Stability Analysis of Prey-Predator Mathematical Model with Delay and Control of the Prey

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science (M.Sc.) in Mathematics.

By: Dawit Getachew

Advisor: Dr. Chernet Tuge

Co-advisor: Dinka Tilahun (M.Sc)

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Abstract

Dynamics of interacting biological species has been studied in the past few decades from various angles. Many species become extinct and many others are at the verge of extinction due to several reasons like, over exploitation, over predation, environmental pollution, mismanagement of natural resources etc. Establishing the conditions for the stability of ecosystems and for stable coexistence of interacting populations is a problem of the highest priority in mathematical ecology. Bearing this in mind, in this study the stability of prey predator in the absence of delay was clearly stated. The minimum cutoff value at which the system loses its stability was also pointed out. Furthermore, the existence of global stability without linearizing the model was proved with an appropriate condition. Finally, non-existence of limit cycle at positive equilibrium was proved by Dulac's criterion.

Key words: *Prey-predator model, local stability, global stability, limit cycle*

Contents

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Predator-prey model is the first model to illustrate the interaction between predators and prey. It is a topic of great interest for many ecologists and mathematicians. This model assumes that the predator populations have negative effects on the prey populations. The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance (Berryman, 1992). The central goal in ecology is to understand the dynamical relationship between predators and prey (Kot, 2001). The most significant factor of the prey predator relationship is the predator's rate of feeding upon prey, known as predator's functional response, which is the average number of prey killed per individual predator per unit of time.

In recent years, predator-prey models are arguably the most fundamental building blocks of any biological and ecosystems as all biomasses are grown out of their resource masses. Species compete, evolve and disperse often simply for the purpose of seeking resources to sustain their struggle for their very existence. Their extinctions are often the results of their failure in obtaining the minimum level of resources needed for their subsistence. Mathematical models in terms of ordinary differential equation (ODE) have been widely used to model physical phenomena, engineering systems, economic behavior, biological and biomedical processes. In particular, ODE models have recently played a prominent role in describing the dynamic behavior of predator-prey systems. The study of population phenomena or growth phenomena or competition between two species is really dominated problem in the biological system. The first prey-predator model with aftereffect was proposed by Volterra (Kolmanovskii and Myshkis, 1999) as follows:

$$\begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) \\ y'(t) = -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) \end{cases}, \tag{1.1}$$

where $x(t)$ – Population density of prey at time t , $y(t)$ – Population density of predator at time t

a_1 – intrinsic growth rate of prey, a_4 – death rate of predators,

a_2 & a_5 – Interspecific competition, a_3 – intraspecific competition rate of prey,

τ – the average time delay between death of prey and birth of subsequent number of predators and $a_i > 0\ (i = 1, 2, ..., 5)$.

In the real world, there is sometimes a need to control a population at a reasonable level because otherwise this population may cause decrease or even extinction of other populations. Bearing this in mind, if a control u, taking into account some purposeful action of various factors on the system, acts only on predators, then model (1.1) modified to the following model (Kolmanovskii and Myshkis, 1999).

$$\begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - u x(t) \\ y'(t) = -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) \end{cases}, \qquad (1.2)$$

$$u > 0$$

Stability of dynamical systems plays a very important role in control system analysis and design. Unlike the case of linear systems, proving stability of equilibria of nonlinear systems is more complicated. A sufficient condition is the existence of a Lyapunov function: a positive definite function V defined in some region of the state space containing the equilibrium point whose derivative along the system trajectories is negative semi-definite. This is Lyapunov's direct method, which even though addresses exactly and in a simple way the important issue of stability, it does not provide any coherent methodology for constructing such a function. Lyapunov's indirect method that investigates the local stability of the equilibria, is inconclusive when the linearized system has imaginary axis eigenvalues (Hassan, 2002).

Chernet Tuge and Mitiku Daba, (2017), investigated the stability analysis of delayed nonlinear cournot model in the sense of Lyapunov. One of the finding of this investigation indicates that the presence of time delay in a given model causes oscillation process in the system and doesn't affect the qualitative behavior of the solution (no effect on the stability of the equilibrium point), but only changes the transition process. In other words it delays stability as delay parameter increases.

Mathematical control theory is the area of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object means to influence its behavior so as to achieve a desired goal. In order to implement this influence, control engineers build devices that incorporate various mathematical techniques. Mathematical control theory is a rapidly growing field which provides theoretical and computational tools for dealing with a variety of problems arising in electrical and aerospace engineering, automatics, robotics, management, economics, applied chemistry, biology, ecology, medicine, etc. Selected such problems, to mention but a few, are the following : stable performance of motors and machinery, optimal guidance of rockets, optimal exploitation of natural resources, optimal investment or production strategies, regulation of physiological functions, and fight against insects, epidemics (Remsing, 2006).

A dynamical system is one which changes in time (in some well-defined way); what changes is the state of the system. For such systems, the basic problem is to predict the future behavior. For this purpose the differential equations are exactly tailored. The differential equation itself represents the (physical or otherwise) law governing the evolution of the system; this plus the initial conditions should determine uniquely the future evolution of the system.

To date, many authors have studied the dynamics of predator-prey models with time delay and obtained complex dynamic behavior, such as stability of equilibrium, Hopf bifurcation, and limit cycle. For example, Song and Wei (2005) investigated further the dynamics of the system prey-predator model by considering the time delay as the bifurcation parameter and they obtained that, under certain conditions, the unique positive equilibrium of the model is absolute stable while it is conditionally stable and there exist k switches from stability to instability to stability under other conditions. Further, by using normal form theory and the center manifold theorem, they obtained the formulae for determining the direction of Hopf bifurcations and the stability of bifurcating periodic solutions. Yan and Li (2006) studied the properties of Hopf bifurcation for prey-predator model by using normal form theory and the center manifold theorem, which is different from that used by Song and Wei (2005).

