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BRIGHT ENTANGLED SQUEEZED PHOTONS WITH THE SUPERPOSED SIGNAL LIGHT BEAMS

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Abstract

In this paper, we have studied the statistical and squeezing properties of a pair of superposed signal-signal light beams employing the density operator and a slightly modified definition of entanglement condition. We have found that the mean photon number of the superposed light beams to be the sum of the mean photon numbers of the separate light beams. And a large part of the mean photon number is confined in a relatively small frequency interval. Moreover, the superposed bright light beams have a maximum squeezing of 75% below the vacuum state level and occurs in ± 0.01 frequency interval. We have also clearly shown that a pair of superposed light beams are entangled at steady-state and the entanglement turned out be observed in the squeezed photons.

1. Introduction

One-mode subharmonic generation is one of the most interesting and widely studied quantum optical process. In this process a pump photon of frequency 2ω is down converted into a pair of signal photons each of frequency ω. A theoretical analysis of the statistical and squeezing properties of the signal mode produced by one-mode subharmonic generation has been made by a number of authors [1-7].

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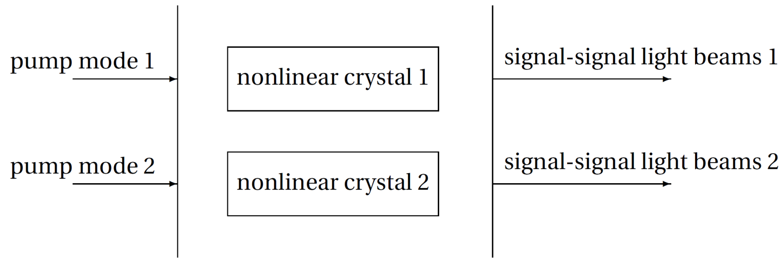
Among other things, it has been predicted that the signal mode has a maximum squeezing of 50% below the vacuum-state level [4-7].

It is to be recalled that the Hamiltonian describing the process of subharmonic generation consists of the operators $\hat{a}^2$ and $\hat{a}^{\dagger 2}$. And the quantum analysis of the signal mode is usually carried out employing the operators $\hat{a}$ and $\hat{a}^{\dagger}$ with the commutation relation $[\hat{a}.\hat{a}^{\dagger}] = 1$. However, such analysis leads, among others, to one-half of the mean photon number of the signal mode [1-7]. This is surely the mean number of one set of the signal photons, consisting of one photon from each pair [6-7]. Since the other set of the signal photons is not included in such analysis, the problem is resolved in [8] by applying the commutation relation $[\hat{a}.\hat{a}^{\dagger}] = 2$.

Furthermore, employing the usual definition for the quadrature variance, some authors have arrived at the conclusion that the superposition of coherent light beam with some other light beam does not affect the quadrature variance of the other light beam [10]. However, it has been established that the presence of the coherent light indeed affects the quadrature variance of the other light beam by using a slightly modified definition for the quadrature variance [9]. For instance, the squeezing properties of superposed coherent and squeezed light produced by onemode subharmonic generator in the same cavity has been studied applying a slightly modified definition for the quadrature variance of a pair of superposed light beams [9, 11]. It has been found that the quadrature squeezing of the superposed light beams is half of the squeezed light. This is just the average quadrature squeezing of the separate light beams. On the other hand, applying the usual definition for quadrature squeezing, one can readily verify that the quadrature squeezing of superposed triple signal light beams is the sum of the quadrature squeezing of the separate light beam. This implies that the quadrature squeezing of superposed triple squeezed light beams, each having 50% squeezing, would be 150%. Evidently, there cannot be any justification for this unrealistic result. In light of this, the definition for quadrature squeezing of the superposed light beams should somehow be slightly modified [9].

Eventhough Einstein, along with his colleagues Podolsky and Rosen, was first to recognize the criterion for analyzing entanglement condition for a single light beam [12], a significant number of works have not been devoted on superposed light beams. In this paper, we present a slightly modified definition of entanglement analysis for a pair of superposed light beams.

Moreover, we also seek to analyze the photon statistics and quadrature squeezing of a pair of superposed signal-signal light beams produced by a one-mode subharmonic generators. We first obtain the density operators for a pair of superposed signal-signal light beams.



Dig. 1. A pair of superposed one-mode subharmonic generators.

Using the resulting density operator and Q functions, we calculate the mean photon number and the variance of the photon number for a pair of superposed light beams. And making use of a slightly modified definition of the quadrature variance and entanglement of a pair of superposed light beams, we analyze the quadrature squeezing and the entanglement condition.

