Analysis of photon Entanglement of Non degenerate Three-Level Laser coupled with thermal Reservoir

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DECLARATION

I hereby declare that this Msc. Thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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Abstract

In this thesis, we have studied the squeezing and statistical properties of the cavity light beams produced by a coherently driven non degenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir via a single-port mirror. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we have calculated the mean and variance of the photon number as well as the quadrature squeezing of the cavity light. We find the maximum quadrature squeezing to be the same in the presence as well as in the absence of spontaneous emission. The quadrature squeezing when $\gamma = 0$ is greater than when $\gamma = 0.2$ in the interval $0 < \Omega < 0.4545$ and the quadrature squeezing when $\gamma = 0$ is less than when $\gamma = 0.2$ for $\Omega > 0.4545$. And the quadrature squeezing when $\gamma = 0$ is greater than when $\gamma = 0.1$ in the interval $0 < \Omega < 0.4242$ and the maximum quadrature squeezing when $\gamma = 0$ is less than when $\gamma = 0.1$ for $\Omega > 0.4242$. Moreover, the plots in the same figure show that the quadrature squeezing when $\gamma = 0.1$ is greater than when $\gamma = 0.2$ in the interval $0 < \Omega < 0.5253$ and the quadrature squeezing when $\gamma = 0.1$ is less than when $\gamma = 0.2$ for $\Omega > 0.5253$. Furthermore, from the same plots the maximum squeezing is found to be 58.08% for $\gamma = 0.2$ (dashed curve), for $\gamma = 0.1$ (dotted curve), and for $\gamma = 0$ (solid curve) below the thermal-state level.

Contents

1

Introduction

Light has played a special role in our attempt to understand nature quantum mechanically. Squeezing is one of the nonclassical features of light that has attracted a great deal of interest [1-8]. In squeezed light the noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [7,8]. Squeezed light has potential applications in low-noise optical communications and weak signal detection [8-11].The squeezing does not exist in single modes but in the correlated state formed by the two modes. In general, as a result of the strong correlation between the modes, a two-mode squeezed state violates certain classical inequalities and hence can be applied in preparing Einstein- Podolsky-Rosen (EPR)-type entanglement [12]. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot described independently instead, a quantum state may be given for the system as a whole.

A three level laser may be defined as a quantum optical system in which three level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels, are injected into a cavity coupled to vacuum reservoir via a single-port mirror have been studied by different authors [13]. These atoms are removed from the cavity after some time. When three level atoms make a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, the quantum optical system is said to be a degenerate three level atom; otherwise it is called a non degenerate three level atom. The two photons are highly correlated and this correlation is responsible for the squeezing of light generated by a three level laser. In a cascade three-level atom the top, intermediate, and bottom levels are conveniently denoted by $|a\rangle$,$|b\rangle$, and $|c\rangle$. We hence realize that a non degenerate three-level laser is a two photon device in which squeezing properties are expected to occur due to the correlation between these two photons [14-15]. Some authors have studied the squeezing and statistical properties of the light produced by three-level laser in which the crucial role is played by the superposition of the top and bottom levels . Ansari [17] has predicted that such a laser can generate under certain conditions squeezed light. S . Tesfa has studied the squeezing and statistical properties of the light generated by a non-degenerate three-level laser coupled to squeezed vacuum reservoir [18]. Furthermore, Lu and Zhu have considered a non degenerate three level laser with the atoms initially prepared in coherent superposition of the top and bottom levels.The coherent superposition of the top and bottom levels of injected atoms shows that the quantum optical system can generate light in a squeezed state under certain conditions [19-21]. Moreover, Fesseha [22] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, is found to be 50% below the vacuum-state level. On the other hand, this study shows that the local quadrature squeezing is greater than the global quadrature squeezing. He has also found that a large part of the total mean photon number is confined in a relatively small frequency interval. In addition, Fesseha [22] has studied the squeezing and the statistical properties of the light produced by a degenerate three-level laser with the atoms in a closed cavity and pumped by coherent light. He has shown that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found with electron bombardment.

This Msc thesis, we wish to study the squeezing and statistical properties of the light generated by a coherently driven nondegenerate three-level laser with an open cavity coupled to a two-mode thermal reservoir via a single-port mirror. We carry out our calculation by putting the noise operators associated with the thermal reservoir in normal order. We thus first determine the master equation for a coherently driven nondegenerate three-level laser in an open cavity coupled to a two-mode thermal reservoir and the quantum Langevin equations for the cavity mode operators.Then, employing the master equation and the large-time approximation scheme, we obtain evolution of the expectation values of atomic operators. Moreover, we determine the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators. Then applying the resulting solutions, we calculate the photon statistics and the quadrature variances of the single-mode cavity light beams. Furthermore, applying the same solutions, we obtain the mean and variance of the two-mode cavity light. Finally, we determine the quadrature squeezing of the two-mode cavity light.

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Operator Dynamics

In this chapter we consider a nondegenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode thermal reservoir via a single-port mirror as shown in Fig. (2.1). We first obtain the master equation for a coherently driven nondegenerate three-level atom with the cavity modes and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation and the large-time approximation scheme, we drive the equations of evolution of the expectation values of the atomic oprators. Finally, we determine the steady-state solutions of the resulting equations of evolution. Here we carry out our calculation by putting the noise operators associated with the thermal reservoir in normal order.

2.1 Master equation

We consider here the case in which N nondegenerate three-level atoms in cascade configuration are available in an open cavity. We denote the top, intermediate, and bottom levels of the three-level atom by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively.As shown in Fig. (2.1)for nondegenerate cascade configuration, when the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and from levels $|b\rangle_k$ to $|c\rangle_k$ two photons with different frequencies are emit-

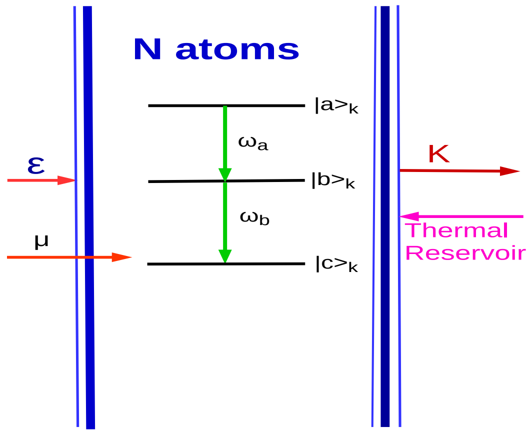

Figure 2.1: Schematic representation of a coherently driven nondegenerate three-level laser coupled to a two-mode thermal reservoir.

ted. The emission of light when the atoms makes the transition from the top level to the intermediate level is light mode a and the emission of light when the atoms makes the transition from the intermediate level to the bottom level is light mode b. We assume that the cavity mode a is at resonance with transition $|a\rangle_k \rightarrow |b\rangle_k$ and the cavity mode b is at resonance with the transition $|b\rangle_k \rightarrow |c\rangle_k$, with top and bottom levels of the three-level atom coupled by coherent light. The coupling of the top and bottom levels of a non degenerate three-level atom by coherent light can be described by the Hamiltonian

$$\hat{H}_3 = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big], \tag{2.1}$$

$$\hat{H}_1 = \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big], \tag{2.2}$$

$$\hat{H}_2 = i\varepsilon\big[\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}\big], \tag{2.3}$$

is lowering atomic operator and

$$\Omega = 2\mu\lambda. \tag{2.4}$$

Here μ, considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the driving coherent light and the three-level atom. In addition, the interaction of a three-level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$$

from which follows

$$\begin{aligned} \hat{H} &= ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big] \\ &+ \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big] + i\varepsilon\big[\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}\big] \end{aligned} \tag{2.5}$$

where $\hat{H}$ is the Hamiltonian of the system , $\hat{H}_3$ describes the interaction between atom and the cavity,$\hat{H}_1$ is the Hamiltonian of the coupling of the top and bottom levels of a non degenerate three-level atom by coherent light and $\hat{H}_2$ is the Hamiltonian of the cavity.

$$\hat{\sigma}_a^k = |b\rangle_{k\,k}\langle a|, \tag{2.6}$$

$$\hat{\sigma}_b^k = |c\rangle_{k\,k}\langle b|, \tag{2.7}$$

$$\hat{\sigma_c^k} = |c\rangle_{k\,k}\langle a|, \tag{2.8}$$

g is the coupling constant between the atom and cavity mode a or b, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b.

The quantum analysis of the interaction of a system such as a cavity mode or a three-level atom with the external environment is a relatively complex problem. The external environment, usually referred to as a reservoir, can be thermal light, ordinary or squeezed vacuum. We are interested in the dynamics of the system and this is describable by the master equation, the Fokker-Planck equation, or quantum Langevin equations. Here, we obtain the above set of dynamical equations for a cavity mode coupled to a thermal reservoir via a single-port mirror. The resulting equations are easily adaptable to the case when the external environment is either a thermal or a vacuum reservoir. We then focus our study when the cavity mode is couple to a thermal reservoir. A system coupled with a thermal reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3, \tag{2.9}$$

Where $\hat{H}$ is the Hamiltonian of the system and $\hat{H}_3$ describes the interaction between cavity and the atoms. Suppose $\hat{\chi}(t)$ is the density operator for the system and the reservoir. Then the equation of evolution of this density operator is given by

$$\frac{d}{dt}\hat{\chi}(t) = -i\big[\hat{H}_S(t) + \hat{H}_{SR}, \hat{\chi}(t)\big]. \tag{2.10}$$

We are interested in the quantum dynamics of the system alone. Hence taking into account (2.10), we see that the density operator for the system, also known as the the reduced density operator,

$$\hat{\rho}(t) = Tr_R\hat{\chi}(t) \tag{2.11}$$

evolves in time according to

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}(t), \hat{\rho}(t)\big] - iTr\big[\hat{H}_{ca}(t), \hat{\chi}(t)\big], \tag{2.12}$$

In which Tr_R indicates the trace over the reservoirs variables only. On the other hand, a formal solution of Eq. (2.10) can be written as

$$\hat{\chi}(t) = \hat{\chi}(0) - i\int_0^t \big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\chi}(t')\big]dt'. \tag{2.13}$$

In order to obtain mathematically manageable that $\hat{\chi}(t')$ by some approximately valid expression. Then, in the first place, we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. This can be achieved by letting a beam of thermal light (or light in a vacuum state) of constant intensity fall continuously on the system. Moreover, we decouple the system and reservoirs density operators, so that

$$\hat{\chi}(t') = \hat{\rho}(t')\hat{R}. \tag{2.14}$$

Therefore, with the aid of this, one can rewrite Eq. (2.13) as

$$\hat{\chi}(t') = \hat{\rho}(t')\hat{R} - \int_0^t \big[\hat{H}_S(t') + \hat{H}_{ca}(t'), \hat{\rho}(t')\hat{R}\big]dt'. \tag{2.15}$$

Now on substituting Eq .(2.15) in to Eq. (2.12) there follows

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i\big[\hat{H}_{SR}(t), \hat{\rho}(t)\big] - i\big[\langle \hat{H}_{SR}(t)\rangle_R, \hat{\rho}(0)\big]\\ &\quad - \int_0^t \big[\langle \hat{\rho}_{SR}(t)\rangle_R, \big[\hat{H}_S(t'), \hat{\rho}(t')\big]\big]dt'\\ &\quad - \int_0^t Tr_R\big[\hat{H}_{SR}(t'), \big[\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\big]\big]dt',\end{aligned} \tag{2.16}$$

where the subscript R indicates that the expectation value is to be calculated using the reservoirs density operator $\hat{R}$. Furthermore, the master equation for a system coupled to a reservoir takes the form

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -iTr_A[\hat{H}_S, \hat{\rho}_{AR}(t,t')] - h\langle \hat{H}_{SR}^2\hat{R}\rangle_R\hat{\rho}(t)\\ &\quad +2hTr_R(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}) - h\hat{\rho}(t)\langle \hat{H}_{SR}^2\hat{R}\rangle_R,\end{aligned} \tag{2.17}$$

A light mode confined in a cavity, usually formed by two mirrors, is called a cavity mode. A commonly used cavity has a single-port mirror. One side of each cavity is a mirror through which light can enter or leave the cavity. We now proceed to obtain the equation of evolution of the reduced density operator, in short the master equation, for the atoms coupled to a two-mode thermal reservoir via a single port-mirror. We consider the reservoirs to be composed of large number of submodes. Thus, the interaction Hamilitonian for N nondegenerate three-level atoms coupled to thermal reservoir is written as

$$\hat{H}_3 = i\lambda(\hat{\sigma}_a^{\dagger k}\hat{a}_{in} - \hat{a}_{in}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}_{in} - \hat{b}_{in}^{\dagger}\hat{\sigma}_b^k), \tag{2.18}$$

where λ is the coupling constant, $\hat{a}_{in}$ and $\hat{b}_{in}$ are the annihilation operators of the two-mode thermal reservoir. By employing Eq. (2.18), we then see that

$$hTr_R(\hat{H}_{SR}^2\hat{R}) = hTr_R\langle(i\lambda(\hat{\sigma}_a^{\dagger k}\hat{a}_{in} - \hat{a}_{in}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}_{in} - \hat{b}_{in}^{\dagger}\hat{\sigma}_b^k)^2\rangle. \tag{2.19}$$

This can be rewritten as

$$\begin{aligned} hTr_R(\hat{H}_{SR}^2\hat{R}) \;=\; & -h\lambda^2 Tr_R\big[(\hat{\sigma}_a^{\dagger k}\hat{a}_{in}\hat{\sigma}_a^{\dagger k}\hat{a}_{in})_R - (\hat{\sigma}_a^{\dagger k}\hat{a}_{in}\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k)_R + (\hat{\sigma}_a^{\dagger k}\hat{a}_{in}\hat{\sigma}_b^{\dagger k}\hat{b}_{in})_R \\ & -(\hat{\sigma}_a^{\dagger k}\hat{a}_{in}\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k)_R - (\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k\hat{a}^{\dagger k}\hat{a}_{in})_R + (\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k)_R - (\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k\hat{\sigma}_b^{\dagger k}\hat{b}_{in})_R \\ & +(\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k)_R + (\hat{\sigma}_b^{\dagger k}\hat{b}_{in}\hat{\sigma}_a^{\dagger k}\hat{a}_{in})_R - (\hat{\sigma}_b^{\dagger k}\hat{b}_{in}\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k)_R + (\hat{\sigma}_b^{\dagger k}\hat{b}_{in}\hat{\sigma}_b^{\dagger k}\hat{b}_{in})_R \\ & -(\hat{\sigma}_b^{\dagger k}\hat{b}_{in}\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k)_R - (\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k\hat{\sigma}_a^{\dagger k}\hat{a}_{in})_R + (\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k\hat{a}_{in}^{\dagger}\hat{\sigma}_a^k)_R - (\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k}\hat{b}_{in})_R \\ & +(\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k\hat{b}_{in}^{\dagger}\hat{\sigma}_b^k)_R\big]. \end{aligned} \tag{2.20}$$

The atomic operators with operators of the reservoir are commute to each other. Then we observe that

$$
\begin{aligned}
hTr_R(\hat{H}_{SR}^2\hat{R}) \;=\; & -h\lambda^2\big[\hat{\sigma}_a^{\dagger k2}\langle\hat{a}_{in}^2\rangle_R - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^{k}\langle\hat{a}_{in}\hat{a}_{in}^{\dagger}\rangle_R + \hat{\sigma}_a^{\dagger k}\hat{\sigma}_b^{\dagger k}\langle\hat{a}_{in}\hat{b}_{in}\rangle_R - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_b^{k}\langle\hat{a}_{in}\hat{b}_{in}^{\dagger}\rangle_R \\
& -\hat{\sigma}_a^{k}\hat{\sigma}_a^{\dagger k}\langle\hat{a}_{in}^{\dagger}\hat{a}_{in}\rangle_R + \hat{\sigma}_a^{\dagger k2}\langle\hat{a}_{in}^{\dagger 2}\rangle_R - \hat{\sigma}_a^{k}\hat{\sigma}_b^{\dagger k}\langle\hat{a}_{in}^{\dagger}\hat{b}_{in}\rangle_R + \hat{\sigma}_a^{k}\hat{\sigma}_a^{k}\langle\hat{a}_{in}^{\dagger}\hat{b}_{in}^{\dagger}\rangle_R \\
& +\hat{\sigma}_b^{\dagger k}\hat{\sigma}_a^{\dagger k}\langle\hat{b}_{in}\hat{a}_{in}\rangle_R - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_a^{k}\langle\hat{b}_{in}\hat{a}_{in}^{\dagger}\rangle_R + \hat{\sigma}_b^{\dagger k2}\langle\hat{b}_{in}^2\rangle_R - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\langle\hat{b}_{in}\hat{b}_{in}^{\dagger}\rangle_R \\
& -\hat{\sigma}_b^{k}\hat{\sigma}_a^{\dagger k}\langle\hat{b}_{in}^{\dagger}\hat{a}_{in}\rangle_R + \hat{\sigma}_b^{k}\hat{\sigma}_a^{k}\langle\hat{b}_{in}^{\dagger}\hat{a}_{in}^{\dagger}\rangle_R - \hat{\sigma}_b^{k}\hat{\sigma}_b^{\dagger k}\langle\hat{b}_{in}^{\dagger}\hat{b}_{in}\rangle_R + \hat{\sigma}_b^{\dagger k2}\langle\hat{b}_{in}^{\dagger 2}\rangle_R\Big]. \quad (2.21)
\end{aligned}
$$

Now using the density operator of the thermal reservoir

$$\hat{R} = \sum_{n=0}^{\infty}\frac{\bar{n}_{th}^n}{(1+\bar{n}_{th})^{n+1})}|n\rangle\langle n|, \quad (2.22)$$

one can easily check that

$$\langle\hat{a}_{in}^2\rangle_R = \sum_{n=0}^{\infty}\frac{\bar{n}_{th}^n}{(1+\bar{n}_{th})^{n+1}}Tr_R(|n\rangle\langle n|\hat{a}_{in}^2). \quad (2.23)$$