In 2017, Ahmed Buseri studied global asymptotic stability analysis of predator-prey model. In 2016, Murthy studied stability analysis of prey-predator harvesting model using biological parameters. In 2015, Soliman and Jarallah, studied asymptotic stability of solutions of lotka-volterra predator-prey model for four species. In 2014, Yue and Qingling studied stability and bifurcation analysis of a singular delayed predator-prey bioeconomic model with stochastic fluctuations. In 2013, Liu et al., studied global stability analysis and optimal control of a harvested eco-epidemiological prey predator model with vaccination and taxation. In 2012, Debasis studied the bifurcation and stability analysis of prey-predator model with a reserved area. In 2012, Debasis studied the bifurcation and stability analysis of prey-predator model with a reserved area. In 2011, Xu et al., studied stability and Hopf bifurcation analysis for a lotka-volterra predator–prey model with two time delays.

However, the dynamical behavior of prey-predator model represented by equation (1.2) is not yet studied. Therefore, the main objective of this study is to investigate the dynamic behavior like local stability, global stability and existence of limit cycle of prey-predator model represented by equation (1.2).

1.2 Statement of the Problem

This research mainly focuses on the following problems related to prey-predator mathematical model given by equation (1.2).

- Local stability analysis,
- Global stability analysis,
- To check existence and non-existence of limit cycle.

1.3 Objective of the Study

1.3.1 General Objective of the Study

The general objective of this study is to investigate stability analysis of prey-predator mathematical model with delay and control of the prey represented by equation (1.2).

1.3.2 Specific Objectives of the Study

The specific objectives of the study are to:

- Determine local stability condition of prey-predator model given by equation (1.2) using Routh Hurwitz Criterion.
- Analyze global stability of prey-predator model given by equation (1.2) by constructing appropriate Lyapunov function.
- Check the existence and non-existence of limit cycle using Dulac's criterion.

1.4. Significance of the Study

The outcomes of this study have the following importance:

- Provides evidence on stabilizing prey-predator coexistence in a given real ecosystem without extinction of the preys by applying a necessary control on prey.
- The result and the method can be used as bench mark for other researchers in related areas.

1.5 Delimitation of the Study

This study is delimited to investigate local stability, global stability and existence of limit cycle of prey-predator mathematical model represented by equation (1.2).

CHAPTER TWO

LITERATURE REVIEW

2.1 Historical background

Mathematical models in terms of ordinary differential equations (ODE's) have been widely used to model physical phenomena, engineering systems, economic behavior, biological and biomedical processes. In particular, ODE models have recently played a prominent role in describing the dynamic behavior of predator-prey systems. To study the dynamic behavior of a model, mathematical modeling is used as an effective tool to describe and analyze the model. Mathematical population models have been used to study the dynamics of prey predator systems since Lotka and Volterra proposed the simple model of prey-predator interactions now called the Lotka-Volterra model. Since then, many mathematical models, some reviewed in this study, have been constructed based on more realistic explicit and implicit biological assumptions. Modeling is a frequently evolving process, to gain a deep understanding of the mathematical aspects of the problem and to yield non trivial biological insights; one must carefully construct biologically meaningful and mathematically tractable population models (Kuang, 2002).

Inter species or Intra species competition models have been the subjects of central discussions in ecological and biological systems. Among the competition models, Lotka-Volterra inter-specific competition model occupies the top role to discuss the competitive behavior of the biological species which determines the present state in terms of past state and changes with the period of time. The competition models are used in forecasting of species growth rate, maximum and minimum consumption of resource, food pre- serving, environment capacities and many others applications. The study of population phenomena or growth phenomena or competition between two species is really dominated problem in the biological system. Volterra (1926) first developed a competition model between a predator and a prey (Brauer and Soudack, 1979). In the ecology system, the predator-prey model is among the oldest studies and also the first model to illustrate the interaction between predators and prey.

This model assume that the predator populations have negative effects on the prey populations and this system was formulated by Vito Volterra who is an Italian mathematician and Alfred Lotka who is an American mathematical biologist in 1925 (Boyce, 2010).

2.2 Models of Prey Predator with time delay

Predator-prey interaction is the fundamental structure in population dynamics. Understanding the dynamics of predator-prey models will be very helpful for investigating multiple species interactions. Delayed predator-prey models were first proposed by Volterra (Volterra, 1931) in 1925 to study fish population under harvesting. Since then delayed differential equations have been extensively used to model population dynamics, including predator-prey interactions. The original delayed predator-prey models proposed by Volterra (Volterra, 1931), are described by integro differential equations; such delays are also called distributed delays Cushing (Cushing, 1977) and Mac Donald (MacDonald, 1978), and discrete delays are special cases of the distributed delays when the kernels are taken as delta functions.

In nature, populations do not reproduce instantaneously; rather, it is mediated by certain time delay required for gestation, maturation time, capturing time, or other reasons. Thus, time delays of one type or another have been incorporated into mathematical models of population dynamics. The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology because of its universal existence and importance (Yue and Qingling, 2014). The time delay is considered into the population dynamics when the rate of change of the population is not only a function of the present population but also depends on the past population.

2.3 Models of Prey Predator with control

In the context of predator-prey interaction, some studies that treat population can be extended by Martin and Ruan have analyzed generalized Gause predator prey models where the prey is harvested with constant rate while Kar considered the predator-prey model with the predator harvested and suggested that it is ideal to study the combined harvesting of predator and prey population models (Kar, 2003). The effect of constant rate of harvesting has been studied by Holmberg and the results showed that the constant catch quota can lead to both oscillations and chaos and an increased risk for over exploitation (Homberg, 1995).

Brauer and Soudack have analyzed the global behavior of a predator-prey system under constant rate predator harvesting. They showed how to classify the possibilities and determine the region of stability. They found that if the equilibrium point is asymptotically stable which is determined by a local linearization, then every solution whose initial value is in some neighborhood of the stable equilibrium point tends to it as the time approaches infinity.

