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Strong Convergence of Mann Iteration for a Hybrid Pair of Mappings in a Banach Space

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Abstract

We prove the strong convergence of Mann iteration for a hybrid pair of maps to a common fixed point of a selfmap f and a multi-valued f-nonexpansive mapping T in Banach space E. Our result extend Theorem 2.3 of Song and Wang [Y. Song, H. Wang, Convergence of iterative algorithms for multi-valued mappings in Banach spaces, Nonlinear Analysis, 70 (2009), 1547–1556] to a hybrid pair of maps.

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1 Introduction

Let E be a Banach space and K, a nonempty subset of E. We denote by 2^E, the family of all subsets of E; $CB(E)$, the family of nonempty closed and bounded subsets of E and $C(E)$, the family of nonempty compact subsets of E. Let $f : K \to K$ be a selfmap. Let H be a Hausdorff metric on $CB(E)$. That is, for $A, B \in CB(E)$,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\},$$

where

$$d(x, B) = \inf\{\|x - y\| : y \in B\}.$$

A multi-valued mapping $T : K \to 2^K$ is called *f-nonexpansive* if

$$H(Tx, Ty) \leq \|fx - fy\|,$$

for all $x, y \in K$.

If $f = I_K$, the identity mapping on K , then we call T is a *multi-valued nonexpansive* mapping.

A point x is *a fixed point* of T if $x \in Tx$. A point x is called *a common fixed point* of f and T if $fx = x \in Tx$. $F(T) = \{x \in K : x \in Tx\}$ stands for the fixed point set of a mapping T and $F = F(T) \cap F(f) = \{x \in K : fx = x \in Tx\}$ stands for the common fixed point set of maps f and T.

Recently, Song and Wang [2] introduced the following Mann iterates of a Multi-valued mapping T:

Let K be a nonempty convex subset of E, $\alpha_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Let $T : K \to CB(K)$ be a multi-valued mapping. Let $x_0 \in K$, and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \quad (1)$$

where $y_n \in Tx_n$ such that $\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n, n = 0, 1, 2, \cdots$.

Song and Wang [2] established the following theorems on the convergence of Mann iteration.

Theorem 1.1 *(Theorem 2.3, Song and Wang [2]). Let K be a nonempty, compact and convex subset of a Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued nonexpansive mappings for which $F(T) \neq \emptyset$ and for which $T(y) = \{y\}$ for each $y \in F(T)$. For $x_0 \in K$, let $\{x_n\}$ be the Mann iteration defined by (1). Assume that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point of T.

The aim of this paper is to prove the strong and weak convergence of Mann iteration for a hybrid pair of maps to a common fixed point of a selfmap f and a multi-valued f-nonexpansive mapping T in Banach space E. Our results extend the results of Song and Wang [2] to a hybrid pair of maps.

2 Preliminary Notes

Throughout this paper E denotes real Banach space. We denote the strong convergence of $\{x_n\}$ to x in E by $x_n \to x$.

Lemma 2.1 *(Nadler [1]). Let (E, d) be a complete metric space, and $A, B \in CB(E)$ and $a \in A$. Then for each positive number ε, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Lemma 2.2 *(Suzuki [3]). Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in [0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$ for all integers $n \geq 1$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

We will construct the following iteration.
Let K be a nonempty subset of a metric space X. Let $f : K \to K$, $T : K \to CB(K)$ with $f(K)$ is convex and $Tx \subseteq f(K)$ for all $x \in K$. Let $\alpha_n \in [0, 1]$, and $\gamma_n \in (0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$ and $y_0 \in Tx_0$. Let $z_0 = fx_0$ and

$$\begin{aligned} z_1 = fx_1 &= (1 - \alpha_0)fx_0 + \alpha_0 y_0 \\ &= (1 - \alpha_0)z_0 + \alpha_0 y_0. \end{aligned}$$

From Lemma 2.1, there exists $y_1 \in Tx_1$ such that

$$\|y_1 - y_0\| \leq H(Tx_1, Tx_0) + \gamma_0.$$

Let

$$z_2 = fx_2 = (1 - \alpha_1)z_1 + \alpha_1 y_1.$$

Inductively, we have

$$z_{n+1} = fx_{n+1} = (1 - \alpha_n)z_n + \alpha_n y_n, \tag{2}$$

where $y_n \in Tx_n$ such that

$$\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n, \ n = 0, 1, 2, \cdots.$$

3 Main Results

These are the main results of the paper.

Proposition 3.1 *Let K be a nonempty subset of a Banach space E. Let $f : K \to K$ be a selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multi-valued f-nonexpansive mapping and $Tx \subseteq f(K)$ for all $x \in K$. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps T and f, defined by (2) and assume also that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then $\lim_{n\to\infty} \|z_n - y_n\| = 0$ *and* $\lim_{n\to\infty} d(z_n, Tx_n) = 0$.