2. The Density Operator

Here we seek to determine the density operator for a pair of superposed signal-signal light beams. The density operator for the first signal-signal light beams is expressible as

$$\hat{\rho}'(\hat{a}_1^\dagger, \hat{a}_1, t) = \int d^2\alpha_1 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)|\alpha_1\rangle\langle\alpha_1|. \tag{1}$$

This expression, for density operator, can be put in the form

$$\hat{\rho}'(\hat{a}_1^\dagger, \hat{a}_1, t) = \int d^2\alpha_1 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\hat{D}(\alpha_1)\hat{\rho}_0\hat{D}(-\alpha_2) \tag{2}$$

in which

$$\hat{\rho}_0 = |0\rangle\langle 0|. \tag{3}$$

Now we realize that the density operator for the superposition of the first signal-

signal light beams and another one is expressible as

$$\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t) = \frac{1}{\pi}\int d^2\alpha_2 \sum_{mn} C_{mn}\alpha_2^{*m}\left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}\right)^n \hat{D}(\alpha_2)\hat{\rho}'(t)\hat{D}(-\alpha_2), \quad (4)$$

so that in view of (1), we have

$$\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t) = \int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)$$

$$\times Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)|\alpha_1 + \alpha_2\rangle\langle\alpha_2 + \alpha_1|, \quad (5)$$

in which

$$Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) = \frac{1}{\pi}\sum_{kl} C_{kl}\alpha_1^{*k}\left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}\right)^l \quad (6)$$

and

$$Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) = \frac{1}{\pi}\sum_{mn} C_{mn}\alpha_2^{*m}\left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}\right)^n, \quad (7)$$

are Q functions associated with the two signal-signal light beams.

3. Photon Statistics

In this section, we seek to study the statistical properties of a pair of superposed signal-signal light beams.

3.1. The mean photon number

We next obtain the global and local mean photon number employing the density operator.

A. The global mean photon number

The global mean photon number can be expressed in terms of the density operator as

$$\bar{n} = Tr(\hat{\rho}(t)\hat{a}^{\dagger}(0)\hat{a}(0)), \quad (8)$$

where we assume that $\hat{a}$ represents the annihilation operator for a pair of superposed signal-signal light beams. Thus introducing (5) into (8), we have

$$\bar{n} = \int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)$$

$$\times [\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 + \alpha_1^*\alpha_2 + \alpha_1\alpha_2^*]. \tag{9}$$

It then follows that

$$\bar{n} = \int d^2\alpha_1 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1^*\alpha_1 + \int d^2\alpha_2 Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2^*\alpha_2$$

$$+ \int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_1^*\alpha_2$$

$$+ \int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_1\alpha_2^*. \tag{10}$$

Hence equation (10) can be put in the form

$$\bar{n} = \langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + \langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle + \langle \hat{a}_1^\dagger(t)\rangle\langle \hat{a}_2(t)\rangle + \langle \hat{a}_1(t)\rangle\langle \hat{a}_2^\dagger(t)\rangle \tag{11}$$

in which $\hat{a}_1$ and $\hat{a}_2$ represent the annihilation operators for the first and second signal-signal light beams with commutation relations $[\hat{a}_1, \hat{a}_1^\dagger] = 2$ and $[\hat{a}_2, \hat{a}_2^\dagger] = 2$, respectively. In view of the fact that $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are Guassian operators with zero mean, we see that

$$\langle \hat{a}_1(t)\rangle = \langle \hat{a}_2(t)\rangle = \langle \hat{a}_1^\dagger(t)\rangle = \langle \hat{a}_2^\dagger(t)\rangle = 0. \tag{12}$$

Thus on account of (12), we have

$$\bar{n} = \sum_{i=1}^{2} \langle \hat{a}_1^\dagger(t)\hat{a}_i(t)\rangle. \tag{13}$$

Then with the aid of [9], the global mean photon number at steady state turns out to be

$$\bar{n}_{ss} = \sum_{i=1}^{2} \frac{4\varepsilon_i^2}{\kappa^2 - 4\varepsilon_i^2}. \tag{14}$$

We observe that the global mean photon number of a pair of superposed signal-signal light beams is the sum of the mean photon numbers of the separate light beams.

B. Local mean photon number

We calculate the local mean photon number in a given frequency interval using the power spectrum. The power spectrum with central frequency ω_0 is expressible as

$$P(\omega) = \frac{1}{\pi} Re\int_0^\infty \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau. \tag{15}$$

The two-time correlation function for cavity light beams can be written as

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = Tr[\hat{\rho}(t)\hat{a}^\dagger(0)\hat{a}(\tau)]. \tag{16}$$

Now introducing (5) into (16), we have

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)$$

$$\times Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)(\alpha_1^* + \alpha_2^*)Tr[\hat{\rho}(0)\hat{a}(\tau)], \tag{17}$$

in which

$$\hat{\rho}(0) = |\alpha_1 + \alpha_2\rangle\langle\alpha_2 + \alpha_1|. \tag{18}$$

We note that

$$Tr[\hat{\rho}(0)\hat{a}(\tau)] = Tr[\hat{\rho}(\tau)\hat{a}(0)]. \tag{19}$$

Now replacing $(\alpha_1, \alpha_1^*, t)$ by $(\lambda_1, \lambda_1^*, \tau)$ and $(\alpha_2, \alpha_2^*, t)$ by $(\lambda_2, \lambda_2^*, \tau)$ in (5), the density operator $\hat{\rho}(\tau)$ can be written as

$$\hat{\rho}(\tau) = \int d^2\lambda_1 d^2\lambda_2 Q'\left(\lambda_1^*, \lambda_1 + \frac{\partial}{\partial\lambda_1^*}, \tau\right)$$

$$\times Q''\left(\lambda_2^*, \lambda_2 + \frac{\partial}{\partial\lambda_2^*}, \tau\right)|\lambda_1 + \lambda_2\rangle\langle\lambda_2 + \lambda_1|. \tag{20}$$