There exists an asymptotically stable limit cycle when the constant rate is small and the equilibrium point is unstable (Brauer and Soudack, 1979).

A predator-prey model with Holling type using harvesting efforts as control has been presented by Srinivasu et al. and showed that with harvesting, it is possible to break the cyclic behavior of the system and introduces a globally stable limit cycle in the system (Srinivasu *et al.*, 2001). The effect of constant rate of harvesting on the dynamics of predator prey systems has been investigated by many authors, for example, Brauer, Soudack and Myerscough et al. Some interesting dynamical behaviors have been observed such as the stability of the equilibria, existence of Hopf bifurcation and limit cycles.

It is well known that harvesting has a strong impact on the dynamic evolution of a population. Evidence shows that many species have already become extinct and many others are at the verge of extinction due to several natural or man-made reasons like over exploitation, indiscriminate harvesting and mismanagement of natural resources, and so forth. The severity of this impact that may range from rapid depletion to complete preservation of a population depends on the implemented harvesting agency (Kar, 2003). Due to practical and economic utilization, biological resources in the prey predator system are extensively harvested nowadays. Furthermore, exploitation of biological resources has been increased by people's multifarious material needs, which attract a global concern to protect the limited biological resources.

Therefore, regulation of exploitation of biological resources has become a problem of major concern in view of dwindling resource stocks and the deteriorating environment. It is necessary to establish a constructive management of commercial exploitation of the biological resources. The techniques and issues associated with bio-economic exploitation have been discussed in details (Clark, 1990).

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted in Jimma University under the department of Mathematics from September 2017 to June 2018 G.C.

3.2. Study Design

This study employed mixed-design (documentary review design and experimental design) on prey-predator model given by equation (1.2).

3.3. Source of Information

The relevant sources of information for this study were books, published articles & related studies from internet.

3.4. Mathematical Procedures

This study was conducted based on the following procedures

- **i.** Determining the steady state point of the model.
- **ii.** Linearizing the mathematical model of prey-predator under consideration.
- **iii.** Determining the local stability condition of the model.
- **iv.** Analyzing the global stability of the model.
- **v.** Checking existence and non-existence of limit cycle.

CHAPTER FOUR

RESULT AND DISCUSSION

4. Mathematical Model for prey predator

Studying and modeling the interaction between predators and prey have been one of the central topics in ecology and evolutionary biology. The first prey-predator model with aftereffect and control on prey was proposed by Volterra as given by equation (4.1). In this study the dynamic behavior like equilibrium point, local stability in the absence and presence of delay, global stability and existence of limit cycle were studied as follows.

$$\begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - u x(t) \\ y'(t) = -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) \end{cases} \tag{4.1}$$

4.1 Equilibrium point

To find equilibrium point, equate the right hand side of equation (4.1) with zero,

$$\begin{cases} a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - u x(t) = 0 \\ -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) = 0 \end{cases} \tag{4.2}$$

Since the time delay has no effect on the equilibrium point,

$$\begin{cases} x(t)\left[a_1 - a_2 y(t) - a_3 x(t) - u x(t)\right] = 0 \\ y(t)\left[-a_4 + a_5 x(t)\right] = 0 \end{cases},$$

$$\begin{cases} x(t) = 0 \quad or \quad \left[a_1 - a_2 y(t) - a_3 x(t) - u x(t)\right] = 0 \\ y(t) = 0 \quad or \quad \left[-a_4 + a_5 x(t)\right] = 0 \end{cases}$$

$$E_1 = (0,0)$$

$$\begin{cases} a_1 - a_2 y(t) - a_3 x(t) - u x(t) = 0 \\ -a_4 + a_5 x(t) = 0 \end{cases} \tag{4.3}$$

$$\begin{cases} -a_2 y(t) - a_3 x(t) = ux(t) - a_1 \\ a_5 x(t) = a_4 \end{cases}$$

$$x(t) = \frac{a_4}{a_5} \tag{4.4}$$

To find $y(t)$ plugging equation (4.4) into equation (4.3) we get,

$$y(t) = \frac{a_1 a_5 - a_3 a_4 - u a_4}{a_2 a_5}$$

Therefore, $x(t) = \dfrac{a_4}{a_5}$ *and* $y(t) = \dfrac{a_1 a_5 - a_3 a_4 - u a_4}{a_2 a_5}$

$E_2 = (x^*, y^*)$

Where, $x^* = \dfrac{a_4}{a_5}$ and $y^* = \dfrac{a_1 a_5 - a_3 a_4 - u a_4}{a_2 a_5}$

Providing that $a_1 a_5 - a_3 a_4 - u a_4 > 0$

To find equilibrium point in the absence of prey and presence of predator, substitute $(0, y^*)$ in equation (4.2). Then we get,

$x = 0 \ \Rightarrow y = 0 \quad E_1 = (0,0)$

To find equilibrium point in the absence of predator and presence of prey, substitute $(x^*, 0)$ in equation (4.2). Then we get,

$$\begin{cases} a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - ux(t) = 0 \\ -a_4 y(t) + a_5 x(t) y(t) = 0 \end{cases} \tag{4.5}$$

$$a_1 x(t) - a_2 x(t)(0) - a_3 x^2(t) - ux(t) = 0$$

$$x(t)\left[a_1 - a_3 x(t) - u\right] = 0$$

$$x(t)=0 \quad \text{or} \quad a_1-a_3x(t)-u=0$$

$$x(t)=\frac{1}{a_3}\left[a_1-u\right]$$

Therefore $E_3=(\frac{1}{a_3}\left[a_1-u\right],0)$.

