Stability and Bifurcation Analysis of Prey-Predator Mathematical Model with Delay and Control of the Predator

A Thesis Submitted to the Department of Mathematics, Jimma University, in Partial Fulfillment for the Requirements of the Degree of Masters of Science (M.Sc.) in Mathematics.

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DECLARATION

Here, I submit the thesis entitled "**Stability and Bifurcation Analysis of Prey-Predator Mathematical Model with Delay and Control of the Predator**" for the award of degree of Master of Science in Mathematics. I, the undersigned declare that, this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or the like, where other sources of information have been used, they have been acknowledged.

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ABSTRACT

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes of research in applied mathematics and ecology. In this thesis, mathematical model of prey predator with delay was studied. Firstly, local stability of the model in the absence and presence of delay was studied at the positive equilibrium point by linearizing the model. Secondly, the existence of global stability was proved by the aid of Lyapunov theorem. Thirdly, non-existence of limit cycle at positive equilibrium was checked by Dulac's criterion and the limit cycle exists if there is no intraspecific competition rate of prey. Finally, Hopf bifurcation condition was well spelled out.

Key words: *Prey-predator model, local stability, global stability, limit cycle, Hopf bifurcation.*

Contents

DECLARATION i

ACKNOWLEDGMENT ii

ABSTRACT iii

CHAPTER ONE 1

INTRODUCTION 1

1.1 Background of the Study 1

1.2 Statement of the Problem 4

1.3 Objective of the Study 4

1.3.1 General Objective of the Study 4

1.3.2 Specific Objectives of the Study 4

1.4. Significance of the Study 5

1.5 Delimitation of the Study 5

CHAPTER TWO 6

LITERATURE REVIEW 6

2.1 Historical background 6

2.2 Models of Prey Predator with time delay 6

2.3 Models of Prey Predator with control 7

CHAPTER THREE 9

METHODOLOGY 9

3.1. Study Area and Period 9

3.2. Study Design 9

3.3. Source of Information 9

3.4. Mathematical Procedures 9

CHAPTER FOUR 10

RESULT AND DISCUSSION 10
4. Mathematical Model of prey predator 10
4.1 Equilibrium point 10
4.2 Linearization 12
4.3 Local Stability 14
4.4 Global Stability 21
4.4.1 Global Stability Without time delay 21
4.5 Limit Cycle 24
4.4.1 Limit Cycle without delay 24
4.6 Hopf Bifurcation 25
CHAPTER FIVE 28
CONCLUSION AND FUTURE WORK 28
5.1. Conclusions 28
5.2 Future Work 28
References 29

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and applied mathematics due to its universal existence and importance (Berryman, 1992). These problems may appear to be mathematically simple but actually, they are challenging and complicated. Although the predator–prey theory has seen in much progress in the last 40 years, many long standing mathematical and ecological problems remain open (Berryman, 1992).The central goal in ecology is to understand the dynamical relationship between predator and prey (Kot, 2001). The most significant factor of the prey predator relationship is the predator's rate of feeding upon prey, known as predator's functional response, which is the average number of prey killed per individual predator per unit of time.

Prey-predator interactions abound in the biological world, and are one of the most important topics in theoretical ecology. The study of predation has long history, beginning with the work of Lotka and Volterra and continuing to be of interest today. In most ecological models the growth rate of species does not only depend on the instantaneous population size but also on the past history of the population. For example, in the prey-predator model the loss of prey by predator will affect the growth rate of predators at the future time. The first prey-predator model with aftereffect was proposed by Volterra(Kolmanovskii and Myshkis, 1999) as follows:

$$\begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) \\ y'(t) = -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) \end{cases}, \tag{1.1}$$

where $x(t)-$Population density of prey at time t, $y(t)-$Population density of predator at time t

$a_1 -$intrinsic growth rate of prey, $a_4 -$death rate of predators,

a_2 & $a_5 -$interspecific competition, $a_3 -$intraspecific competition rate of prey,

$\tau -$the average time delay between death of prey and birth of subsequent number of predators and $a_i > 0$ $(i = 1, 2, ..., 5)$.

In the real world, there is sometimes a need to control population at a reasonable level because otherwise this population may cause decrease or even extinction of other populations. Bearing this in mind, if a control u, taking into account some purposeful action of various factors on the system, acts only on predators, then model (1.1) modified to the following model (Kolmanovskii and Myshkis, 1999).

$$\begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) \\ y'(t) = -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) - u y(t) \end{cases} . \qquad (1.2)$$

$$u > 0$$

Stability of dynamical systems plays a very important role in control system analysis and design. Unlike the case of linear systems, proving stability of equilibria of nonlinear systems is more complicated. A sufficient condition is the existence of a Lyapunov function: a positive definite function V defined in some region of the state space containing the equilibrium point whose derivative along the system trajectories is negative semi-definite. This is Lyapunov's direct method, which even though addresses exactly and in a simple way the important issue of stability, it does not provide any coherent methodology for constructing such a function. Lyapunov's indirect method that investigates the local stability of the equilibria is inconclusive when the linearized system has imaginary axis eigenvalues (Hassan, 2002).

Chernet Tuge and Mitiku Daba (2017) investigated the stability analysis of delayed nonlinear cournot model in the sense of Lyapunov. One of the finding of this investigation indicates that the presence of equal information time delay in the given model causes oscillation process in the system and doesn't affect the qualitative behavior of the solution (no effect on the stability of the equilibrium point), but only changes the transition process. In other words it delays stability as delay parameter increases. On the other hand, when one of the firms has implementation delay and the rival player makes decision without delay, it leads to instability of the dynamic system at least locally.

In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoretical ecology, there is the application of what is referred to as bifurcation analysis; the analysis of a system of ordinary differential equations (ODE's) under parameter variation (Kuznetsov, 2000).

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given dynamical systems. Bifurcation occurs when a small change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behavior (Blanchard,2006).The name "bifurcation" was first introduced by Henri Poincaré in 1885 in the first paper in mathematics showing such a behavior (Kuznetsov, 2000)**.** Local bifurcation occurs when a parameter change causes the stability of equilibrium to change.

To date, many authors have studied the dynamics of predator-prey models with time delay and obtained complex dynamic behavior, such as stability of equilibrium, Hopf bifurcation, and limit cycle. For example, Song and Wei (2005) investigated further the dynamics of the system prey-predator model by considering the time delay as the bifurcation parameter and they obtained that, under certain conditions, the unique positive equilibrium of the model is absolute stable while it is conditionally stable and there exist k switches from stability to instability to stability under other conditions. Further, by using normal form theory and the center manifold theorem, they obtained the formulae for determining the direction of Hopf bifurcations and the stability of bifurcating periodic solutions. Yan and Li (2006) studied the properties of Hopf bifurcation for prey-predator model by using normal form theory and the center manifold theorem, which is different from that used by Song and Wei(2005).

