THE SUPER STABILITY OF THIRD ORDER LINEAR ORDINARY DIFFERENTIAL HOMOGENEOUS EQUATION WITH BOUNDARY CONDITION.

BY

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DECLARATION

I, the undersigned declare that the thesis entitled “The super stability of third order linear ordinary differential homogeneous equations with boundary condition” is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

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Abstract

The stability problem is a fundamental issue in the design of any distributed systems like local area networks, multiprocessor systems, distribution computation and multidimensional queuing systems. In Mathematics stability theory addresses the stability solutions of differential, integral and other equations, and trajectories of dynamical systems under small perturbations of initial conditions. Differential equations describe many mathematical models of a great interest in Economics, Control theory, Engineering, Biology, Physics and to many areas of interest.

In this study the recent work of Jinghao Huang, Qusuay.H. Alqifiary, and Yongjin Li in establishing the super stability of differential equation of second order with boundary condition was extended to establish the super stability of differential equation third order with boundary condition.

CHAPTER ONE

Introduction

1.1. Back ground of the study

In recent years, a great deal of work has been done on various aspects of differential equations of third order. Third order differential equations describe many mathematical models of great iterest in engineering, biology and physics. Equation of the form $x''' + a(x)x'' + b(x)x' + c(x)x = f(t)$ arise in the study of entry-flow phenomena ,a problem of hydrodynamics which is of considerable importance in many branches of engineering.

There are different problems concerning third order differential equations which have drawn the attention of researchers throughout the world. [25]

In mathematics stability theory addresses the stability of solutions of differential equations, Integral equations, including other equations and trajectories of dynamical systems under small perturbations. Following this, stability means that the trajectories do not change too much under small perturbations [11].The stability problem is a fundamental issue in the design of any distributed systems like local area networks, multiprocessor systems, mega computations and multidimensional queuing systems and others. In the field of economics, stability is achieved by avoiding or limiting fluctuations in production, employment and price.

For many decades, a great deal of work has been done on the stability of various aspects of differential equations, because it describes many mathematical models of great interest in economics, control theory, engineering, biology, physics and other areas of interest[20].

The stability problem in mathematics started by Poland mathematician Stan Ulam for functional equations around 1940; and the partial solution of Hyers to the Ulam's problem [5] and [29].
In 1940, Ulam [30] posed a problem concerning the stability of functional equations:
"Give conditions in order for a linear function near an approximately linear function to exist."
A year later, Hyers answered to the problem of Ulam for additive functions defined on Banach space: let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function

$$f : X_1 \to X_2 \text{ Satisfying } \quad \| f(x+y) - f(x) - f(y) \| \le \varepsilon \qquad x, y \in X_1$$

There exists a unique additive function $A : X_1 \to X_2$ with the property

$$\| f(x) - A(x) \| \le \varepsilon \qquad x \in X_1$$

Thereafter, Rassias [22] attempted to solve the stability problem of the Cauchy additive functional equations in more general setting. A generalization of Ulam's problem is recently proposed by replacing functional equations with differential equations $\varphi(f, y, y', ... y^{(n)}) = 0$ and has the Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function y such that

$$\left|\varphi\left(f, y, y', ... y^{(n)}\right)\right| \le \varepsilon$$

There exists a solution y_0 of the differential equation such that

$$\left|y(t) - y_0(t)\right| \le k(\varepsilon) \quad \text{And} \quad \lim_{\varepsilon\to 0} k(\varepsilon) = 0$$

Obloza seems to be the first author who has investigated the Hyers- Ulam stability of linear differential equation [14] and [15]. Thereafter, Alsina and Ger published their first paper which handles the Hyers –Ulam stability of linear differential equations $y'(t) = y(t)$

If adifferentiable function $y(t)$ is asolution of the inequality $\left|y'(t) - y(t)\right| \le \varepsilon$ for any $t \in (a, \infty)$,

there exists a constant c such that $\left|y(t) - c e^{t}\right| \le 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [12, 27] and in [28] respectively.

Rus investigated the Hyers-Ulam stability of differential and integral equations using the Granwall lemma and the technique of weakly Picard operators [23, 24].

Miura et al [13] proved the Hyers-Ulam stability of the first-order linear differential equations

$$y'(t) + g(t)y(t) = 0,$$

Where $g(t)$ is a continuous function, while Jung (8) proved the Hyers-Ulam stability of differential equations of the form

$$\varphi(t)y'(t) = y(t)$$

Motivation of this study comes from the work of Li [9] where he established the stability of linear differential equations of second order in the sense of the Hyers and Ulam.

$$y' = \lambda y$$

Li and Shen [10] proved the stability of non-homogeneous linear differential equation of second order in the sense of the Hyers and Ulam

$$y'' + p(x)y' + q(x)y + r(x) = 0$$

While Gavaruta et al [3] proved the Hyers- Ulam stability of the equation

$$y'' + \beta(x)y(x) = 0$$

with boundary and initial conditions.

The recently, introduced notion of super stability [16,17,18,19] is utilized in numerous applications of the automatic control theory such as robust analysis, design of static output feedback, simultaneous stabilization, robust stabilization, and disturbance attenuation.

In 1979, Baker, Lawrence, and Zorzitto [1] investigated the super stability, which states that if f is a function from a vector space to R satisfying

$$\left| f(x+y) - f(x)f(y) \right| \leq \varepsilon$$

For some fixed, $\varepsilon > 0$ then either f is bounded or f satisfies the exponential functional equation

$$f(x+y) = f(x)f(y)$$

Gavruta [2] proved the super stability of the Lobachevski equation under the condition bounded by a constant.