Thus applying (20) in (19), we get

$$Tr[\rho(\tau)\hat{a}(0)] = \int d^2\lambda_1 Q'\left(\lambda_1^*, \lambda_1 + \frac{\partial}{\partial\lambda_1^*}, \tau\right)\lambda_1$$

$$+ \int d^2\lambda_2 Q''\left(\lambda_2^*, \lambda_2 + \frac{\partial}{\partial\lambda_2^*}, \tau\right)\lambda_2. \quad (21)$$

Then expression (21) can be rewritten as

$$\langle\hat{a}(\tau)\rangle = Tr[\hat{\rho}(\tau)\hat{a}(0)] = \sum_{i=1}^{2}\langle\hat{a}_i(\tau)\rangle. \quad (22)$$

Thus combination of (15), (17), and (22) leads to

$$P(\omega) = \frac{1}{\pi} Re\int_0^\infty d\tau e^{i(\omega-\omega_0)\tau}\int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)$$

$$\times Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)(\alpha_1^* + \alpha_2^*)[\langle\hat{a}_1(\tau)\rangle + \langle\hat{a}_2(\tau)\rangle]. \quad (23)$$

Therefore, on the basis of [9], the power spectrum turns out to be

$$P(\omega) = \sum_{i=1}^{2}\bar{n}_i \frac{(\kappa^2 - 4\varepsilon_i^2)}{8\pi\varepsilon_i}\left[\left(\frac{1}{\Omega^2 + \left(\frac{\kappa - 2\varepsilon_i}{2}\right)^2}\right) - \left(\frac{1}{\Omega^2 + \left(\frac{\kappa + 2\varepsilon_i}{2}\right)^2}\right)\right]\Bigg\}. \quad (24)$$

We see that the power spectrum of a pair of superposed signal-signal light beams is the sum of the power spectrum of the separate light beams. Upon integrating both sides of (24) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega)d\omega = \bar{n}_{ss}. \quad (25)$$

Furthermore, on the basis of (24), we observe that $P(\omega)d\omega$ represents the steady-state mean photon number in the interval between ω and $\omega + d\omega$. We thus realize that the steady-state local mean photon number in the interval between $\omega' = -\lambda$ and $\omega = \lambda$ can be written as

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \quad (26)$$

where $\omega' = \omega - \omega_0$. Therefore, using (24) and (26), we readily obtain

$$\bar{n}_{\pm\lambda} = \sum_{i=1}^{2} \bar{n}_i z_i(\lambda), \tag{27}$$

where

$$z_i(\lambda) = \frac{1}{2\pi\varepsilon_i}\left[(\kappa + 2\varepsilon_i)\tan^{-1}\left(\frac{2\lambda}{\kappa - 2\varepsilon_i}\right) - (\kappa - 2\varepsilon_i)\tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon_i}\right)\right]. \tag{28}$$

We observe that the local mean photon number of a pair of superposed light beams is the sum of the local mean photon number of the separate light beams.

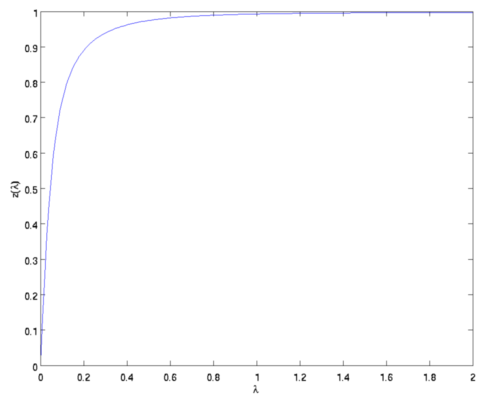


Figure 1. A plot of $z_i(\lambda)$ [eq. (28)] versus λ for $\kappa = 0.8$ and $\varepsilon = 0.35$.

One can easily get from Figure 1 that $z(0.4) = 0.9636$, $z(0.6) = 0.9824$, $z(0.8) = 0.9895$, and $z(1) = 0.9930$. Then combination of these results with (27) yields $\bar{n}_{\pm 0.4} = 0.9636\bar{n}$, $\bar{n}_{\pm 0.6} = 0.9824\bar{n}$, $\bar{n}_{\pm 0.8} = 0.9895\bar{n}$, and $\bar{n}_{\pm 1} = 0.9930\bar{n}$. We immediately see that a large part of the total mean photon number is confined in a relatively small frequency interval.