4.2 Linearization

$$\begin{cases} x'(t)=a_1x(t)-a_2x(t)y(t)-a_3x^2(t)-ux(t) \\ y'(t)=-a_4y(t)+a_5x(t-\tau)y(t-\tau) \end{cases} \tag{4.6}$$

Let $x_1(t)=x(t)-x^*$ and $y_1(t)=y(t)-y^*$

$$\left.\begin{aligned} x(t)=x_1(t)+x^* \\ y(t)=y_1(t)+y^* \end{aligned}\right\} \Rightarrow \begin{cases} x'(t)=x_1'(t) \\ y'(t)=y_1'(t) \end{cases} \tag{4.7}$$

Now plugging equation (4.7) in to equation (4.6)

$$\begin{cases} x_1'(t)=a_1\left[x_1(t)+x^*\right]-a_2\left[x_1(t)+x^*\right]\left[y_1(t)+y^*\right]-a_3\left[x_1(t)+x^*\right]^2-u\left[x_1(t)+x^*\right] \\ y_1'(t)=-a_4\left[y_1(t)+y^*\right]+a_5\left[x_1(t-\tau)+x^*\right]\left[y_1(t-\tau)+y^*\right] \end{cases}$$

$$\begin{cases} x_1'(t)=a_1x_1(t)+a_1x^*-a_2\left[x_1(t)y_1(t)+x_1(t)y^*+x^*y_1(t)+x^*y^*\right]-a_3\left[x_1^2(t)+2x^*x_1(t)+(x^*)^2\right]-ux_1(t)-ux^* \\ y_1'(t)=-a_4y_1(t)-a_4y^*+a_5\left[x_1(t-\tau)y_1(t-\tau)+x_1(t-\tau)y^*+x^*y_1(t-\tau)+x^*y^*\right] \end{cases}$$

$$\begin{cases} x_1'(t)=a_1x_1(t)+a_1x^*-a_2x_1(t)y_1(t)-a_2x_1(t)y^*-a_2x^*y_1(t)-a_2x^*y^*-a_3x_1^2(t)-2a_3x^*x_1(t)-a_3(x^*)^2-ux_1(t)-ux^* \\ y_1'(t)=-a_4y_1(t)-a_4y^*+a_5x_1(t-\tau)y_1(t-\tau)+a_5x_1(t-\tau)y^*+a_5x^*y_1(t-\tau)+a_5x^*y^* \end{cases}$$

$$\begin{cases} x_1'(t)=a_1x^*-a_2x^*y^*-a_3(x^*)^2-ux^*+a_1x_1(t)-a_2x_1(t)y^*-a_2x^*y_1(t)-a_3x_1^2(t)-2a_3x^*x_1(t)-a_2x_1(t)y_1(t)-ux_1(t) \\ y_1'(t)=-a_4y^*+a_5x^*y^*-a_4y_1(t)+a_5x_1(t-\tau)y_1(t-\tau)+a_5x_1(t-\tau)y^*+a_5x^*y_1(t-\tau) \end{cases}$$

$$a_1x^* - a_2x^*y^* - a_3(x^*)^2 - ux^* = 0 \quad \text{and} \quad -a_4y^* + a_5x^*y^* = 0$$

$$\begin{cases} x_1'(t) = a_1x_1(t) - a_2x_1(t)y^* - a_2x^*y_1(t) - a_3x_1^2(t) - 2a_3x^*x_1(t) - a_2x_1(t)y_1(t) - ux_1(t) \\ y_1'(t) = -a_4y_1(t) + a_5x_1(t-\tau)y_1(t-\tau) + a_5x_1(t-\tau)y^* + a_5x^*y_1(t-\tau) \end{cases} \tag{4.8}$$

$$\begin{cases} x_1'(t) = \left[a_1 - a_2y^* - 2a_3x^* - u\right]x_1(t) - a_2x^*y_1(t) - a_2x_1(t)y_1(t) - a_3x_1^2(t) \\ y_1'(t) = a_5y^*x_1(t-\tau) - a_4y_1(t) + a_5x^*y_1(t-\tau) + a_5x_1(t-\tau)y_1(t-\tau) \end{cases}$$

$$\begin{cases} x_1'(t) = m_1x_1(t) + m_2y_1(t) + m_3x_1(t)y_1(t) + m_4x_1^2(t) \\ y_1'(t) = n_1x_1(t-\tau) + n_2y_1(t) + n_3y_1(t-\tau) + n_4x_1(t-\tau)y_1(t-\tau) \end{cases}, \tag{4.9}$$

where $m_1 = a_1 - a_2y^* - 2a_3x^* - u$,

$m_2 = -a_2x^*$, $m_3 = -a_2$, $m_4 = -a_3$,

$n_1 = a_5y^*$, $n_2 = -a_4$, $n_3 = a_5x^*$, $n_4 = a_5$.

However, $x_1(t)$ and $y_1(t)$ are small perturbation hence, its product as well as any higher order greater or equal to two goes to zero.

$$x_1(t)\ y_1(t) \longrightarrow 0,\ x_1^2(t) \longrightarrow 0,\ x_1(t-\tau)y_1(t-\tau) \longrightarrow 0$$

Therefore, equation (4.9) reduced to

$$\begin{cases} x_1'(t) = m_1x_1(t) + m_2y_1(t) \\ y_1'(t) = n_1x_1(t-\tau) + n_2y_1(t) + n_3y_1(t-\tau) \end{cases}, \tag{4.10}$$

which is the linearized form.