Recently, Jinghao Huang, Qusuay H. Alqifiary, Yongjin Li[6] established the super stability of differential equations of second order with boundary conditions or with initial conditions as well as the super stability of differential equations of higher order in the form $y^{(n)}(x) + \beta(x)y(x) = 0$ with initial conditions, $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$

This study aimed to extend the super stability of second order to the third order linear ordinary differential homogeneous equations with boundary condition.

1.2 Statement of the problem

Following the work of Stan Ulam many mathematicians extend the work of him to establish the stability and super stability of different types of functional equations. And recently such Ulam's stability problem in functional equation replaced by differential equation. The researchers Jinghao Huang, Qusuay H. Alqifiary, Yongjin Li investigate the super stability for homogeneous second order differential equation of higher order with initial condition and second order with boundary condition. But the super stability of third order linear ordinary differential homogeneous equation with boundary condition has not been discussed.

So the purpose of this study was establishing the super stability of third order linear ordinary differential homogeneous equations with boundary condition by answering the following questions.

- How can we determine the super stability of third order linear ordinary differential homogeneous equations with boundary condition?
- What kind of example can we construct which satisfies the super stability of linear ordinary differential homogenous equations with boundary condition?

1.3 Objective of the study

1.3.1 General objective of the study

The general objective of this study was to establish the super stability of third order linear ordinary differential homogeneous equations with boundary condition.

1.3.2 Specific objectives

The following are the specific objectives of this study

- Determine the super stability of third order linear ordinary differential homogeneous equation with boundary condition.
- Construct an example that satisfies the super stability of linear ordinary differential homogeneous equations with boundary condition.

1.4 Significance of the study

On completion the study is expected to have the following significances:

- Develop ideas with regard to the subject matter and
- Apply in those areas which need the super stability of the system

1.5 Delimitation of the study

- This study was delimited to super stability of linear third order ordinary differential homogeneous equations with boundary conditions.

CHAPTER- TWO

Review Literature

In 1892 Liapunov introduced the concept of stability of determined dynamical systems and established his second method of stability theory. The advantage of his method conjugated the stability of the systems without the knowledge of the solutions of the systems, and therefore, it was a hot pot in the study of the stability theory in the last century[11],[7].

Studying the stability of a system is a very crucial problem in the theory and application of mathematical systems [26]. The stability problem in mathematics started by Poland mathematician Stan Ulam for functional equations around 1940; and the partial solution of Hyers to the Ulam's problem [5] and [29].

In 1940, Ulam [30] posed a problem concerning the stability of functional equations:

"Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers answered to the problem of Ulam for additive functions defined on Banach space:

Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function

$f : X_1 \to X_2$ Satisfying:

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon \qquad x, y \in X_1$$

There exists a unique additive function $A: X_1 \to X_2$ with the property

$$\| f(x) - A(x) \| \leq \varepsilon \qquad x \in X_1$$

Thereafter, Rassias [22] attempted to solve the stability problem of the Cauchy additive functional equations in more general setting. A generalization of Ulam's problem is recently proposed by replacing functional equations with differential equation $\varphi(f, y, y', \ldots y^{(n)}) = 0$ and has the Hyer-Ulam stability if for a given $\varepsilon > 0$ and a function y such that

$$\left| \varphi(f, y, y', \ldots y^{(n)}) \right| \leq \varepsilon$$

There exists a solution y_0 of the differential equation such that

$$\left| y(t) - y_0(t) \right| \leq k\varepsilon \text{ And } \lim_{\varepsilon \to 0} k(\varepsilon) = 0$$

In 1978 Themistocles M.Rassias [21]. Considered the unbounded Cauchy difference Inequality

$$\| f(x+y) - f(x) - f(y)\| \le \varepsilon \|x\|^p + \|y\|^p$$

where $\varepsilon > 0$ and $P \in [0,1]$ and proved the stability of linear mappings between Banach spaces. Thus the Ulam problem for approximate homeomorphisms is brought in to a new general context. This has led to the stability phenomena that are known these days by the Hyers–Ulam stability. After Hyers–Rassias stability result a generalization of Ulam's problem is recently proposed by replacing functional equations with differential equations by Obloza [14], [15].

In 1979, Baker, Lawrence Zorzitto [1] proved a new type of stability of the exponential equation

$$f(x+y) = f(x) f(y)$$

More precisely they prove that if a complex valued mapping f defined on a normed vector space satisfies the inequality

$$|f(x+y) - f(x) f(y)| \le \delta$$

For some given $\delta > 0$ and for all x, y then either f is bounded or f is exponential such phenomenon is called the super stability of the exponential equation which is special kind of Hyers –Ulam stability. It seems that the result of Gavruta, Jung and Li [4].

And recently, Jinghao Huang, Qusuay H. Alqifiary, Yongjin Li[6] proved the super stability of the linear differential equation of second order with initial condition and boundary condition as well as linear differential equation of higher order in the form of $y^{(n)}(x) + \beta(x) y = 0$ with initial condition, $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$. Now in this paper we establish the super stability of third order linear ordinary differential homogeneous equation with boundary condition.

CHAPTER-THREE

Methodology

3.1 Study area and period

This study was conducted in Jimma University Department of Mathematics from December 2014 to June, 2015.

3.2 Study design

The study design for this study was analytical method.

3.3 Source of information

The source of information for this study was secondary data like reference books, internet, reading on line books, and different published research articles(journals) that leads to the investigation of super stability for third order linear ordinary differential homogeneous equation with boundary condition.

3.4 Study procedures

In order to achieve the above mentioned objectives the following were used

- The super stability of second order differential equation was extended to the super stability of linear third order ordinary differential homogeneous equation. .
- Boundary condition was considered

3.5 Ethical issues

Ethical Clearance was obtained from Jimma University, department of Mathematics. So the researcher was made appropriate communication with responsible bodies and has got permission and in addition concerned bodies were informed about the purpose of the study.