3.2. The global photon-number variance

We next proceed to obtain the global variance of the photon number. The global photon-number variance is defined by

$$(\Delta n)^2 = \langle(\hat{a}^\dagger(t)\hat{a}(t))^2\rangle - \langle(\hat{a}^\dagger(t)\hat{a}(t))^2\rangle. \tag{29}$$

Using the commutation relation

$$[\hat{a}, \hat{a}^{\dagger}] = 4, \tag{30}$$

which holds for a pair of superposed signal-signal light beams, we find

$$(\Delta n)^2 = \langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle + 4\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle^2. \tag{31}$$

Then the expectation value of $\hat{a}^{\dagger 2}(t)\hat{a}^2(t)$ is expressible in terms of the density operator as

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = Tr[\hat{\rho}(t)\hat{a}^{\dagger 2}(0)\hat{a}^2(0)]. \tag{32}$$

With the aid of (5), expression (32) can be written as

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \int d^2\alpha_1 d^2\alpha_2 Q'\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)Q''\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)$$
$$\times\Big[\alpha_1^{*2}\alpha_1^2 + 2\alpha_1^{*2}\alpha_1\alpha_2 + \alpha_1^{*2}\alpha_2^2 + \alpha_1^2\alpha_2^{*2} + 2\alpha_1^*\alpha_1^2\alpha_2^*$$
$$+ 2\alpha_1^*\alpha_2^*\alpha_2^2 + 2\alpha_1\alpha_2^{*2}\alpha_2 + 4\alpha_1^*\alpha_1\alpha_2^*\alpha_2 + \alpha_2^{*2}\alpha_2^2\Big]. \tag{33}$$

One can put (33) in the form

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle + \langle \hat{a}_1^{\dagger 2}(t)\rangle\langle \hat{a}_2^2(t)\rangle + \langle \hat{a}_1^2(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\rangle$$
$$+ \langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle + 4\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle$$
$$+ 2\langle \hat{a}_2(t)\rangle\langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1(t)\rangle + 2\langle \hat{a}_2^{\dagger}(t)\rangle\langle \hat{a}_1^{\dagger}(t)\hat{a}_1^2(t)\rangle$$
$$+ 2\langle \hat{a}_1(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2(t)\rangle + 2\langle \hat{a}_1^{\dagger}(t)\rangle\langle \hat{a}_2^{\dagger}(t)\hat{a}_2^2(t)\rangle. \tag{34}$$

Then on the basis of (12), we get

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle + \langle \hat{a}_1^{\dagger 2}(t)\rangle\langle \hat{a}_2^2(t)\rangle$$
$$+ \langle \hat{a}_1^2(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\rangle + \langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle$$
$$+ 4\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle. \tag{35}$$

Employing (13) and (31) together with (35), we find

$$(\Delta n(t))^2 = \langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle + 4\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle - \langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle^2$$

$$\langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle + 4\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle - \langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle^2$$

$$+ \langle \hat{a}_1^{\dagger 2}(t)\rangle\langle \hat{a}_2^2(t)\rangle + \langle \hat{a}_1^2(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\rangle$$

$$+ 2\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle. \tag{36}$$

Moreover, if $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are Guassian operators with zero mean, we see that

$$\langle \hat{a}_1^{\dagger 2}(t)\hat{a}_1^2(t)\rangle = 2\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle^2 + \langle \hat{a}_1^{\dagger 2}(t)\rangle\langle \hat{a}_1^2(t)\rangle \tag{37}$$

and

$$\langle \hat{a}_2^{\dagger 2}(t)\hat{a}_2^2(t)\rangle = 2\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle^2 + \langle \hat{a}_2^{\dagger 2}(t)\rangle\langle \hat{a}_2^2(t)\rangle. \tag{38}$$

Thus in view of (37) and (38), (36) takes the form

$$(\Delta n(t))^2 = \langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle^2 + 4\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle + \langle \hat{a}_1^2(t)\rangle\langle \hat{a}_1^{\dagger 2}(t)\rangle$$

$$\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle^2 + 4\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle + \langle \hat{a}_2^2(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\rangle$$

$$+ \langle \hat{a}_1^{\dagger 2}(t)\rangle\langle \hat{a}_2^2(t)\rangle + \langle \hat{a}_1^2(t)\rangle\langle \hat{a}_2^{\dagger 2}(t)\rangle$$

$$+ 2\langle \hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle \hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle. \tag{39}$$

Now making use of [9], the global photon number variance at steady state found to be

$$(\Delta n)^2 = \left[\frac{2\kappa^2\varepsilon_1^2}{(\kappa^2 - 4\varepsilon_1^2)^2} + \frac{2\kappa^2\varepsilon_2^2}{(\kappa^2 - 4\varepsilon_2^2)^2} + \frac{16\varepsilon_1^2}{(\kappa^2 - 4\varepsilon_1^2)}\right.$$

$$+ \frac{16\varepsilon_2^2}{(\kappa^2 - 4\varepsilon_1^2)} + \frac{16\varepsilon_1^4}{(\kappa^2 - 4\varepsilon_1^2)^2} + \frac{16\varepsilon_2^4}{(\kappa^2 - 4\varepsilon_2^2)^2}$$

$$\left. + \frac{4\kappa^2\varepsilon_1\varepsilon_2}{(\kappa^2 - 4\varepsilon_1^2)(\kappa^2 - 4\varepsilon_2^2)} + \frac{32\varepsilon_1^2\varepsilon_2^2}{(\kappa^2 - 4\varepsilon_1^2)(\kappa^2 - 4\varepsilon_2^2)}\right]. \tag{40}$$

This shows that unlike that of the global mean photon number, the global photon number variance of a pair of superposed light beams is not the sum of that of the separate light beams.

4. Quadrature Squeezing

We wish here to study the squeezing properties for a pair of superposed signal-signal light beams.

