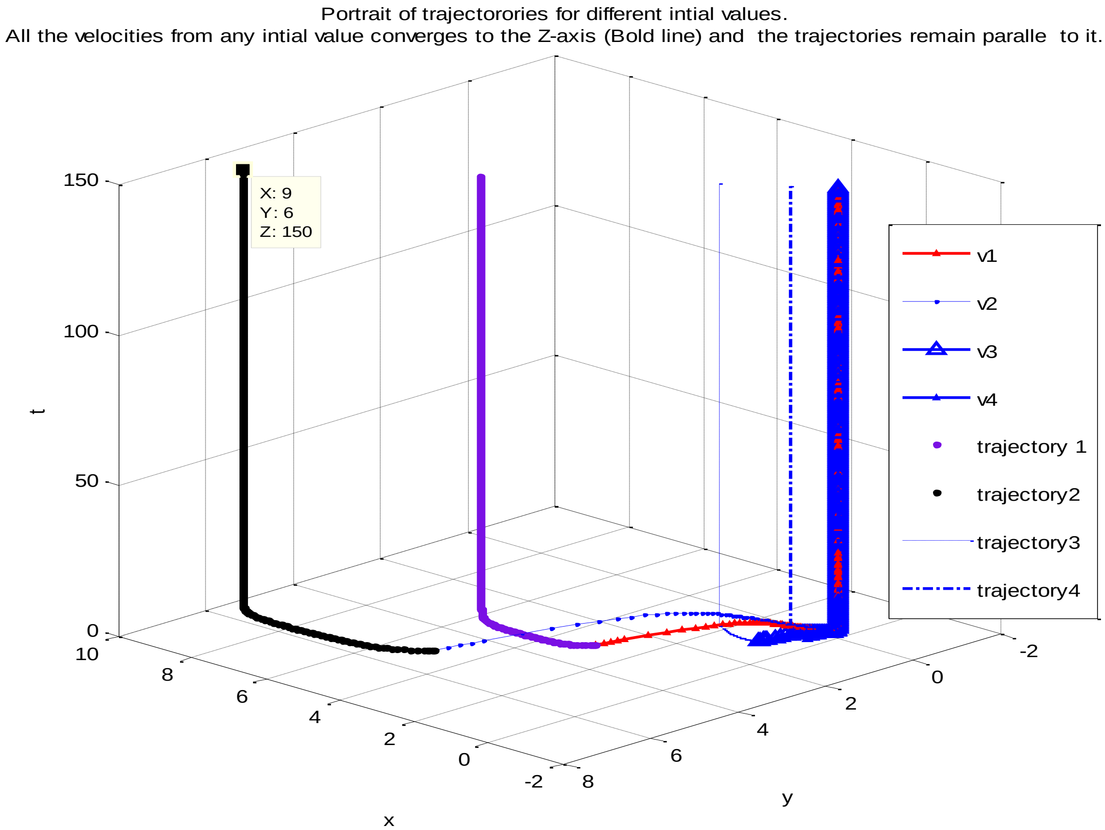

fig. 1

All the velocities which starts from different initial values converge to the Z-axis (which is bold line in the figure above) and the trajectories remain parallel to it. It is observed from the above 3D graph that, all the trajectories converge to a line in space given by the equation

$$\begin{cases} x = 1.8 + 1.8\lambda \\ y = 1.2 + 1.2\lambda \ , \\ z = t \end{cases}$$

where $t, \lambda \epsilon \mathbb{R}, and\ t \geq 0$, provided that the initial condition of the trajectories and the velocities are the same. Moreover it can be shown that this equilibrium line (with the fourth component zero) is a subspace of the null space of the coefficient matrix A. Moreover all the velocity curves converge to the Z-axis shown in bold line in the figure above. The portrait of the velocity direction is shown in the figure below (see appendix B for a Syntax).