CHAPTER-FOUR

RESULTS AND DISCUSSION

4.1 Super stability with boundary condition

Definition : Assume that for any function $y \in c^n[a,b]$,if y satisfies the differential inequalities

$$\left|\varphi\left(f, y, y', ..., y^{(n)}\right)\right| \leq \varepsilon \ldots ......(1)$$

for all $x \in [a,\ b]$and for some $\varepsilon \geq 0$ with boundary conditions, then either y is a solution of the differential equation

$$\varphi(f, y, y', ..., y^{(n)}) = 0 \ldots\ldots \quad \ldots\ldots(2)$$

or $|y(x)| \leq k\varepsilon$ for any $x \in [a,b]$, where k is a constant. Then, we say that (2) has super stability with boundary conditions.

Preliminaries

Lemma 1[6]. Let $y \in c^2[a,b]$, $y(a) = 0 = y(b)$,then

$$\max |y(x)| \leq \frac{(b-a)^2}{8} \max |y''(x)|.$$

Proof

Let M = $\max\{|y(x)| : x \in [a,b]\}$ Since $y(a) = 0 = y(b)$, there exists $x_0 \in (a,b)$ such that

$|y(x_0)| = M$. By Taylor's formula, we have

$$y(a) = y(x_0) + y'(x_0)(x_0 - a) + \frac{y''(\delta)}{2!}(x_0 - a)^2,$$

$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(\eta)}{2!}(b - x_0)^2,$$

Where δ , $\eta \in (a,b)$

Thus $|y''(\delta)| = \frac{2M}{(x_0 - a)^2}$,

$$\left|y''(\eta)\right| = \frac{2M}{\left(b-x_0\right)^2}$$

In the case $x_0 \in \left(a, \frac{(a+b)}{2}\right]$, we have

$$\frac{2M}{\left(x_0-a\right)^2} \geq \frac{2M}{\frac{\left(b-a\right)^2}{4}} = \frac{8M}{\left(b-a\right)^2}$$

In the case $x_0 \in \left[\frac{a+b}{2}, b\right)$, we have

$$\frac{2M}{\left(b-x_0\right)^2} \geq \frac{2M}{\frac{\left(b-a\right)^2}{4}} = \frac{8M}{\left(b-a\right)^2}$$

So, $\max\left|y''(x)\right| \geq \frac{8M}{\left(b-a\right)^2} = \frac{8}{\left(b-a\right)^2}\max\left|y(x)\right|$

Therefore, $\max\left|y(x)\right| \leq \frac{\left(b-a\right)^2}{8}\max\left|y''(x)\right|$

In (2011) the three researchers Pasc Gavaruta, Soon-Mo, Jung and Yongjin Li, investigate the Hyers-Ulam stability of second order linear differential equation

$$y''(x) + \beta(x)y(x) = 0 \quad \ldots\ldots\ldots\ldots.. (3)$$

with boundary conditions $y(a) = y(b) = 0$

where, $y \in c^2\left[a,b\right], \beta(x) \in c\left[a,b\right], -\infty < a < b < +\infty$

Definition: We say (3) has the Hyers-Ulam stability with boundary conditions

$y(a) = y(b) = 0$ if there exists a positive constant K with the following property:

For every $\varepsilon > 0, y \in c^2\left[a,b\right],$ if

$$\left|y''(x) + \beta(x)y(x)\right| \leq \varepsilon,$$

And $y(a)=y(b)=0$, then there exists some $z\in c^{2}\left[a,b\right]$ satisfying

$$z''(x)+\beta(x)z(x)=0$$

And $z(a)=0=z(b)$, such that $\left|y(x)-z(x)\right|<K\varepsilon$

Theorem 1[3]. Consider the differential equation

$$y''(x)+\beta(x)y(x)=0 \text{---------------------} (4)$$

With boundary conditions $y(a)=y(b)=0$

Where $y\in c^{2}\left[a,b\right],\beta(x)\in c\left[a,b\right],-\infty<a<b<+\infty$

If $\max\left|\beta(x)\right|<\dfrac{8}{\left(b-a\right)^{2}}$

then the equation (4) above has the super stability with boundary condition

$y(a)=y(b)=0$

Proof:

For every $\varepsilon>0, y\in C^{2}\left[a,b\right]$, if $\left|y''(x)+\beta(x)y(x)\right|\le\varepsilon$ and $y(a)=0=y(b)$

Let M = $\max\left\{\left|y(x)\right|\right\}: x\in\left[a,b\right]$, since $y(a)=0=y(b)$, there exists $x_0\in(a,b)$

Such that $\left|y(x_0)\right| = \text{M}$. By Taylor formula, we have

$$y(a)=y(x_0)+y'(x_0)(x_0-a)+\frac{y''(\delta)}{2!}(x_0-a)^{2},$$

$$y(b)=y(x_0)+y'(x_0)(b-x_0)+\frac{y''(\eta)}{2!}(b-x_0)^{2},$$

Where $\delta,\eta\in\left(a,b\right)$

Thus $\left|y''(\delta)\right|=\dfrac{2M}{\left(x_0-a\right)^{2}},$

$$\left|y''\left(\eta\right)\right|=\frac{2M}{(x_0-b)^{2}}$$

On the case $x_0 \in \left(a, \dfrac{(a+b)}{2} \right]$, we have

$$\frac{2M}{(x_0-a)^2} \geq \frac{2M}{\frac{(b-a)^2}{4}} = \frac{8M}{(b-a)^2}$$

On the case $x_0 \in \left[\dfrac{(a+b)}{2}, b \right)$ we have

$$\frac{2M}{(b-x_0)^2} \geq \frac{2M}{\frac{(b-a)^2}{4}} = \frac{8M}{(b-a)^2}$$

So $\max|y''(x)| \geq \dfrac{8M}{(b-a)^2} = \dfrac{8}{(b-a)^2} max|y(x)|$

Therefore $max|y(x)| \leq \dfrac{(b-a)^2}{8} max|y''(x)|$

Thus $max|y(x)| \leq \dfrac{(b-a)^2}{8} \left[max|y''(x) - \beta(x)y(x)| + max|\beta(x)| max|y(x)| \right]$

$$\leq \frac{(b-a)^2}{8}\varepsilon + \frac{(b-a)^2}{8} \max|\beta(x)| \max|y(x)|$$