Proof. From the definition of the Mann iteration $\{z_n\}$ given by (2), it follows that $z_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n$, where $y_n \in Tx_n$ such that

$$\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n$$

$$\leq \|z_{n+1} - z_{n+1}\| + \gamma_n,\ n = 0, 1, 2, \cdots.$$

Therefore,

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|z_{n+1} - z_n\|) \leq \limsup_{n\to\infty} \gamma_n = 0.$$

Hence, all conditions of Lemma 2.2 are satisfied. Hence, by Lemma 2.2, we obtain $\lim_{n\to\infty} \|z_n - y_n\| = 0$.

Since $y_n \in Tx_n$ for all $n = 0, 1, 2, \cdots$, we have $d(z_n, Tx_n) \leq \|z_n - y_n\|$.

Hence, $\lim_{n\to\infty} d(z_n, Tx_n) = 0$.

Theorem 3.2 *Let K be a nonempty compact subset of a Banach space E. Let $f : K \to K$ be a continuous selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multi-valued f-nonexpansive mapping for which $Tx \subseteq f(K)$ for all $x \in K$; $F(T) \cap F(f) \neq \emptyset$, and $d(x, Tx) \leq d(fx, Tx)$ for all $x, y \in K$. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps T and f, defined by (2) and assume also that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

If $T(y) = \{y\}$ for each $y \in F(T)$, then the Mann iteration $\{z_n\}$ strongly converges to a common fixed point of f and T.

Proof. It follows from Proposition 3.1 that $\lim\limits_{n\to\infty} d(z_n, Tx_n) = 0$.
Further, since $d(x_n, Tx_n) \leq d(z_n, Tx_n)$ we get $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$.
Now let $p \in F(T) \cap F(f)$. Then,

$$\begin{aligned}\|z_{n+1} - p\| &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n H(Tx_n, Tp) \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \\ &= \|z_n - p\|,\ n = 0, 1, 2, \cdots.\end{aligned}$$

Then the sequence $\{\|z_n - p\|\}$ is a decresing sequence of nonnegative reals and hence $\lim\limits_{n\to\infty} \|z_n - p\|$ exists for each $p \in F(T) \cap F(f)$.

From the compactness of K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim\limits_{k\to\infty} x_{n_k} = u$ for some $u \in K$. By the continuity of f, we have $\lim\limits_{k\to\infty} z_{n_k} = fu = q$ (say). Now

$$\begin{aligned}d(q, Tu) &\leq \|q - z_{n_k}\| + d(z_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tu) \\ &\leq \|q - z_{n_k}\| + d(z_{n_k}, Tx_{n_k}) + \|fu - fx_{n_k}\| \\ &= 2\|q - z_{n_k}\| + d(z_{n_k}, Tx_{n_k}) \to 0 \ as\ k \to \infty.\end{aligned}$$

Hence, $fu = q \in Tu$.
Also,

$$\begin{aligned}d(u, Tu) &\leq \|u - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tu) \\ &\leq \|u - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \|fu - fx_{n_k}\| \\ &= \|u - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \|z_{n_k} - q\| \to 0 \ as\ k \to \infty.\end{aligned}$$

Hence, $u \in Tu$ so that $Tu = \{u\}$.
Hence, $fq = q \in Tq$.
Thus, q is a common fixed point of f and T.
Now replacing q in place of p, we get that $\lim\limits_{n\to\infty} \|z_n - q\|$ exists and hence $\lim\limits_{n\to\infty} \|z_n - q\| = 0$.
Hence the conclusion follows.

Corollary 3.3 *If $f = I_K$, the identity mapping on K, we get Theorem 1.1. Hence, Theorem 3.2 extends Theorem 1.1 to a hybrid pair of maps.*

The following is an example in support of Theorem 3.2.

Example 3.4 *Let $E = R$, the set of all real numbers, with the usual norm and $K = [\frac{1}{3}, 1]$. We define mappings $f : K \to K$ by $fx = 1 - \frac{1}{2}x$ and $T : K \to CB(K)$ by $Tx = [\frac{2}{3},\ \frac{5}{6}x + \frac{1}{3}]$.*

Here $f(K) = [\frac{1}{2}, \frac{2}{3}]$, $Tx \subseteq f(K)$ for all $x \in K$, and $F(f) \cap F(T) = \{\frac{2}{3}\} \neq \emptyset$.

Now we consider the following two cases.

Case (i): $x \in [\frac{1}{3}, \frac{2}{3}]$.