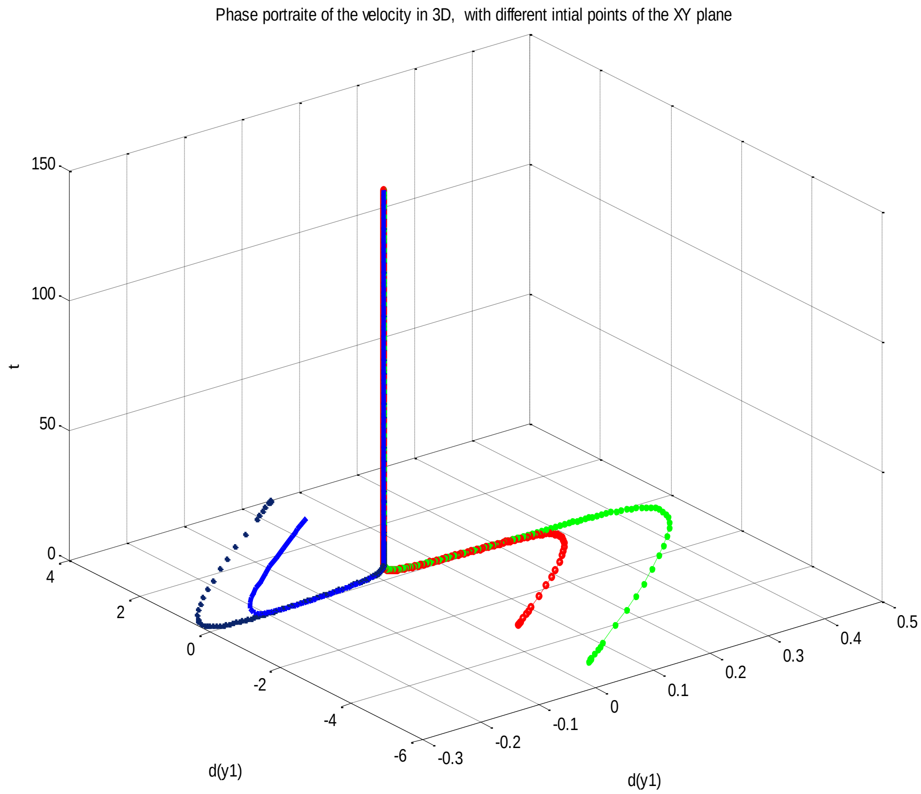

fig.2

Starting from any initial point in the XY plane the velocity trajectories are stable to the Z-axis.

The stability of the coordinates and the velocity curves are also simulated in 2D as shown in the figure below (see appendix C and D for a Syntax).