In 2017, Ahmed Buseri studied global asymptotic stability analysis of predator-prey model. Also, in 2017, Peng et al., studied hybrid control of Hopf bifurcation in a Lotka-Volterra predator-prey model with two time delays. In 2016, Murthy studied stability analysis of prey-predator harvesting model using biological parameters. In 2015, Soliman and Jarallah, studied asymptotic stability of solutions of lotka-volterra predator-prey model for four species. In 2014, Yue and Qingling studied stability and bifurcation analysis of a singular delayed predator-prey bio-economic model with stochastic fluctuations. In 2013, Liu et al., studied global stability analysis and optimal control of a harvested eco-epidemiological prey predator model with vaccination and taxation.In 2012, Debasis studied the bifurcation and stability analysis of prey-predator model with a reserved area. In 2012, Debasis studied the bifurcation and stability analysis of prey-predator model with a reserved area. In 2011, Xu et al., studied stability and Hopf bifurcation analysis for a Lotka-Volterra predator–prey model with two time delays.

However, to the best knowledge of the author the stability and bifurcation analysis of the mathematical model of prey-predator represented by equation (1.2) is not yet investigated. Therefore, the central goal of this study is to investigate the dynamic behavior such as local stability, global stability, and existence of limit cycle and Hopf bifurcation of prey-predator model represented by equation (1.2).

1.2 Statement of the Problem

This research mainly focuses on the following problems related to prey-predator mathematical model given by equation (1.2).

- Local stability analysis,
- Global stability analysis,
- Non-existence and existence of limit cycle,
- Conditions for Hopf bifurcation.

1.3 Objective of the Study

1.3.1 General Objective of the Study

The general objective of this study is to investigate the stability and bifurcation analysis of prey-predator mathematical model with time delay and control of the predator represented by equation (1.2).

1.3.2 Specific Objectives of the Study

The specific objectives of the study are to:

- Determine local stability condition of prey-predator model given by equation (1.2) using different methods such as Lyapunov's indirect method.
- Analyze global stability of prey-predator model given by equation (1.2) by constructing appropriate Lyapunov function.
- Check non-existence and existence of limit cycle using Dulac's criterion.
- Establish conditions of Hopf bifurcation of prey-predator model given by equation (1.2).

1.4. Significance of the Study

The outcomes of this study have the following importance:

- Provides evidence on stabilizing prey-predator coexistence in a given real ecosystem without extinction of the preys by applying a necessary control on predators
- The result and the method can used as bench mark for other researchers in related areas

1.5 Delimitation of the Study

This study is delimited to investigate the local stability, global stability, existence of limit cycle and Hopf bifurcation of prey-predator model represented by equation (1.2).

CHAPTER TWO

LITERATURE REVIEW

2.1 Historical background

Mathematical population models have been used to study the dynamics of prey predator systems since Lotka and Volterra proposed the simple model of prey-predator interactions now called the Lotka-Volterra model. Since then, many mathematical models, some reviewed in this study, have been constructed based on more realistic explicit and implicit biological assumptions. Modeling is a frequently evolving process, to gain a deep understanding of the mathematical aspects of the problem and to yield non trivial biological insights; one must carefully construct biologically meaningful and mathematically tractable population models (Kuang, 2002). Some of the aspects that need to be critically considered in a realistic and plausible mathematical model include; carrying capacity which is the maximum number of prey that the ecosystem can sustain in absence of predator, competition among prey and predators which can be intraspecific or interspecific, harvesting of prey or predators and functional responses of predators. The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics and is one of the most popular models in mathematical ecology. In both the analysis and experiment, the predator and prey can coexist by reducing the frequency of contact between them (Luckinbill, 1973).

2.2 Models of Prey Predator with time delay

In nature, populations do not reproduce instantaneously; rather it is mediated by certain time delay required for gestation, maturation time, capturing time, or other reasons. Thus, time delays of one type or another have been incorporated into mathematical models of population dynamics. Another factor to be taken into account when modeling dynamical systems is the presence of delays. Delay is a general concept that can represent different phenomena such as the time it takes for the progeniture to reach maturity or the finite gestation period of a species.

Mathematical delays are input in model to correct the classical logistic model, which assumes that the growth rate of a population at time t is determined by the number of individual at that time. Of course biological delays are complex and the mathematical representation is often a simplification of reality. The time delay is considered into the population dynamics when the rate of change of the population is not only a function of the present population but also depends on the past population.

2.3 Models of Prey Predator with control

Dynamics of interacting biological species has been studied in the past few decades from various angles. Many species become extinct and many others are at the verge of extinction due to several reasons like, over exploitation, over predation, environmental pollution, mismanagement of natural resources etc. To save these species, suitable measures such as restriction on harvesting, creating reserve zones/refuges etc. have to be considered. For example, in order to eliminate algal bloom, an effective way is to introduce suitable fish species (chub *etc*.) that usually feed on plankton such that algal bloom can be controlled (Peng*et al.*, 2017).

It is well known that harvesting has a strong impact on the dynamic evolution of a population. The severity of this impact that may range from rapid depletion to complete preservation of a population depends on the implemented harvesting agency (Kar, 2003).Due to practical and economic utilization and biological resources in the prey predator system are extensively harvested nowadays. Furthermore, exploitation of biological resources has been increased by people's multifarious material needs, which attract a global concern to protect the limited biological resources.

Therefore, regulation of exploitation of biological resources has become a problem of major concern in view of dwindling resource stocks and the deteriorating environment. It is necessary to establish a constructive management of commercial exploitation of the biological resources. The techniques and issues associated with bio economic exploitation have been discussed in details (Clark, 1990). Taxation, license fees, lease of property rights, seasonal harvesting, and so forth are usually considered as possible governing instruments in regulation for harvesting as well as regulatory mechanisms to keep the damage to the ecosystem minimal.

Out of these regulating options, taxation is considered to be superior because of its economic flexibility, and economists are particularly attracted to taxation because a competitive system can be better maintained under taxation rather than other regulatory methods. There has been considerable interest in the modeling of harvesting of biological resources in recent decades, where harvest effort is considered to be a dynamic variable, and optimal harvesting policies with taxation are discussed.

Brauer and Soudack, (1979) have analyzed the global behavior of a predator-prey system under constant rate predator harvesting. They showed how to classify the possibilities and determine the region of stability. They found that if the equilibrium point is asymptotically stable, which is determined by a local linearization, then every solution whose initial value is in some neighborhood of the stable equilibrium point tends to it as the time approaches infinity. There exists an asymptotically stable limit cycle when the constant rate is small and the equilibrium point is unstable. A predator-prey model with Holling type using harvesting efforts as control has been presented by (Srinivasu*et al.*, 2001) and showed that with harvesting, it is possible to break the cyclic behavior of the system and introduces a globally stable limit cycle in the system. In 2017, Peng et al., studied hybrid control of Hopf bifurcation in a Lotka-Volterra predator-prey model with two time delays.