4.3 Local stability

Local stability of the model is predicted from the linearized part

$$\begin{cases} x_1'(t) = m_1 x_1(t) + m_2 y_1(t) \\ y_1'(t) = n_1 x_1(t-\tau) + n_2 y_1(t) + n_3 y_1(t-\tau) \end{cases} \tag{4.11}$$

The characteristics equation

Let $x_1(t) = re^{\lambda t}$ then $x'_1(t) = r\lambda e^{\lambda t}$ and $y_1(t) = se^{\lambda t}$ then $y'_1(t) = s\lambda e^{\lambda t}$ (4.12)

Plugging equation (4.12) into equation (4.11)

$$re^{\lambda t} = m_1 re^{\lambda t} + m_2 se^{\lambda t}$$
$$s\lambda e^{\lambda t} = n_1 re^{\lambda(t-\tau)} + n_2 se^{\lambda t} + n_3 se^{\lambda(t-\tau)}$$

$$r\lambda e^{\lambda t} = m_1 re^{\lambda t} + m_2 se^{\lambda t}$$
$$s\lambda e^{\lambda t} = n_1 re^{\lambda t} e^{-\lambda t} + n_2 se^{\lambda t} + n_3 se^{\lambda t} e^{-\lambda t}$$

Since $e^{\lambda t} \neq 0$

$$r\lambda = m_1 r + m_2 s \qquad m_1 r + m_2 s - r\lambda = 0$$
$$s\lambda = n_1 re^{-\lambda t} + n_2 s + n_3 se^{-\lambda t} \qquad n_1 re^{-\lambda t} + n_2 s + n_3 se^{-\lambda t} - s\lambda = 0$$

$$\begin{aligned} &r(m_1 - \lambda) + m_2 s = 0 \\ &n_1 re^{-\lambda t} + s(n_2 + n_3 e^{-\lambda t} - \lambda) = 0 \end{aligned} \tag{4.13}$$

For equation (4.13) to have non trivial solution the determinant of coefficient matrix must be zero.

$$\begin{vmatrix} m_1 - \lambda & m_2 \\ n_1 e^{-\lambda\tau} & n_2 + n_3 e^{-\lambda\tau} - \lambda \end{vmatrix} = 0$$

$$(m_1-\lambda)(n_2+n_3e^{-\lambda\tau}-\lambda)-m_2n_1e^{-\lambda\tau}=0$$

$$\lambda^2-\lambda(m_1+n_2)+m_1n_2+(-n_3\lambda+m_1n_3-m_2n_1)e^{-\lambda\tau}=0 \quad , \qquad (4.14)$$

which is the characteristics equation of equation (4.11).

Case 1 If $\tau=0$, in the absence of time delay characteristic equation (4.14) reduced to:

$$\lambda^2-\lambda(m_1+n_2)+m_1n_2+(-n_3\lambda+m_1n_3-m_2n_1)=0$$

$$\lambda^2-(m_1+n_2+n_3)\lambda+m_1n_2+m_1n_3-m_2n_1=0 \qquad (4.15)$$

Claim 1 (i) $m_1+n_2+n_3<0$

(ii) $m_1n_2+m_1n_3-m_2n_1>0$

Proof:

(i) $m_1+n_2+n_3=-a_3x^*-a_4+a_5x^*$

$$=-a_3\left(\frac{a_4}{a_5}\right)-a_4+a_5\left(\frac{a_4}{a_5}\right)$$

$$m_1+n_2+n_3=\frac{-a_3a_4}{a_5}<0$$

(ii) $m_1n_2+m_1n_3-m_2n_1=(-a_3x^*)(-a_4)+(-a_3x^*)(a_5x^*)-(-a_2x^*)(a_5y^*)$

$$=(-a_3x^*)(-a_4)+(-a_3x^*)(a_4)+a_2a_5x^*y^*$$

$$m_1n_2+m_1n_3-m_2n_1=(-a_3x^*)(-a_4)+(-a_3x^*)(a_4)+a_2a_5x^*y^*$$

$$=a_3a_4x^*-a_3a_4x^*+a_2a_5x^*y^*$$

$$m_1n_2+m_1n_3-m_2n_1=a_2a_5x^*y^*$$

$m_1 n_2 + m_1 n_3 - m_2 n_1 > 0,$ Since $x*$ and $y*$ are positive.

As a result, the positive equilibrium point in the absence of delay is stable by Routh Hurwitz criterion.

Case 2 if $\tau \neq 0,$

Equation (4.14) is reduced to

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0 \,, \tag{4.16}$$

where $p = -(m_1 + n_2)$, $r = m_1 n_2$, $s = -n_3$ and $q = m_1 n_3 - m_2 n_1$

Suppose $\lambda = \omega i$ where $\omega > 0$ is a root of equation (4.16)

$$(\omega i)^2 + p(\omega i) + r + (s(\omega i) + q)e^{-\tau(\omega i)} = 0$$

$$(\omega i)^2 + p(\omega i) + r + (s\omega i + q)(\cos\omega\tau - i\sin\omega\tau) = 0$$

$$-\omega^2 + p\omega i + r + is\omega\cos\omega\tau + s\omega\sin\omega\tau + q\cos\omega\tau - qi\sin\omega\tau = 0$$

$$(s\omega\sin\omega\tau + q\cos\omega\tau) + i(s\omega\cos\omega\tau - q\sin\omega\tau) = \omega^2 - p\omega i - r$$

Equating the real and imaginary part

$$\begin{aligned} s\omega\sin\omega\tau + q\cos\omega\tau &= \omega^2 - r \\ s\omega\cos\omega\tau - q\sin\omega\tau &= -p\omega \end{aligned} \tag{4.17}$$

Squaring both sides of equation (4.17)

$$\begin{aligned} (s\omega\sin\omega\tau + q\cos\omega\tau)^2 &= (\omega^2 - r)^2 \\ (s\omega\cos\omega\tau - q\sin\omega\tau)^2 &= (-p\omega)^2 \end{aligned}$$

$$\left.\begin{aligned} s^2\omega^2\sin^2\omega\tau - 2s\omega q\sin\omega\tau\cos\omega\tau + q^2\cos^2\omega\tau &= \omega^4 - 2\omega^2 r + r^2 \\ s^2\omega^2\cos^2\omega\tau + 2s\omega q\sin\omega\tau\cos\omega\tau + q^2\sin^2\omega\tau &= p^2\omega^2 \end{aligned}\right\} \tag{4.18}$$