Let $k = \dfrac{(b-a)^2}{8\left(1 - \dfrac{(b-a)^2}{8} \max|\beta(x)|\right)}$

Obviously, $z_0(x) = 0$ is a solution of $y''(x) - \beta(x)y(x) = 0$ with the Boundary conditions $y(a) = 0 = y(b)$.

$$|y(x) - z_0(x)| \leq K\varepsilon$$

Hence the differential equation $y''(x) + \beta(x)y(x) = 0$ has the super stability with boundary condition $y(a) = 0 = y(b)$

In (2014) the researchers J.Huang, Q.H.Aliqifiary, Y.Li established the super stability of the linear differential equations.

$$y''(x)+p(x)y'(x)+q(x)y(x)=0$$

With boundary conditions $y(a)=0=y(b)$

Where $y\in c^2\left[a,b\right], p\in c^1\left[a,b\right], q\in c^0\left[a,b\right], -\infty<a<b<+\infty$

Then the aim of this paper is to investigate the super stability of third-order linear differential homogeneous equations by extending the work of J.Huang, Q.H.Aliqifiary and Y.Li using the standard procedures of them.

Theorem 2. If $\max\left|q(x)-\frac{1}{2}p'(x)-\frac{p^2(x)}{4}\right|\le\frac{8}{(b-a)^2}$ ---------- (5)

Then the equation $y''(x)+p(x)y'(x)+q(x)y(x)=0$ has the super stability with boundary conditions $y(a)=0=y(b)$

Proof

Suppose that $y\in c^2\left[a,b\right]$ satisfies the inequality

$$\left|y''(x)+p(x)y'(x)+q(x)y(x)\right|\le\varepsilon \text{ ---------- (6) for some } \varepsilon>0$$

$$\text{Let } U(x)= y''(x)+p(x)y'(x)+q(x)y(x) \text{ ----------- (7)}$$

For all $x\in\left[a,b\right]$, and defined by $z(x)$ by

$$y(x)=z(x)e^{-\frac{1}{2}\int_a^x p(\tau)d(\tau)} \text{ -------(8)}$$

By substitution (8) in (7), we obtain

$$z''(x)+(q(x)-\frac{1}{2}p'(x)-\frac{1}{4}p^2(x))z(x)=U(x)e^{\frac{1}{2}\int_a^x p(\tau)d(\tau)}$$

Then it follows from (6) that

$$\left|z''(x)+\left(q(x)-\frac{1}{2}p'(x)-\frac{1}{4}p^2(x)\right)z(x)\right|=\left|U(x)\exp(\frac{1}{2}\int_a^x p(\tau)d(\tau)\right|\le\varepsilon e^{\frac{1}{2}\int_a^x p(\tau)d(\tau)}$$

From the boundary condition $y(a) = 0 = y(b)$ and (8) we have

$$z(a) = z(b) --- (9)$$

Define $\beta(x) = q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p^2$, then it follows from (5) and Lemma (4.2)

$$\max|z(x)| \le \frac{(b-a)^2}{8}\max|z''(x)|$$

$$\le \frac{(b-a)^2}{8}\left[\max|z''(x)+\beta(x)z(x)| + \max|\beta(x)|\max|z(x)|\right]$$

$$\le \frac{(b-a)^2}{8}\max \varepsilon e^{\frac{1}{2}p(\tau)d(\tau)} + \frac{(b-a)^2}{8}\max|\beta(x)|\max|z(x)|$$

Obviously, $\max e^{\frac{1}{2}\int_a^x p(\tau)d(\tau)} < \infty$ on $[a,b]$

Which implies that there exists a constant $K' > 0$ such that

$$|y(x)| = \left| z(x)e^{-\frac{1}{2}\int_a^x p(\tau)d(\tau)} \right|$$

$$\le \max\left\{\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right\}K\varepsilon$$

$$\le K'\varepsilon$$

Which implies $|y(x)| \le K'\varepsilon$ which fulfills the definition of super stability.

Thus $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ has super stability with the given boundary condition.