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted in Jimma University under the department of Mathematics from September 2017 to June 2018 G.C.

3.2. Study Design

This study employed analytical (documentary review design) on prey-predator model given by equation (1.2).

3.3. Source of Information

The relevant sources of information for this study were books, published articles & related studies from internet.

3.4. Mathematical Procedures

This study was conducted based on the following procedures

- ✓ Determining the steady state point of the model.
- ✓ Linearizing the mathematical model of prey-predator under consideration.
- ✓ Determining the local stability condition of the model.
- ✓ Analyzing the global stability of the system.
- ✓ Checking non-existence and existence of limit cycle.
- ✓ Establishing conditions of Hopf bifurcation.

CHAPTER FOUR

RESULT AND DISCUSSION

4. Mathematical Model of prey predator

Understanding predator-prey interaction has been one of the central topics in ecology and conservation biology. In an ecological community, most species depend on successful predation to survive because prey serves as food resources which provide predators with energy.

Consequences of converting prey biomass into predator biomass through direct predation are referred to as a direct effect between predators and prey, and have been the focus of modeling predator-prey interactions (Murray, 2007).

$$\left.\begin{aligned} x'(t) &= a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) \\ y'(t) &= -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) - u y(t) \end{aligned}\right\} \tag{1}$$

4.1 Equilibrium point

In dynamical system theory, equilibrium solutions are solutions which do not change with time (Meiss, 2007). Studying equilibrium solutions is important in mathematical biology because it predicts long-term behaviors of a system.

To find the equilibrium point, equate the right hand side of equation (1) with zero.

$$\begin{cases} a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t) = 0 \\ -a_4 y(t) + a_5 x(t-\tau) y(t-\tau) - u y(t) = 0 \end{cases}$$

Since the time delay does not affect the equilibrium point,

$$x(t)\left[a_1 - a_2 y(t) - a_3 x(t)\right] = 0$$
$$y(t)\left[-a_4 + a_5 x(t) - u\right] = 0$$

$x(t) = 0$ or $a_1 - a_2 y(t) - a_3 x(t) = 0$ and $y(t) = 0$ or $-a_4 + a_5 x(t) - u = 0$

$\mathrm{E}_1 = (0,0)$, meaning equilibrium point in the absence of the two species.

$$\begin{cases} a_1 - a_2 y(t) - a_3 x(t) = 0 \\ -a_4 + a_5 x(t) - u = 0 \end{cases} \tag{2}$$

Solving equation (2) one can obtain that,

$$-a_4 + a_5 x(t) - u = 0 \quad \Rightarrow a_5 x(t) = a_4 + u \quad \Rightarrow x(t) = \frac{a_4 + u}{a_5}$$

$$\begin{aligned} &a_5 \left[a_1 - a_2 y(t) - a_3 x(t) = 0 \right] \\ &a_3 \left[-a_4 + a_5 x(t) - u = 0 \right] \end{aligned} \tag{3}$$

Adding equation (3) and solving for y(t),

$$y(t) = \frac{a_1 a_5 - a_3 a_4 - a_3 u}{a_2 a_5}$$

$E_2 = (x^*, y^*)$, where $x^* = \dfrac{a_4 + u}{a_5}$, $\quad y^* = \dfrac{a_1 a_5 - a_3 a_4 - a_3 u}{a_2 a_5}$

$E_2 = (x^*, y^*)$, indicates the presence of the two species. It is positive whenever,

$a_1 a_5 - a_3 a_4 - a_3 u > 0$

To find equilibrium point in the absence of prey, $x(t) = 0$ and presence of predator:

From equation (1) one can easily obtain that $y(t) = 0$

which implies in the absence of prey, predator cannot live.

In the absence of predator that is $y(t) = 0$, one can obtain that $x(t)(a_1 - a_3 x(t)) = 0$.

Solving we have $x(t) = 0$ or $x(t) = \dfrac{a_1}{a_3}$.

Thus, the equilibrium becomes $\quad E_3 = (\dfrac{a_1}{a_3}, 0)$

4.2 Linearization

In mathematics, linearization is finding the linear approximation to a function at a given point. In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of nonlinear differential equations or discrete dynamical systems.

Linearization can be used to give important information about how the system behaves in the neighborhood of equilibrium points. Linearization makes it possible to use tools for studying linear systems to analyze the behavior of a nonlinear function near a given point.

$$\text{Let } \begin{array}{ll} x_1(t) = x(t) - x^* & x(t) = x_1(t) + x^* \\ y_1(t) = y(t) - y^* & y(t) = y_1(t) + y^* \end{array} \Longrightarrow \begin{array}{l} x'(t) = x'_1(t) \\ y'(t) = y'_1(t) \end{array} \tag{4}$$

Plugging equation (4) in to equation (1)

$$\begin{aligned} x'_1(t) &= a_1(x_1(t) + x^*) - a_2(x_1(t) + x^*)(y_1(t) + y^*) - a_3(x_1(t) + x^*)^2 \\ y'_1(t) &= -a_4(y_1(t) + y^*) + a_5(x_1(t-\tau) + x^*)(y_1(t-\tau) + y^*) - u(y_1(t) + y^*) \end{aligned}$$

$$\begin{aligned} x'_1(t) &= a_1x_1(t) + a_1x^* - a_2\left[x_1(t)y_1(t) + x^*y_1(t) + y^*x_1(t) + x^*y^*\right] - a_3\left[x_1^{\,2}(t) + 2x^*x_1(t) + (x^*)^2\right] \\ y'_1(t) &= -a_4y_1(t) - a_4y^* + a_5\left[x_1(t-\tau)y_1(t-\tau) + x_1(t-\tau)y^* + x^*y_1(t-\tau) + x^*y^*\right] - uy_1(t) - uy^* \end{aligned}$$

$$\begin{aligned} x'_1(t) &= a_1x^* - a_2x^*y^* - a_3(x^*)^2 + a_1x_1(t) - a_2x_1(t)y_1(t) - a_2x^*y_1(t) - a_2y^*x_1(t) - a_3x_1^{\,2}(t) - 2a_3x^*x_1(t) \\ y'_1(t) &= -a_4y^* + a_5x^*y^* - uy^* - a_4y_1(t) + a_5x_1(t-\tau)y_1(t-\tau) + a_5y^*x_1(t-\tau) + a_5x^*y_1(t-\tau) - uy_1(t) \end{aligned}$$