Adding equation (4.18) together

$$s^2\omega^2(\sin^2\omega\tau+\cos^2\omega\tau)+q^2(\cos^2\omega\tau+\sin^2\omega\tau)=\omega^4+p^2\omega^2-2\omega^2 r+r^2$$

$$=\omega^4+(p^2-s^2-2r)\omega^2+r^2-q^2=0$$

$$\omega^4+\alpha\omega^2+\beta=0, \tag{4.19}$$

where $\alpha=p^2-2r-s^2$ and $\beta=r^2-q^2$

Claim 2: $\alpha>0$

Proof:

$$\alpha=p^2-2r-s^2 \tag{4.20}$$

$$p=-(m_1+n_2)$$

$$p^2=(a_3x^*+a_4)^2$$

$$p^2=a_3{}^2(x^*)^2+2a_3a_4x^*+(a_4)^2 \tag{4.21}$$

$$r=m_1n_2=(-a_3x^*)(-a_4)$$

$$r=a_3x^*a_4 \tag{4.22}$$

$$s=-n_3=-a_5x^*$$

$$s^2=a_5{}^2(x^*)^2 \tag{4.23}$$

Plugging equation (4.21), (4.22) and (4.23) into (4.20),

$$\alpha = a_3{}^2 (x^*)^2 + 2a_3 a_4 x^* + a_4{}^2 - a_5{}^2 (x^*)^2 - 2a_3 x^* a_4$$

$$\alpha = a_3{}^2 (x^*)^2 + a_4{}^2 - a_5{}^2 (\frac{a_4}{a_5})^2$$

$\alpha = (a_3 x^*)^2 > 0$ hence, proved.

From equation (4.19),

$$\omega^2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$$

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$$\omega = \sqrt{\frac{-\alpha + \sqrt{\alpha^2 - 4}\beta}{2}}$$

Case 1: If $\beta < 0$, we can find $\omega > 0$ such that $\lambda = i\omega$. This indicates that the system becomes unstable when $\beta < 0$.

$$\beta = r^2 - q^2$$

$$\beta = (m_1 n_2)^2 - (m_1 n_3 - m_2 n_1)^2$$

$$\beta = m_1{}^2 n_2{}^2 - m_1{}^2 n_3{}^2 - m_2{}^2 n_1{}^2 + 2m_1 n_3 m_2 n_1$$

$$\beta = (-a_3 x^*)^2 (-a_4)^2 - (-a_3 x^*)^2 (a_5 x^*)^2 - (-a_2 x^*)^2 (a_5 y^*)^2 + 2(-a_3 x^*)(a_5 x^*)(-a_2 x^*)(a_5 y^*)$$

$$= (a_3 x^*)^2 (a_4)^2 - (a_3 x^*)^2 (a_4)^2 - (a_2 x^*)^2 (a_5 y^*)^2 + 2a_2 a_3 a_5{}^2 (x^*)^3 y^*$$

$$= 2a_2 a_3 (a_5)^2 (x^*)^3 y^* - (a_2 x^*)^2 (a_5 y^*)^2$$

$$= a_2 (a_5)^2 (x^*)^2 y^* \left[2a_3 x^* - a_2 y^* \right]$$

$$\beta < 0, \Rightarrow 2a_3 x^* - a_2 y^* < 0$$

$$\Rightarrow 2a_3\left(\frac{a_4}{a_5}\right) < a_2\left(\frac{a_1a_5 - a_3a_4 - ua_4}{a_2a_5}\right)$$

$$\Rightarrow \frac{2a_3a_4}{a_5} < \frac{a_1a_5 - a_3a_4 - ua_4}{a_5}$$

$$\Rightarrow \quad 2a_3a_4 < a_1a_5 - a_3a_4 - a_4u$$

$$\Rightarrow \quad 2a_3a_4 - a_1a_5 + a_3a_4 < -a_4u$$

$$\Rightarrow \quad u < \frac{a_1a_5}{a_4} - 2a_3 - a_3 \tag{4.24}$$

As a result, in the presence of delay the system become unstable when condition (4.24) is satisfied.

Case 2: If $\beta > 0$, then the characteristic equation (4.16) has negative real part root. Hence, the system becomes stable when $\beta > 0$ in the presence of delay.

$$\beta > 0 \quad \Rightarrow \quad u > \frac{a_1a_5}{a_4} - 2a_3 - a_3$$

To find the minimum value of τ, for which the stability of the system lost.

Substitute ω_0 into equation (4.17) and solve for τ

$$\begin{aligned} &s\omega_0 \sin\omega_0\tau + q\cos\omega_0\tau = \omega_0^{\;2} - r \\ &s\omega_0 \cos\omega_0\tau - q\sin\omega_0\tau = -p\omega_0 \end{aligned} \tag{4.25}$$

$$\begin{aligned} &s\omega_0\left[s\omega_0 \sin\omega_0\tau + q\cos\omega_0\tau = \omega_0^{\;2} - r\right] \\ &-q\left[s\omega_0 \cos\omega_0\tau - q\sin\omega_0\tau = -p\omega_0\right] \end{aligned}$$

$$\begin{aligned} &s^2\omega_0^{\;2} \sin\omega_0\tau + sq\omega_0 \cos\omega_0\tau = s\omega_0(\omega_0^{\;2} - r) \\ &q^2 \sin\omega_0\tau - sq\omega_0 \cos\omega_0\tau = pq\omega_0 \end{aligned} \tag{4.26}$$