4.1. The global quadrature squeezing

We define the quadrature variance by

$$(\Delta a_{\pm})^2 = \langle \hat{a}_{\pm}(t),\, \hat{a}_{\pm}(t)\rangle, \tag{41}$$

where

$$\hat{a}_{+}(t) = \hat{a}^{\dagger}(t) + \hat{a}(t) \tag{42}$$

and

$$\hat{a}_{-}(t) = i(\hat{a}^{\dagger}(t) - \hat{a}(t)), \tag{43}$$

are the plus and minus quadrature operators for a pair of superposed signal-signal light beams. Using the commutation relation described by (30), eq. (41) can be put in the form

$$(\Delta a_{\pm})^2 = 4 + \langle :\hat{a}_{\pm}(t),\, \hat{a}_{\pm}(t):\rangle. \tag{44}$$

We can take the first term on the right hand side of (44) to be the quadrature variance of a pair of superposed cavity vacuum-vacuum states. Then with the aid of (85) and (86), eq. (44) can be put in the form

$$(\Delta a_{\pm}(t))^2 = 4 + \Big[2\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle \pm \langle \hat{a}^{\dagger 2}(t)\rangle \pm \langle \hat{a}^{2}(t)\rangle$$
$$\mp \langle \hat{a}^{\dagger}(t)\rangle^2 \mp \langle \hat{a}(t)\rangle^2 - 2\langle \hat{a}^{\dagger}(t)\rangle\langle \hat{a}(t)\rangle\Big]. \tag{45}$$

Hence the expectation value of $\hat{a}(t)$ can be expressed in terms of the density operator as

$$\langle \hat{a}(t)\rangle = Tr(\hat{\rho}(t)\hat{a}(0)). \tag{46}$$

Thus introducing (5) into (46), we have

$$\langle \hat{a}(t)\rangle = \sum_{i=1}^{2}\int d^2\alpha_i Q\left(\alpha_i^*,\, \alpha_i + \frac{\partial}{\partial\alpha_i^*},\, t\right)\alpha_i. \tag{47}$$

It then follows

$$\langle \hat{a}(t) \rangle = \sum_{i=1}^{2} \langle \hat{a}_i(t) \rangle. \tag{48}$$

In view of (12), we have

$$\langle \hat{a}(t) \rangle = 0. \tag{49}$$

Moreover, the expectation value of $\hat{a}^2(t)$ can be written as

$$\langle \hat{a}^2(t) \rangle = \sum_{i=1}^{2} \int d^2\alpha_i Q\left(\alpha_i^*, \alpha_i + \frac{\partial}{\partial \alpha_i^*}, t\right)\alpha_i$$

$$+ \sum_{i=1}^{2} \int d^2\alpha_i Q\left(\alpha_i^*, \alpha_i + \frac{\partial}{\partial \alpha_i^*}, t\right)\alpha_i^2. \tag{50}$$

Then on account of (12), we see that

$$\langle \hat{a}^2(t) \rangle = \sum_{i=1}^{2} \langle \hat{a}_i^2(t) \rangle. \tag{51}$$

Thus substitution of (13), (49), and (51) into (45) results in

$$(\Delta a_{\pm}(t))^2 = 4 + 2\left[\sum_{i=1}^{2} \langle \hat{a}_i^{\dagger}(t)\hat{a}_i(t) \rangle \pm \sum_{i=1}^{2} \langle \hat{a}_i^{\dagger 2}(t) \rangle\right]. \tag{52}$$

Thereore, employing [9], the quadrature variance takes the form

$$(\Delta a_{\pm})^2 = 4 \mp \sum_{i=1}^{2} \left\{\frac{4\varepsilon_i}{(\kappa \pm 2\varepsilon_i)}\left[1 - e^{-(\kappa \pm 2\varepsilon_i)t}\right]\right\}. \tag{53}$$

At steady state, this turns out to be

$$(\Delta a_{\pm})^2 = 4 \mp \sum_{i=1}^{2} \left\{\frac{4\varepsilon_i}{(\kappa \pm 2\varepsilon_i)}\right\}. \tag{54}$$

We observe that a pair of superposed light beams are in a squeezed state and the squeezing occurs in the plus quadrature. Moreover, we see that the global quadrature variance of a pair of superposed light beams is the sum of the quadrature variance of the individual light beams.

Furthermore, for the case in which the signal-signal light beams are identical, one can easily see that

$$(\Delta a_{\pm})^2 = 4\left\{1 \mp \frac{2\varepsilon}{(\kappa \pm 2\varepsilon)}\left[1 - e^{-(\kappa \pm 2\varepsilon)t}\right]\right\}. \tag{55}$$

Then it follows that

$$(\Delta a_{\pm})^2 = 4\left[1 \mp \frac{2\varepsilon}{(\kappa \pm 2\varepsilon)}\right]. \tag{56}$$

This shows that the global quadrature variance of a pair of superposed identical signal-signal light beams is twice that of one of the separate light beams.

On the other hand, upon setting $\varepsilon_2 = 0$ in (54), we see that

$$(\Delta a_{\pm})^2 = 2 + \left[2 \mp \frac{4\varepsilon}{(\kappa \pm 2\varepsilon)}\right]. \tag{57}$$

We clearly see that the presence of the vacuum-state indeed affects the quadrature variance of the superposed light beams.