Since (x^*, y^*) is an equilibrium point:

$$\begin{aligned} &a_1x^* - a_2x^*y^* - a_3(x^*)^2 = 0 \\ &-a_4y^* + a_5x^*y^* - uy^* = 0 \end{aligned}$$

$$\begin{aligned} x'_1(t) &= \left(a_1 - a_2y^* - 2a_3x^*\right)x_1(t) - a_2x^*y_1(t) - a_2x_1(t)y_1(t) - a_3x^2_1(t) \\ y'_1(t) &= a_5y^*x_1(t-\tau) + (-a_4 - u)y_1(t) + a_5x^*y_1(t-\tau) + a_5x_1(t-\tau)y_1(t-\tau) \end{aligned}$$

$$\begin{aligned} x'_1(t) &= m_1x_1(t) + m_2y_1(t) + m_3x_1(t)y_1(t) + m_4x^2_1(t) \\ y'_1(t) &= n_1x_1(t-\tau) + n_2y_1(t) + n_3y_1(t-\tau) + n_4x_1(t-\tau)y_1(t-\tau) \end{aligned},$$

where,
$$\begin{array}{ll} m_1 = a_1 - a_2 y^* - 2a_3 x^* & n_1 = a_5 y^* \\ m_2 = -a_2 x^* & n_2 = -a_4 - u \\ m_3 = -a_2 & n_3 = a_5 x^* \\ m_4 = -a_3 & n_4 = a_5 \end{array}$$

However, since $x_1(t)$ and $y_1(t)$ are small perturbation their product and any of its higher order greater or equal to two goes to zero.

$$\begin{aligned} & m_3 x_1(t) y_1(t) \rightarrow 0 \\ & m_4 x^2{}_1(t) \rightarrow 0 \\ & n_4 x_1(t-\tau) y_1(t-\tau) \rightarrow 0 \end{aligned}$$

$$\begin{aligned} x_1{}'(t) &= m_1 x_1(t) + m_2 y_1(t) \\ y_1{}'(t) &= n_1 x_1(t-\tau) + n_2 y_1(t) + n_3 y_1(t-\tau) \end{aligned} \tag{5}$$

is the linearized form of the system.

4.3 Local Stability

The stability of equilibrium may be local or global, depending on the basin of attraction of the equilibrium. To determine the local stability of equilibrium, linearization of a system at the equilibrium is a useful tool. The equilibrium is locally asymptotically stable if all eigenvalues of the Jacbian matrix evaluated at this point have negative real parts and is unstable if at least one eigenvalue has a positive real part (Meiss, 2007). If one of the eigenvalues has zero real part, then the linearized system is not enough to capture dynamical behaviors near by the equilibrium and therefore, higher order approximation is required.

To find the characteristic equation of equation (5),

Let
$$\begin{aligned} x(t) &= re^{\lambda t} \quad \Rightarrow x'(t) = r\lambda e^{\lambda t} \\ y(t) &= se^{\lambda t} \quad \Rightarrow y'(t) = s\lambda e^{\lambda t} \end{aligned} \tag{6}$$

Plugging equation (6) in to (5)

$$\begin{aligned} r\lambda e^{\lambda t} &= m_1 re^{\lambda t} + m_2 se^{\lambda t} \\ s\lambda e^{\lambda t} &= n_1 re^{\lambda(t-\tau)} + n_2 se^{\lambda t} + n_3 se^{\lambda(t-\tau)} \end{aligned}$$

$$\begin{aligned} r\lambda e^{\lambda t} &= m_1 re^{\lambda t} + m_2 se^{\lambda t} \\ s\lambda e^{\lambda t} &= n_1 re^{-\lambda\tau} e^{\lambda t} + n_2 se^{\lambda t} + n_3 se^{-\lambda\tau} e^{\lambda t} \end{aligned}$$

Since, $e^{\lambda t} \neq 0$
$$\begin{aligned} r\lambda &= m_1 r + m_2 s \\ s\lambda &= n_1 re^{-\lambda\tau} + n_2 s + n_3 se^{-\lambda\tau} \end{aligned}$$

$$\begin{aligned} &(m_1 - \lambda)r + m_2 s = 0 \\ &n_1 e^{-\lambda\tau} r + (n_2 + n_3 e^{-\lambda\tau} - \lambda)s = 0 \end{aligned}$$

For non-trivial solution to exist, the determinant of coefficient matrix must be zero.

$$\begin{vmatrix} m_1 - \lambda & m_2 \\ n_1 e^{-\lambda\tau} & n_2 + n_3 e^{-\lambda\tau} - \lambda \end{vmatrix} = 0$$

$$(m_1 - \lambda)(n_2 + n_3 e^{-\lambda\tau} - \lambda) - m_2 n_1 e^{-\lambda\tau} = 0$$

$$\lambda^2-(m_1+n_2)\lambda+(m_1n_3-m_2n_1-n_3\lambda)e^{-\lambda\tau}+m_1n_2=0$$

$$\lambda^2-(m_1+n_2)\lambda+m_1n_2-(n_3\lambda+m_2n_1-m_1n_3)e^{-\lambda\tau}=0$$

is the characteristic equation for equation (5).

Case 1: If $\tau=0$, then the characteristic equation becomes:

$$\lambda^2-(m_1+n_2+n_3)\lambda+m_1n_2+m_1n_3-m_2n_1=0 \tag{7}$$

Theorem A steady state with characteristic equation (7) is locally stable in the absence of time delay if condition (i) and (ii) holds.

$$(i)\ m_1+n_2+n_3<0$$

$$(ii)\ m_1n_2+m_1n_3-m_2n_1>0$$

Proof: $(i)\ m_1+n_2+n_3=-a_3x^*-a_4-u+a_5x^*$

$$m_1+n_2+n_3=-a_3x^*-a_4-u+a_5(\frac{a_4+u}{a_5})$$

$$=-a_3(\frac{a_4+u}{a_5})-a_4-u+a_4+u$$

$$m_1+n_2+n_3=-a_3(\frac{a_4+u}{a_5})<0$$

$(ii)\ m_1n_2+m_1n_3-m_2n_1=-a_3x^*(-a_4-u)+(-a_3x^*)(a_5x^*)-(-a_2x^*)(a_5y^*)$

$$\Rightarrow m_1n_2+m_1n_3-m_2n_1=(-a_3x^*)(-a_5x^*)-(a_3x^*)(a_5x^*)+(a_2x^*)(a_5y^*)$$

$$\Rightarrow m_1n_2+m_1n_3-m_2n_1=a_3x^*(a_5x^*)-(a_3x^*)(a_5x^*)+(a_2x^*)(a_5y^*)$$

$$m_1n_2+m_1n_3-m_2n_1=(a_2x^*)(a_5y^*)>0$$

Since, x^* and y^* are positive for $a_1a_5-a_3(a_4+u)>0$.

Consequently, the positive equilibrium point in the absence of delay is locally stable by Routh Hurtz criteria.