4.2 Main Results

Lemma(2) .Let $y \in c^3[a,b]$ and $y(a)=0=y(b)$, then

$$\max|y(x)| \le \frac{(b-a)^3}{48}\max|y'''(x)|$$

Proof

Let $M=\max\{|y(x)|: x\in[a,b]\}$ since $y(a)=0=y(b)$ there exists:

$x_0 \in (a,b)$ Such that $|y(x_0)|=M$

By Taylor's formula we have

$$y(a)=y(x_0)+y'(x_0)(x_0-a)+\frac{y''(x_0)}{2!}(x_0-a)^2+\frac{y'''(\delta)}{3!}(x_0-a)^3,$$

$$y(b)=y(x_0)+y'(x_0)(b-x_0)+\frac{y''(x_0)}{2!}(b-x_0)^2+\frac{y'''(\eta)}{3!}(b-x_0)^3,$$

$$\Rightarrow |y(a)| \le |y(x_0)|+|y'(x_0)|(x_0-a)+\left|\frac{y''(x_0)}{2!}\right|(x_0-a)^2+\left|\frac{y'''(\delta)}{3!}\right|(x_0-a)^3,$$

And $$\Rightarrow |y(b)| \le |y(x_0)|+|y'(x_0)|(b-x_0)+\left|\frac{y''(x_0)}{2!}\right|(b-x_0)^2+\left|\frac{y'''(\eta)}{3!}\right|(b-x_0)^3,$$

Where $\delta,\eta \in (a,b)$

Thus $|y'''(\delta)|=\dfrac{3!M}{(x_0-a)^3}=\dfrac{6M}{(x_0-a)^3}$

And $|y'''(\eta)|=\dfrac{3!M}{(b-x_0)^3}=\dfrac{6M}{(b-x_0)^3}$

For the case $x_0 \in \left(a,\dfrac{a+b}{2}\right]$ that is $a<x_0\le\dfrac{a+b}{2}$,we have

$$\frac{6M}{(x_0-a)^3} \ge \frac{6M}{\left(\frac{a+b}{2}-a\right)^3}=\frac{6M}{\frac{(b-a)^3}{8}}=\frac{48M}{(b-a)^3}$$

And for the case $x_0 \in \left[\frac{a+b}{2}, b\right)$,that is $\frac{a+b}{2} \le x_0 < b$ we have

$$\frac{6M}{(b-x_0)^3} \ge \frac{6M}{\left(\frac{b-a}{2}\right)^3} = \frac{6M}{\frac{(b-a)^3}{8}} = \frac{48M}{(b-a)^3}$$

$$\Rightarrow \frac{6M}{(b-x_0)^3} \ge \frac{48M}{(b-a)^3}$$

Thus $\max|y'''(x)| \ge \frac{48M}{(b-a)^3} = \frac{48}{(b-a)^3}\max|y(x)|$

from $\max|y'''(x)| \ge \frac{48}{(b-a)^3}\max|y(x)|$

$$\Rightarrow \max|y(x)| \le \frac{(b-a)^3}{48}\max|y'''(x)|$$

Theorem 3. Consider the differential equation

$$y'''(x) + \beta(x)y(x) = 0$$

with boundary conditions $y(a) = 0 = y(b)$ --------------- (*)

where $y \in c^3[a,b], \beta(x) \in c[a,b], -\infty < a < b < +\infty,$ if $\max|\beta(x)| < \frac{48}{(b-a)^3}$

Proof

Suppose that $y \in c^3[a,b]$ satisfies inequality $|y'''(x) + \beta(x)y| \le \varepsilon$ for some $\varepsilon > 0$

Let $M = \max\{|y(x)| : x \in [a,b]\}$, since $y(a) = 0 = y(b)$ there exists:
$x_0 \in (a,b)$ such that $|y(x_0)| = M$

Using Taylor formula, we have

$$y(a) = y(x_0) + y'(x_0)(x_0 - a) + y''(x_0)\frac{(x_0-a)^2}{2!} + y'''(\delta)\frac{(x_0-a)^3}{3!}$$

and $y(b) = y(x_0) + y'(x_0)(b-x_0) + \frac{y''(x_0)}{2!}(b-x_0)^2 + \frac{y'''(\eta)}{3!}(b-x_0)^3$

where $\delta, \eta \in (a,b)$

$$\text{thus } \left|y'''(\delta)\right| = \frac{3!M}{(x_0-a)^3} = \frac{6M}{(x_0-a)^3}$$

$$\text{and } \left|y'''(\eta)\right| = \frac{3!M}{(b-x_0)^3} = \frac{6M}{(b-x_0)^3}$$

from the existence of x_0 within the interval $a < x_0 < b$

Let $x_0 \in \left(a, \frac{a+b}{2}\right]$ that is $a < x_0 \le \frac{a+b}{2}$ then we have

$$\frac{6M}{(x_0-a)^3} \ge \frac{6M}{\left(\frac{b-a}{2}\right)^3} = \frac{6M}{\frac{(b-a)^3}{8}} = \frac{48M}{(b-a)^3}$$

And for the case $x_0 \in \left[\frac{a+b}{2}, b\right)$ that is $\frac{a+b}{2} \le x_0 < b$, then we have

$$\frac{6M}{(b-x_0)^3} \ge \frac{6M}{\frac{(b-a)^3}{8}} = \frac{48M}{(b-a)^3}$$

Then using **lemma (2)**

$$\max\left|y'''(x)\right| \ge \frac{48M}{(b-a)^3} = \frac{48}{(b-a)^3}\max\left|y(x)\right|$$

and there fore $\max\left|y(x)\right| \le \frac{(b-a)^3}{48}\max\left|y'''(x)\right|$

$$\le \frac{(b-a)^3}{48}\left[\max\left|y'''(x)+\beta(x)y(x)\right| + \max\left|\beta(x)\right|\max\left|y(x)\right|\right]$$

$$\le \frac{(b-a)^3}{48}\varepsilon + \frac{(b-a)^3}{48}\max\left|\beta(x)\right|\max\left|y(x)\right|$$

Let $k = \dfrac{(b-a)^3}{48\left(1-\frac{(b-a)^3}{48}\max\left|\beta(x)\right|\right)}$

Then $\left|y(x)\right| \le K\varepsilon$

Hence the differential equation $y'''(x)+\beta(x)y(x)=0$ has the super stability with boundary condition.