Next we calculate the quadrature squeezing for a pair of superposed light beams relative to the quadrature variance of a pair of superposed cavity vacuum-states. Thus in view of (52), we define the quadrature squeezing by

$$S_{+} = \frac{4 - (\Delta a_{\pm})^2}{4}, \tag{58}$$

so that on account of (53), there follows

$$S_{+} = \frac{1}{4}\sum_{i=1}^{2}\frac{4\varepsilon_i}{(\kappa + 2\varepsilon_i)}\left[1 - e^{-(\kappa + 2\varepsilon_i)t}\right]. \tag{59}$$

At steady state, we have

$$S_{+} = \frac{1}{4}\sum_{i=1}^{2}\frac{4\varepsilon_i}{(\kappa + 2\varepsilon_i)}. \tag{60}$$

This shows that the global quadrature squeezing of a pair of superposed light beams is the average of the quadrature squeezing of the separate light beams. We note that at steady state and at threshold there is a 50% squeezing of the superposed light beams below the vacuum-state level. Upon setting $\varepsilon_1 = \varepsilon_2$, we find

$$S_+ = \frac{2\varepsilon_1}{(\kappa + 2\varepsilon_1)}\left[1 - e^{-(\kappa+2\varepsilon_1)t}\right]. \tag{61}$$

We note that the quadrature squeezing of the combined light beams is exactly the same as that of one of the light beams. For the case in which $\varepsilon_2 = 0$, we observe that

$$S_+ = \frac{1}{4}\left\{\frac{4\varepsilon_1}{(\kappa + 2\varepsilon_1)}\left[1 - e^{-(\kappa+2\varepsilon_1)t}\right]\right\}. \tag{62}$$

This is half of the global quadrature squeezing of one of the individual light beams.

4.2. Local quadrature squeezing

To this end, we first obtain the spectrum of quadrature fluctuations. We define the spectrum of quadrature fluctuations with central frequency ω_0 by

$$S_\pm(\omega) = \frac{1}{\pi} Re\int_0^\infty \langle \hat{a}_\pm(t),\ \hat{a}_\pm(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau, \tag{63}$$

in which

$$\hat{a}_+(t+\tau) = (\hat{a}^\dagger(t+\tau) + \hat{a}(t+\tau)) \tag{64}$$

and

$$\hat{a}_-(t+\tau) = i(\hat{a}^\dagger(t+\tau) - \hat{a}(t+\tau)). \tag{65}$$

Then in view of (85), (86), (64), and (65), eq. (63) can be put in the form

$$S_\pm(\omega) = \frac{1}{\pi} Re\int_0^\infty [\pm\langle \hat{a}(t)\hat{a}(t+\tau)\rangle + \langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle + \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle \pm \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle] e^{i(\omega-\omega_0)\tau} d\tau. \tag{66}$$

Now with the aid of [9], the spectrum of quadrature fluctuations turns out to be

$$S_\pm(\omega) = \sum_{i=1}^{2} (\Delta a_\pm)_i^2 \left(\frac{\left(\frac{\kappa}{2} \pm \varepsilon_i\right)/\pi}{\Omega^2 + \left[\frac{\kappa}{2} \pm \varepsilon_i\right]^2}\right). \tag{67}$$

We clearly see that the spectrum of quadrature fluctuations for a pair of superposed light beams is the sum of that of the individual light beams.

Upon integrating both sides of (67) over ω, we get

$$\int_{-\infty}^{\infty} S_{\pm}(\omega)d\omega = (\Delta a_{\pm})^2. \tag{68}$$

On the basis of eq. (68), we observe that $S_{\pm}(\omega)d\omega$ is the quadrature variance of the light in the interval between ω and $\omega + d\omega$. Then the local quadrature variance in the interval $\omega' = -\lambda$ and $\omega' = \lambda$ can then be written as

$$(\Delta a_{\pm\lambda})^2 = \int_{-\lambda}^{\lambda} (S_{\pm}(\omega'))^2 d\omega', \tag{69}$$

in which $\omega' = \omega - \omega_0$.

Furthermore, upon integrating eq. (67) in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$, we readily get

$$(\Delta a_{\pm\lambda})^2 = \sum_{i=1}^{2} (\Delta a_{\pm})_i^2 z_{i\pm}(\lambda), \tag{70}$$

in which

$$z_{i\pm}(\lambda) = \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{\frac{\kappa}{2} \pm \varepsilon_i}\right). \tag{71}$$

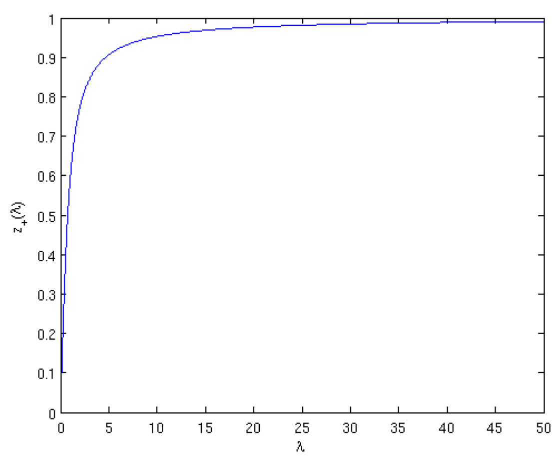


Figure 2. A plot of $z_{i+}(\lambda)$ [Eq. 71] versus λ for $\kappa = 0.8$ and $\varepsilon = 0.35$.