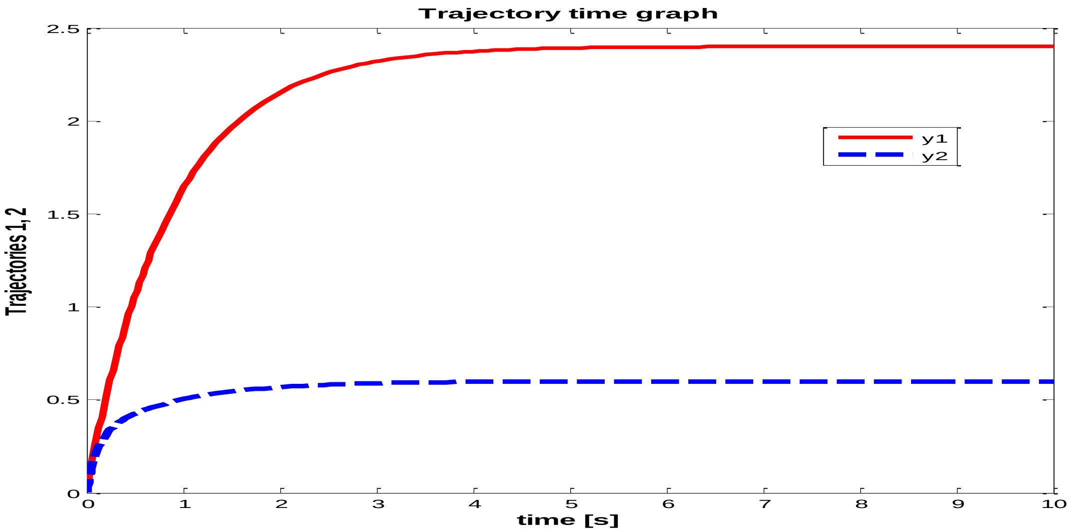

fig.3: trajectory time graph

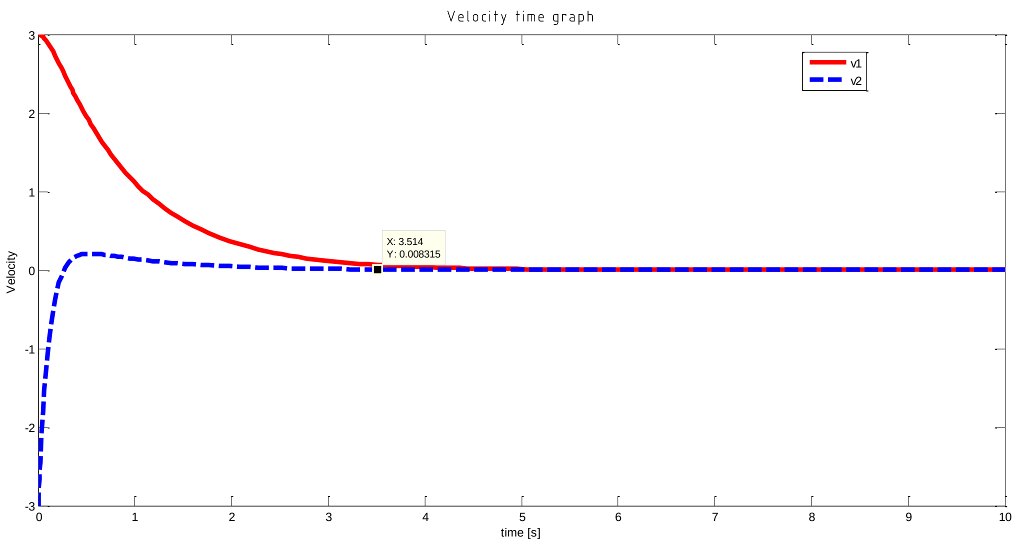

fig.4: velocity time graph

CHAPTER FIVE

5. CONCLUSION AND FUTURE SCOPE

5.1 Conclusion

In this paper the stability condition of Meshcherskii dynamic system with variable mass for different trajectories is analyzed. For the stability Lyapunov's direct Method is used. The result obtained based on Lyapunov function construction for different trajectories is summarized in the table below.

No.	General trajectory	Lyapunov function	Stability criteria	Remark
1	$x = \varphi(t)$ and $z = \psi(t)$	$V = \alpha(t){x_2}^2 + \beta(t){x_4}^2$	$\frac{\ddot{\varphi}}{\dot{\varphi}} < -\frac{k_2\dot{\psi}}{m\dot{\varphi}}$ $\frac{\ddot{\psi}}{\dot{\psi}} + \frac{g}{\dot{\psi}} - \frac{k_2\dot{\varphi}}{m\dot{\psi}} < 0$ $\alpha - \beta = 0$	$\alpha = \beta$ positive constant
			$\frac{2k_2\dot{\psi}}{m\dot{\varphi}^3} < 0$ $\frac{2g}{\dot{\psi}} - \frac{2k_2\dot{\varphi}}{m\dot{\psi}} < 0$ $\dot{\varphi}^2 = \dot{\psi}^2$	$\alpha = \frac{1}{\dot{\varphi}^2}$ and $\beta = \frac{1}{\dot{\psi}^2}$
			$\frac{\ddot{\varphi}}{\dot{\varphi}} < \frac{-1}{2}\dot{m} - \frac{k_2\dot{\psi}}{m\dot{\varphi}}$ $\frac{\ddot{\psi}}{\dot{\psi}} < -\frac{1}{2}\dot{m} - \frac{g}{\dot{\psi}} + \frac{k_2\dot{\varphi}}{m\dot{\psi}}$	$\alpha = e^m$ $= \beta$