Case 2: If $\tau \neq 0$

$$\lambda^2 - (m_1 + n_2)\lambda + m_1 n_2 - (n_3\lambda + m_2 n_1 - m_1 n_3)e^{-\lambda\tau} = 0$$

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0 \tag{8}$$

where
$$\begin{aligned} p &= -(m_1 + n_2) \\ r &= m_1 n_2 \\ s &= -n_3 \\ q &= m_1 n_3 - m_2 n_1 \end{aligned}$$

For $\omega > 0$. Suppose $\lambda = \omega i$ is a root of equation (8) it follows,

$$(i\omega)^2 + pi\omega + r + (s(i\omega) + q)e^{-i\omega\tau} = 0$$

$$-\omega^2 + ip\omega + r + (is\omega + q)(\cos\omega\tau - i\sin\omega\tau) = 0$$

$$-\omega^2 + ip\omega + r + (s\omega\cos\omega\tau)i + s\omega\sin\omega\tau + q\cos\omega\tau - (q\sin\omega\tau)i = 0$$

$$(s\omega\sin\omega\tau + q\cos\omega\tau) + (s\omega\cos\omega\tau - q\sin\omega\tau)i = \omega^2 - r - ip\omega$$

Equating real and imaginary part:-

$$\begin{aligned} s\omega\sin\omega\tau + q\cos\omega\tau &= \omega^2 - r \\ s\omega\cos\omega\tau - q\sin\omega\tau &= -p\omega \end{aligned} \tag{9}$$

Squaring both sides of equation (9),

$$\begin{aligned} s^2\omega^2\sin^2\omega\tau + 2s\omega q\sin\omega\tau\cos\omega\tau + q^2\cos^2\omega\tau &= \omega^4 - 2\omega^2 r + r^2 \\ s^2\omega^2\cos^2\omega\tau - 2s\omega q\sin\omega\tau\cos\omega\tau + q^2\sin^2\omega\tau &= p^2\omega^2 \end{aligned} \tag{10}$$

Adding the first and the second equation of equation (10)

$$s^2\omega^2 + q^2 = \omega^4 + p^2\omega^2 - 2\omega^2 r + r^2$$

$$\omega^4 + (p^2 - 2r - s^2)\omega^2 + r^2 - q^2 = 0 \, , \quad \omega^4 + \alpha\omega^2 + \beta = 0 \, , \qquad (11)$$

where $\alpha = p^2 - 2r - s^2$ and $\beta = r^2 - q^2$

Remark1: $\alpha > 0$

Proof: $\alpha = p^2 - 2r - s^2$ (12)

$$p = -(-a_3 x^* - a_4 - u) \Rightarrow p = a_3 x^* + a_4 + u$$

$$p^2 = (a_3 x^* + a_4 + u)^2$$

$$p^2 = (a_3)^2 (x^*)^2 + 2a_3 a_4 x^* + 2a_3 u x^* + (a_4)^2 + 2a_4 u + u^2 \qquad (13)$$

$$r = m_1 n_2 = a_3 x^* (a_4 + u) \qquad (14)$$

$$s = -n_3 = -a_5 x^* \Rightarrow s^2 = (a_5)^2 (x^*)^2 \qquad (15)$$

Substituting (13), (14), and (15) in equation (12), we get,

$$p^2 - 2r - s^2 = (a_3)^2 (x^*)^2 + 2a_3 a_4 x^* + 2a_3 u x^* + (a_4)^2 + 2a_4 u + u^2 - 2a_3 a_4 x^* - 2a_3 u x^* - (a_5)^2 (x^*)^2$$

$$= (a_3)^2 (x^*)^2 + (a_4)^2 + 2a_4 u + u^2 - (a_5)^2 (x^*)^2$$

$$= (a_3)^2 (\frac{a_4 + u}{a_5})^2 - (a_5)^2 (\frac{a_4 + u}{a_5})^2 + (a_4)^2 + 2a_4 u + u^2$$

$$= \frac{(a_3)^2}{(a_5)^2} (a_4 + u)^2 - (a_4 + u)^2 + (a_4 + u)^2$$

$$= \frac{(a_3)^2}{(a_5)^2} ((a_4)^2 + 2a_4 u + u^2)$$

$$= \frac{(a_3)^2}{(a_5)^2} ((a_4)^2 + 2a_4 u + u^2) > 0$$

this completes the proof.

From equation (11)

$$\omega^2 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$$

$$\omega = \sqrt{\frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}}$$

If $\beta < 0$, then there is $\omega > 0$ such that $\lambda = i\omega$.

Therefore, the system becomes unstable when $\beta < 0$.

$$\beta = r^2 - q^2 \;\; \beta = (m_1 n_2)^2 - (m_1 n_3 - m_2 n_1)^2$$

$$\beta = m_1^{\ 2} n_2^{\ 2} - m_1^{\ 2} n_3^{\ 2} - m_2^{\ 2} n_1^{\ 2} + 2 m_1 m_2 n_1 n_3$$

$$\beta = (-a_3 x^*)^2 (-a_4 - u)^2 - (-a_3 x^*)^2 (a_5 x^*)^2 - (-a_2 x^*)^2 (a_5 y^*)^2 + 2(-a_3 x^*)(-a_2 x^*)(a_5 y^*)(a_5 y^*)$$

$$\beta = (a_3 x^*)^2 (a_5 x^*)^2 - (a_3 x^*)^2 (a_5 x^*)^2 - (a_2 x^*)^2 (a_5 y^*)^2 + 2(a_3 x^*)(a_2 x^*)(a_5 y^*)(a_5 y^*)$$

$$\beta = 2(a_3 x^*)(a_2 x^*)(a_5 y^*)(a_5 x^*) - (a_2 x^*)^2 (a_5 y^*)^2$$

$$\beta = (a_2 x^*)(a_5 y^*)[2(a_3 x^*)(a_5 x^*) - (a_2 x^*)(a_5 y^*)]$$

$$\beta = a_2 a_5 (x^*)^2 y^* [2 a_3 a_5 x^* - a_2 a_5 y^*]$$

$$\beta = a_2 (a_5)^2 (x^*)^2 y^* [2 a_3 x^* - a_2 y^*]$$

For $\beta < 0$, $2a_3 x^* - a_2 y^* < 0$

$$2a_3 \left(\frac{a_4 + u}{a_5}\right) - a_2 \left(\frac{a_1 a_5 - a_3 (a_4 + u)}{a_2 a_5}\right) < 0$$

$$2a_3 \left(\frac{a_4 + u}{a_5}\right) - \left(\frac{a_1 a_5 - a_3 (a_4 + u)}{a_5}\right) < 0$$

$$\frac{3a_3(a_4+u)-a_1a_5}{a_5}<0$$

$$3a_3(a_4+u)-a_1a_5<0 \quad \Rightarrow 3a_3a_4+3a_3u<a_1a_5$$

$$3a_3u<a_1a_5-3a_3a_4 \Rightarrow u<\frac{a_1a_5-3a_3a_4}{3a_3}$$

$$u<\frac{a_1a_5}{3a_3}-a_4 \tag{16}$$

As a result, the positive equilibrium in the presence of delay becomes unstable when condition (16) is satisfied.