Then, $fx = 1 - \frac{1}{2}x \geq \frac{2}{3}$, $Tx = [\frac{1}{2}x + \frac{1}{3},\ \frac{2}{3}]$.

Thus we have $d(x, Tx) = \frac{1}{2}(\frac{2}{3} - x) = d(fx, Tx)$.

Case (ii): $x \in [\frac{2}{3}, 1]$.

Then, $fx = 1 - \frac{1}{2}x \leq \frac{2}{3}$, $Tx = [\frac{2}{3},\ \frac{1}{2}x + \frac{1}{3}]$.

Thus we have $d(x, Tx) = \frac{1}{2}(x - \frac{2}{3}) = d(fx, Tx)$.

Hence, from case (i) and case (ii), it follows that

$$d(x, Tx) = d(fx, Tx) \ forall\ x \in K.$$

Also, T is f-nonexpansive on K, for, proceeding as in the above, we get

$$H(Tx, Ty) = \max\{\sup_{a \in Ty} d(Tx, a),\ \sup_{a \in Tx} d(a, Ty)\}$$

$$= |fx - fy| \ forall\ x, y \in K;$$

and $Ty = \{y\}$ for each $y \in F(T) = \{\frac{2}{3}\}$.

Next we show that for any $x_0 \in K$, the Mann iteration defined by (2) converges to the unique common fixed point of f and T, which is the conclusion of Theorem 3.2.

Let $x_0 \in K$ be arbitrary. Let $\alpha_n \in [0,\ 1]$ be such that

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.$$

If $x_0 \in [\frac{1}{3}, \frac{2}{3}]$. Then $fx_0 = 1 - \frac{1}{2}x_0$ and $Tx_0 = [\frac{1}{2}x_0 + \frac{1}{3},\ \frac{2}{3}]$. Choose $y_0 = \frac{1}{2}x_0 + \frac{1}{3}$. Then $y_0 \in Tx_0$, and $fx_1 = \frac{2}{3} + (\frac{1}{2} - \alpha_0)\ (\frac{2}{3} - x_0)$.

On continuing this process, inductively we get a sequence $\{x_n\}$ in K such that

$$fx_{n+1} = \frac{2}{3} + \frac{1}{2}(\frac{2}{3} - x_0)\ \prod_{j=0}^{n}(1 - 2\alpha_j),\ n = 0, 1, 2, \cdots. \tag{3}$$

If $x_0 \in [\frac{2}{3}, 1]$. Then $fx_0 = 1 - \frac{1}{2}x_0$ and $Tx_0 = [\ \frac{2}{3},\ \frac{1}{2}x_0 + \frac{1}{3}]$. Again, choose $y_0 = \frac{1}{2}x_0 + \frac{1}{3}$. Then $y_0 \in Tx_0$, and $fx_1 = \frac{2}{3} - (\frac{1}{2} - \alpha_0)\ (x_0 - \frac{2}{3})$.

On continuing this process, inductively we get a sequence $\{x_n\}$ in K such that

$$fx_{n+1} = \frac{2}{3} - \frac{1}{2}(x_0 - \frac{2}{3})\ \prod_{j=0}^{n}(1 - 2\alpha_j),\ n = 0, 1, 2, \cdots. \tag{4}$$

Since $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, there exist real numbers $0 < \gamma, \eta < 1$ such that $0 < \gamma \leq \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n \leq \eta < 1$, and hence there exists a positive integer N such that $\gamma \leq \alpha_n \leq \eta$ for all $n \geq N$.

Hence, $\beta = \sup_{j\geq N} |2\alpha_j - 1| \leq \max\{|2\gamma - 1|,\ |2\eta - 1|\} < 1$.

Now, by using (3) and (4) for $x_0 \in K$, we get

$$fx_{n+1} = \frac{2}{3} + \frac{1}{2}(\frac{2}{3} - x_0) \prod_{j=0}^{N-1}(1 - 2\alpha_j) \prod_{j=N}^{n}(1 - 2\alpha_j),\ n \geq N. \tag{5}$$

Hence,

$$|fx_{n+1} - \frac{2}{3}| \leq \frac{1}{2}|\frac{2}{3} - x_0| \prod_{j=0}^{N-1}|1 - 2\alpha_j| \prod_{j=N}^{n}|1 - 2\alpha_j| \tag{6}$$

$$\leq \frac{1}{2}|\frac{2}{3} - x_0| \prod_{j=0}^{N-1}|1 - 2\alpha_j|\ \beta^{n-N+1},\ n \geq N. \tag{7}$$

Hence, $fx_n \to \frac{2}{3}$ strongly as $n \to \infty$, and $\frac{2}{3}$ is a common fixed point of f and T.

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