5.2 Future Scope

Now a day stability theory is the most desirable area of study. In this work the stability criteria of Meshcherskii's equation of variable mass system was done. The upcoming post graduate student and other researchers who are interested in this area to use the result of this as a stepping stone and make further investigations. For instance the results of this paper can extended to control design for the dynamic system Rockets motion and spin stabilization of modern space crafts.

References

1. Arnold V.I. (1992).*Ordinary Differential Equations*. MIT press or springer verlag.
2. Perko L. (2001).*Differential Equation and Dynamical System*. Third Edition, *Text in Applied Mathematics7*, Springer, New York, ISBN 0-387-95116-4.
3. Thomson W. T. (1966).Equation of motion for variable mass. *Journal of American Institute of Aeronautics and Astronautics*, Vol. 4, No. 4, p. 766.
4. Simmons G.F. (1972).*Ordinary Differential Equations with application and historical notes.* McGrawal-Hall Book Company, New York.
5. Smith P. and Smith R. C. (1993).*Mechanics. S*econd edition, John Wiley and sons, New York.
6. Peraire J. and Widnall S. (2008).L*ecture on Variable Mass Systems: The Rocket Equation*.
7. Signell P. (2007).*Mass changing With Time. The Vertical Rocket*: State University, Michigan.
8. Hans I. and Alexander B. K. (2014).*Dynamics of Mechanical Systems with Variable Mass*. Springer, ISBN 978-3-7091-1809-2.
9. Cveticanin L. (2012).A Review on Dynamics of Mass Variable Systems. *Journal of the Serbian Society for Computational Mechanics*, Vol. 6, No. 1, pp. 56-73.
10. Allen S. M. (1967).*Equation of Motion for Variable Mass. Inertial Guided System*, Journal of Spacecraft. Vol. 4, No. 10, p. 1366.
11. Meshcherskii I.V. (1904).Equation of Motion of Variable Mass Point in the General Case.*St. Petersburg Polytechnic University News*, Vol. 1 pp77-118.
12. Meriam J.L. and Kraige L.G. (2002).*Engineering Mechanics.* Fifth Edition, by John Wiley & Sons, Inc.
13. Fidelis E. O. (1998).Dynamics of variable mass system. *NASA Research grant NAG 2-4003.* Final Technical Report Submitted by Department of Mechanical and Aeronautical Engineering University of California One Shields Avenue Davis, Ca 95616.
14. Samardzija B. and Segan S. (2012).Movement of a Body with Variable Mass. *Publication of the Astronomical Observatory*, Belgrade, No. 91, pp. 97-104.
15. George H. W. and Donald H. E. (1980).*Applied Mechanics Dynamics*. Division of Engineering California Institute of Technology.