It then follows that

$$\langle\hat{a}_{in}^2\rangle_R = \sum_{n=0}^{\infty}\frac{\bar{n}_{th}^n}{(1+\bar{n}_{th})^{n+1}}\langle n|n-2\rangle = 0, \quad (2.24)$$

where $\langle n|n-2\rangle = 0$. Following the same procedure, we obtain

$$\langle\hat{a}_{in}^2\rangle = \langle\hat{b}_{in}^2\rangle = \langle\hat{a}_{in}^{\dagger 2}\rangle = \langle\hat{b}_{in}^{\dagger 2}\rangle = 0, \quad (2.25)$$

$$\langle\hat{a}_{in}^{\dagger}\hat{b}_{in}\rangle_R = \langle\hat{a}_{in}\hat{b}_{in}^{\dagger}\rangle_R = \langle\hat{b}_{in}^{\dagger}\hat{a}_{in}\rangle_R = \langle\hat{b}_{in}\hat{a}_{in}^{\dagger}\rangle_R = 0, \quad (2.26)$$

$$\langle\hat{a}_{in}\hat{b}_{in}\rangle_R = \langle\hat{b}_{in}\hat{a}_{in}\rangle_R = \langle\hat{b}_{in}^{\dagger}\hat{a}_{in}^{\dagger}\rangle_R = \langle\hat{a}_{in}^{\dagger}\hat{b}_{in}^{\dagger}\rangle_R = 0. \quad (2.27)$$

In addition, applying the commutation relation $[\hat{a}_{in},\hat{a}_{in}^{\dagger}] = 1$, we then note that

$$\langle\hat{a}_{in},\hat{a}_{in}^{\dagger}\rangle = \bar{n}_{th} + 1, \quad (2.28)$$

$$\langle\hat{a}_{in}^{\dagger},\hat{a}_{in}\rangle = \bar{n}_{th}, \quad (2.29)$$

where $\bar{n}_a = \bar{n}_b = \bar{n}_{th}$ is the mean photon number for the thermal reservoir. Hence on account of Eqs. (2.25), (2.26), (2.27), (2.28), and (2.29) into Eq. (2.20), there follows

$$hTr_R(\hat{H}_{SR}^2\hat{R})\hat{\rho}(t) = h\lambda^2[(\bar{n}_{th}+1)(\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho}) + \bar{n}_{th}(\hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\hat{\rho} + \hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k}\hat{\rho})]. \quad (2.30)$$

In the same manner, one can readily verify that

$$h\hat{\rho}(t)Tr_R(\hat{H}_{SR}^2\hat{R}) = h\lambda^2[(\bar{n}_{th}+1)(\hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k) + \bar{n}_{th}(\hat{\rho}\hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k} + \hat{\rho}\hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k})]. \quad (2.31)$$

In addition, one can readily find

$$\begin{aligned}
2hTr_R[\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}] &= -2h\lambda^2\big[\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\langle\hat{a}_{in}^2\rangle_R - \hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^k\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle_R + \hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_b^{\dagger k}\langle\hat{b}_{in}\hat{a}_{in}\rangle_R \\
&\quad -\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_b^k\langle\hat{b}_{in}^\dagger\hat{a}_{in}\rangle_R - \hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k}\langle\hat{a}_{in}\hat{a}_{in}^\dagger\rangle_R + \hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^k\langle\hat{a}_{in}^{\dagger 2}\rangle_R \\
&\quad -\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_b^{\dagger k}\langle\hat{b}_{in}\hat{a}_{in}^\dagger\rangle_R + \hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_b^k\langle\hat{b}_{in}^\dagger\hat{a}_{in}^\dagger\rangle_R + \hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_a^{\dagger k}\langle\hat{a}_{in}\hat{b}_{in}\rangle_R \\
&\quad -\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_a^k\langle\hat{a}_{in}^\dagger\hat{b}_{in}\rangle_R + \hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^{\dagger k}\langle\hat{b}_{in}^2\rangle_R - \hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^k\langle\hat{b}_{in}^\dagger\hat{b}_{in}\rangle_R \\
&\quad -\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_a^{\dagger k}\langle\hat{a}_{in}\hat{b}_{in}^\dagger\rangle_R + \hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_a^k\langle\hat{a}_{in}^\dagger\hat{b}_{in}^\dagger\rangle_R - \hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k}\langle\hat{b}_{in}\hat{b}_{in}^\dagger\rangle_R \\
&\quad +\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^k\langle\hat{b}_{in}^{\dagger 2}\rangle_R\big], \qquad (2.32)
\end{aligned}$$

So that applying Eqs. (2.25), (2.26), (2.27), (2.28), and (2.29) in Eq. (2.32) leads to

$$2hTr_R[\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}] = 2\lambda^2 h[\bar{n}_{th}(\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_a^k) + (\bar{n}_{th}+1)(\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k})]. \quad (2.33)$$

Taking into account Eq. (2.30), (2.31), and (2.33), the expression in Eq. (3.17) takes the form

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) &= -i\big[\hat{H},\hat{\rho}(t)\big] + \frac{\gamma}{2}(\bar{n}_{th}+1)\big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\big] \\
&\quad +\frac{\gamma}{2}\bar{n}_{th}\big[2\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^k - \hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\big] + \frac{\gamma}{2}\bar{n}_{th}\big[2\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^k - \hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k}\big] \\
&\quad +\frac{\gamma}{2}(\bar{n}_{th}+1)\big[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\big], \qquad (2.34)
\end{aligned}$$

where $\gamma_a = \gamma_b = \gamma = 2h\lambda^2$, considered to be the same for levels $|a\rangle$ and $|b\rangle$, is the spontaneous emission decay constant. In addition, a nondegenerate three-level atom in an open cavity is coupled to a two-mode thermal reservoir. Therefore, the master equation for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode thermal reservoir, with the aid of (2.17), is found to be

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= g\big[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\big] \\ &+\varepsilon\big[\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger - \hat{a}\hat{\rho} + \hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger - \hat{b}\hat{\rho} + \hat{\rho}\hat{b}\big] \\ &+\frac{\Omega}{2}\big[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\sigma}_c^k\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} + \hat{\rho}\hat{\sigma}_c^k\big] + \frac{\gamma}{2}\bar{n}_{th}\big[2\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^k - \hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^k\hat{\sigma}_a^{\dagger k}\big] \\ &+\frac{\gamma}{2}(\bar{n}_{th}+1)\big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\big] + \frac{\gamma}{2}\bar{n}_{th}\big[2\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^k - \hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^k\hat{\sigma}_b^{\dagger k}\big] \\ &+\frac{\gamma}{2}(\bar{n}_{th}+1)\big[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\big]. \end{aligned} \tag{2.35}$$

This is the master equation for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode thermal reservoir.

2.2 Quantum Langevin Equations

We recall that the laser cavity is coupled to a two-mode thermal reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the thermal reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [7,8]. We can therefore drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}] \tag{2.36}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}] \tag{2.37}$$

where κ is the cavity damping constant. Then in view of eq.(2.35),the quantum Langevin equations for cavity mode operators $\hat{a}$ and $\hat{b}$ turns out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + g\hat{\sigma}_a^k + \varepsilon, \tag{2.38}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + g\hat{\sigma}_b^k + \varepsilon, \tag{2.39}$$

2.3 Evolutions of the atomic operators

Here we start to derive the equations of evolution of the expectation values of the atomic oprators by applying the master equation and the large-time approximation scheme. Moreover, we find the steady-state solutions of the equations of evolution of the atomic oprators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{2.40}$$

Along with the master equation (2.35), one can readily establish that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle &= g[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle \\ &+\varepsilon[\langle\sigma_a^k\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle \\ &+\langle\hat{a}\hat{\sigma}_a^k\rangle - \langle\hat{\sigma}_a^k\hat{a}\rangle + \langle\hat{\sigma}_a^k\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{\sigma}_a^k\rangle + \langle\hat{b}\hat{\sigma}_a^k\rangle \\ &-\langle\hat{\sigma}_a^k\hat{b}\rangle] + \gamma[\frac{3}{2}\bar{n}_{th} + 1]\langle\hat{\sigma}_a^k\rangle\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle &= g[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle] \\ &-\frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle + \varepsilon[\langle\sigma_b^k\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\sigma}_b^k\rangle \\ &+\langle\hat{a}\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_b^k\hat{a}\rangle \\ &+\langle\hat{\sigma}_b^k\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{b}\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_b^k\hat{b}\rangle] + \gamma[\frac{3}{2}\bar{n}_{th} + \frac{1}{2}]\langle\hat{\sigma}_b^k\rangle,\end{aligned} \tag{2.42}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle &= g[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle] + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big] \\ &-\gamma[\bar{n}_{th} + \frac{1}{2})]\langle\hat{\sigma}_c^k\rangle + \varepsilon[\langle\hat{\sigma}_c^k\hat{a}^\dagger\rangle, -\langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle + \langle\hat{a}\hat{\sigma}_c^k\rangle - \langle\hat{\sigma}_c^k\hat{a}\rangle \\ &+\langle\hat{\sigma}_c^k\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle + \langle\hat{b}\hat{\sigma}_c^k\rangle - \langle\hat{\sigma}_c^k\hat{b}\rangle],\end{aligned} \tag{2.43}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_a^k\rangle &= g[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle \\ &+\langle\hat{\sigma}_c^{\dagger k}\rangle\big] - \gamma[\bar{n}_{th} + 1]\langle\hat{\eta}_a^k\rangle \\ &+\varepsilon[\langle\hat{\eta}_a^k\hat{a}^\dagger\rangle, -\langle\hat{a}^\dagger\hat{\eta}_a^k\rangle + \langle\hat{a}\hat{\eta}_a^k\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle \\ &+\langle\hat{\eta}_a^k\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{\eta}_a^k\rangle + \langle\hat{b}\hat{\eta}_a^k\rangle - \langle\hat{\eta}_a^k\hat{b}\rangle],\end{aligned} \tag{2.44}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b^k\rangle &= g[\langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] \\ &+\gamma(2\bar{n}_{th} + 1)[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle] + \varepsilon[\langle\hat{\eta}_b^k\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\eta}_b^k\rangle \\ &+\langle\hat{a}\hat{\eta}_b^k\rangle - \langle\hat{\eta}_b^k\hat{a}\rangle + \langle\hat{\eta}_b^k\hat{b}^\dagger\rangle \\ &-\langle\hat{b}^\dagger\hat{\eta}_b^k\rangle + \langle\hat{b}\hat{\eta}_b^k\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle],\end{aligned} \tag{2.45}$$

Where

$$\hat{\eta}_a^k = a\rangle_{k\ k}\langle a|,, \tag{2.46}$$

$$\hat{\eta}_b^k = |b\rangle_{k\ k}\langle b|,, \tag{2.47}$$

$$\hat{\eta}_c^k = |c\rangle_{k\ k}\langle c|. \tag{2.48}$$

We see that Eqs. (2.41)-(2.45) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [7,8]. Therefore, employing this approximation scheme, we get from Eqs. (2.38) and (2.39) the approximately valid relations

$$\hat{a} = \frac{2g\sigma_a^\kappa}{\kappa} + \frac{2\varepsilon}{\kappa} \quad , \tag{2.49}$$

$$\hat{b} = \frac{2g\sigma_b^\kappa}{\kappa} + \frac{2\varepsilon}{\kappa} \tag{2.50}$$

Evidently, these turn out to be exact relations at steady-state. Now introducing Eqs. (2.49) and (2.50) into Eqs. (2.41) - (2.45), the equations of evolution of the atomic operators take the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -(\gamma+\gamma_c)(\frac{3}{2}\bar{n}_{th}+1)\langle\hat{\sigma}_a^k\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{2.51}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{1}{2}(\gamma+\gamma_c)(3\bar{n}_{th}+1)\langle\hat{\sigma}_b^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{2.52}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{1}{2}(\gamma+\gamma_c)(2\bar{n}_{th}+1)\langle\hat{\sigma}_c^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{2.53}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -(\gamma+\gamma_c)(\bar{n}_{th}+1)\langle\hat{\eta}_a^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.54}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -(\gamma+\gamma_c)(2\bar{n}_{th}+1)[\langle\hat{\eta}_b^k\rangle - \langle\hat{\eta}_a^k\rangle], \tag{2.55}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.56}$$

Is the stimulated emission decay constant.

We next sum Eqs. (2.51) -(2.55) over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -(\gamma+\gamma_c)(\frac{3}{2}\bar{n}_{th}+1)\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^\dagger\rangle, \tag{2.57}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{1}{2}(\gamma+\gamma_c)(3\bar{n}_{th}+1)\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^\dagger\rangle, \tag{2.58}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{1}{2}(\gamma+\gamma_c)(2\bar{n}_{th}+1)\langle\hat{m}_c\rangle + \frac{\Omega}{2}\big[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\big], \tag{2.59}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma+\gamma_c)(\bar{n}_{th}+1)\langle\hat{N}_a\rangle + \frac{\Omega}{2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^\dagger\rangle\big], \tag{2.60}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -(\gamma+\gamma_c)(2\bar{n}_{th}+1)[\langle\hat{N}_b\rangle - \langle\hat{N}_a\rangle], \tag{2.61}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N} \hat{\sigma}_a^k, \tag{2.62}$$

$$\hat{m}_b = \sum_{k=1}^{N} \hat{\sigma}_b^k, \tag{2.63}$$

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k, \tag{2.64}$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \tag{2.65}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{2.66}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k, \tag{2.67}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels, respectively. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{2.68}$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.69}$$

Furthermore, using the definition,

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{2.70}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|, \tag{2.71}$$

following the same procedure, one can also easily establish that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{2.72}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{2.73}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{2.74}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.75}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{2.76}$$

Using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b, \tag{2.77}$$

and taking into account Eqs. (2.71)-(2.77)

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{2.78}$$

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{2.79}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{2.80}$$

In the presence of N three-level atoms, we rewrite Eqs. (2.38) and (2.39)

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda\hat{m}_a + \varepsilon, \tag{2.81}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \beta\hat{m}_b + \varepsilon, \tag{2.82}$$

In which λ and β are constants whose values remain to be fixed. We note that the steady-state solutions of Eqs.(2.49) and (2.50) are

$$\hat{a} = \frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2\varepsilon}{\kappa} \tag{2.83}$$

$$\hat{b} = \frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2\varepsilon}{\kappa}. \tag{2.84}$$

Now employing Eqs. (2.83) and (2.84), the commutation relations for the cavity mode operators are found to be

$$[\hat{a},\hat{a}^{\dagger}]_k = \frac{\gamma_c}{\kappa}\big[\hat{\eta}_b^k - \hat{\eta}_a^k\big], \tag{2.85}$$

$$[\hat{b},\hat{b}^{\dagger}]_k = \frac{\gamma_c}{\kappa}\big[\hat{\eta}_c^k - \hat{\eta}_b^k\big], \tag{2.86}$$

and on summing over all atoms, we have

$$[\hat{a},\hat{a}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_a\big], \tag{2.87}$$

$$[\hat{b},\hat{b}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_b\big], \tag{2.88}$$

where

$$[\hat{a},\hat{a}^{\dagger}] = \sum_{k=1}^{N}[\hat{a},\hat{a}^{\dagger}]_k, \tag{2.89}$$

$$[\hat{b},\hat{b}^{\dagger}] = \sum_{k=1}^{N}[\hat{b},\hat{b}^{\dagger}]_k. \tag{2.90}$$

We note that Eqs. (2.89) and Eqs. (2.90) stand for the commutators $\hat{a}$ and $\hat{a}^{\dagger}$, and for $\hat{b}$ and $\hat{b}^{\dagger}$ when the light modes a and b are interacting with all the N three-level atoms. On the other hand, using the steady-state solutions of Eqs. (2.81) and (2.82), one can easily verify that

$$[\hat{a},\hat{a}^{\dagger}] = N\left(\frac{2\lambda}{\kappa}\right)^2\left(\hat{N}_b - \hat{N}_a\right) \tag{2.91}$$

and

$$[\hat{b},\hat{b}^{\dagger}] = N\left(\frac{2\beta}{\kappa}\right)^2\left(\hat{N}_c - \hat{N}_b\right). \tag{2.92}$$

Thus on account of Eqs. (2.87) and, we see that

$$\lambda = \pm\frac{g}{\sqrt{N}}. \tag{2.93}$$

Similarly, inspection of Eqs. (2.88) and shows that

$$\beta = \pm \frac{g}{\sqrt{N}}. \tag{2.94}$$

Hence in view of these two results, the equations of evolution of the light modes a and b operators given by Eqs. (2.83) and (2.84) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \varepsilon \tag{2.95}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \varepsilon. \tag{2.96}$$

Now adding Eqs. (2.83) and (2.84) as well as Eqs. (2.77),(2.78) and (2.80), we get

$$[\hat{c}, \hat{c}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_a\big], \tag{2.97}$$

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m} + 2\varepsilon, \tag{2.98}$$

in which

$$\hat{c} = \hat{a} + \hat{b}. \tag{2.99}$$

We next proceed to obtain the expectation value of the cavity mode operators. One can rewrite Eq. (2.57) and the adjoint of (3.56) as

$$\frac{d}{dt}\langle\hat{m}_a(t)\rangle = -(\gamma + \gamma_c)(\frac{3}{2}\bar{n}_{th} + 1)\langle\hat{m}_a(t)\rangle + \frac{\Omega}{2}\langle\hat{m}_b^\dagger(t)\rangle \tag{2.100}$$

and

$$\frac{d}{dt}\langle\hat{m}_b^\dagger(t)\rangle = -\frac{\Omega}{2}\langle\hat{m}_a(t)\rangle - \frac{1}{2}(\gamma + \gamma_c(3\bar{n}_{th} + 1)\langle\hat{m}_b^\dagger(t)\rangle. \tag{2.101}$$

To solve the coupled differential equations (2.100) and (2.101), we write the single-matrix equation

$$\frac{d}{dt}\begin{pmatrix} \langle\hat{m}_a(t)\rangle \\ \langle\hat{m}_b^\dagger(t)\rangle \end{pmatrix} = M \begin{pmatrix} \langle\hat{m}_a(t)\rangle \\ \langle\hat{m}_b^\dagger(t)\rangle \end{pmatrix}, \tag{2.102}$$

with

$$M = \begin{pmatrix} -(\gamma+\gamma_c)(\frac{3}{2}\bar{n}_{th}+1) & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & -\frac{1}{2}(\gamma+\gamma_c)(3\bar{n}_{th}+1) \end{pmatrix}. \tag{2.103}$$