Adding equation (4.26)

$$(s^2\omega_0{}^2 + q^2)\sin\omega_0\tau = pq\omega_0 + s\omega_0(\omega_0{}^2 - r)$$

$$\sin\omega_0\tau = \frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{s^2\omega_0{}^2 + q^2}$$

$$\sin(\omega_0\tau - 2\pi k) = \frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{s^2\omega_0{}^2 + q^2} \qquad (4.27)$$

Similarly,

$$q\left[s\omega_0\sin\omega_0\tau + q\cos\omega_0\tau = \omega_0{}^2 - r\right]$$
$$s\omega_0\left[s\omega_0\cos\omega_0\tau - q\sin\omega_0\tau = -p\omega_0\right]$$

$$\left.\begin{array}{l} qs\omega_0\sin\omega_0\tau + q^2\cos\omega_0\tau = q(\omega_0{}^2 - r) \\ -qs\omega_0\sin\omega_0\tau + s^2\omega_0{}^2\cos\omega_0\tau = -ps\omega_0{}^2 \end{array}\right\} \qquad (4.28)$$

Adding equation (4.28)

$$(s^2\omega_0{}^2 + q^2)\cos\omega_0\tau = q(\omega_0{}^2 - r) - ps\omega_0{}^2$$

$$\cos\omega_0\tau = \frac{q(\omega_0{}^2 - r) - ps\omega_0{}^2}{(s^2\omega_0{}^2 + q^2)}$$

$$\cos(\omega_0\tau - 2k\pi) = \frac{q(\omega_0{}^2 - r) - ps\omega_0{}^2}{(s^2\omega_0{}^2 + q^2)} \qquad (4.29)$$

Dividing equation (4.27) by equation (4.29)

$$\frac{\sin(\omega_0\tau - 2\pi k)}{\cos(\omega_0\tau - 2k\pi)} = \frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r^2) - ps\omega_0{}^2}$$

$$\frac{\sin(\omega_0\tau - 2\pi k)}{\cos(\omega_0\tau - 2k\pi)} = \frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r^2) - ps\omega_0{}^2}$$

$$\tan(\omega_0\tau - 2k\pi) = \frac{pq\omega_0 + s\omega_0(\omega_0^2 - r)}{q(\omega_0^2 - r^2) - ps\omega_0^2}$$

$$\omega_0\tau - 2k\pi = \arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0^2 - r)}{q(\omega_0^2 - r^2) - ps\omega_0^2}\right)$$

$$\omega_0\tau = \arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0^2 - r)}{q(\omega_0^2 - r^2) - ps\omega_0^2}\right) + 2k\pi$$

$$\tau = \frac{1}{\omega_0}\arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0^2 - r)}{q(\omega_0^2 - r^2) - ps\omega_0^2}\right) + \frac{2k\pi}{\omega_0}, \qquad (4.30)$$

where $k = 0, 1, 2, 3, \ldots$

If k = 0, then

$$\tau_0 = \frac{1}{\omega_0}\arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0^2 - r)}{q(\omega_0^2 - r^2) - ps\omega_0^2}\right)$$

This value is called cut off value. It is the smallest value of time delay when stability loses and never regained in the future time.

4.4 Global Stability

4.4.1 Global stability without time delay

$$\left.\begin{aligned} x'(t) &= a_1 x(t) - a_2 x(t)y(t) - a_3 x^2(t) - ux(t) \\ y'(t) &= -a_4 y(t) + a_5 x(t)y(t) \end{aligned}\right\} \qquad (4.31)$$

Let $V(x,y) = \frac{1}{2}(x - x*)^2 + \frac{1}{2}(y - y*)^2$ be a candidate Lyapunov function

i. $V(x^*, y^*) = 0$, the function is zero at positive equilibrium

ii. $V(x,y) > 0$, because the square of any number is positive. The function is positive definite

iii. Differentiating $V(x,y)$ along the solution of equation (4.31)

$$\frac{dv}{dt} = (x - x^*)\frac{dx}{dt} + (y - y^*)\frac{dy}{dt}$$

$$\frac{dv}{dt} = (x - x^*)\left[a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - ux(t)\right] + \left[(y - y^*)(-a_4 y(t) + a_5 x(t) y(t)\right]$$

Let $f(x, y) = (x - x^*)\left[a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - ux(t)\right] + \left[(y - y^*)(-a_4 y(t) + a_5 x(t) y(t)\right]$

Expanding $\frac{dv}{dt}$ by Taylor series about the positive equilibrium point and ignoring cubic and higher derivatives we get the following

$$f(x, y) = (x - x^*)(a_1 x - a_2 xy - a_3 x^2 - ux) + (y - y^*)(-a_4 y + a_5 xy)$$

$$f_x = (a_1 x - a_2 xy - a_3 x^2 - ux) + (x - x^*)(a_1 x - a_2 y - 2a_3 x - u) + a_5 y(y - y^*)$$

$$f_{xx} = 2(a_1 - a_2 y - a_3 x - u) - 2a_3 x - 2a_3(x - x^*)$$

$$f_{xy} = -2a_2 x - a_2(x - x^*) + a_5 y + a_5(y - y^*)$$

$$f_y = (x - x^*)(-a_2 x) + (-a_4 y - a_5 xy) + (y - y^*)(-a_4 + a_5 x)$$

$$f_{yy} = -2a_4$$

Evaluating the partial derivatives at positive equilibrium points

$$f_x(x^*, y^*) = 0, \quad f_{xx}(x^*, y^*) = -2a_3 x^*, \quad f_{xy}(x^*, y^*) = a_5 y^* - a_2 x^*, \quad f_y(x^*, y^*) = 0$$

$$f_{yy}(x^*, y^*) = -2a_4$$

$$\frac{dv}{dt} = \frac{1}{2}\left[(x - x^*)^2 f_{xx}(x^*, y^*) + 2(x - x^*)(y - y^*) f_{xy}(x^*, y^*) + (y - y^*)^2 f_{yy}(x^*, y^*)\right]$$