16. Horst L. (1987).A*n Introduction to the Stability of Dynamic Systems and Rigid Bodies*. Second Edition, University Of Waterloo, Ontario, Canada.
17. Chutiphon P. (2011).A Review of Fundamentals of Lyapunov Theory. T*he Journal of Applied Science*, Vol. 10 No. 2, pp. 55-61.
18. Remco I. (2010). The Historical Development of Classical Stability Concepts: Lagrange, Poisson and Lyapunov Stability. *Nonlinear Dynamics. Vol.59* Pp. 173–182, Doi 10.1007.
19. Gerhard B. (2003).*Selected Topics in M214*. University of Bergen.
20. Guanrong C. (2004). Stability of nonlinear system, John Wiley, New York, Pp. 4881-4896.
21. Hassan K. (2002).*Nonlinear Systems*. Third Edition Michigan State University, Prentice Hall.
22. Vidyasar M. (1993).*Nonlinear System Analysis*. Second Edition, Prentice Hall, Englewood Cliffs, New York.
23. Shanker S. (1999). *Nonlinear System Analysis, Stability and Control.* Spring-Verlag, New York.
24. Alexander B. and Irina F. (2006). Stability Concepts, *Control System, Robotics and Automation. Vol. 1.*
25. Hahn W. (1967).S*tability of motion*. Translated by Arne P. B. springer-Verlag, New York Inc.
26. Slotine L. (1991). *Applied Nonlinear Control*. Prentice-Hall, Inc. A Division of Simon & Schuster Englewood Cliffs, New Jersey.
27. Brauer F. and Nohel J. A. (1969*). The Qualitative Theory of Ordinary Differential Equations,* Novel Publications, Inc., New York.
28. Peter G. (2010).Review On Computational Methods for Lyapunov Functions, *Discrete and Continuous Dynamical Systems Series B*, Volume 20, Number 10.
29. Deo S.G. and Tagavandra V. (1980). *Ordinary differential equation and stability theory*. Tata McGrawal-Hill, New York.
30. Nicolas M. Master PSPI (2009-2010). *Control of Nonlinear Systems.* Gipsa-lab Grenoble, France.

31. Matousek R. et al. (2009). Simple Methods for Stability Analysis of Nonlinear Control Systems. *Proceedings of the World Congress on Engineering and Computer Science Vol. II*, San Francisco, USA.
32. Bountis T. (2006). Stability of Motion: From Lyapunov to the Dynamics of *N*-Degree of Freedom Hamiltonian Systems. *Nonlinear Phenomena in Complex Systems Vol. 9, No. 3.*

Appendix 1

A: MATLAB code for fig.1

```
tspan=[0 150];
y0=[3;3;3;3];
[t,y]=ode45(@sta1,tspan,y0);
xlabel('x'),ylabel('y'),zlabel('t');
plot3(y(:,2),y(:,4),t,'r');
hold on;
tspan=[0 150];
y0=[5;5;5;5];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'g');
hold on;
y0=[1;1;1;1];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'--');
grid  on;
hold on;
y0=[0.4;0.4;0.4;0.4];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'-');
hold on ;
tspan=[0 150];
y0=[3;3;3;3];
[t,y]=ode45(@sta1,tspan,y0);
xlabel('x'),ylabel('y'),zlabel('t');
plot3(y(:,1),y(:,3),t,'g');
hold on;
tspan=[0 150];
y0=[5;5;5;5];
[t,y]=ode45(@sta1,tspan,y0);
```

```
plot3(y(:,1),y(:,3),t,'g');
hold on;
y0=[1;1;1;1];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,1),y(:,3),t,'--');
grid  on;
hold on;
y0=[0.4;0.4;0.4;0.4];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,1),y(:,3),t,'-');
```

B: MATLAB code for fig.2

```
tspan=[0 150];
y0=[-3;0.05;0.05;-3];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'r');
xlabel('x'),ylabel('y'),zlabel('t');
hold on;
tspan=[0 150];
y0=[-5;0.05;0.05;-5];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'g');
hold on;
y0=[3;0.05;0.05;3];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'--');
grid  on;
hold on;
y0=[0.4;0.05;0.05;4];
[t,y]=ode45(@sta1,tspan,y0);
plot3(y(:,2),y(:,4),t,'-');
```

C MATLAB code for fig.3

```
tspan=[0 10];
y0=[0.00005;3;0.00005;3];
[t,y]=ode45(@sta1,tspan,y0);
xlabel('t'),ylabel('y');
plot(t,y(:,1),'r');
hold on;
plot(t,y(:,3),'g');
```

D MATLAB code for fig.4

```
tspan=[0 10];
y0=[0.00005;1.5;0.00005;1.5];
[t,y]=ode45(@sta1,tspan,y0);
xlabel('t'),ylabel('y');
plot(t,y(:,2),'r');
hold on;
plot(t,y(:,4),'g');
```