In order to solve Eq. (2.103), we need the eigenvalues and eigenvectors of M such that

$$MV_i = \lambda_i V_i, \tag{2.104}$$

with $i = 1, 2$, and the eigenvectors

$$V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \tag{2.105}$$

subject to the normalization condition

$$x_i^2 + y_i^2 = 1. \tag{2.106}$$

The eigenvalue equation (2.103) has nontrivial solution, provided that

$$det(M - \lambda I) = 0, \tag{2.107}$$

so that applying Eq. (2.107), the eigenvalues are found to be

$$\lambda_1 = -\frac{3}{4}(\gamma+\gamma_c)(2\bar{n}_{th}+1) + \frac{1}{2}p \tag{2.108}$$

and

$$\lambda_2 = -\frac{3}{4}(\gamma+\gamma_c)(2\bar{n}_{th}+1) - \frac{1}{2}p, \tag{2.109}$$

where

$$p = \sqrt{\frac{1}{4}(\gamma+\gamma_c)^2 - \Omega^2}. \tag{2.110}$$

We next seek to obtain the eigenvectors of M. To this end, the eigenvector corresponding to λ_1 is expressible as

$$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.111}$$

Then employing Eqs. (2.103) and (2.104), we write the matrix equation

$$M = \begin{pmatrix} -(\gamma + \gamma_c)(\frac{3}{2}\bar{n}_{th} + 1) & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & -\frac{1}{2}(\gamma + \gamma_c)(3\bar{n}_{th} + 1) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{2.112}$$

Taking into account this equation and the normalization condition

$$x_1^2 + y_1^2 = 1, \tag{2.113}$$

we get

$$V_1 = \frac{1}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}} \begin{pmatrix} \frac{\Omega}{2} \\ \lambda_1 + \gamma + \gamma_c \end{pmatrix}. \tag{2.114}$$

The eigenvector corresponding to λ_2 can also be established following a similar procedure that

$$V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}} \begin{pmatrix} \frac{\Omega}{2} \\ \lambda_2 + \gamma + \gamma_c \end{pmatrix}. \tag{2.115}$$

Finally, we construct a matrix V consisting of the eigenvectors of the matrix M as column matrices

$$V = \begin{pmatrix} \frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4}+(\lambda_1+\gamma+\gamma_c)^2}} & \frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4}+(\lambda_2+\gamma+\gamma_c)^2}} \\ \frac{\lambda_1+\gamma+\gamma_c}{\sqrt{\frac{\Omega^2}{4}+(\lambda_1+\gamma+\gamma_c)^2}} & \frac{\lambda_2+\gamma+\gamma_c}{\sqrt{\frac{\Omega^2}{4}+(\lambda_2+\gamma+\gamma_c)^2}} \end{pmatrix}. \tag{2.116}$$

We next proceed to determine the inverse of the matrix V . To this end, it can be readily verified that the characteristic equation

$$det(V - \lambda I) = 0 \tag{2.117}$$

Has explicit form

$$\lambda^2 - \left[\frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}} + \frac{\lambda_2 + \gamma + \gamma_c}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}}\right]\lambda$$
$$- \frac{\frac{\Omega}{2}(\lambda_1 - \lambda_2)}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}} I = 0. \tag{2.118}$$

Thus applying the Cayley-Hamilton theorem that a matrix satisfies its own characteristic equation, we have

$$V^2 - \left[\frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}} + \frac{\lambda_2 + \gamma + \gamma_c}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}}\right]V$$
$$- \frac{\frac{\Omega}{2}(\lambda_1 - \lambda_2)}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}} I = 0. \tag{2.119}$$

In view of this, we obtain

$$V^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \frac{-(\lambda_2+\gamma+\gamma_c)}{\frac{\Omega}{2}}\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2} & \sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2} \\ \frac{(\lambda_1+\gamma+\gamma_c)}{\frac{\Omega}{2}}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2} & -\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2} \end{pmatrix}. \tag{2.120}$$

Using the fact that $VV^{-1} = I$, Eq. (2.116) can be rewritten as

$$\frac{d}{dt}\langle\hat{U}(t)\rangle = VV^{-1}MVV^{-1}\langle\hat{U}(t)\rangle, \tag{2.121}$$

in which

$$\langle\hat{U}(t)\rangle = \begin{pmatrix} \langle\hat{m}_a(t)\rangle \\ \langle\hat{m}_b^\dagger(t)\rangle \end{pmatrix}. \tag{2.122}$$

Multiplying Eq. (2.122) by V^{-1} from the left, we get

$$\frac{d}{dt}(V^{-1}\langle\hat{U}(t)\rangle) = DV^{-1}\langle\hat{U}(t)\rangle, \tag{2.123}$$

where

$$D = V^{-1}MV = \begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}, \tag{2.124}$$

in which $\beta = \gamma + \gamma_c$. The formal solution of Eq. (2.124) can be written as

$$V^{-1}\langle\hat{U}(t)\rangle = e^{Dt}V^{-1}\langle\hat{U}(0)\rangle, \tag{2.125}$$

from which follows

$$\langle\hat{U}(t)\rangle = Ve^{Dt}V^{-1}\langle\hat{U}(0)\rangle. \tag{2.126}$$

In view of the fact that D is diagonal, we have

$$e^{Dt} = \begin{pmatrix} e^{-\beta t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix}. \tag{2.127}$$

Therefore, on account of Eq. (2.126) along with (2.122), and (2.127) we obtain

$$Ve^{Dt}V^{-1}\langle\hat{U}(0)\rangle = \begin{bmatrix} e^{-\beta t}\langle\hat{m}_a(0)\rangle & 0 \\ 0 & e^{-\beta t}\langle\hat{m}_b^\dagger(0)\rangle \end{bmatrix}, \tag{2.128}$$

In view of Eqs. (2.122) and (2.127) along with (2.128), we see that

$$\begin{pmatrix} \langle\hat{m}_a(t)\rangle \\ \langle\hat{m}_b^\dagger(t)\rangle \end{pmatrix} = \begin{bmatrix} \langle\hat{m}_a(0)\rangle e^{-\beta t} & 0 \\ 0 & \langle\hat{m}_b^\dagger(0)\rangle e^{-\beta t} \end{bmatrix}. \tag{2.129}$$

It then follows that

$$\langle\hat{m}_a(t)\rangle = \langle\hat{m}_a(0)\rangle e^{-\beta t} \tag{2.130}$$

and

$$\langle \hat{m}_b^\dagger(t)\rangle = \langle \hat{m}_b^\dagger(0)\rangle e^{-\beta t}. \tag{2.131}$$

Furthermore, the adjoint of Eq. (2.131) can be written as

$$\langle \hat{m}_b(t)\rangle = \langle \hat{m}_b(0)\rangle e^{-\beta t}. \tag{2.132}$$

With the atoms considered to be initially in the bottom level, Eqs. (2.130) and (2.131) reduce to

$$\langle \hat{m}_a(t)\rangle = 0 \tag{2.133}$$

and

$$\langle \hat{m}_b(t)\rangle = 0. \tag{2.134}$$

The expectation value of the solution of Eq. (2.38) is expressible as

$$\langle \hat{a}(t)\rangle = \langle \hat{a}(0)\rangle e^{-\kappa t/2} + \frac{g}{\sqrt{N}}\int_0^t e^{\kappa t'/2}\langle \hat{m}_a(t')\rangle dt' + 2\varepsilon\int_0^t e^{\kappa t'/2} dt' \tag{2.135}$$

With the help of Eq. (2.133) and the assumption that the cavity light is initially in a vacuum state, Eq. (2.135) turns out to be

$$\langle \hat{a}(t)\rangle = \frac{2\varepsilon}{\kappa}, \tag{2.136}$$

Following a similar procedure, one can readily obtain the expectation value of the solution of Eq. (2.39) to be

$$\langle \hat{b}(t)\rangle = \frac{2\varepsilon}{\kappa}, \tag{2.137}$$

Now with the aid of Eqs. (2.136) and (2.137) togather with (2.99), we have

$$\langle \hat{c}(t) \rangle = \frac{4\varepsilon}{\kappa} \tag{2.138}$$

Finally, we seek to determine the steady-state solutions of the expectation values of the atomic operators. We note that the steady-state solutions of Eqs. (2.59) and (2.60)are given by

$$\langle \hat{m}_c \rangle = \left(\frac{\Omega}{\gamma + \gamma_c(2\bar{n}_{th} + 1})) \right) \left[\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle \right], \tag{2.139}$$

$$\langle \hat{N}_a \rangle = \frac{1}{2} \left(\frac{\Omega}{\gamma + \gamma_c(\bar{n}_{th} + 1)} \right) \left[\langle \hat{m}_c \rangle + \langle \hat{m}_c^\dagger \rangle \right], \tag{2.140}$$

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle. \tag{2.141}$$

Furthermore, with the help of Eq. (2.69) togather with (2.141), we see that

$$\langle \hat{N}_c \rangle = N - 2\langle \hat{N}_a \rangle. \tag{2.142}$$

With the aid of Eq. (2.142) and Eq. (2.139) can be written as

$$\langle \hat{m}_c \rangle = \left(\frac{\Omega}{\gamma + \gamma_c(2\bar{n}_{th} + 1)} \right) \left[N - 3\langle \hat{N}_a \rangle \right] \tag{2.143}$$

and in view of Eq. (2.60), we observe that

$$\langle \hat{m}_c \rangle = \langle \hat{m}_c^\dagger \rangle. \tag{2.144}$$

Now taking into consideration this result, Eq. (2.140) can be put in the form

$$\langle \hat{N}_a \rangle = \left(\frac{\Omega}{\gamma + \gamma_c(\bar{n}_{th} + 1)} \right) \langle \hat{m}_c \rangle. \tag{2.145}$$

Using Eqs. (2.140) and (2.143), one readily gets

$$\langle \hat{N}_a \rangle = \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2(\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N. \tag{2.146}$$

Substitution of Eq. (2.58) into Eqs. (2.59), (2.60), and (2.66) results in

$$\langle \hat{N}_b \rangle = \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 (\bar{n}_{th} 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N, \tag{2.147}$$

$$\langle \hat{N}_c \rangle = \left[\frac{(\gamma_c + \gamma)^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + \Omega^2}{(\gamma_c + \gamma)^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N, \tag{2.148}$$

$$\langle \hat{m}_c \rangle = \left[\frac{\Omega(\gamma_c + \gamma)(\bar{n}_{th} + 1)}{(\gamma_c + \gamma)^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N. \tag{2.149}$$

These equations represent the steady-state solutions of the equations of evolution of the atomic operators. Furthermore, upon setting $\gamma = 0$, for the case in which spontaneous emission is absent, the steady-state solutions described by Eqs. (2.146)-(2.149) take the form

$$\langle \hat{N}_a \rangle = \left[\frac{\Omega^2}{\gamma_c^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N, \tag{2.150}$$

$$\langle \hat{N}_b \rangle = \left[\frac{\Omega^2}{\gamma_c^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N, \tag{2.151}$$

$$\langle \hat{N}_c \rangle = \left[\frac{\gamma_c^2 (\bar{n}_{th} 1)(2\bar{n}_{th} + 1) + \Omega^2}{\gamma_c^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2} \right] N, \tag{2.152}$$

$$\langle \hat{m}_c \rangle = \left[\frac{\Omega \gamma_c (\bar{n}_{th} + 1)}{\gamma_c^2 (\bar{n}_{th} + 1)(2\bar{n}_{th} 1) + 3\Omega^2} \right] N. \tag{2.153}$$

The results described by Eqs. (2.146)-(2.149) are exactly the same as those obtained In addition, we note that for$\Omega >> \gamma_c$ Eqs. (2.150)-(2.153) reduce to

$$\langle \hat{N}_b \rangle = \frac{1}{3} N, \tag{2.154}$$

$$\langle \hat{N}_b \rangle = \frac{1}{3} N, \tag{2.155}$$

$$\langle \hat{N}_c \rangle = \frac{1}{3} N, \tag{2.156}$$

$$\langle \hat{m}_c \rangle = 0 \tag{2.157}$$

3

Photon statistics

In this chapter we proceed to study the statistical properties of the light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir via a single-port mirror. Applying the solutions of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we obtain the global photon statistics for light modes a and b. olso, we determine the global photon statistics of the two-mode cavity light.

3.1 Single-mode photon statistics

In this section we find the global mean and variance of the photon numbers for light modes a and b.

3.2 Global mean photon number

Here we start to calculate the global mean photon numbers of light modes a and b produced by the coherently driven non degenerate three level laser with an open cavity and coupled to a two mode thermal reservoir.

3.2.1 Global photon number of light mode a

we know start to find the mean photon number of light mode a in the entire frequency interval.The mean photon number of light mode a,represented by the operators $\hat{a}$ and $\hat{a}^\dagger$, is defined by

$$\overline{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle \tag{3.1}$$

we note that the steady-state solution of Eq.(2.95),

$$\hat{a} = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a + \frac{2\varepsilon}{\kappa}, \tag{3.2}$$

$$\hat{a}^\dagger = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a^\dagger + \frac{2\varepsilon}{\kappa} \tag{3.3}$$

so that introducing eq(3.2) and eq.(3.3) into (3.1), we see that

$$\overline{n}_a = [\frac{2g}{\kappa\sqrt{N}}\hat{m}_a^\dagger + \frac{2\varepsilon}{\kappa}][\frac{2g}{\kappa\sqrt{N}}\hat{m}_a + \frac{2\varepsilon}{\kappa}], \tag{3.4}$$

$$\overline{n}_a = \frac{4g^2}{\kappa^2 N}\langle \hat{m}_a^\dagger \hat{m}_a \rangle + \frac{4\varepsilon^2}{\kappa^2} \tag{3.5}$$

with the help of eq.(2.71), Eq.(3.4) can be expressed as

$$\overline{n}_a = \frac{\gamma_c}{\kappa}[\frac{\Omega^2}{(\gamma + \gamma_c)^2(2\bar{n}_{th} + 1)(\bar{n}_{th} + 1) + 3\Omega^2}]N + \frac{4\varepsilon^2}{\kappa^2} \tag{3.6}$$

in view of Eq.(3.6),there follows

$$\overline{n}_a = \frac{\gamma_c N}{\kappa}[\frac{\Omega^2}{(\gamma + \gamma_c)^2(2\bar{n}_{th} + 1)(\bar{n}_{th} + 1) + 3\Omega^2}] + \frac{4\varepsilon^2}{\kappa^2} \tag{3.7}$$

This is the steady-state mean photon number of light mode a. moreover, we consider the case in which spontaneous emission is absent(γ=0).Then the mean photon number of light mode a for this case has the form

$$\overline{n}_a = \frac{\gamma_c}{\kappa}[\frac{\Omega^2}{(\gamma_c)^2(2\bar{n}_{th} + 1)(\bar{n}_{th} + 1) + 3\Omega^2}]N + \frac{4\varepsilon^2}{\kappa^2} \tag{3.8}$$

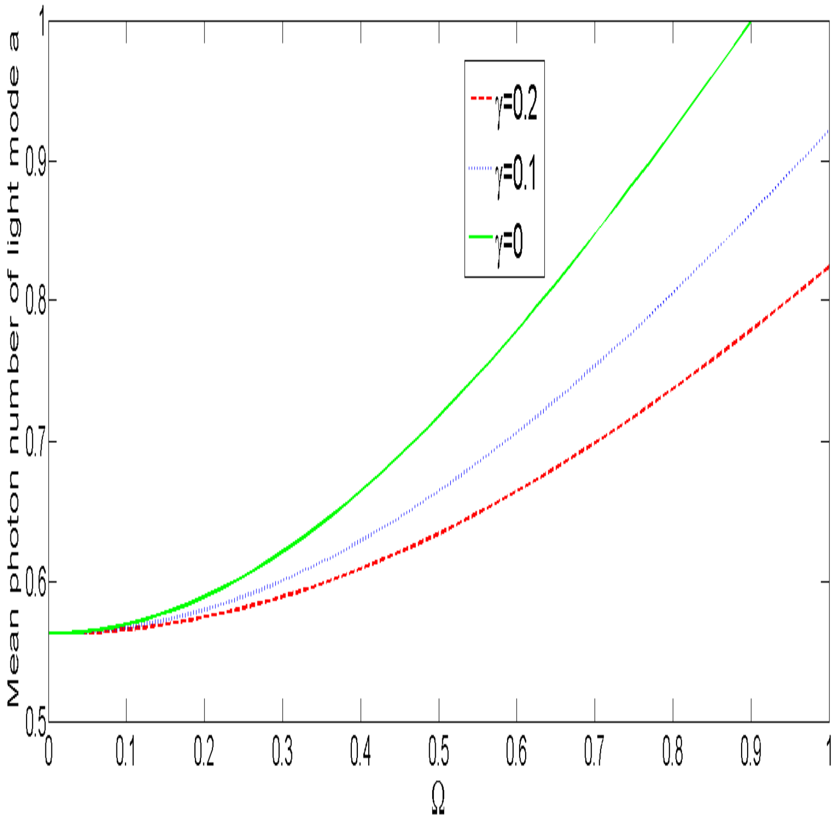

Figure 3.1: The plots of the mean photon number of light mode a [Eq. (3.7)] versus Ω for $\gamma_c = 0.4$,$\kappa = 0.8$, $N = 50$,$\bar{n}_{th}$=5, and for different values of γ.

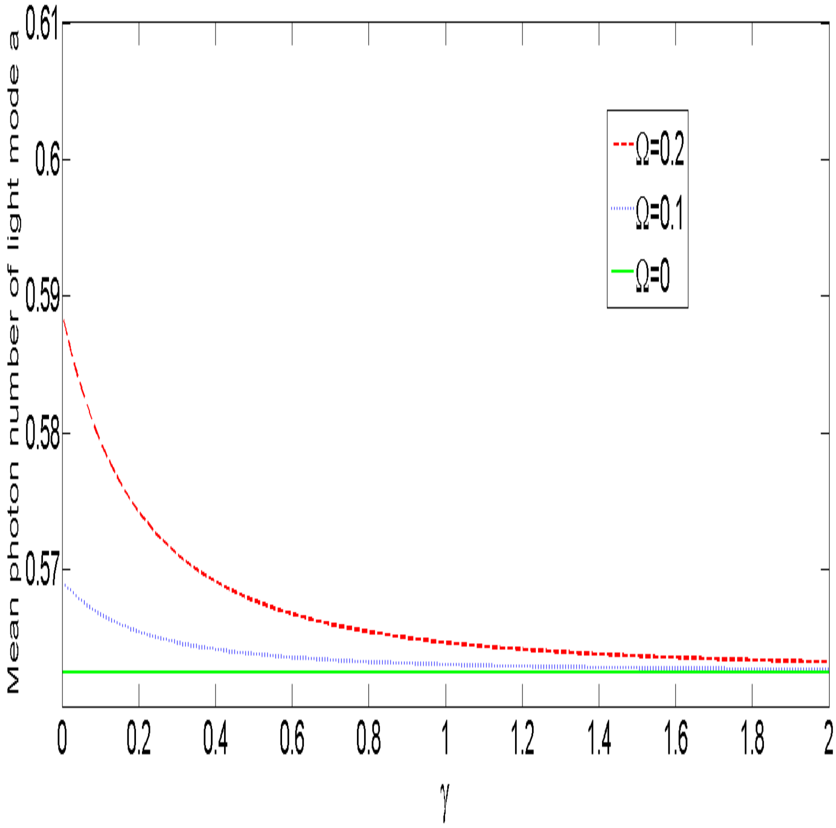

Figure 3.2: The plots of the mean photon number of light mode a [Eq. (3.7)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $N = 50$, $\bar{n}_{th}$=5, and for different values of Ω

.