If $\beta>0$,then we cannot find $\omega>0$

As a result, the characteristic equation (11) has negative real part root and hence stable for $\beta>0$.

$$\beta>0 \;\Rightarrow\; u>\frac{a_1a_5}{3a_3}-a_4$$

To find the minimum value of τ for which the stability of the system lost, substitute ω_0 in equation (9) we obtain.

$$\begin{aligned} &s\omega_0\sin\omega_0\tau+q\cos\omega_0\tau=\omega_0{}^2-r\\ &s\omega_0\cos\omega_0\tau_0-q\sin\omega_0\tau=-p\omega_0 \end{aligned}$$

Solving this equation for τ ,

$$\begin{aligned} &s\omega_0[s\omega_0\sin\omega_0\tau+q\cos\omega_0\tau=\omega_0{}^2-r]\\ &-q[s\omega_0\cos\omega_0\tau_0-q\sin\omega_0\tau=-p\omega_0] \end{aligned}$$

$$\begin{aligned} &s^2\omega_0{}^2\sin\omega_0\tau+qs\omega_0\cos\omega_0\tau=s\omega_0(\omega_0{}^2-r)\\ &-qs\omega_0\cos\omega_0\tau_0+q^2\sin\omega_0\tau=qp\omega_0 \end{aligned} \tag{17}$$

Adding equation (17) together,

$$(q^2 + s^2\omega_0{}^2)\sin\omega_0\tau = qp\omega_0 + s\omega_0(\omega_0{}^2 - r)$$

$$\sin\omega_0\tau = \frac{qp\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q^2 + s^2\omega_0{}^2}$$

$$\sin(\omega_0\tau - 2n\pi) = \frac{qp\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q^2 + s^2\omega_0{}^2} \tag{18}$$

$$q[s\omega_0 \sin\omega_0\tau + q\cos\omega_0\tau = \omega_0{}^2 - r]$$
$$s\omega_0[s\omega_0 \cos\omega_0\tau - q\sin\omega_0\tau = -p\omega_0]$$

$$\begin{aligned} qs\omega_0 \sin\omega_0\tau + q^2\cos\omega_0\tau &= q(\omega_0{}^2 - r) \\ s^2\omega_0{}^2\cos\omega_0\tau - qs\omega_0\sin\omega_0\tau &= -sp\omega_0{}^2 \end{aligned} \tag{19}$$

Adding equation (19) together,

$$(q^2 + s^2\omega_0{}^2)\cos\omega_0\tau = q(\omega_0{}^2 - r) - sp\omega_0{}^2$$

$$\cos\omega_0\tau = \frac{q(\omega_0{}^2 - r) - sp\omega_0{}^2}{q^2 + s^2\omega_0{}^2}$$

$$\cos(\omega_0\tau - 2n\pi) = \frac{q(\omega_0{}^2 - r) - sp\omega_0{}^2}{q^2 + s^2\omega_0{}^2} \tag{20}$$

Dividing equation (18) by (20)

$$\frac{\sin(\omega_0\tau - 2n\pi)}{\cos(\omega_0\tau - 2n\pi)} = \frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r) - ps\omega_0{}^2}$$

$$\tan(\omega_0\tau - 2n\pi) = \frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r) - ps\omega_0{}^2}$$

$$\omega_0\tau - 2n\pi = \arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r) - ps\omega_0{}^2}\right)$$

$$\omega_0\tau = \arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r) - ps\omega_0{}^2}\right) + 2n\pi$$

$$\tau = \frac{1}{\omega_0}\arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r) - ps\omega_0{}^2}\right) + \frac{2n\pi}{\omega_0}$$

where, $n = 0,1,2,3,\ldots$

If $n = 0$ $\tau_0 = \frac{1}{\omega_0}\arctan\left(\frac{pq\omega_0 + s\omega_0(\omega_0{}^2 - r)}{q(\omega_0{}^2 - r) - ps\omega_0{}^2}\right)$ is the smallest cut off value at which stability of the equilibrium point is lost and never be regained in the future time.

4.4 Global Stability

Without linearizing the system, it is possible to study global stability of a given non-linear model with the aid of Lyapunov function.

4.4.1 Global Stability Without time delay

$$\begin{aligned} x'(t) &= x(t)[a_1 - a_2 y(t) - a_3 x(t)] \\ y'(t) &= y(t)[-a_4 + a_5 x(t) - u] \end{aligned} \tag{21}$$

Let $v(x, y) = \frac{1}{2}(x - x^*)^2 + \frac{1}{2}(y - y^*)^2$ be a candidate Lyapunov function.

(i) $v(x^*, y^*) = 0$

(ii) $v(x, y) > 0$, Since the square of any number is positive.

(iii) Differentiating $v(x, y)$ along the solution of equation (21),

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$

$$\frac{dv}{dt} = (x - x^*)\frac{dx}{dt} + (y - y^*)\frac{dy}{dt}$$

$$\frac{dv}{dt} = (x - x^*)(xa_1 - a_2xy - a_3x^2) + (y - y^*)(-a_4y + a_5xy - uy)$$

Let $f(x, y) = \frac{dv}{dt}$

$$f(x,y)=(x-x^*)(a_1x-a_2xy-a_3x^2)+(y-y^*)(-a_4y+a_5xy-uy)$$

Expanding $\frac{dv}{dt}$ by Taylor Series about positive equilibrium point we get,

$$f(x,y)=(x-x^*)(a_1x-a_2xy-a_3x^2)+(y-y^*)(-a_4y+a_5xy-uy)$$

$$f_x=(a_1x-a_2xy-a_3x^2)+(x-x^*)(a_1-a_2y-2a_3x)+(y-y^*)(a_5y)$$

$$f_{xx}=(a_1-a_2y-2a_3x)+(a_1-a_2y-2a_3x)+(x-x^*)(-2a_3)$$

$$=2(a_1-a_2y-a_3x)-2a_3x+(x-x^*)(-2a_3)$$

$$f_{xy}=-a_2x+(x-x^*)(-a_2)+a_5(y-y^*)+a_5y$$

$$=-a_2x-a_2(x-x^*)+a_5(y-y^*)+a_5y$$

$$f_y=(x-x^*)(-a_2x)+(-a_4y+a_5xy-uy)+(y-y^*)(-a_4+a_5x-u)$$

$$f_{yy}=(-a_4+a_5x-u)+(-a_4+a_5x-u)=2(-a_4+a_5x-u)$$

Evaluating the partial derivatives at equilibrium point

$f(x^*,y^*)=0$, $f_x(x^*,y^*)=0$, $f_y(x^*,y^*)=0$, $f_{xx}(x^*,y^*)=-2a_3x^*$, $f_{yy}=0$