$$\frac{dv}{dt} = \frac{1}{2}\left[-2a_3 x^*(x - x^*)^2 + 2(a_5 y^* - a_2 x^*)(x - x^*)(y - y^*) - 2a_4(y - y^*)^2\right]$$

$$\frac{dv}{dt} = -a_3 x^*(x - x^*)^2 + (a_5 y^* - a_2 x^*)(x - x^*)(y - y^*) - a_4(y - y^*)^2$$

$$\frac{dv}{dt} = 0 \text{ when } x = x^* \text{ and } y = y^*$$

$$\frac{dv}{dt} = -a_3 x^* (x - x^*)^2 + (a_5 y^* - a_2 x^*)(x - x^*)(y - y^*) - a_4 (y - y^*)^2$$

$$\frac{dv}{dt} = -a_3 x^* (x - x^*)^2 - a_4 (y - y^*)^2 + (a_5 y^* - a_2 x^*)(x - x^*)(y - y^*)$$

$$\frac{dv}{dt} = -a_3 x^* (x - x^*)^2 - a_4 (y - y^*)^2, \text{ if}$$

$$(a_5 y^* - a_2 x^*) = 0 \Rightarrow a_5 y^* = a_2 x^* \Rightarrow \frac{x^*}{y^*} = \frac{a_5}{a_2}$$

$$\frac{a_4}{a_5}\left(\frac{a_2 a_5}{a_1 a_5 - a_5 u - a_3 a_4}\right) = \frac{a_5}{a_2} \Rightarrow \frac{a_2 a_4}{a_1 a_5 - a_5 u - a_3 a_4} = \frac{a_5}{a_2}$$

$$a_1 (a_5)^2 - (a_5)^2 u - a_3 a_4 a_5 = (a_2)^2 a_4$$

$$(a_5)^2 u = a_1 (a_5)^2 + (a_2)^2 a_4 - a_3 a_4 a_5$$

$$u = \frac{a_1 (a_5)^2 - (a_2)^2 a_4 - a_3 a_4 a_5}{(a_5)^2} \tag{4.32}$$

$$\frac{dv}{dt} < 0, \text{ if condition (4.32) is satisfied}$$

Therefore, by Lyapunov theorem there exist global stability of the system in the absence of delay when condition (4.32) is satisfied

4.5 Limit cycle

4.5.1 Limit cycle without delay

$$\left.\begin{aligned} x'(t) &= a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) - u x(t) \\ y'(t) &= -a_4 y(t) + a_5 x(t) y(t) \end{aligned}\right\}$$

Let $\frac{dx}{dt} = f(x,y) = a_1 x - a_2\, xy - a_3 x^2 - ux$

$\frac{dy}{dt} = g(x,y) = -a_4 y + a_5\, xy$

$\phi(x,y) = \frac{1}{xy}$ be candidate Dulac's function

Since x, y>0 then $\phi(x,y) = \frac{1}{xy}$ is continuously differentiable function. Hence, $\phi(x,y) = \frac{1}{xy}$ is an appropriate Dulac's function.

$$\frac{\partial}{\partial x}(\phi(x,y), f(x,y)) + \frac{\partial}{\partial y}(\phi(x,y) g(x,y)) = \frac{\partial}{\partial x}(\frac{1}{xy}(a_1 x - a_2\, xy - a_3 x^2 - ux)) + \frac{\partial}{\partial y}(\frac{1}{xy}(-a_4 y + a_5\, xy))$$

$$= \frac{\partial}{\partial x}(\frac{a_1}{y} - a_2 - \frac{a_3 x}{y} - \frac{u}{y}) + \frac{\partial}{\partial y}(-\frac{a_4}{x} + a_5\ ) \quad = \frac{-a_3}{y}$$

$$\frac{\partial}{\partial x}(\phi(x,y), f(x,y)) + \frac{\partial}{\partial y}(\phi(x,y) g(x,y)) = \frac{-a_3}{y}$$

Since $\frac{\partial}{\partial x}(\phi(x,y), f(x,y)) + \frac{\partial}{\partial y}(\phi(x,y) g(x,y))$ is different from zero and does not change the sign, by Dulac's criterion there is no limit cycle for the system.

$$\frac{\partial}{\partial x}(\phi(x,y), f(x,y)) + \frac{\partial}{\partial y}(\phi(x,y) g(x,y)) = 0, \text{ when } \ a_3 = 0$$

$$a_3 = 0 \Rightarrow m_1 + n_2 + n_3 = 0,$$

The characteristics equation (4.15) reduced to

$$\lambda^2 + m_1 n_2 + m_1 n_3 - m_2 n_1 = 0$$

Since $m_1 n_2 + m_1 n_3 - m_2 n_1 > 0$, then the characteristics equation has pure imaginary root and the system is center consequently the system has limit cycle when $a_3 = 0$

CHAPTER FIVE

CONCLUSION AND FUTURE WORK

5.1. Conclusions

In this thesis, mathematical model of prey predator with delay when control parameter applied on prey was studied. From the result of the study the positive equilibrium point in the absence of delay is stable. In the presence of delay the system becomes stable with specific condition and loses its stability at cutoff value which is the minimum value for time. Furthermore, the existence of global stability in the absence of delay was proved using Lyapunov theorem by constructing appropriate Lyapunov function. Nonexistence of limit cycle at positive equilibrium point was also proved by using Dulac's criterion. However, if there is no intraspecific competition rate of prey there exist limit cycle for the system.

5.2 Future Work

One can carry out further study on the following issues, global stability with delay, direction of stability and Hopf bifurcation, Persistence of the prey predator, global existence of periodic solution of the model and other related things. Furthermore, it is possible to consider control that change with time rather than control parameter and different time delay on the two equations. It is also possible to develop a new mathematical model than describe prey predator model by considering different assumption and then followed by the study of its qualitative behavior.

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