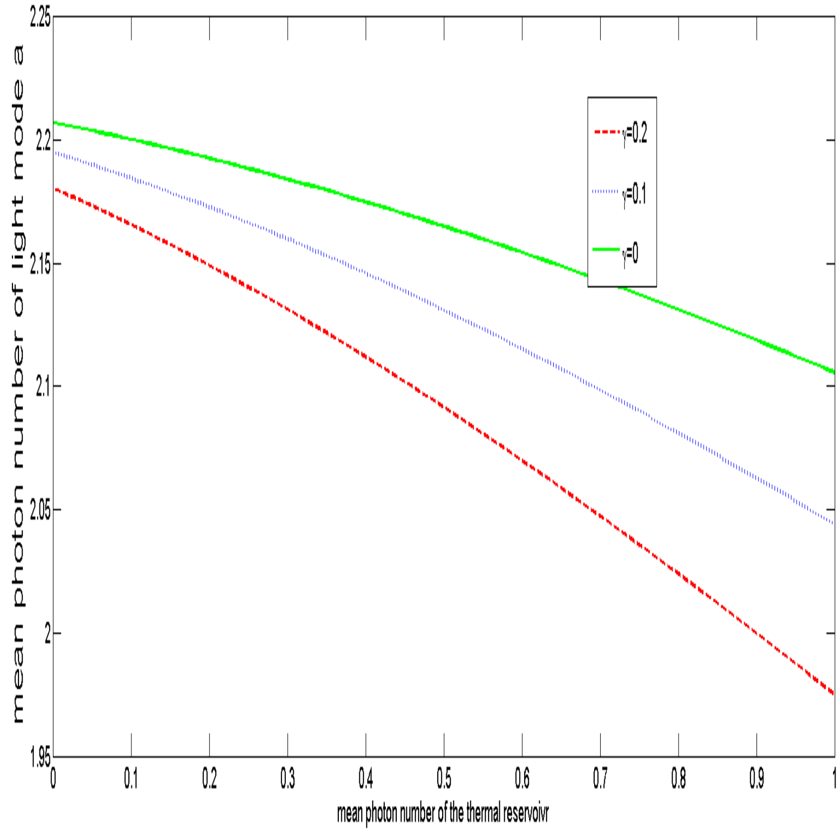

Figure 3.3: The plots of the mean photon number of light mode a [Eq. (3.7)] versus mean photon number for the thermal reservoir for γ_c = 0.4,κ = 0.8, $N = 50$,Ω=2, and for different values of γ

.

The plots in Fig. (3.1) show that the steady-state mean photon number of light mode a in the absence of spontaneous emission when $(\gamma = 0)$ is greater than in the presence of spontaneous emission (when $\gamma \neq 0$). Moreover, the mean photon number of light mode a decreases when γ increases.

3.2.2 Global photon number of light mode b

we know proceed to obtain the mean photon number of light mode b in the entire frequency interval.the mean photon number of light mode b,represented by the operators $\hat{b}$ and $\hat{a}^\dagger$, is defined by

$$\overline{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle \tag{3.9}$$

we note that the steady-state solution of Eq.(2.96)

$$\hat{b} = \frac{2g}{\kappa\sqrt{N}}\hat{m}_b + \frac{2\varepsilon}{\kappa} \tag{3.10}$$

$$\hat{b}^\dagger = \frac{2g}{\kappa\sqrt{N}}\hat{m}_b^\dagger + \frac{2\varepsilon}{\kappa} \tag{3.11}$$

so that introducing eq(3.10) and eq.(3.11) into (3.9), we see that

$$\overline{n}_b = [\frac{2g}{\kappa\sqrt{N}}\hat{m}_b^\dagger + \frac{2\varepsilon}{\kappa}][\frac{2g}{\kappa\sqrt{N}}\hat{m}_b + \frac{2\varepsilon}{\kappa}], \tag{3.12}$$

$$\overline{n}_b = \frac{4g^2}{\kappa^2 N}\langle \hat{m}_b^\dagger \hat{m}_b \rangle + \frac{4\varepsilon^2}{\kappa^2} \tag{3.13}$$

with the help of eq.(2.72), Eq.(3.13) can be expressed as

$$\overline{n}_b = \frac{\gamma_c}{\kappa}[\frac{\Omega^2}{(\gamma+\gamma_c)^2(2\bar{n}_{th}+1)(\bar{n}_{th}+1)+3\Omega^2}]N + \frac{4\varepsilon^2}{\kappa^2} \tag{3.14}$$

in view of Eq.(3.14),there follows

$$\overline{n}_b = \frac{\gamma_c N}{\kappa}[\frac{\Omega^2}{(\gamma+\gamma_c)^2(2\bar{n}_{th}+1)(\bar{n}_{th}+1)+3\Omega^2}] + \frac{4\varepsilon^2}{\kappa^2} \tag{3.15}$$

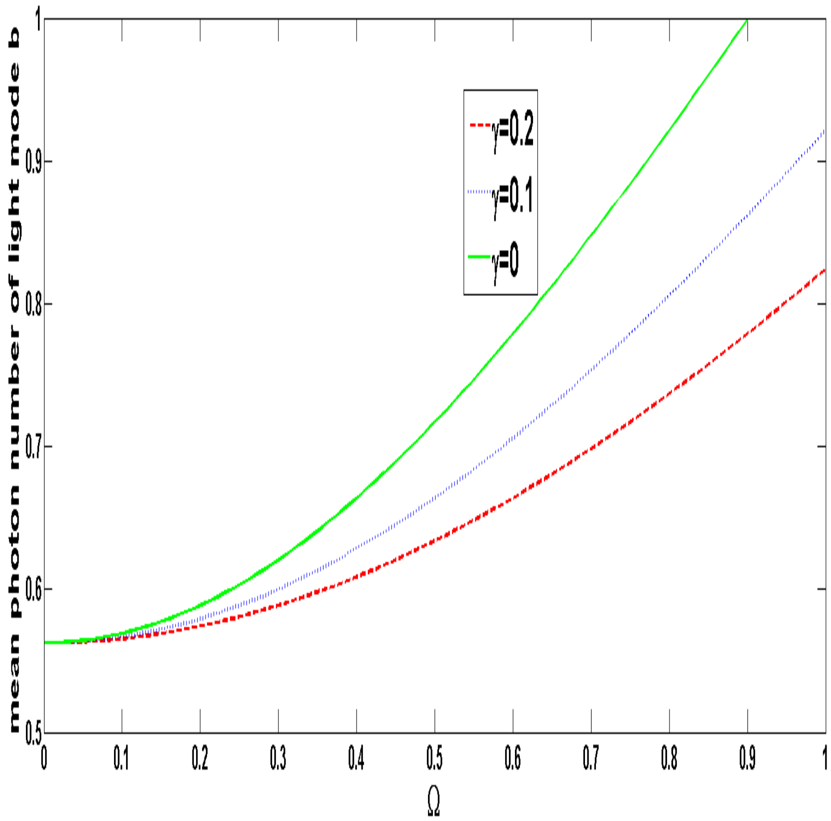

Figure 3.4: plots of the mean photon number of light mode b [Eq. (3.15)] versus Ω for γ_c = 0.4,κ = 0.8, $N = 50$,$\bar{n}_{th} = 5$, and for different values of γ

.

this is the steady-state mean photon number of light mode b. moreover, we consider the case in which spontaneous emission is absent(γ=0).Then the mean photon number of light mode b for this case has the form

$$\overline{n}_b = \frac{\gamma_c}{\kappa}[\frac{\Omega^2}{(\gamma_c)^2(2\bar{n}_{th}+1)(\bar{n}_{th}+1)+3\Omega^2}]N + \frac{4\varepsilon^2}{\kappa^2} \tag{3.16}$$

The plots in Fig. (3.4) show that the steady-state mean photon number of light mode b in the absence of spontaneous emission (when $\gamma = 0$) is greater than in the presence of spontaneous emission (when $\gamma \neq 0$). Moreover, the mean photon number of light mode b decreases when γ increases.

3.3 Local mean photon number

Here we seek to determine the local mean photon numbers of of light modes a and b,produced by the coherently driven nondegenerate three level laser with open cavity and coupled to a two mode thermal reservoir.

3.3.1 Local mean photon number of light mode a

we now proceed to obtain the mean photon number of light mode a in a given frequency interval.To determine the local mean photon number of light mode a, we need to consider the power spectrum of light mode a.the power spectrum of light mode a with central frequency ω_0 is expressible as [22]

$$p_a(\omega) = \frac{1}{\pi} R_e \int_0^{\infty} d\tau e^{i(\omega-\omega_o)\tau} \langle \hat{a}^\dagger(t)\hat{a}(\tau+t)\rangle_{ss} \tag{3.17}$$

upon integrating both sides of Eq (3.17) over ω,we readily get

$$\int_{-\infty}^{\infty} P_a(\omega) d\omega = \overline{n}_a \tag{3.18}$$

In which $\overline{n}_a$ is the steady- state mean photon number of light mode a.From this result,we observe that $P_a(\omega)d\omega$ is the steady- state mean photon number of light mode a in the frequency interval between ω and ω+$d\omega$ we now proceed to determine the total correlation function that appears in Eq.(3.17). To this end, we realize that the solution of Eq.(2.71)

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}} e^{-\kappa\frac{\tau}{2}} \int_0^{\infty} e^{\kappa\frac{\tau'}{2}} \hat{m}_a(t+\tau')d\tau' + \varepsilon \int_0^{\infty} e^{\kappa\frac{\tau'}{2}} d\tau' \tag{3.19}$$

on the basis of Eq.(2.76)and Eq.(2.77), we see that

$$\frac{d}{dt}\hat{m}_a(t) = -(\gamma+\gamma_c)(\frac{3}{2}\overline{n}+1)m_a(t) + \frac{\Omega}{2}\hat{m}_b^\dagger(t) + \hat{F}_a(t) \tag{3.20}$$

and

$$\frac{d}{dt}\hat{m}_b^\dagger(t) = -\frac{\Omega}{2}\hat{m}_a(t) - \frac{1}{2}(\gamma+\gamma_c(3\overline{n}+1)\hat{m}_b^\dagger(t) + \hat{F}_b(t). \tag{3.21}$$

where $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise force operators of the two light mode. to solve the coupled differential equations (3.21),we write the single matrix equation

$$\frac{d}{dt}\hat{U}(t) = M\hat{U}(t) + \hat{F}(t), \tag{3.22}$$

in which

$$\hat{U}(t) = \begin{pmatrix} \hat{m}_a(t) \\ \hat{m}_b^\dagger(t) \end{pmatrix}, \tag{3.23}$$

$$\hat{F}(t) = \begin{pmatrix} \hat{F}_a(t) \\ \hat{F}_b^\dagger(t) \end{pmatrix} \tag{3.24}$$

and M is given Eq.(2.79).using the fact that VV^{-1}=I, Eq.(3.22)can be rewritten as

$$\frac{d}{dt}\hat{U}(t) = VV^{-1}MVV^{-1}\hat{U}(t) + \hat{F}(t) \tag{3.25}$$

multiplying this equation by V^{-1} from the left, we get

$$\frac{d}{dt}(V^{-1}\hat{U}(t) = DV^{-1}\hat{U}(t) + V^{-1}\hat{F}(t) \tag{3.26}$$

where D is given by Eq.(2.100).Now the formal solution of Eq.(3.26) can be rewritten as

$$V^{-1}\hat{U}(t+\tau) = e^{D\tau}V^{-1}U(t) + \int_0^\tau e^{(\tau-\tau')}\hat{F}(t+\tau')d\tau' \tag{3.27}$$

from which follows

$$\hat{U}(t+\tau) = Ve^{D\tau}V^{-1}U(t) + \int_0^{\tau} Ve^{(\tau-\tau')}\hat{F}(t+\tau')d\tau' \tag{3.28}$$

in view of the fact that D is diagonal, we have

$$e^{D\tau} = \begin{pmatrix} e^{\lambda_1\tau} & 0 \\ 0 & e^{\lambda_2\tau} \end{pmatrix}. \tag{3.29}$$

and

$$e^{D\tau-\tau'} = \begin{pmatrix} e^{\lambda_1\tau-\tau'} & 0 \\ 0 & e^{\lambda_2\tau-\tau'} \end{pmatrix}. \tag{3.30}$$

therefore, on account of Eq.(3.28)along with (3.23),(3.24),(2.92), (2.96), (3.28), and (3.29), we obtain

$$Ve^{D\tau}V^{-1}\hat{U}(\tau) = \begin{bmatrix} p_1(\tau)\hat{m}_a(\tau) & p_1(\tau)\hat{m}_b^{\dagger}(\tau) \\ p_2(\tau)\hat{m}_a(\tau) & p_2(\tau)\hat{m}_b^{\dagger}(\tau) \end{bmatrix}. \tag{3.31}$$

$$\int_0^{\tau-\tau'} V^{-1}\hat{F}(t+\tau')d\tau' = \begin{bmatrix} \int_0^{\tau} p_1(\tau-\tau')\hat{F}_a(t+\tau')d\tau' & \int_0^{\tau} q_1(\tau-\tau')\hat{F}_b^{\dagger}(t+\tau')d\tau' \\ \int_0^{\tau} q_2(\tau-\tau')\hat{F}_a(t+\tau')d\tau' & \int_0^{\tau} p_2(\tau-\tau')\hat{F}_b^{\dagger}(t+\tau')d\tau' \end{bmatrix} \tag{3.32}$$

where

$$p_1(\tau) = \frac{\lambda_1 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_2 \tau} - \frac{\lambda_2 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_1 \tau}, \tag{3.33}$$

$$p_2(\tau) = \frac{\lambda_1 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_1 \tau} - \frac{\lambda_2 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_2 \tau}, \tag{3.34}$$

$$q_1(\tau) = \frac{\frac{\Omega}{2}(e^{\lambda_1 \tau} - e^{\lambda_2 \tau})}{\lambda_1 - \lambda_2}, \tag{3.35}$$

$$q2(\tau) = -\frac{\frac{\Omega}{2}(e^{\lambda_1 \tau} - e^{\gamma_2 \tau})}{\lambda_1 - \lambda_2}, \tag{3.36}$$

$$p_1(\tau - \tau') = \frac{\lambda_1 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_2 (\tau - \tau')} - \frac{\lambda_2 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_1 (\tau - \tau')} \tag{3.37}$$

$$p_2(\tau - \tau') = \frac{\lambda_1 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_1 (\tau - \tau')} - \frac{\lambda_2 + \gamma + \gamma_c}{\lambda_1 - \lambda_2} e^{\lambda_2 (\tau - \tau')}, \tag{3.38}$$

$$q_1(\tau - \tau') = \frac{\frac{\Omega}{2}(e^{\lambda_1 (\tau - \tau')} - e^{\gamma_2 (\tau - \tau')})}{\lambda_1 - \lambda_2}, \tag{3.39}$$

$$q_2(\tau - \tau') = \frac{-\frac{\Omega}{2}(e^{\lambda_1 (\tau - \tau')} - e^{\gamma_2 (\tau - \tau')})}{\lambda_1 - \lambda_2} \tag{3.40}$$

finally, application of Eqs. (3.31),(3.32) into (3.28) results in

$$\hat{m}_a(t + \tau) = p_1(\tau)\hat{m}_a(t) + q_1(\tau)\hat{m}_b^\dagger(t) + G_1(t + \tau) \tag{3.41}$$

and

$$\hat{m}_b^\dagger(t + \tau) = p_2(\tau)\hat{m}_b^\dagger(t) + q_2(\tau)\hat{m}_a(t) + G_2(t + \tau) \tag{3.42}$$

Where

$$G_1(t + \tau) = \int_0^\tau [p_2(\tau - \tau')\hat{F}_b^\dagger(t + \tau') + q_1(\tau - \tau')\hat{F}_a(t + \tau')]d\tau' \tag{3.43}$$

and

$$G_2(t + \tau) = \int_0^\tau [p_1(\tau - \tau')\hat{F}_a(t + \tau') + q_1(\tau - \tau')\hat{F}_b^\dagger(t + \tau')]d\tau' \tag{3.44}$$

we now proceed to determine the two-time correlation function that appears in Eq.(3.17).to this end,we realize that the solution of Eq.(2.71)can be written as

$$\hat{a}(t + \tau) = \hat{a}(t)e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}} e^{-\kappa\frac{\tau}{2}} \int_0^\infty e^{\kappa\frac{\tau'}{2}} \hat{m}_a(t + \tau')d\tau' + \varepsilon \int_0^\infty e^{\kappa\frac{\tau'}{2}} d\tau' \tag{3.45}$$

so that on introducing Eq.(3.41)into Eq.(3.45),there follows

$$\begin{aligned}
\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}\frac{\hat{m}_a(t)(\lambda_1+\gamma+\gamma_c)}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}] \\
&-\frac{g}{\sqrt{N}}\Big[\frac{\hat{m}_a(t)}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{(\lambda_2+\gamma+\gamma_c)}{\lambda_1-\lambda_2}\Big]\big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\big] \\
&+\frac{g}{\sqrt{N}}\Big[\frac{\hat{m}_b^\dagger(t)}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\big] \\
&-\frac{g}{\sqrt{N}}\Big[\frac{\hat{m}_b^\dagger(t)}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\big] \\
&+\frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\Big\{\int_0^\tau d\tau'\int_0^\tau\Big[\Big(\frac{\lambda_1+\gamma+\gamma_c}{\lambda_1-\lambda_2}\Big)e^{(\kappa\frac{\tau'}{2})\tau'-\lambda_2\tau''} \\
&-\Big(\frac{\lambda_1+\gamma+\gamma_c}{\lambda_1-\lambda_2}\Big)e^{(\kappa\frac{\tau'}{2}+\lambda_1)\tau'-\lambda_1\tau''}\Big]\hat{F}_a(t+\tau'')d\tau''\Big\} \\
&+\frac{g}{\sqrt{N}}\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\int_0^\tau d\tau'\int_0^{\tau'}\Big[e^{(\kappa\frac{\tau'}{2}+\lambda_1)\tau'-\lambda_1\tau''}-e^{(\kappa\frac{\tau'}{2}+\lambda_2)\tau'-\lambda_2\tau''}\Big] \\
&\times\hat{F}_b^\dagger(t+\tau'')d\tau''
\end{aligned} \tag{3.46}$$