We observe that the quadrature variance of the superposed light in a given frequency interval is the sum of the separate light beams. We easily obtain from Figure 2 that $z(+5) = 0.906$, $z(+15) = 0.968$, $z(+25) = 0.981$, and $z(+50) = 0.990$. Then combination of this results with (70) yields $(\Delta a_{\pm 5})^2 = 0.906(\Delta a_+)^2$, $(\Delta a_{\pm 15})^2 = 0.968(\Delta a_+)^2$, $(\Delta a_{\pm 25})^2 = 0.981(\Delta a_+)^2$, and $(\Delta a_{\pm 50})^2 = 0.990$ $(\Delta a_+)^2$. We immediately see that a large part of the quadrature variance is confined in a relatively small frequency interval.

Moreover, we note that the quadrature variance of a pair of superposed vacuum-vacuum states in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ can be obtained by setting $\varepsilon_1 = \varepsilon_2 = 0$ in (70). We then get

$$(\Delta a_{\pm\lambda})_v^2 = (\Delta a_{\pm v})^2 z_v(\lambda), \tag{72}$$

where

$$z_v(\lambda) = \frac{2}{\pi}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{73}$$

The plot in Figure 3 shows as λ increases, $z_v(\lambda)$ approaches to 1.

To this end, we define the local quadrature squeezing of a pair of superposed cavity light beams in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ by

$$S_{\pm\lambda} = \left[\frac{(\Delta a_{+\lambda})_v^2 - (\Delta a_{+\lambda})^2}{(\Delta a_{+\lambda})_v^2}\right]. \tag{74}$$

Then combination of (70), (72), and (74) leads to

$$S_{\pm\lambda} = 1 - \frac{1}{4}\left[\sum_{i=1}^{2}\frac{(\Delta a_+)_i^2\, z_{i+}(\lambda)}{z_v(\lambda)}\right], \tag{75}$$

where we have used the fact that

$$(\Delta a_{\pm vac.})^2 = 4. \tag{76}$$

On account of (70) and (72) together with (54), we see that

$$S_{\pm\lambda} = 1 - \frac{1}{4}\left[\frac{\left(\frac{2\kappa}{\kappa + 2\varepsilon_1}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon_1}\right) + \left(\frac{2\kappa}{\kappa + 2\varepsilon_2}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon_2}\right)}{\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}\right]. \quad (77)$$

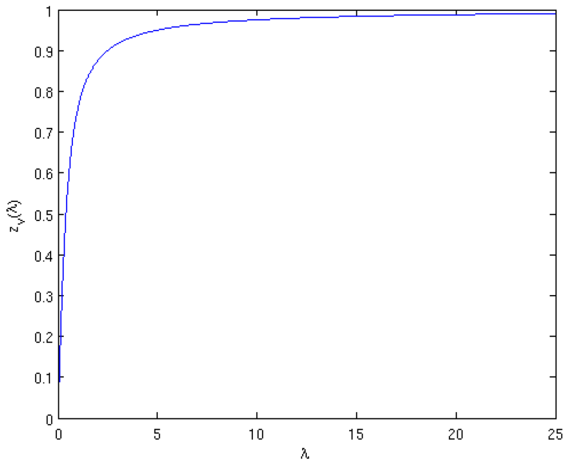


Figure 3. A plot of $z_v(\lambda)$ [Eq. 73] versus λ for $\kappa = 0.8$.

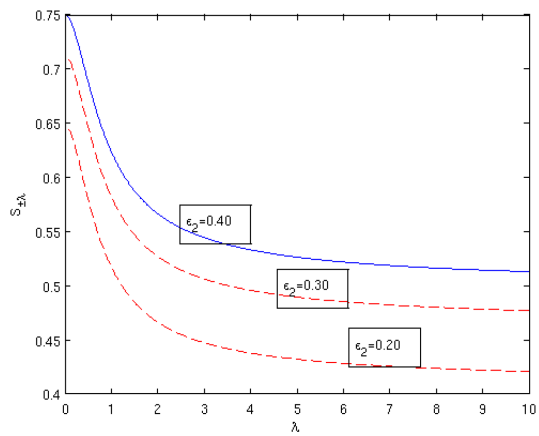


Figure 4. A plot of $S_{\pm\lambda}$ [Eq. 77] versus λ for $\kappa = 0.8$ and $\varepsilon_1 = 0.4$.

We immediately notice that the quadrature squeezing of the superposed light beams in a given frequency interval is not equal to that of the combined light in the entire frequency interval. We see from the plot in Figure 4 that the maximum local quadrature squeezing is 75% and occurs in the ± 0.01 frequency interval. In addition, we observe that the local quadrature squeezing approaches to the global quadrature squeezing as λ increases.