Now, we investigate the super stability of the following theorem:

Theorem 3. Consider $y'''(x)+m(x)y''(x)+p(x)y'(x)+q(x)y(x)=0$ ------------------ (10)

with boundary conditions $y(\mathrm{a})$ =0= $\mathrm{y}(\mathrm{b})$

where $y\in c^3[a,b], m\in c^2[a,b], p\in c'[a,b], q\in c^0[a,b] -\infty<a<b<+\infty$

$$\text{If}\quad \max\left|q(x)+\frac{1}{2}(p-p'')+\frac{1}{4}(pp'-2mp'+mp^2-2p^2)-\frac{1}{8}p^3\right|<\frac{48}{(b-a)^3}---(11)$$

Then (10) has the super stability with boundary conditions $y(a)=0=y(b)$

Proof

Suppose that $y\in c^3[a,b]$ satisfies the inequality:

$$\left|y'''(x)+m(x)y''(x)+p(x)y'(x)+q(x)y(x)\right|\le\varepsilon \text{ ------------ (12) for some } \varepsilon>0$$

Let $\mathrm{U}(\mathrm{x})=$ $y'''(x)+m(x)y''(x)+p(x)y'(x)q(x)y(x)$ --------- (13)

For all $x\in[a,b]$ and define $z(x)$ by

$$y(x)=z(x)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau} \text{ --------- (14)}$$

And by taking the first, second and third derivative of (14)

That is

$$y'(x)=\left(z'-\frac{1}{2}zp\right)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$y''(x) = \left(z'' - z'p - \frac{1}{2} zp' + \frac{1}{4} zp^2 \right) e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$y'''(x) = \left(z''' - \frac{1}{2} z'' p - z'' p - z' p' + \frac{1}{2} z'p^2 - \frac{1}{2} z'p' - \frac{1}{2} zp'' + \frac{1}{4} zpp' + \frac{1}{4} z'p^2 + \frac{1}{2} zp - \frac{1}{8} zp^3 \right) e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

And by substituting (14) and its first, second and third derivatives in (13) we get

$$u(x) = \Big[z''' - \frac{1}{2} z'' p - z'' p - z' p' + \frac{1}{2} z' p^2 - \frac{1}{2} z' p' - \frac{1}{2} zp'' + \frac{1}{4} zpp' + \frac{1}{4} z' p^2 + \frac{1}{2} pz - \frac{1}{8} zp^3 +$$

$$m\left(z'' - z' p - \frac{1}{2} zp' + \frac{1}{4} zp^2 \right) + p\left(z' - \frac{1}{2} zp \right) + zq \quad \Big] e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$= \Big[z''' - \frac{1}{2} z'' p - z'' p - z' p' + \frac{1}{2} z' p^2 - \frac{1}{2} z' p' - \frac{1}{2} zp'' + \frac{1}{4} zpp' + \frac{1}{4} z' p^2 + \frac{1}{2} zp - \frac{1}{8} zp^3 +$$

$$mz'' - z' mp - \frac{1}{2} zmp' + \frac{1}{4} zmp^2 + pz' - \frac{1}{2} zp^2 + qz \quad \Big] e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$= \Big[z''' + mz'' - \frac{3}{2} z'' p + z' p - z' mp - \frac{3}{2} z' p' + \frac{3}{4} z' p^2 + qz + \frac{1}{2} zp + \frac{1}{4} zpp' - \frac{1}{2} zmp' - \frac{1}{2} zp'' +$$

$$\frac{1}{4} zmp^2 - \frac{1}{2} zp^2 - \frac{1}{8} zp^3 \quad \Big] e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$= \Big[\quad z''' + \left(m - \frac{3}{2} p \right) z'' + \left(p - mp - \frac{3}{2} p' + \frac{3}{4} p^2 \right) z' +$$

$$\left(q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8} p^3 \right) z \quad \Big] e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

Now choose $m = \frac{3}{2} p$ and $m = \dfrac{\frac{3}{4} p^2 - \frac{3}{2} p' + p}{p}$

By doing this the coefficients of $z''(x)$ and $z'(x)$ will vanish.

To check whether the relation holds or not, for

$$\frac{3}{2}p=\frac{\frac{3}{4}p^2-\frac{3}{2}p'+p}{p}$$

$$\Rightarrow \frac{3}{2}p^2=\frac{3}{4}p^2-\frac{3}{2}p'+p$$

$$\Rightarrow \frac{3}{2}p'=p-\frac{3}{4}p^2$$

$$\Rightarrow \frac{3}{2}p'=p\left(1-\frac{3}{4}p\right)$$

$$\frac{3}{2}\frac{dp}{dx}=p\left(1-\frac{3}{4}p\right)$$

$$\Rightarrow \frac{dp}{p\left(1-\frac{3}{4}p\right)}=\frac{2}{3}dx$$ using partial fraction

Since $\frac{1}{p}+\frac{\frac{3}{4}}{1-\frac{3}{4}p}=\frac{1}{p\left(1-\frac{3}{4}p\right)}$

$$\left(\frac{1}{p}+\frac{\frac{3}{4}}{1-\frac{3}{4}p}\right)dp=\frac{2}{3}dx \quad \text{---------(*)}$$