Now multiplying on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned}
\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}\frac{\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle(\lambda_1+\gamma+\gamma_c)}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}] \\
&-\frac{g}{\sqrt{N}}\Big[\frac{\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{(\lambda_2+\gamma+\gamma_c)}{\lambda_1-\lambda_2}\Big]\big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\big] \\
&+\frac{g}{\sqrt{N}}\Big[\frac{\langle\hat{a}^\dagger(t)\hat{m}_b^\dagger(t)\rangle}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\big] \\
&-\frac{g}{\sqrt{N}}\Big[\frac{\langle\hat{a}^\dagger(t)\hat{m}_b^\dagger(t)\rangle}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\big] \\
&+\frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\Big\{\int_0^\tau d\tau'\int_0^\tau\Big[\Big(\frac{\lambda_1+\gamma+\gamma_c}{\lambda_1-\lambda_2}\Big)e^{(\kappa\frac{\tau'}{2})\tau'-\lambda_2\tau''} \\
&-\Big(\frac{\lambda_1+\gamma+\gamma_c}{\lambda_1-\lambda_2}\Big)e^{(\kappa\frac{\tau'}{2}+\lambda_1)\tau'-\lambda_1\tau''}\Big]\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle d\tau''\Big\} \\
&+\frac{g}{\sqrt{N}}\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\int_0^\tau d\tau'\int_0^{\tau'}\Big[e^{(\kappa\frac{\tau'}{2}+\lambda_1)\tau'-\lambda_1\tau''}-e^{(\kappa\frac{\tau'}{2}+\lambda_2)\tau'-\lambda_2\tau''}\Big] \\
&\times\langle\hat{a}^\dagger(t)\hat{F}_b^\dagger(t+\tau'')\rangle d\tau''
\end{aligned} \tag{3.47}$$

since a noise operator at a certain time should not affect a light mode operator at an earlier time , we note that

$$\langle\hat{a}^{\dagger}(t)\hat{F}_a^{\dagger}(t+\tau'')\rangle = \langle\hat{a}^{\dagger}(t)\hat{F}_b^{\dagger}(t+\tau'')\rangle = 0 \tag{3.48}$$

It then follows that

$$\begin{aligned}\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle &= \langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}\frac{\langle\hat{a}^{\dagger}(t)\hat{m}_a(t)\rangle(\lambda_1+\gamma+\gamma_c)}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2}[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}] \\ &-\frac{g}{\sqrt{N}}\Big[\frac{\langle\hat{a}^{\dagger}(t)\hat{m}_a(t)\rangle}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{(\lambda_2+\gamma+\gamma_c)}{\lambda_1-\lambda_2}\Big]\Big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\Big] \\ &+\frac{g}{\sqrt{N}}\Big[\frac{\langle\hat{a}^{\dagger}(t)\hat{m}_b^{\dagger}(t)\rangle}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\Big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\Big] \\ &-\frac{g}{\sqrt{N}}\Big[\frac{\langle\hat{a}^{\dagger}(t)\hat{m}_b^{\dagger}(t)\rangle}{\lambda_1+\frac{\kappa}{2}}\Big]\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\Big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\Big]\end{aligned} \tag{3.49}$$

applying the large- time approximation scheme,one get from Eq.(2.95)

$$\hat{a}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a(t) + \frac{2\varepsilon}{\kappa} \tag{3.50}$$

so that in view of this result, we get

$$\hat{m}_a(t) = \frac{\kappa\sqrt{N}}{2g}(\hat{a}(t) - \frac{2\varepsilon}{\kappa}) \tag{3.51}$$

similarly,applying the large- time approximation scheme,one get from Eq.(2.96)

$$\hat{b}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_b(t) + \frac{2\varepsilon}{\kappa} \tag{3.52}$$

and

$$\hat{b}^{\dagger}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_b^{\dagger}(t) + \frac{2\varepsilon}{\kappa}, \tag{3.53}$$

$$\hat{m}_b^{\dagger}(t) = \frac{\kappa\sqrt{N}}{2g}(\hat{b}^{\dagger}(t) - \frac{2\varepsilon}{\kappa}) \tag{3.54}$$

on account of Eqs(3.51)and (3.54), we get

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle[Re^{-\frac{\lambda\tau}{2}} + Se^{\lambda_2\tau} - Te^{\lambda_1\tau}]$$
$$\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle[Ae^{-\frac{\lambda\tau}{2}} - Be^{\lambda_2\tau} + Ce^{\lambda_1\tau}] \quad (3.55)$$

in which

$$R = \frac{\frac{\kappa}{2}(\frac{\kappa}{2} + \lambda_1 + \lambda_2 + \gamma + \gamma_c)}{(\kappa 2 + \lambda_1)(\kappa 2 + \lambda_2)}, \quad (3.56)$$

$$S = \frac{\frac{\kappa}{2}(\lambda_1 + \gamma + \gamma_c)}{(\frac{\kappa}{2} + \lambda_1)(\lambda_1 - \lambda_2)}, \quad (3.57)$$

$$T = \frac{\frac{\kappa}{2}(\lambda_2 + \gamma + \gamma_c)}{(\frac{\kappa}{2} + \lambda_1)(\lambda_1 - \lambda_2)}, \quad (3.58)$$

$$A = \frac{(\frac{\kappa}{2})(\frac{\Omega}{2})}{(\frac{\kappa}{2} + \lambda_1)(\frac{\kappa}{2} + \lambda_2)}, \quad (3.59)$$

$$B = \frac{(\frac{\kappa}{2})(\frac{\Omega}{2})}{(\frac{\kappa}{2} + \lambda_2)(\lambda_1 - \lambda_2)}, \quad (3.60)$$

$$C = \frac{(\frac{\kappa}{2})(\frac{\Omega}{2})}{(\frac{\kappa}{2} + \lambda_1)(\lambda_1 - \lambda_2)}, \quad (3.61)$$

At steady-state, we have

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle ss = \overline{n}_a[Re^{-\frac{\kappa\tau}{2}} + Se^{\lambda_2\tau} - Te^{\lambda_1\tau}] + \frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle[Ae^{-\frac{\kappa\tau}{2}} - Be^{\lambda_2\tau} + Ce^{\lambda_1\tau}] \quad (3.62)$$

To determine the total mean photon number of light mode a,we need to consider the power spectrum of mode a.the power spectrum of light mode a with central frequency ω_0 is expressible as [22]

$$p_a(\omega) = \frac{1}{\pi}R_e\int_0^\infty d\tau e^{i(\omega-\omega_o)\tau}\langle\hat{a}^\dagger(t)\hat{a}(\tau+t)\rangle_{ss} \quad (3.63)$$

Thus on combining Eq.(3.62) with equation (3.63),the power spectrum of light mode a with central frequency ω_0 is expressible as

$$\begin{aligned} p_a(\omega) &= [\frac{\overline{n}_a R}{\pi}]Re\int_0^{\infty} d\tau e^{-[\frac{\kappa}{2}-i(\omega-\omega_o)]\tau} \\ &+[\frac{\overline{n}_a s}{\pi}]Re\int_0^{\infty} d\tau e^{-[\lambda_2-i(\omega-\omega_o)]\tau} \\ &-[\frac{\overline{n}_a T}{\pi}]Re\int_0^{\infty} d\tau e^{-[\lambda_1-i(\omega-\omega_o)]\tau} \\ &+[\frac{A}{\pi}\frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle]R_e\int_0^{\infty} d\tau e^{-[-\frac{\kappa}{2}-i(\omega-\omega_o)]\tau} \\ &-[\frac{B}{\pi}\frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle]R_e\int_0^{\infty} d\tau e^{-[\lambda_2-i(\omega-\omega_o)]\tau} \\ &+[\frac{C}{\pi}\frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle]R_e\int_0^{\infty} d\tau e^{-[\lambda_1-i(\omega-\omega_o)]\tau} \end{aligned} \quad (3.64)$$

So that on carrying out the integration, we readily arrive at

$$\begin{aligned} p_a(\omega) &= \overline{n}_a\big(R[\frac{\frac{\kappa}{2}\pi}{[\frac{\kappa}{2}]^2+(\omega-\omega_o)^2}]+S\big[\frac{\frac{\lambda_2}{\pi}}{\lambda_2^2+(\omega-\omega_o)^2}\big]-T\big[\frac{\frac{\lambda_1}{\pi}}{\lambda_1^2+(\omega-\omega_o)^2}\big]\big) \\ &+\frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle\big(A\big[\frac{\frac{\kappa}{2\pi}}{[\frac{\kappa}{2}]^2+(\omega-\omega_0)^2}\big]-B\big[\frac{\frac{\lambda_2}{\pi}}{\lambda_2^2+(\omega-\omega_o)^2}\big]+C\big[\frac{\frac{\lambda_1}{\pi}}{\lambda_1^2+(\omega-\omega_o)^2}\big]\big) \end{aligned} \quad (3.65)$$

We realize that the mean photon number of light mode a in the interval between ω'=-λ and ω'=λ is expressible [13]

$$\overline{n}_a \underline{+} \lambda = \int_{-\lambda}^{\lambda} P_a(\omega')d\omega' \quad (3.66)$$

In which ω'=ω-ω_o. Therefore, upon substituting Eq. (3.65) into Eq.(3.66) and carrying out the integration by employing the relation

$$\int_{-\lambda}^{\lambda}\frac{dx}{x^2+a^2}=\frac{2}{a}\tan^{-1}(\frac{\lambda}{a}) \quad (3.67)$$

The local mean photon number of light mode a produced by the coherently driven non-degenerate three level laser with an open cavity and coupled to a two-mode thermal reservoir is found to be

$$\overline{n}_a\underline{+}\lambda = \overline{n}_a z_a(\lambda) + \frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle z'_a(\lambda) \tag{3.68}$$

Where

$$z_a(\lambda) = [\frac{2R}{\pi}]\tan^{-1}(\frac{2\lambda}{\kappa}) + [\frac{2S}{\pi}]\tan^{-1}(\frac{\lambda}{\lambda_2}) + [\frac{2T}{\pi}]\tan^{-1}(\frac{\lambda}{\lambda_1}) \tag{3.69}$$

and

$$z'_a(\lambda) = [\frac{2A}{\pi}]\tan^{-1}(\frac{2\lambda}{\kappa}) - [\frac{2B}{\pi}]\tan^{-1}(\frac{\lambda}{\lambda_2}) + [\frac{2C}{\pi}]\tan^{-1}(\frac{\lambda}{\lambda_1}) \tag{3.70}$$

we see from Eq.(3.68) along with the plot of z_a that $\overline{n}_a\underline{+}\lambda$ increase with λ until it reaches the maximum value of the global mean photon number

3.3.2 Local mean photon number of light mode b

We now proceed to obtain the mean photon number of light mode b in a given frequency interval produced by the system consideration.To determine the local mean photon number of light mode b, we need to consider the power spectrum of light mode b. The power spectrum of light mode b with central frequency ω_0 is expressible as

$$p_b(\omega) = \frac{1}{\pi}R_e\int_0^{\infty} d\tau e^{i(\omega-\omega_o)\tau}\langle\hat{b}^{\dagger}(t)\hat{b}(\tau+t)\rangle_{ss} \tag{3.71}$$

Upon integrating both sides of Eq (3.71) over ω,we readily get

$$\int_{-\infty}^{\infty} P_b(\omega)d\omega = \overline{n}_b \tag{3.72}$$

In which $\overline{n}_b$ is the steady- state mean photon number of light mode b.From this result,we observe that $P_b(\omega)d\omega$ is the steady- state mean photon number of light mode a in the frequency interval between ω and ω+$d\omega$ we now proceed to determine the total correlation function that appears in Eq.(3.71). To this end, we realize that the solution of Eq.(3.71)

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\int_0^{\tau} e^{\kappa\frac{\tau'}{2}}\hat{m}_b(t+\tau')d\tau' + \varepsilon\int_0^{\infty} e^{\kappa\frac{\tau'}{2}}d\tau' \tag{3.73}$$

in view of Eq.(3.43) can be put in the form

$$\hat{m}_b(t+\tau) = p_2(\tau)\hat{m}_b(t) + q_2\hat{m}_a^{\dagger}(t) + \hat{G}_2^{\dagger}(t+\tau) \tag{3.74}$$

so that on introducing this into Eq. (3.73),we have

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\hat{m}_b(t)\int_0^{\tau} p_2(\tau')e^{\kappa\frac{\tau'}{2}}d\tau'$$

$$+\frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\int_0^{\tau} e^{\kappa\frac{\tau'}{2}}\big[\hat{m}_a^{\dagger}(t)q_2(\tau') + \hat{G}_2^{\dagger}(t+\tau)\big]d\tau' \tag{3.75}$$

Thus on carrying out the first two integrations, we arrive at

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}\hat{m}_b(t)\big[\frac{\lambda_1+\gamma+\gamma_c}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_2\tau} - e^{-\kappa\frac{\tau}{2}}\big]$$

$$-\frac{g}{\sqrt{N}}\hat{m}_b(t)\big[\frac{\lambda_2+\gamma+\gamma_c}{(\lambda_1+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_1\tau} - e^{-\kappa\frac{\tau}{2}}\big]$$

$$-\frac{g}{\sqrt{N}}\hat{m}_b^{\dagger}(t)\big[\frac{\frac{\Omega}{2}}{(\lambda_1+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_2\tau} - e^{-\kappa\frac{\tau}{2}}\big]$$

$$+\frac{g}{\sqrt{N}}\hat{m}_b^{\dagger}(t)\big[\frac{\frac{\Omega}{2}}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_2\tau} - e^{-\kappa\frac{\tau}{2}}\big]$$

$$+\frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\big\{\int_0^{\tau} d\tau'\int_0^{\tau'}\big[\big(\frac{\lambda_1+\gamma+\gamma_c}{\lambda_1-\lambda_2}\big)e^{(\kappa\frac{\tau'}{2})\tau'-\lambda_1\tau''}$$

$$-\big(\frac{\lambda_2+\gamma+\gamma_c}{\lambda_1-\lambda_2}\big)e^{(\kappa\frac{\tau'}{2}+\lambda_2)\tau'-\lambda_2\tau''}\big]\hat{F}_b(t+\tau'')d\tau''\big\}$$

$$-\frac{g}{\sqrt{N}}\big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\big]\int_0^{\tau} d\tau'\int_0^{\tau'}\big[e^{(\kappa\frac{\tau'}{2}+\lambda_1)\tau'-\lambda_1\tau''} - e^{(\kappa\frac{\tau'}{2}+\lambda_2)\tau'-\lambda_2\tau''}\big]$$

$$\times \hat{F}_b^{\dagger}(t+\tau'')d\tau'' \tag{3.76}$$

Now multiplying on the left by $\hat{b}^{\dagger}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{b}^{\dagger}(t)\hat{b}(t+\tau)\rangle = \langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b(t)\rangle\Big[\frac{\lambda_1+\gamma+\gamma_c}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\Big]\Big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\Big]$$

$$-\frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b(t)\rangle\Big[\frac{\lambda_2+\gamma+\gamma_c}{(\lambda_1+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\Big]\Big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\Big]$$

$$-\frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b^{\dagger}(t)\rangle\Big[\frac{\frac{\Omega}{2}}{(\lambda_1+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\Big]\Big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\Big]$$

$$+\frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b^{\dagger}(t)\rangle\Big[\frac{\frac{\Omega}{2}}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\Big]\Big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\Big]$$

$$+\frac{g}{\sqrt{N}}e^{-\kappa\frac{\tau}{2}}\Big\{\int_0^{\tau}d\tau'\int_0^{\tau'}\Big[\Big(\frac{\lambda_1+\gamma+\gamma_c}{\lambda_1-\lambda_2}\Big)e^{(\kappa\frac{\tau'}{2})\tau'-\lambda_1\tau''}$$

$$-\Big(\frac{\lambda_2+\gamma+\gamma_c}{\lambda_1-\lambda_2}\Big)e^{(\kappa\frac{\tau'}{2}+\lambda_2)\tau'-\lambda_2\tau''}\Big]\hat{F}_b(t+\tau'')d\tau''\Big\}$$

$$-\frac{g}{\sqrt{N}}\Big[\frac{\frac{\Omega}{2}}{\lambda_1-\lambda_2}\Big]\int_0^{\tau}d\tau'\int_0^{\tau'}\Big[e^{(\kappa\frac{\tau'}{2}+\lambda_1)\tau'-\lambda_1\tau''}-e^{(\kappa\frac{\tau'}{2}+\lambda_2)\tau'-\lambda_2\tau''}\Big]$$

$$\times\langle\hat{b}^{\dagger}(t)\hat{F}_b^{\dagger}(t+\tau'')\rangle d\tau'' \tag{3.77}$$