For $\varepsilon_1 = \varepsilon_2$, we easily obtain

$$S_{\pm\lambda} = 1 - \frac{1}{2}\left[\frac{\left(\frac{2\kappa}{\kappa + 2\varepsilon_1}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon_1}\right)}{\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}\right]. \tag{78}$$

We immediately observe that the local quadrature squeezing of a pair of superposed light beams is exactly equal to that of the separate light beams.

On the other hand, upon setting $\varepsilon_2 = 0$ in eq. (77), we see that

$$S_{\pm\lambda} = \frac{1}{2}\left\{1 - \frac{1}{2}\left[\frac{\left(\frac{2\kappa}{\kappa + 2\varepsilon_1}\right)\tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon_1}\right)}{\tan^{-1}\left(\frac{2\lambda}{\kappa}\right)}\right]\right\}. \tag{79}$$

This is half of the local quadrature squeezing of one of the constituent light beams.

5. The Entanglement Condition

In this section, we seek to study the entanglement condition for a pair of superposed signal-signal light beams generated by one-mode subharmonic generators. Hence in order to show the entanglement of a pair of superposed cavity light beams, we apply the criterion presented in [12]. According to this criterion, a pair of superposed cavity light beams is said to be entangled if the sum of the variance of the two EPR-like operators $\hat{s}$ and $\hat{t}$ satisfies the inequality

$$(\Delta s)^2 + (\Delta t)^2 < 4, \tag{80}$$

where

$$\hat{s} = \frac{1}{\sqrt{2}}(\hat{a}_{1+} - \hat{a}_{2+}), \tag{81}$$

$$\hat{t} = \frac{1}{\sqrt{2}}(\hat{a}_{1-} + \hat{a}_{2-}), \tag{82}$$

with

$$\hat{a}_{1+}(t) = \hat{a}_1^{\dagger}(t) + \hat{a}_1(t), \tag{83}$$

$$\hat{a}_{1-}(t) = i(\hat{a}_1^\dagger(t) - \hat{a}_1(t)), \tag{84}$$

$$\hat{a}_{2+}(t) = \hat{a}_2^\dagger(t) + \hat{a}_2(t), \tag{85}$$

and

$$\hat{a}_{2-}(t) = i(\hat{a}_2^\dagger(t) - \hat{a}_2(t)). \tag{86}$$

The steady-state variance of the operators $\hat{s}$ and $\hat{t}$ can be expressed as

$$(\Delta s)^2 = \langle \hat{s}^2 \rangle - \langle \hat{s} \rangle^2 \tag{87}$$

and

$$(\Delta t)^2 = \langle \hat{t}^2 \rangle - \langle \hat{t} \rangle^2. \tag{88}$$

Thus employing (81), (83), and (85), one can readily obtains

$$(\Delta s)^2 = \left[2 + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_1^{\dagger 2} \rangle + \langle \hat{a}_2^{\dagger 2} \rangle\right]. \tag{89}$$

Following the same procedure, we get

$$(\Delta t)^2 = \left[2 + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_1^{\dagger 2} \rangle + \langle \hat{a}_2^{\dagger 2} \rangle\right]. \tag{90}$$

On account of (89) and (90) along with (52), we have

$$(\Delta s)^2 + (\Delta t)^2 = (\Delta a_+)^2. \tag{91}$$

Moreover, in view of (91) and (54) at steady-state and threshold, the sum of the variance of the two EPR-like operators to be

$$(\Delta s)^2 + (\Delta t)^2 = 2. \tag{92}$$

On the basis of the criteria (80), we clearly see that a pair of superposed signal-signal light beams are entangled at steady-state and the entanglement turned out be observed in the squeezed photons.

6. Conclusion

We have analyzed the statistical and squeezing properties of a pair of superposed signal-signal light beams applying the density operator, the Q function, and a slightly modified definition of the entanglement condition. Employing the density operator as well as the Q function, we have found that the mean photon number of a pair of superposed light beams to be the sum of the mean photon numbers of the constituent light beams. However, the photon number variance of a pair of superposed light beams does not happen to be the sum of the photon number variances of the separate light beams. We have also observed that a large part of the mean photon number is confined in a relatively small frequency interval.

Furthermore, applying a slightly modified definition of the quadrature variance, we have obtained that the quadrature variance of a pair of superposed light beams to be the sum of the quadrature variances of the individual light beams and the superposed light beams are in a squeezed state and the squeezing occurs in the plus quadrature. Moreover, the global quadrature squeezing of a pair of superposed light beams turned out to be the average of the global quadrature squeezing of the component light beams. In light of this, the vacuum state indeed affects the quadrature variance and squeezing of a pair of superposed light beams. Besides, our analysis shows that at steady state and at threshold, the superposed light beams have a maximum squeezing of 50% below the vacuum state level in the entire frequency interval.

On the other hand, we have observed that the local quadrature squeezing of a pair of superposed light beams is in general greater than the global quadrature squeezing and approaches to the global quadrature squeezing as λ increases. We have noticed that the maximum local quadrature squeezing is 75% below the vacuum state level and occurs in the ± 0.01 frequency interval. We have also clearly shown that a pair of superposed light beams are entangled at steady-state and the entanglement turned out be observed in the squeezed photons.

To this end, we would like to mention that the predictions made in this paper concerning the local mean photon number and the local quadrature squeezing to be experimentally verified.

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