$f_{xy}(x^*,y^*)=a_5y^*-a_2x^*$

$$\frac{dv}{dt}=\frac{1}{2}[(x-x^*)^2 f_{xx}(x^*,y^*)+2(x-x^*)(y-y^*)f_{xy}(x^*,y^*)]$$

$$\frac{dv}{dt}=\frac{1}{2}[(x-x^*)^2(-2a_3x^*)+2(x-x^*)(y-y^*)(a_5y^*-a_2x^*)]$$

$$\frac{dv}{dt}=-a_3x*(x-x^*)^2+(a_5y^*-a_2x^*)(x-x^*)(y-y^*)$$

If $a_5y^*-a_2x^*=0$ then $\frac{dv}{dt}=-a_3x*(x-x^*)^2$

$$a_5 y^* = a_2 x^* \quad \frac{x^*}{y^*} = \frac{a_5}{a_2}$$

$$(\frac{a_4 + u}{a_5})(\frac{a_2 a_5}{a_1 a_5 - a_3(a_4 + u)}) = \frac{a_5}{a_2}$$

$$\frac{a_2(a_4 + u)}{a_1 a_5 - a_3(a_4 + u)} = \frac{a_5}{a_2}$$

$$a_2{}^2(a_4 + u) = a_1(a_5)^2 - a_3 a_5(a_4 + u)$$

$$a_2{}^2(a_4 + u) + a_3 a_5(a_4 + u) = a_1(a_5)^2$$

$$(a_4 + u)((a_2)^2 + a_3 a_5) = a_1(a_5)^2$$

$$a_4 + u = \frac{a_1(a_5)^2}{(a_2)^2 + a_3 a_5}$$

$$u = \frac{a_1(a_5)^2}{(a_2)^2 + a_3 a_5} - a_4 \tag{22}$$

$$\frac{dv}{dt} = 0 \text{ if } x = x^* \text{ or } x = x^* \ \& \ y = y^*$$

$\frac{dv}{dt}$ is negative definite when (22) is satisfied .

As the result, in the absence of delay there exist global stability of positive equilibrium point when condition (22) is satisfied by Lyapunov theorem.

4.5 Limit Cycle

For predator–prey systems, the existence and stability of a limit cycle is related to the existence and stability of a positive equilibrium. We assume that a positive equilibrium exists, for otherwise the predator population tends to extinction.

4.4.1 Limit Cycle without delay

Consider
$$x'(t) = a_1 x(t) - a_2 x(t) y(t) - a_3 x^2(t)$$
$$y'(t) = -a_4 y(t) + a_5 x(t) y(t) - u y(t)$$

Let
$$\frac{dx}{dt} = f(x,y) = a_1 x - a_2 xy - a_3 x^2$$
$$\frac{dy}{dt} = g(x,y) = -a_4 y + a_5 xy - uy$$

$\phi(x,y) = \dfrac{1}{xy}$ be a candidate Dulac's function. Since $x, y > 0$ then $\phi(x,y)$ is continuously differentiable function. Hence, $\phi(x,y) = \dfrac{1}{xy}$ is a suitable Dulac's function.

$$\frac{\partial}{\partial x}(\phi(x,y) f(x,y)) + \frac{\partial}{\partial y}(\phi(x,y) g(x,y)) =$$
$$\frac{\partial}{\partial x}\left[\frac{1}{xy}(a_1 x - a_2 xy - a_3 x^2)\right] + \frac{\partial}{\partial y}\left[\frac{1}{xy}(-a_4 y + a_5 xy - uy)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{a_1}{y} - a_2 - \frac{a_3 x}{y}\right) + \frac{\partial}{\partial y}\left(\frac{-a_4}{x} + a_5 - \frac{u}{x}\right)$$

$$= -\frac{a_3}{y}$$

$$\frac{\partial}{\partial x}(\phi(x,y) f(x,y)) + \frac{\partial}{\partial y}(\phi(x,y) g(x,y)) = -\frac{a_3}{y}$$

Since $\dfrac{\partial}{\partial x}(\phi(x,y) f(x,y)) + \dfrac{\partial}{\partial y}(\phi(x,y) g(x,y))$ is different from zero and does not change sign, by Dulac's criterion there is no limit cycle for the system.

$$\frac{\partial}{\partial x}(\phi(x,y)f(x,y))+\frac{\partial}{\partial y}(\phi(x,y)g(x,y))=0 \text{ ,when } a_3=0$$

$$a_3=0 \Rightarrow m_1+n_2+n_3=0$$

Therefore, the characteristic equation (7) reduced to

$$\lambda^2+m_1n_2+m_1n_3-m_2n_1=0 \tag{23}$$

Since $m_1n_2+m_1n_3-m_2n_1>0$, then the system has pure imaginary roots and the system is center. Therefore, the system has limit cycle when $a_3=0$

4.6 Hopf Bifurcation

Bifurcation study is a powerful tool in understanding an ecological community because bifurcation implies an abrupt change from one state to the other. For predator-prey systems, the population of prey and predators may stay at a steady state or oscillate periodically. The bifurcation parameter considered in the model is time delay.

1. From the characteristic equation (8), suppose it has a simple pair of pure imaginary eigenvalues $\lambda=\pm i\omega,\ \omega>0$. By the same analysis made for local stability with delay, there exist $\omega>0$ when condition (16) is satisfied.
2. Transeversality condition

$$\lambda^2+p\lambda+r+(s\lambda+q)e^{-\lambda\tau}=0 \tag{24}$$

Differentiate both sides of (24) with respect to τ where λ is a function of τ

$$2\lambda\frac{d\lambda}{d\tau}+p\frac{d\lambda}{d\tau}+se^{-\lambda\tau}\frac{d\lambda}{d\tau}+(s\lambda+q)e^{-\lambda\tau}(-\tau\frac{d\lambda}{d\tau}-\lambda)=0$$

$$\frac{d\lambda}{d\tau}(2\lambda+p+se^{-\lambda\tau}-\tau(s\lambda+q)e^{-\lambda\tau})-(s\lambda+q)\lambda e^{-\lambda\tau}=0$$

$$\frac{d\lambda}{d\tau}(2\lambda+p+(s-\tau s\lambda-\tau q)e^{-\lambda\tau})=(s\lambda+q)\lambda e^{-\lambda\tau}$$

$$\frac{d\lambda}{d\tau} = \frac{(s\lambda + q)\lambda e^{-\lambda\tau}}{(2\lambda + p + (s - \tau s\lambda - \tau q)e^{-\lambda\tau})}$$

$$\frac{d\tau}{d\lambda} = \frac{(2\lambda + p) + (s - \tau s\lambda - \tau q)e^{-\lambda\tau})}{(s\lambda + q)\lambda e^{-\lambda\tau}}$$

$$\frac{d\tau}{d\lambda} = \frac{(2\lambda + p)}{\lambda(s\lambda + q)e^{-\lambda\tau}} + \frac{se^{-\lambda\tau}}{(s\lambda + q)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}$$