And taking account the fact that

$$\langle\hat{b}^{\dagger}(t)\hat{F}_b^{\dagger}(t+\tau'')\rangle = \langle\hat{b}^{\dagger}(t)\hat{F}_a^{\dagger}(t+\tau'')\rangle = 0 \tag{3.78}$$

we arrive at

$$\langle\hat{b}^{\dagger}(t)\hat{b}(t+\tau)\rangle = \langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle e^{-\kappa\frac{\tau}{2}} + \frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b(t)\rangle\Big[\frac{\lambda_1+\gamma+\gamma_c}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\Big]\Big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\Big]$$

$$-\frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b(t)\rangle\big[\frac{\lambda_2+\gamma+\gamma_c}{(\lambda_1+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_1\tau}-e^{-\kappa\frac{\tau}{2}}\big]$$

$$-\frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b^{\dagger}(t)\rangle\big[\frac{\frac{\Omega}{2}}{(\lambda_1+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\big]$$

$$+\frac{g}{\sqrt{N}}\langle\hat{b}^{\dagger}(t)\hat{m}_b^{\dagger}(t)\rangle\big[\frac{\frac{\Omega}{2}}{(\lambda_2+\frac{\kappa}{2})(\lambda_1-\lambda_2)}\big]\big[e^{\lambda_2\tau}-e^{-\kappa\frac{\tau}{2}}\big] \tag{3.79}$$

In view of the adjoint of Eqs.(3.52) and (3.54), we see that

$$\langle\hat{b}^{\dagger}(t)\hat{b}^{\dagger}(t)\rangle\big[Ke^{-\frac{\lambda\tau}{2}}+Me^{\lambda_2\tau}-Ne^{\lambda_1\tau}\big] \tag{3.80}$$

where

$$K=\frac{\lambda_1\lambda_2-\frac{\kappa}{2}(\gamma+\gamma_c)}{(\frac{\kappa}{2}+\lambda_1)(\frac{\kappa}{2}+\lambda_2)} \tag{3.81}$$

$$M=\frac{\frac{\kappa}{2}(\lambda_1+\gamma+\gamma_c)}{(\frac{\kappa}{2}+\lambda_2)(\lambda_1-\lambda_2)} \tag{3.82}$$

$$N=\frac{\frac{\kappa}{2}(\lambda_2+\gamma+\gamma_c)}{(\frac{\kappa}{2}+\lambda_1)(\lambda_1-\lambda_2)} \tag{3.83}$$

therefore, at the steady-state, Eq.(3.80)takes the form

$$\langle\hat{b}^{\dagger}(t)\hat{b}(t+\tau)\rangle ss=\overline{n}_b\big[Ke^{-\frac{\kappa\tau}{2}}+Me^{\lambda_2\tau}-Ne^{\lambda_1\tau}\big] \tag{3.84}$$

Thus on combining Eq.(3.80) with equation (3.71),the power spectrum of light mode b with central frequency ω_0 is takes the form

$$p_b(\omega)=\big[\frac{\overline{n}_bK}{\pi}\big]Re\int_0^{\infty}d\tau e^{-[\frac{\kappa}{2}-i(\omega-\omega_o)]\tau}$$

$$+[\frac{\overline{n}_bM}{\pi}]Re\int_0^{\infty}d\tau e^{-[-\lambda_2-i(\omega-\omega_o)]\tau}$$

$$-[\frac{\overline{n}_b N}{\pi}]Re\int_0^{\infty} d\tau e^{-[-\lambda_1 - i(\omega-\omega_o)]\tau} \tag{3.85}$$

So that on carrying out the integration, we readily arrive at

$$p_b(\omega) = \overline{n}_b\big(K[\frac{\frac{\kappa}{2\pi}}{[\frac{\kappa}{2}]^2 + (\omega-\omega_o)^2}] + M\big[\frac{\frac{\lambda_2}{\pi}}{\lambda_2^2 + (\omega-\omega_o)^2}\big] - N\big[\frac{\frac{\lambda_1}{\pi}}{\lambda_1^2 + (\omega-\omega_o)^2}\big] big) \tag{3.86}$$

we realize that the mean photon number of light mode b in the interval between ω'=-λ and ω'=λ is expressible [22]

$$\overline{n}_b\underline{+}\lambda = \int_{-\lambda}^{\lambda} P_b(\omega')d\omega' \tag{3.87}$$

In which ω'=ω-ω_o. Therefore, upon substituting Eq. (3.86) into Eq.(3.87) and carrying out the integration by employing the relation given by (3.67),the local mean photon number of light mode b is found to be

$$\overline{n}_b\underline{+}\lambda = \overline{n}_b\underline{z_b}(\lambda) \tag{3.88}$$

where

$$z_b(\lambda) = [\frac{2K}{\pi}]\tan^{-1}\big(\frac{2\lambda}{\kappa}\big) + [\frac{2M}{\pi}]\tan^{-1}\big(\frac{\lambda}{\lambda_2}\big) + [\frac{2N}{\pi}]\tan^{-1}\big(\frac{\lambda}{\lambda_1}\big) \tag{3.89}$$

3.4 Two-mode photon number

In this section, applying the steady-state solution of the equations of evolution of the expectation value of the atomic operators and the quantum langavin equations for the cavity mode operators, we seek to obtain the mean photon numbers for two mode light beam.

3.4.1 Two mode mean photon number

Here we seek to calculate the steady-state mean photon numbers two mode light cavity light beam.The mean photon numbers two mode light cavity light beam, represented by the operators $\hat{c}$ and $\hat{c}^{\dagger}$, is defined by

$$\overline{n} = \langle \hat{c}^{\dagger}\hat{c} \rangle \tag{3.90}$$

The steady state solution of Eq.(2.99)

$$\hat{c} = \frac{2g}{\kappa\sqrt{N}}\hat{m} + \frac{4\varepsilon}{\kappa} \tag{3.91}$$

$$\hat{c}^{\dagger} = \frac{2g}{\kappa\sqrt{N}}\hat{m}^{\dagger} + \frac{4\varepsilon}{\kappa} \tag{3.92}$$

$$\overline{n} = \langle \frac{2g}{\kappa\sqrt{N}}\hat{m}^{\dagger} + \frac{4\varepsilon}{\kappa} \rangle \langle \frac{2g}{\kappa\sqrt{N}}\hat{m} + \frac{4\varepsilon}{\kappa} \rangle \tag{3.93}$$

Hence at steady state the mean photon number goes over into

$$\overline{n} = \frac{\gamma_c}{\kappa}[\langle N_a \rangle + \langle N_b \rangle] + \frac{16\varepsilon^2}{\kappa^2} \tag{3.94}$$

$$\overline{n} = \frac{\gamma_c}{\kappa}[2\langle N_a \rangle] + \frac{16\varepsilon^2}{\kappa^2} \tag{3.95}$$

$$\langle \hat{N}_a \rangle = \left[\frac{\Omega^2}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2} \right] N, \tag{3.96}$$

$$\overline{n} = \frac{\gamma_c}{\kappa}(N)\left[\frac{2\Omega^2}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2} \right] + \frac{16\varepsilon^2}{\kappa^2} \tag{3.97}$$

The plots in Fig. (3.5) show that the steady-state mean photon number of light beam in the absence of spontaneous emission (when $\gamma = 0$) is greater than in the presence of spontaneous emission (when $\gamma \neq 0$). Moreover, the mean photon number decreases when γ increases.

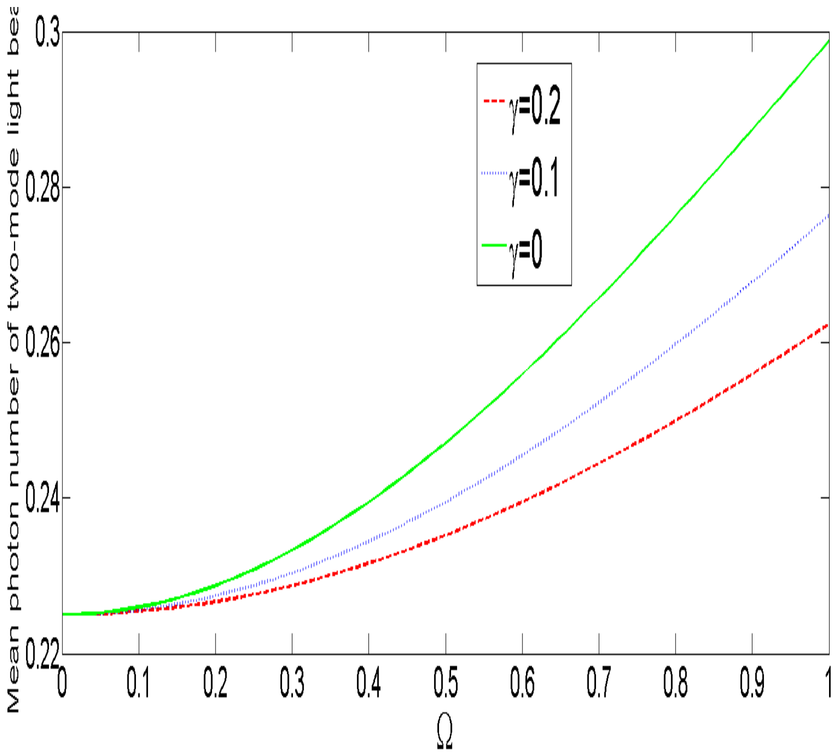

Figure 3.5: Plots of the mean photon number Eq. (3.97) versus Ω for $\gamma_c = 0.4$,$\kappa = 0.8$, $N = 50$,$\bar{n}_{th} = 5$, and for different values of γ. The sum of the mean photon numbers of the separate single-mode light beams

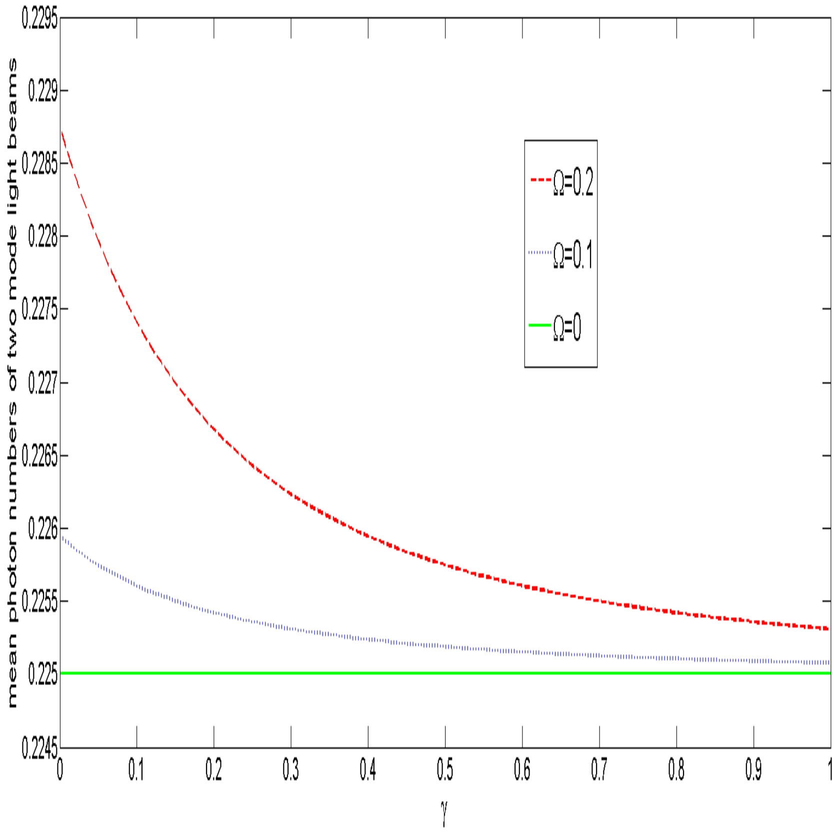

Figure 3.6: Plots of the mean photon number.Eq. (3.97) versus γ for γ_c = 0.4,κ = 0.8, $N = 50$,$\bar{n}_{th} = 5$, and for different values of Ω. The sum of the mean photon numbers of the separate single-mode light beams

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3.5 Photon number correlation

The photon number correlation for two modes of a radiation can be defined as

$$g^2(_{\hat{a},\hat{b}})^{(0)} = \frac{\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}, \tag{3.98}$$

in which

$$\langle \overline{n}_a \overline{n}_b \rangle = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle, \tag{3.99}$$

$$g^2(_{\hat{a},\hat{b}})^{(0)} = 1 + \frac{\langle \hat{b}\hat{a} \rangle \langle \hat{a}^\dagger \hat{b}^\dagger \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}, \tag{3.100}$$

$$\hat{a} = \frac{2gm_a}{\kappa\sqrt{N}} + \frac{2\varepsilon}{\kappa}, \tag{3.101}$$

$$\hat{b} = \frac{2gm_b}{\kappa\sqrt{N}} + \frac{2\varepsilon}{\kappa}, \tag{3.102}$$

$$\hat{a}^\dagger = \frac{2gm_a^\dagger}{\kappa\sqrt{N}} + \frac{2\varepsilon}{\kappa}, \tag{3.103}$$

$$\hat{b}^\dagger = \frac{2gm_b^\dagger}{\kappa\sqrt{N}} + \frac{2\varepsilon}{\kappa}, \tag{3.104}$$

$$g^2(_{\hat{a}}, \hat{b})^{(0)} = 1 + \frac{\langle \hat{m}_c \rangle^2 + \frac{16e^4}{\kappa^4}}{\langle N_a \rangle \langle N_b \rangle + \frac{16e^4}{\kappa^4}}, \tag{3.105}$$

thus on combining Eq.(2.146) and equation (2.149),with Eq.(2.141)

$$g^2(_{\hat{a}}, \hat{b})^{(0)} = 1 + \frac{(\frac{\gamma_c N}{\kappa})^2 \Big(\Big[\frac{\Omega(\gamma_c+\gamma)(\bar{n}_{th}+1)(2\bar{n}_{th}+1)}{(\gamma_c+\gamma)^2(\overline{n}+1)(2\bar{n}_{th}+1)+3\Omega^2} \Big] \Big)^2 + \frac{16e^4}{\kappa^4}}{(\frac{\gamma_c N}{\kappa})^2 \Big(\Big[\frac{\Omega^2}{(\gamma_c+\gamma)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2} \Big] \Big)^2 + \frac{16e^4}{\kappa^4}}. \tag{3.106}$$

The plots in Fig. (3.7) show that The photon number correlation for two modes of a radiation in the absence of spontaneous emission (when $\gamma = 0$) is greater than in the presence of spontaneous emission (when $\gamma \neq 0$). Moreover,The photon number correlation for two modes of a radiation decreases when γ increases.

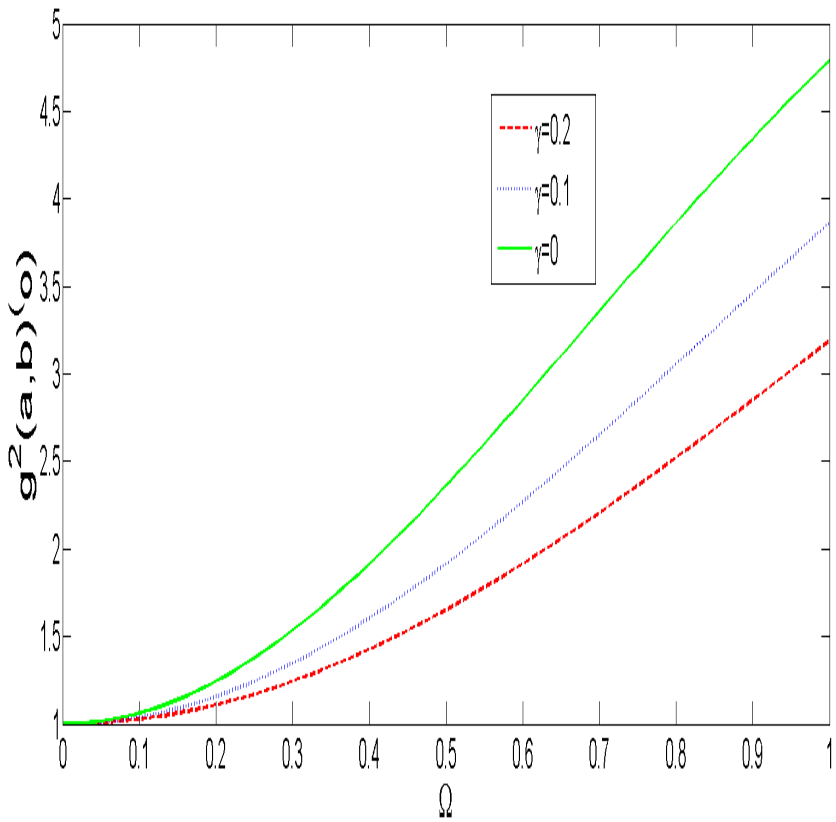

Figure 3.7: Plot of the $g^2(a,b)^{(0)}$ of the two-mode cavity light versus Ω for $\gamma_c = 0.4$,$\kappa = 0.8$, $N = 50$,$\bar{n}_{th} = 5$, and for different values of γ

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4

Quadrature Squeezing

In this chapter we seek to study the quadrature variance and the quadrature squeezing of the light produced by the coherently driven non degenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir via a single-port mirror. Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global quadrature variances for light modes$a$ and b. In addition, we determine the global quadrature squeezing of the two-mode cavity light.

4.1 Single-mode quadrature variance

In this section we obtain the global quadrature variances of light modes a and b, produced by the system under consideration.