By integrating both sides of (*) we have

$$\ln|p|-\ln\left|1-\frac{3}{4}p\right|=\frac{2}{3}x+c_1$$

$$\Rightarrow \ln\left|\frac{p}{1-\frac{3}{4}p}\right|=\frac{2}{3}x+c_1$$

$$\Rightarrow \frac{p}{1-\frac{3}{4}p}=Ce^{\frac{2}{3}x}$$

$$p = Ce^{\frac{2}{3}x}\left(1 - \frac{3}{4}p\right)$$

$$= Ce^{\frac{2}{3}x} - \frac{3}{4}Cpe^{\frac{2}{3}x}$$

$$p + \frac{3}{4}Cpe^{\frac{2}{3}x} = Ce^{\frac{2}{3}x} = p\left(1 + \frac{3}{4}Ce^{\frac{2}{3}x}\right)$$

$$p = \frac{Ce^{\frac{2}{3}x}}{1 + \frac{3}{4}Ce^{\frac{2}{3}x}} \qquad \Rightarrow \qquad m = \frac{3}{2}p$$

$$\text{m} = \frac{3}{2}\left(\frac{Ce^{\frac{2}{3}x}}{1 + \frac{3}{4}Ce^{\frac{2}{3}x}}\right)$$

Then $z''' + \left(q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3\right)ze^{-\frac{1}{2}\int_a^x p(\tau)d\tau} = u(x)$

Then from the inequality (12) we get

$$\left|z''' + \left(q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3\right)z\right| =$$

$$|u(x)|e^{\frac{1}{2}\int_a^x p(\tau)d\tau} \le \varepsilon e^{\frac{1}{2}\int_a^x p(\tau)d\tau}$$

From the boundary condition $y(a) = 0 = y(b)$ and $y(x) = z(x)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$

We have $z(a) = 0 = z(b)$

Define $\beta(x) = q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3$

Then $|z'''(x) + \beta z(x)| = |u(x)|e^{\frac{1}{2}\int_a^x p(\tau)d\tau} \le \varepsilon e^{\frac{1}{2}\int_a^x p(\tau)d\tau}$

Using **lemma (2)**

$$\max|z(x)| \le \frac{(b-a)^3}{48}\max|z'''(x)|$$

$$\le \frac{(b-a)^3}{48}\left[\max|z'''(x)+\beta z(x)|+\max|\beta|\max|z(x)|\right]$$

$$\le \frac{(b-a)^3}{48}\max\left\{e^{\frac{1}{2}\int_a^x p(\tau)d\tau}\right\}\varepsilon+\frac{(b-a)^3}{48}\max|\beta|\max|z(x)|$$

Since $\max\, e^{\frac{1}{2}\int_a^x p(\tau)d\tau} < \infty$ on the interval $[a,b]$

Hence, there exists a constant $K > 0$ such that

$$|z(x)| \le k\varepsilon \quad \text{For all} \quad x \in [a,b]$$

More over $\max e^{-\frac{1}{2}\int_a^x p(\tau)d\tau} < \infty$ on the interval $[a,b]$ which implies that there exists a constant such that $K' > 0$

$$|y(x)| = \left|z(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right|$$

$$\le \max\left\{\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right\}k\varepsilon \le k'\varepsilon$$

$$\Rightarrow |y(x)| \le k'\varepsilon$$

Then $y'''(x)+m(x)y''(x)+p(x)y'(x)+q(x)y(x)=0$ has super stability with boundary conditions

$y(a) = 0 = y(b)$

Example

Consider the differential equation below

$$y'''+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y''+\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'+y=0 \text{ ----------- } (15)$$

With boundary conditions $y(a)=0=y(b)$

Where $y\in c^3[a,b], \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\in c^2[a,b], \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\in c^1[a,b], 1\in c^0[a,b], \quad -\infty<a<b<+\infty$

If $\max\left|1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'+\right.$

$$\left.\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right|<\frac{48}{(b-a)^3} \text{ --------- } (16)$$

Then (15) has super stability with boundary conditions $y(a)=0=y(b)$

Suppose that $y\in c^3[a,b]$ satisfies the inequality

$$\left|y'''+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y''+\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'+y\right|\le\varepsilon \text{ ,for some } \varepsilon>0 \text{ --------- } (17)$$

Let $v(x)=y'''+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y''+\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'+y$ ------- (18)

For all $x \in [a,b]$ and define $z(x)$ by

$$y(x) = z(x) e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} d\tau} \quad \text{-----} \ (19)$$

$$y' = \left(z' - \frac{1}{2} z \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) \right) e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} d\tau}$$

$$y'' = \left(z'' - z' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) - \frac{1}{2} z \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)' + \frac{1}{4} \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)^2 \right) e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}}$$

$$y''' = \left(z''' - \frac{1}{2} z'' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) - z'' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) - z' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)' + \frac{1}{2} z' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)^2 - \right.$$

$$\frac{1}{2} z' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)' - \frac{1}{2} z \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)'' + \frac{1}{4} z \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) +$$

$$\left. \frac{1}{4} z' \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)^2 + \frac{1}{2} \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) z - \frac{1}{8} z \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right)^3 \right) e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} d\tau}$$

By substituting (19) and its first, second and third derivatives in (18) we get

$$\left[z''' + \left(\frac{3}{2} \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) - \frac{3}{2} \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} \right) \right) z'' + \right.$$

$$\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)-\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'+\frac{3}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2\right)z'$$

$$+\left(1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'\right.$$

$$\left.+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right)z\Big]e^{-\frac{1}{2}\int_a^x\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

Since z' and z''

$$v(x)=\Bigg[z'''+\left(1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'$$

$$+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\Big)z\Big]e^{-\frac{1}{2}\int_a^x\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

Then from inequality (17) we get

$$\Bigg|z'''+\left(1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'$$

$$+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{2}-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{2}-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{3}\;)z|$$

$$=|v(x)|\exp\left(\frac{1}{2}\int_{a}^{x}\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\leq\exp\left(\frac{1}{2}\int_{a}^{x}\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\varepsilon$$

From the boundary condition $y(a)=0=y(b)$ and

$$y(x)=z(x)\exp\left(-\frac{1}{2}\int_{a}^{x}\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\quad\text{we have}\quad z(a)=0=z(b)$$