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda + p}{\lambda(s\lambda + q)e^{-\lambda\tau}} + \frac{s}{\lambda(s\lambda + q)} - \frac{\tau}{\lambda}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1} = \operatorname{Re}\left\{\frac{2\lambda + p}{\lambda(s\lambda + q)e^{-\lambda\tau}} + \frac{s}{\lambda(s\lambda + q)}\right\}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1} = \operatorname{Re}\left[\frac{2\omega i + p}{\omega i(i\omega + q)e^{-i\omega\tau}} + \frac{s}{i\omega q - \omega^2 s}\right]$$

$$= \operatorname{Re}\left[\frac{(2\omega i + p)e^{i\omega\tau}}{-\omega^2 + i\omega q} + \frac{s}{i\omega q - \omega^2 s}\right]$$

$$= \operatorname{Re}\left[\frac{(2\omega i + p)(-\omega^2 s - i\omega q)e^{i\omega\tau}}{\omega^4 s^2 + \omega^2 q^2} + \frac{-\omega^2 s^2 - i\omega sq}{\omega^4 s^2 + \omega^2 q^2}\right]$$

$$= \operatorname{Re}\left[\frac{(-2\omega^3 si + 2\omega^2 q - \omega^2 ps - \omega pqi)(\cos\omega\tau + i\sin\omega\tau}{\omega^4 s^2 + \omega^2 q^2}\right]\frac{-\omega^2 s^2}{\omega^4 s^2 + \omega^2 q^2}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1} = \frac{(2\omega^2 q - \omega^2 ps)\cos\omega\tau + (2\omega^3 s + \omega pq)\sin\omega\tau}{\omega^4 s^2 + \omega^2 q^2} - \frac{\omega^2 s^2}{\omega^4 s^2 + \omega^2 q^2} \tag{25}$$

$$\left.\begin{aligned} \cos\omega\tau &= \frac{\omega^2 q - qr - \omega^2 ps}{\omega^2 s^2 + q^2} \\ \sin\omega\tau &= \frac{\omega^3 s + \omega pq - \omega rs}{\omega^2 s^2 + q^2} \end{aligned}\right\} \tag{26}$$

Substitute equation (26) into (25)

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1} = \frac{(2\omega^2 q - \omega^2 qs)\dfrac{\omega^2 q - qr - \omega^2 ps}{\omega^2 s^2 + q^2} + (2\omega^3 s + \omega pq)\dfrac{\omega^3 s + \omega pq - \omega rs}{\omega^2 s^2 + q^2}}{\omega^4 s^2 + \omega^2 q^2} - \frac{\omega^2 s^2}{\omega^4 s^2 + \omega^2 q^2}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right]^{-1} = \frac{2\omega^4 q^2 - 2\omega^2 qr - 2\omega^4 qs - \omega^4 pqs + \omega^2 pqrs + \omega^4 p^2 s^2}{(\omega^4 s^2 + \omega^2 q^2)(\omega^2 s^2 + q^2)} +$$

$$\frac{2\omega^6 s^2 + 2\omega^4 pqs - 2\omega^4 s^2 r + \omega^4 pqs + \omega^2 p^2 q^2 - \omega^2 pqrs}{(\omega^4 s^2 + \omega^2 q^2)(\omega^2 s^2 + q^2)} - \frac{\omega^2 s^2}{\omega^4 s^2 + \omega^2 q^2}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau)}{d\tau}\right]^{-1} = \frac{\begin{pmatrix} 2\omega^4 q^2 - 2\omega^2 q^2 r - 2\omega^4 pqs + \omega^2 pqrs + \omega^4 p^2 s^2 + 2\omega^6 s^2 + \\ 2\omega^4 pqs - 2\omega^4 s^2 r + \omega^4 pqrs + \omega^2 p^2 q^2 - \omega^2 pqrs - \omega^4 s^4 - \omega^2 s^2 q^2 \end{pmatrix}}{(\omega^4 s^2 + \omega^2 s^2)(\omega^2 s^2 + q^2)}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau)}{d\tau}\right]^{-1} = \frac{\omega^4 s^2 (p^2 - 2r - s^2) + \omega^2 q^2 (p^2 - 2r - s^2) + 2\omega^4 q^2 + 2\omega^6 s^2}{(\omega^4 s^2 + \omega^2 s^2)(\omega^2 s^2 + q^2)}$$

$$\left[\frac{d(\operatorname{Re}\lambda(\tau)}{d\tau}\right]^{-1} > 0$$

$$Sign\left[\frac{d(\operatorname{Re}\lambda(\tau)}{d\tau}\right]^{-1} = Sign\left\{\frac{\omega^4 s^2 (p^2 - 2r - s^2) + \omega^2 q^2 (p^2 - 2r - s^2) + 2\omega^4 q^2 + 2\omega^6 s^2}{(\omega^4 s^2 + \omega^2 s^2)(\omega^2 s^2 + q^2)}\right\}$$

$$Sign\left\{\frac{d(\operatorname{Re}\lambda(\tau)}{d\tau}\right\}^{-1} = 1 > 0$$

Hence, transversality condition is satisfied.

As a result, the systems undergo Hopf bifurcation at $\tau = \tau_o$ when condition (16) is satisfied.

CHAPTER FIVE

CONCLUSION AND FUTURE WORK

5.1. Conclusions

In this thesis, mathematical model of prey predator with delay was studied. The result of the study indicates that the positive equilibrium point in the absence of delay is stable. In the presence of delay the system becomes stable with certain conditions and loses its stability at cutoff value. Furthermore, the existence of global stability in the absence of delay was proved using Lyapunov theorem by constructing appropriate Lyapunov function. Nonexistence of limit cycle was also proved by using Dulac’s criterion. However, if there is no intraspecific competition rate of prey there exist limit cycle for the system. Finally, by considering the bifurcation parameter as a time delay the system undergo Hopf bifurcation at cutoff value with certain condition stated by equation (16).

5.2 Future Work

One can investigate the Hopf bifurcation of the system by considering other parameters involved in the model different from time delay. Global stability with delay, direction of stability and Hopf bifurcation, Persistence of the prey predator, global existence of periodic solution of the model are also further investigation. Furthermore, it is possible to consider control that change with time rather than control parameter and different time delay on the two equations.

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