4.1.1 Global quadrature variance of light mode a

We now proceed to calculate the quadrature variance of light mode a in the entire frequency interval. The squeezing properties of light mode a are described by two quadrature operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{4.2}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation oprators for light mode a . With the help of Eqs. (4.1) and (4.2), we can show that the two quadrature operators satisfy the commutation relation

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}\big[\hat{N}_a - \hat{N}_b\big] \tag{4.3}$$

In view of this result, the uncertainity relation for the plus and minus quadrature operators of mode a is expressible as

$$\begin{aligned} \triangle\hat{a}_+\triangle\hat{a}_- \;&\geq\; \frac{1}{2}\big|\langle|\hat{a}_+, \hat{a}_-|\rangle\big| \\ &\geq \big|\langle|\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle\big| \end{aligned} \tag{4.4}$$

so that using Eqs. (3.2) and (3.3), there follows

$$\triangle\hat{a}_+\triangle\hat{a}_- \geq \frac{\gamma_c}{\kappa}\big|\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle\big| \tag{4.5}$$

On account of Eq. (2.141), the uncertainity relation for the quadrature operators can be put in the form

$$\triangle\hat{a}_+\triangle\hat{a}_- \geq 0 \tag{4.6}$$

Next we proceed to calculate the quadrature variance of light mode a. The variance of the plus and minus quadrature operators are defined by

$$(\triangle\hat{a}_+)^2 = \langle\hat{a}_+^2\rangle - \langle\hat{a}_+\rangle^2 \tag{4.7}$$

and

$$(\triangle \hat{a}_{-})^2 = \langle \hat{a}_{-}^2 \rangle - \langle \hat{a}_{-} \rangle^2 \tag{4.8}$$

With the aid of Eq. (4.1), Eq. (4.7) can be expressed in terms of the creation and annihilation operators as

$$(\triangle \hat{a}_{+})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle + \langle \hat{a}_{+}^2 \rangle + \langle \hat{a}_{+}^{\dagger 2} \rangle - \langle \hat{a}_{+}^2 \rangle - \langle \hat{a}_{+}^{\dagger 2} \rangle - 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle \tag{4.9}$$

In addition, on account of Eqs. (4.2) and (4.8), we get

$$(\triangle \hat{a}_{-})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle - \langle \hat{a}_{+}^2 \rangle - \langle \hat{a}_{+}^{\dagger 2} \rangle - \langle \hat{a}_{+}^2 \rangle + \langle \hat{a}_{+}^{\dagger 2} \rangle + 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle \tag{4.10}$$

so that inspection of Eqs. (4.9) and (4.10) shows that

$$(\triangle \hat{a}_{\pm})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle \pm \langle \hat{a}_{+}^2 \rangle \pm \langle \hat{a}_{+}^{\dagger 2} \rangle \mp \langle \hat{a}_{+}^2 \rangle \mp \langle \hat{a}_{+}^{\dagger 2} \rangle \mp 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle \tag{4.11}$$

Moreover, with the help of Eqs. (3.2) and (3.3), we have

$$(\triangle \hat{a}_{\pm})^2 = \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{a}^{\dagger}\hat{a} \rangle \tag{4.12}$$

and in view of Eqs. (3.2) and (3.3), there follows

$$(\triangle \hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa}[\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle] + \frac{8\varepsilon^2}{\kappa^2} \tag{4.13}$$

On account of Eq. (2.141), we see that

$$(\triangle \hat{a}_{\pm})^2 = \frac{2\gamma_c}{\kappa}[\langle \hat{N}_a \rangle] + \frac{8\varepsilon^2}{\kappa^2} \tag{4.14}$$

Now substitution of Eq. (2.146) into Eq. (4.14) results in

$$(\triangle \hat{a}_{\pm})^2 = \Big(\frac{\gamma_c N}{\kappa}\Big)\Big[\frac{2\Omega^2}{(\gamma + \gamma_c)^2(2\bar{n}_{th} + 1)(\bar{n}_{th} + 1) + 3\Omega^2}\Big] + \frac{8\varepsilon^2}{\kappa^2} \tag{4.15}$$

This represents the quadrature variance of light mode a, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. Furthermore,we consider the case in which spontaneous emission is absent $\gamma = 0$ Then the quadrature variance for this case takes the form

$$(\triangle\hat{a}_{\pm})^2 = \big(\frac{\gamma_c N}{\kappa}\big)\big[\frac{2\Omega^2}{\gamma_c^2(2\bar{n}_{th}+1)(\bar{n}_{th}+1)+3\Omega^2}\big] + \frac{8\varepsilon^2}{\kappa^2} \tag{4.16}$$

In addition, we note that for $\Omega >> \gamma_c$Eq. (4.16) reduces to

$$(\triangle\hat{a}_{\pm})^2 = \big(\frac{\gamma_c N}{\kappa}\big)\big(\frac{2\Omega^2}{3\Omega^2}\big) + \frac{8\varepsilon^2}{\kappa^2} \tag{4.17}$$

$$(\triangle\hat{a}_{\pm})^2 = \frac{2\gamma_c N}{3\kappa} + \frac{8\varepsilon^2}{\kappa^2} \tag{4.18}$$

In view of Eq. (3.7), this can be expressed as

$$(\triangle\hat{a}_{\pm})^2 = 2\overline{n_a} \tag{4.19}$$

4.1.2 Global quadrature variance of light mode b

Here we wish to calculate the quadrature variance of light mode b in the entire frequency interval, produced by the system under consideration. The squeezing properties of light mode b are described by two quadrature operators

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b} \tag{4.20}$$

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}) \tag{4.21}$$

where $\hat{b}_+$ and $\hat{b}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{b}^\dagger$ and $\hat{b}$ are the creation and annihilation oprators for light mode b. With the help of Eqs. (4.20) and (4.21), we can show that the two quadrature operators satisfy the commutation relation

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_c\big], \tag{4.22}$$

$$\langle\hat{N}_b\rangle = \left[\frac{\Omega^2}{(\gamma_c+\gamma)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}\right]N, \tag{4.23}$$

$$\langle\hat{N}_c\rangle = \left[\frac{(\gamma_c+\gamma)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+\Omega^2}{(\gamma_c+\gamma)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}\right]N, \tag{4.24}$$

In view of this result, the uncertainity relation for the plus and minus quadrature operators of mode b is expressible as

$$\begin{aligned}\triangle\hat{b}_+\triangle\hat{b}_- \;&\geq\; \frac{1}{2}\big|\langle|\hat{b}_+,\hat{b}_-|\rangle\big| \\ &\geq \big|\langle|\hat{b}\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{b}\rangle\big|\end{aligned} \tag{4.25}$$

so that using Eqs. (3.10) and (3.11), there follows

$$\triangle\hat{b}_+\triangle\hat{b}_- \geq \frac{\gamma_c}{\kappa}\big|\langle\hat{N}_b\rangle - \langle\hat{N}_c|\rangle\big| \tag{4.26}$$

On account of Eqs. (4.23) and (4.24), the uncertainity relation of the quadrature operators can be put the form

$$\triangle\hat{b}_+\triangle\hat{b}_- \geq \frac{\gamma_c N}{\kappa}\Big|\frac{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}\Big| \tag{4.27}$$

Now setting $\gamma = 0$, one finds

$$\triangle\hat{b}_+\triangle\hat{b}_- \geq \frac{\gamma_c N}{\kappa}\Big|\frac{\gamma_c^2(2\bar{n}_{th}+1)}{\gamma_c^2(2\bar{n}_{th}+1)+3\Omega^2}\Big| \tag{4.28}$$

Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.28), we readily get

$$\triangle\hat{b}_+\triangle\hat{b}_- \geq \frac{\gamma_c N(\bar{n}_{th}+1)(2\bar{n}_{th}+1)}{\kappa(\bar{n}_{th}+1)(2\bar{n}_{th}+1)} \tag{4.29}$$

We therefore notice that the product of the uncertainties in the two quadratures satisfies the minimum uncertainty relation. Next we proceed to calculate the quadrature variance of light mode b. The variance of the plus and minus quadrature operators for light mode b are defined by

$$(\triangle\hat{b}_{+})^2 = \langle\hat{b}_{+}^2\rangle - \langle\hat{b}_{+}\rangle^2 \tag{4.30}$$

and

$$(\triangle\hat{b}_{-})^2 = \langle\hat{b}_{-}^2\rangle - \langle\hat{b}_{-}\rangle^2 \tag{4.31}$$

On account of Eq. (4.20), Eq. (4.21) can be expressed in terms of the creation and annihilation operators as

$$(\triangle\hat{b}_{+})^2 = \langle\hat{b}\hat{b}^{\dagger}\rangle + \langle\hat{b}^{\dagger}\hat{b}\rangle + \langle\hat{b}_{+}^2\rangle + \langle\hat{b}_{+}^{\dagger 2}\rangle - \langle\hat{b}_{+}^2\rangle - \langle\hat{b}_{+}^{\dagger 2}\rangle - 2\langle\hat{b}\rangle\langle\hat{b}^{\dagger}\rangle \tag{4.32}$$

In addition, on account of Eqs. (4.21), we get

$$(\triangle\hat{b}_{-})^2 = \langle\hat{b}\hat{b}^{\dagger}\rangle + \langle\hat{b}^{\dagger}\hat{b}\rangle - \langle\hat{b}_{+}^2\rangle - \langle\hat{b}_{+}^{\dagger 2}\rangle - \langle\hat{b}_{+}^2\rangle + \langle\hat{b}_{+}^{\dagger 2}\rangle + 2\langle\hat{b}\rangle\langle\hat{b}^{\dagger}\rangle \tag{4.33}$$

so that inspection of Eqs. (4.32) and (4.33) shows that

$$(\triangle\hat{b}_{\pm})^2 = \langle\hat{b}\hat{b}^{\dagger}\rangle + \langle\hat{b}^{\dagger}\hat{b}\rangle \pm \langle\hat{b}_{+}^2\rangle \pm \langle\hat{b}_{+}^{\dagger 2}\rangle \mp \langle\hat{b}_{+}^2\rangle \mp \langle\hat{b}_{+}^{\dagger 2}\rangle \mp 2\langle\hat{b}\rangle\langle\hat{b}^{\dagger}\rangle \tag{4.34}$$

Moreover, with the aid of Eqs. (4.20) and (4.21), we get

$$(\triangle\hat{b}_{\pm})^2 = \langle\hat{b}\hat{b}^{\dagger}\rangle + \langle\hat{b}^{\dagger}\hat{b}\rangle \tag{4.35}$$

and in view of Eqs. (3.10) and (3.11), there follows

$$(\triangle\hat{b}_{\pm})^2 = \frac{\gamma_c}{\kappa}\big[\hat{N}_b + \hat{N}_c\big], \tag{4.36}$$

Now on account of Eqs. (4.23) and (4.24), the quadrature variance of light mode b takes, at steady-state, the form

$$(\triangle\hat{b}_{\pm})^2 = (\frac{\gamma_c N}{\kappa})[\frac{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+2\Omega^2}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}] + \frac{8\varepsilon^2}{\kappa^2} \tag{4.37}$$

This represents the quadrature variance of light mode b, produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir. Furthermore,we consider the case in which spontaneous emission is absent $\gamma = 0$. Then the quadrature variance for this case has the form

$$(\triangle\hat{b}_{\pm})^2 = (\frac{\gamma_c N}{\kappa})[\frac{\gamma_c^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+2\Omega^2}{\gamma_c^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}] + \frac{8\varepsilon^2}{\kappa^2} \tag{4.38}$$

Furthermore,we consider the case in which spontaneous emission is absent $\Omega \gg \gamma_c$ Then the quadrature variance for this case has the form

$$(\triangle\hat{b}_{\pm})^2 = \frac{2\gamma_c N}{3\kappa} + \frac{8\varepsilon^2}{\kappa^2} \tag{4.39}$$

In view of Eq. (3.15), this can be expressed as

$$(\triangle\hat{b}_{\pm})^2 = 2\overline{n_b} \tag{4.40}$$

which is the normally ordered quadrature variance for chaotic light.

4.2 Two-mode quadrature squeezing

In this section we proceed to study the quadrature variance and the quadrature squeezing of the two-mode light beam produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir. Now we seek to determine the quadrature variances of the two-mode light beam. The squeezing properties of the two-mode cavity light are described by two quadrature operators

$$\hat{c}_{+} = \hat{c}^{\dagger} + \hat{c} \tag{4.41}$$

and

$$\hat{c}_{-} = i(\hat{c}^{\dagger} - \hat{c}) \tag{4.42}$$

where$\hat{c}_{+}$ and $\hat{c}_{-}$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{c}^{\dagger}$ and $\hat{c}$ are the creation and annihilation oprators of the two-mode cavity light. With the aid of Eqs. (4.41) and (4.42), we show that the two quadrature operators satisfy the commutation relation

$$[\hat{c}_{-}, \hat{c}_{+}] = 2i\frac{\gamma_c}{\kappa}\big[N_a \hat{-} \hat{N}_c\big] \tag{4.43}$$

$$\hat{N}_a = (\frac{\Omega^2}{(\gamma + \gamma_c)^2(\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2})N \tag{4.44}$$

and

$$\hat{N}_c = (\frac{\Omega^2 + (\gamma + \gamma_n)^2(\bar{n}_{th} + 1)(2\bar{n}_{th} + 1)}{(\gamma + \gamma_n)^2(\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2})N \tag{4.45}$$

In view of this result, the uncertainity relation for the plus and minus quadrature operators of the two-mode cavity light is expressible as

$$\triangle\hat{c}_{+}\triangle\hat{c}_{-} \geq \frac{1}{2}\big|\langle\hat{c}_{+}, \hat{c}_{-}\rangle\big|$$

$$\geq \big|\langle|\hat{c}\hat{c}^{\dagger}\rangle - \langle\hat{c}^{\dagger}\hat{c}\rangle\big| \tag{4.46}$$

and

$$\triangle\hat{c}_{+}\triangle\hat{c}_{-} \geq \frac{\gamma_c}{\kappa}\big|\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle\big| \tag{4.47}$$

On account of Eqs. (4.44) and (4.45), the uncertainity relation for the plus and minus quadrature operators is found to be

$$\triangle\hat{c}_{+}\triangle\hat{c}_{-} \geq \frac{\gamma_c N}{\kappa}\Big|\frac{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}\Big| \tag{4.48}$$

In addition, we consider the case in which spontaneous emission is absent $\gamma = 0$.Then the uncertainity relation for this case takes the form

$$\triangle\hat{c}_{+}\triangle\hat{c}_{-} \geq \frac{\gamma_c N}{\kappa}\Big|\frac{\gamma_c^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)}{\gamma_c^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2}\Big| \tag{4.49}$$

Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.49), we readily get

$$\triangle\hat{c}_{+}\triangle\hat{c}_{-} \geq \frac{\gamma_c N}{\kappa} \tag{4.50}$$

which is the minimum uncertainty relation for the two-mode cavity thermal state. We therefore notice that the uncertainties in the two quadratures are equal and their product satisfies the minimum uncertainty relation. Next we calculate the quadrature variance of the two-mode cavity light. The variance of the plus and minus quadrature operators of the two-mode cavity light are defined by

$$(\triangle\hat{c}_{+})^2 = \langle\hat{c}_{+}^2\rangle - \langle\hat{c}_{+}\rangle^2 \tag{4.51}$$

and

$$(\triangle\hat{c}_{-})^2 = \langle\hat{c}_{-}^2\rangle - \langle\hat{c}_{-}\rangle^2 \tag{4.52}$$

On account of Eqs. (4.41) and (4.51), the plus quadrature variance can be expressed in terms of the creation and annihilation operators as

$$(\triangle\hat{c}_{+})^2 = \langle\hat{c}\hat{c}^{\dagger}\rangle + \langle\hat{c}^{\dagger}\hat{c}\rangle + \langle\hat{c}_{+}^2\rangle + \langle\hat{c}_{+}^{\dagger 2}\rangle - \langle\hat{c}_{+}^2\rangle - \langle\hat{c}_{+}^{\dagger 2}\rangle - 2\langle\hat{c}\rangle\langle\hat{c}^{\dagger}\rangle \tag{4.53}$$

and with the help of Eqs. (4.42) and (4.52), we get

$$(\triangle\hat{c}_-)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle - \langle\hat{c}_+^2\rangle - \langle\hat{c}_+^{\dagger 2}\rangle - \langle\hat{c}_+^2\rangle + \langle\hat{c}_+^{\dagger 2}\rangle + 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle \tag{4.54}$$

so that inspection of Eqs. (4.53) and (4.54) shows that

$$(\triangle\hat{c}_\pm)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle\pm\langle\hat{c}_+^2\rangle\pm\langle\hat{c}_+^{\dagger 2}\rangle\mp\langle\hat{c}_+^2\rangle\mp\langle\hat{c}_+^{\dagger 2}\rangle\mp 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle \tag{4.55}$$

and

$$(\triangle\hat{c}_\pm)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle\pm\langle\hat{c}_+^2\rangle\pm\langle\hat{c}_+^{\dagger 2}\rangle \tag{4.56}$$

$$\hat{c} = \frac{2g\hat{m}}{\kappa\sqrt{N}} + \frac{4\varepsilon}{\kappa} \tag{4.57}$$

$$\hat{c}^\dagger = \frac{2g\hat{m}^\dagger}{\kappa\sqrt{N}} + \frac{4\varepsilon}{\kappa} \tag{4.58}$$

where

$$\hat{m} = \hat{m}_a + \hat{m}_b, \tag{4.59}$$

$$\hat{m}^\dagger\hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{4.60}$$

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{4.61}$$

$$\hat{m}^2 = N\hat{m}_c, \tag{4.62}$$

$$\hat{N}_a = (\frac{\Omega^2}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2})N \tag{4.63}$$

$$\hat{N}_c = (\frac{\Omega^2+(\gamma+\gamma_n)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)}{(\gamma+\gamma_n)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2})N \tag{4.64}$$

$$\hat{N}_b = (\frac{\Omega^2}{(\gamma+\gamma_c)^2(\bar{n}_{th}1)(2\bar{n}_{th}+1)+3\Omega^2})N \tag{4.65}$$

$$\hat{m}_c = (\frac{\Omega(\gamma+\gamma_c)(\bar{n}_{th}+1)}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1)+3\Omega^2})N \tag{4.66}$$

expression (4.56) goes over into

$$(\triangle\hat{c}_\pm)^2 = \frac{\gamma_c}{\kappa}[\langle\hat{N}_a\rangle + 2\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle\pm\langle\hat{m}_c\rangle\pm\langle\hat{m}_c^\dagger\rangle] \tag{4.67}$$

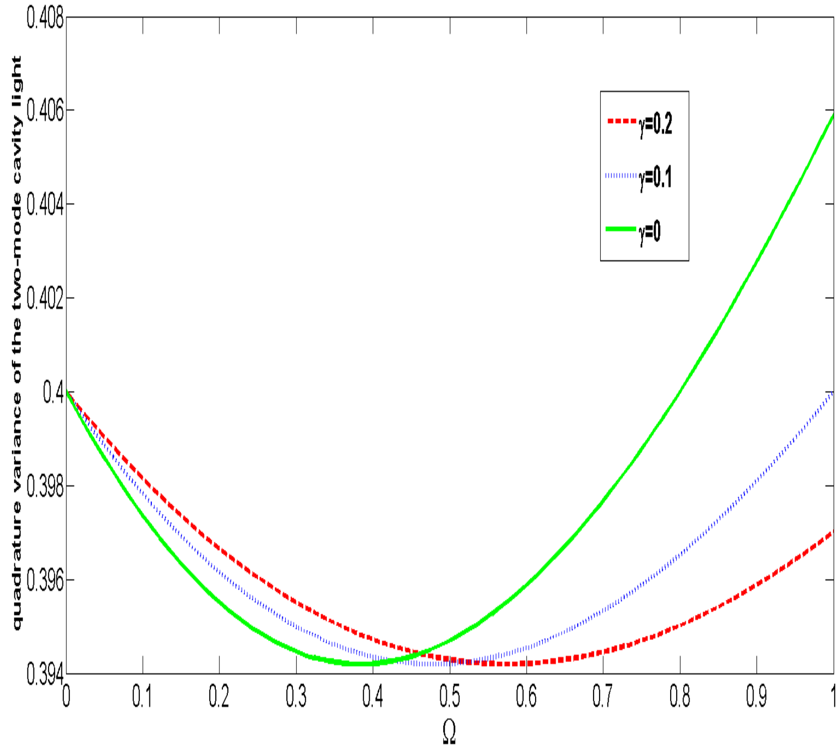

Figure 4.1: the quadrature variance of the two-mode cavity light versus Ω for γ_c = 0.4,κ = 0.8, $N = 50$,$\bar{n}_{th} = 5$, and for different values of γ