Define:

$$\beta=\quad 1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'$$

$$+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{2}-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{2}-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{3}$$

Then $|z'''+\beta z(x)|=|v(x)|\exp\left(\frac{1}{2}\int_{a}^{x}\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\leq\exp\left(\frac{1}{2}\int_{a}^{x}\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\varepsilon$

using **lemma (2)**

$$\max|z(x)|\leq\frac{(b-a)^{3}}{48}\max|z'''(x)|$$

$$\leq \frac{(b-a)^3}{48}\left[\max\left|z''' + \beta z(x)\right| + \max|\beta|\max|z(x)|\right]$$

$$\leq \frac{(b-a)^3}{48}\max\left\{\exp\left(\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right\}\varepsilon + \frac{(b-a)^3}{48}\max|\beta|\max|z(x)|$$

since $\max\left\{\exp\left(\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right\} < \infty$ on the interval $[a,b]$

Hence there exists a constants $k > 0$ such that $|z(x)| \leq k\varepsilon$

$$|y(x)| = \left|z(x)\exp\left(-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right|$$

$$\leq \max\left\{\exp\left(-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right\}k\varepsilon \leq k^{'}\varepsilon$$

$$\Rightarrow |y(x)| \leq k^{'}\varepsilon$$

Then the differential equation $y''' + \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'' + \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y' + y = 0$

Has the super stability with boundary condition $y(a) = 0 = y(b)$ on closed bounded interval $[a,b]$

To show the

$$\max\left\{\left|1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'+\right.\right.$$

$$\left.\left.\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{2}-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{2}-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^{3}\right|\right\}<\frac{48}{(b-a)^{3}}------(**)$$

To make our work simple we simplify the expression (**)

Then the simplified form of (**) is

$$\max\left\{\left|1+\frac{\frac{5}{6}e^{\frac{2}{3}x}+\frac{15}{8}\left(e^{\frac{2}{3}x}\right)^{2}+\frac{35}{32}\left(e^{\frac{2}{3}x}\right)^{3}-\frac{33}{128}\left(e^{\frac{2}{3}x}\right)^{4}}{\left(1+\frac{3}{2}e^{\frac{2}{3}x}\right)^{4}}\right|\right\}<\frac{48}{(b-a)^{3}} \quad ---------- (***)$$

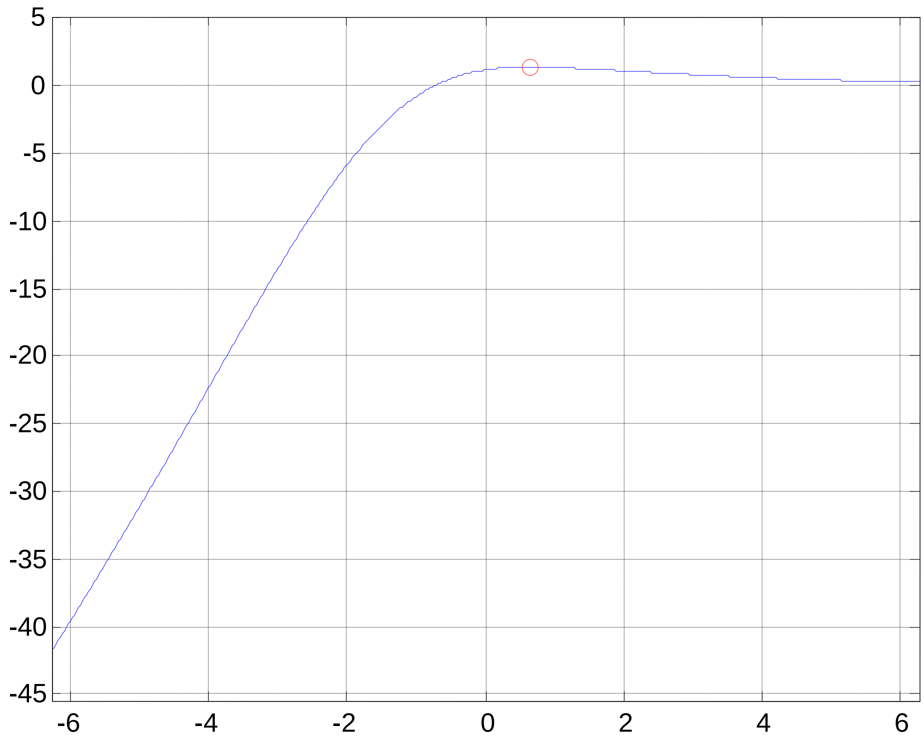

Figure :1 The graph which shows the

$$\max\left\{\left|1+\frac{\frac{5}{6}e^{\frac{2}{3}x}+\frac{15}{8}\left(e^{\frac{2}{3}x}\right)^{2}+\frac{35}{32}\left(e^{\frac{2}{3}x}\right)^{3}-\frac{33}{128}\left(e^{\frac{2}{3}x}\right)^{4}}{\left(1+\frac{3}{2}e^{\frac{2}{3}x}\right)^{4}}\right|\right\}<\frac{48}{(b-a)^{3}}$$

Chapter Five

Conclusion and Future Scopes

In this study, the super stability of third order linear ordinary differential homogeneous equation in the form of $y'''(x)+m(x)y''(x)+p(x)y'(x)+q(x)y(x)=0$ with boundary condition was established. And the standard work of JinghaoHuang, QusuayH.Alqifiary, Yongjin Li on investigating the super stability of second order linear ordinary differential homogeneous equation with boundary condition is extended to the super stability of third order linear ordinary differential homogeneous equation with Dirchilet boundary condition.

Our future scope is to work on super stability of non-linear ordinary differential equation

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