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Now using $\hat{N}_a = \hat{N}_b$and $\hat{N}_a + \hat{N}_b \hat{+} N_c = N$, the quadrature variance of the two-mode cavity light is found to be

$$(\triangle \hat{c}_{\pm})^2 = \frac{\gamma_c}{\kappa}[N + \langle \hat{N}_b \rangle \pm 2 \langle \hat{m}_c \rangle] \tag{4.68}$$

Finally, on account of Eqs. (4.63) - (4.66), the quadrature variance of the two-mode cavity light takes, at steady-state, the form

$$(\triangle \hat{c}_{\pm})^2 = \frac{\gamma_c N}{\kappa}[\frac{(\gamma + \gamma_c)^2(\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 4\Omega^2 + 2\Omega(\gamma + \gamma_c)}{(\gamma + \gamma_c)^2(\bar{n}_{th} + 1)(2\bar{n}_{th} + 1) + 3\Omega^2}] \tag{4.69}$$

This represents the quadrature variance of the two-mode cavity light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir. Furthermore,we consider the case in which spontaneous emission is absent $\gamma = 0$. Thus the quadrature variance for this case has the form

$$(\triangle\hat{c}_{\pm})^2 = \frac{\gamma_c N}{\kappa}\left[\frac{\gamma_c^2 + 4\Omega^2 + 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right] \tag{4.70}$$

In addition, we note that for $\Omega >> \gamma_c$ Eq. (4.70) reduces to

$$(\triangle\hat{c}_{\pm})^2 = \frac{4\gamma_c N}{3\kappa}\frac{\gamma_c^2}{\gamma_c^2} \tag{4.71}$$

$$(\triangle\hat{c}_{\pm})^2 = \frac{4\gamma_c N}{3\kappa} \tag{4.72}$$

This can be rewritten as

$$(\triangle\hat{c}_{\pm})^2 = 2\overline{n} \tag{4.73}$$

where $\overline{n}$ is given by Eq. (3.97). We see that Eq. (4.73) represents the normally ordered quadrature variance for chaotic light. Moreover, we consider the case in which the deriving coherent light is absent. Thus upon setting $\Omega = 0$ in Eq. (4.49), we get

$$(\triangle\hat{c}_{+})^2_{therm} = (\triangle\hat{c}_{-})^2_{therm} = \frac{\gamma_c N}{\kappa} \tag{4.74}$$

which is the normally ordered quadrature variance of the two-mode cavity thermal state. We note that for $\Omega = 0$ the uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation. The plots in Fig. (4.1) show that the minimum value of the quadrature variance for $\gamma = 0.2$, $\gamma = 0.1$, and $\gamma = 0$ is $(\Delta c_{-})^2 = 0.3943$ and occur at $\Omega = 0.4242$, $\Omega = 0.4646$ and $\Omega = 0.5354$, respectively.

Next we proceed to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval relative to the quadrature variance of the two-mode thermal state. We then define the quadrature squeezing of the two-mode cavity light

$$S = \frac{(\triangle\hat{c}_-)^2_{therm} - (\triangle\hat{c}_-)^2}{(\triangle\hat{c}_-)^2_{therm}} \tag{4.75}$$

It then follows that

$$S = 1 - \frac{(\triangle\hat{c}_-)^2}{(\triangle\hat{c}_-)^2_{therm}} \tag{4.76}$$

In view of Eqs. (4.69) and (4.74), the quadrature squeezing of the two-mode cavity light takes, at steady-state, the form

$$S = [\frac{2\Omega(\gamma+\gamma_c) - \Omega^2}{(\gamma+\gamma_c)^2(\bar{n}_{th}+1)(2\bar{n}_{th}+1) + 3\Omega^2}] \tag{4.77}$$

This represents the quadrature squeezing of the two-mode cavity light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir. We observe from this equation that unlike the mean photon number and the quadrature variance, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons. Applying Eqs. (3.2) and (3.10), we find

$$\langle\hat{b}\hat{a}\rangle = \frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle + \frac{4\varepsilon^2}{\kappa^2} \tag{4.78}$$

The two-mode light can be used in experiments involving entangled light modes. In addition, we consider the case in which spontaneous emission is absent $\gamma = 0$. Then the quadrature squeezing for this case takes the form

$$S = [\frac{2\Omega\gamma_c - \Omega^2}{\gamma_c^2(2\bar{n}_{th}+1)(\bar{n}_{th}+1) + 3\Omega^2}] \tag{4.79}$$

This represents the quadrature squeezing of the two-mode cavity light produced by the coherently driven non degenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir From the plots in Fig. (4.2), we find the maximum quadra-

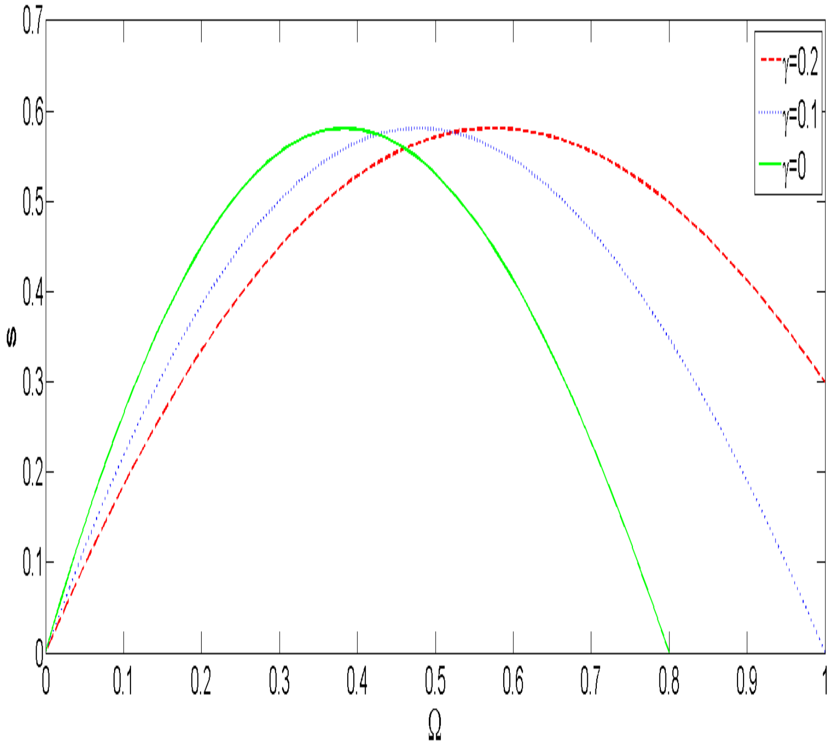

Figure 4.2: Plot of the quadrature squeezing versus Ω for $\gamma_c = 0.4$,$\kappa = 0.8$, $N = 50$,$\bar{n}_{th} = 5$ and for different values of γ

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ture squeezing to be the same in the presence as well as in the absence of spontaneous emission. This plots show that the quadrature squeezing when $\gamma = 0$ is greater than when $\gamma = 0.2$ in the interval $0 < \Omega < 0.4545$ and the quadrature squeezing when $\gamma = 0$ is less than when $\gamma = 0.2$ for $\Omega > 0.4545$. And the quadrature squeezing when $\gamma = 0$ is greater than when $\gamma = 0.1$ in the interval $0 < \Omega < 0.4242$ and the maximum quadrature squeezing when $\gamma = 0$ is less than when $\gamma = 0.1$ for $\Omega > 0.4242$. Moreover, the plots in the same figure show that the quadrature squeezing when $\gamma = 0.1$ is greater than when $\gamma = 0.2$ in the interval $0 < \Omega < 0.5253$ and the quadrature squeezing when $\gamma = 0.1$ is less than when $\gamma = 0.2$ for $\Omega > 0.5253$. Furthermore, from the same plots the maximum squeezing is found to be 58.08% for $\gamma = 0.2$ (dashed curve), for $\gamma = 0.1$ (dotted curve), and for $\gamma = 0$ (solid curve) below the vacuum-state level. These occur when the three-level laser is operating at $\Omega = 0.3838$, $\gamma = 0.4848$, and $\gamma = 0.5758$, respectively.

5

Entanglement

To this end, we prefer to analyze the entanglement of photon-states in the laser cavity. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot described independently instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, polarization, etc. performed on entangled particles are found to be appropriately correlated. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and properties lead to a better understanding of the basic quantum principles. It is in this spirit that this section is devoted to the analysis of the entanglement of the two-mode photon states. In other words, it is a well-known fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product density operators of the constituents , in this section we seek to study the entanglement condition of the two modes in the cavity.

$$\hat{\rho}\neq\sum\hat{\rho}_j^{(1)}\bigotimes\hat{\rho_j^{()2}} \tag{5.1}$$

On the other hand, a maximally entangled continuous variable state can be expressed as a coeigenstate of a pair of EPR-type operators [19]such as $\hat{X}_a - \hat{X}_b$ and $\hat{p}_a + \hat{p}_b$. The total variance of these two operators reduces to zero for maximally entangled continuous variable states. But according to the criteria set by Duan et al. [20]quantum states of the system are entangled if the sum of the variances of a pair of EPR-like operators

$$\hat{u} = \hat{X}_a - \hat{X}_b \tag{5.2}$$

$$\hat{v} = \hat{p}_a + \hat{p}_b \tag{5.3}$$

where$\hat{X}_a = (1/\sqrt{2})(\hat{a} + \hat{a}^\dagger)$,$\hat{X}_b = (1/\sqrt{2})(\hat{b} + \hat{b}^\dagger)$,$\hat{p}_a = (i/\sqrt{2})(\hat{a}^\dagger - \hat{a})$, $\hat{p}_b = (i/\sqrt{2})(\hat{b}^\dagger - \hat{b})$ are quadrature operators for mode a and b, satisfy

$$\triangle u^2 + \triangle v^2 < 2 \tag{5.4}$$

Furthermore, the variance of these quadrature operators can be put, in terms of the c number variables associated with the normal ordering, in the form

$$\triangle u^2 = 1 - \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{b}^\dagger \hat{b}\rangle - 2\langle \hat{a}\hat{b}\rangle, \tag{5.5}$$

$$\hat{a} = \frac{2g}{\kappa\sqrt{N}} m_a + \frac{2\varepsilon}{\kappa}, \tag{5.6}$$

$$\hat{a}^\dagger = \frac{2g}{\kappa\sqrt{N}} {m_a}^\dagger + \frac{2\varepsilon}{\kappa}, \tag{5.7}$$

$$\hat{b} = \frac{2g}{\kappa\sqrt{N}} m_b + \frac{2\varepsilon}{\kappa}, \tag{5.8}$$

$$\hat{b}^\dagger = \frac{2g}{\kappa\sqrt{N}} {m_b}^\dagger + \frac{2\varepsilon}{\kappa}, \tag{5.9}$$

$$\triangle u^2 = 1 - \frac{\gamma_c}{\kappa}[N_a + N_b], \tag{5.10}$$

$$\triangle u^2 = 1 - \frac{\gamma_c}{\kappa}(2N_a), \tag{5.11}$$

$$\triangle u^2 = 1 - \frac{\gamma_c}{\kappa} N\big(\frac{2\Omega^2}{(\gamma+\gamma_c)^2(2\bar{n}_{th}^2 + 3\bar{n}_{th} + 1) + 3\Omega^2}\big), \tag{5.12}$$

$$\triangle u^2 = \triangle v^2 \tag{5.13}$$

$$\begin{aligned} \triangle u^2 + \triangle v^2 \;&=\; 1 - \frac{\gamma_c}{\kappa} N\big(\frac{2\Omega^2}{(\gamma+\gamma_c)^2(2\bar{n}_{th}^2 + 3\bar{n}_{th} + 1) + 3\Omega^2}\big) \\ &\quad + 1 - \frac{\gamma_c}{\kappa} N\big(\frac{2\Omega^2}{(\gamma+\gamma_c)^2(2\bar{n}_{th}^2 + 3\bar{n}_{th} + 1) + 3\Omega^2}\big) \end{aligned} \tag{5.14}$$

$$\triangle v^2 + \triangle u^2 = 2 - \frac{\gamma_c}{\kappa} N\big(\frac{4\Omega^2}{(\gamma+\gamma_c)^2(2\bar{n}_{th}^2 + 3\bar{n}_{th} + 1) + 3\Omega^2}\big) \tag{5.15}$$

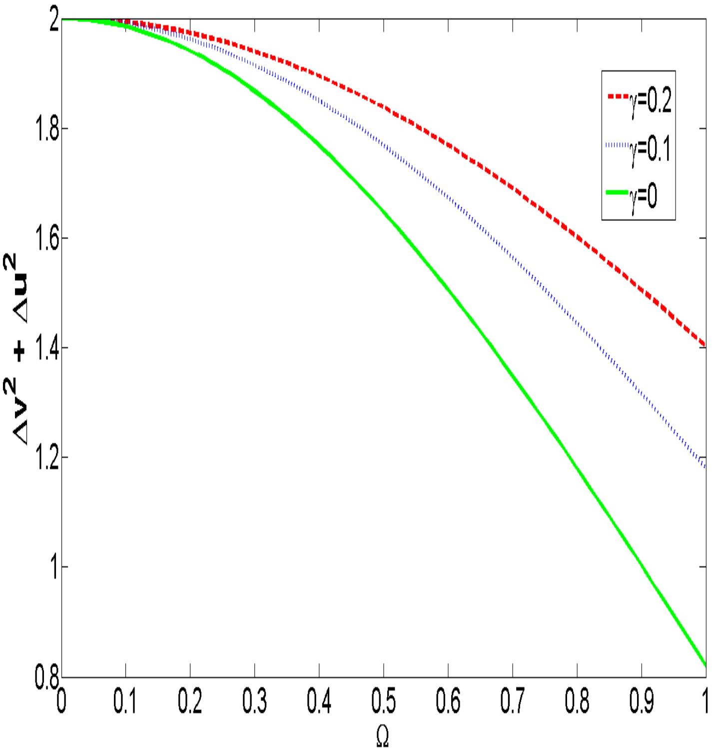

Figure 5.1: Plots of $\triangle u^2 + \triangle v^2$ of the cavity radiation at steady state versus Ω for γ_c = 0.4,κ = 0.8, $N = 50$,$\bar{n}_{th} = 5$, and for different values of γ

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6

Conclusion

In this research ,we have studied the squeezing and statistical properties of the light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir via a single-port mirror. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we have determined the mean and variance of the photon number as well as the quadrature squeezing. We have found that the global mean photon number of light mode a is equal to the global mean photon number of light mode b. We have seen that the mean and variance of the photon numbers of light modes a and b. Moreover, we have shown that the mean photon number of the two-mode light beam is the sum of the mean photon numbers of the separate single-mode light beams.

We have found that the light generated by the three-level laser is in a squeezed state and the squeezing occurs .From the plots in Fig. (4.2), we find the maximum quadrature squeezing to be the same in the presence as well as in the absence of spontaneous emission. Unlike the mean photon number and the quadrature variance, the quadrature squeezing does not depend on the number of atoms.

Reference

[1] M. O. Scully and M. S. Zubairy, Quantum Optics Cambridge University Press,Cambridge,1997).

[2] Fesseha Kassahun, Fundamentals of Quantum Optics (Lulu, Nort Carolina,2008).

[3] C.C.Gerry and P.L.Knight, introductory Quantum Optics(Cambridge universit press,Cambridge2004)

[4] Y. Zhang, H. Wang, X. Li, J. Jing, C. Xie,and K. Peng, Phys. Rev. A 62, 023813(2000).

[5] H. Xiong, M. O.Scully, and M. S. Zubairy, Phys. Rev. Lett. 94, 023601(2005).

[6] H. T. Tan, S. Y. Zhu, and M. S. Zubairy, Phys. Rev. A 72, 022305(2005).

[7] L. M. Duan, G. Giedke, J. J. Cirac, and P. Zoller,Phys. Rev. Lett. 84, 2722 (2000).

[8] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).

[9] Fesseha Kassahun,eprintarXiv:1105.1438v3 quantph (2012).

[10] C. H. Raymond Ooi, eprintar Xiv:quant-ph/0703084.

[11] N. A. Ansari, J. G. Banacloche, and M. S. Zubairy, Phys. Rev.A 41, 5179 (1990) .

[12]T. Abebe.Enhancement of squeezing and entanglement in a non-degenerate three-level cascade laser with coherently driven cavity. Ukr. J. Phys. 63, No. 8, 733 (2018).

[13] C. J.Villas-Boas and M. H. Y. Moussa, Eur. Phys. J. D 32, 147 (2005)

[14] A. Einstein, B. Podolsky, and R. Rosen, Phys. Rev. 47, 777(1935).

[15] N. Lu and S. Y. Zhu, Phys. Rev. A **40**, 5735 (1989).

[16] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics*, (Oxford University Press, New York, 1997).

[17] N. A. Ansari, Phys. Rev.A 46, 1560 (1992).

[18] S. Tesfa. Entanglement amplification in a nondegenerate three-level cascade laser. Phys. Rev. A 74, 043816 (2006)

[19] G. S. Agrawal and G. Adam, Phys. Rev. A **39**, 6259 (1989)

[20] C. J. Villas-Boas and M. H. Y. Moussa, Eur. Phys. J. D 32, 147(2005).

[21] Eyob Alebachew and K. Fesseha, Opt. Commun. **265**, 314 (2006).

[22] Fesseha Kassahun, Refined Quantum Analysis of Light,[Create Space Independent Publishing Platform, 2014