Degenerate Three-Level Laser

Coupled to Thermal Reservoir

Hadush Kiros

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By

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Abstract

In this thesis, we have analyzed the photon statistic and quadrature fluctuations of the degenerate three level laser coupled to thermal reservoir. We have found that the mean photon number of the system under consideration decreases with η but increases with $\overline{n}$. Moreover, the presence of the thermal light indeed affects the squeezing of the degenerate three level laser.

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Introduction

Laser stands for light amplification by the stimulated emission of radiation. Most people are familiar with the atoms which consists of protons, neutrons, and electrons. The energies of the electrons are quantized that is the electrons can only have certain energies. For an electron to move between energy levels it must either gain or lose energy and there are three processes which are absorption, spontaneous emission, and stimulated emission [1]. When a three-level atom in cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have same frequency, the three-level atom is called degenerate [2]. We consider a degenerate three-level laser in which three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate and removed from the laser cavity after some time [2]. The interaction of atoms with photons is one of the central problems in quantum optics. In recent years, a three-level cascade laser has drawn a considerable attention in connection with its potential as a source of squeezed light [5-12]. The squeezing feature of emitted photons is due to atomic coherence that can be induced by preparing atoms in a coherent superposition of the top and bottom levels [5-7]. It has been established that a three level laser under certain conditions generates squeezed light [6-12]. In a cascade three level laser, three level atoms in a cascade configuration were injected into a cavity coupled to vacuum reservoir via a single port mirror. The injected atoms may initially be prepared in a coherent super position of the top and bottom levels. A three-level laser has been studied by some authors [5-7] and the cavity photons are found to be in a squeezed state under certain conditions. In addition, the mean and variance of the photon number for a degenerate three-level cascade laser have been calculated [16]. It is found that the mean photon number of the cavity photons increases with the linear gain coefficient and the cavity radiation exhibits super-Poissonian photon statistics. Ansari [9] has predicted that such a laser can generate under certain conditions squeezed light. A three level laser in which the top and bottom levels of the atoms injected into the cavity are coupled by a strong light has also been studied by different authors [7-9]. Anwar. NA [11] has considered a degenerate three level laser with the atoms initially in the upper level and with the top and bottom levels of the atoms coupled by coherent light. Recently, a three-level laser whose cavity contains a parametric amplifier has been studied [17]. It is found that the parametric amplifier enhances the degree of squeezing of the cavity radiation. Moreover, the cavity light of a degenerate three-level cascade laser in which the top and bottom levels are coupled by strong coherent light has been investigated [11-13].

In this thesis, we study the squeezing and the statistical properties of the light produced by degenerate three level laser coupled to thermal reservoir. We first derive the master equation, stochastic differential equations along with the correlation properties of the noise forces. We then determine the solution of c-number Langevin equation. We also obtain the anti-normally order characteristics function with the aid of which the Q-function is established. The resulting Q-function along with density operator is then used to calculate the mean photon number, the variance of the photon number, the photon number distribution, power spectrum, and quadrature variance.

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The Q Function

In the first three sections of this chapter we focus on developing the master equation, the stochastic differential equations, and the solution of the c-number Langevin equation. Then in the last two sections, we determine the Q function and the density operator.

2.1 The master equation

We consider a degenerate three-level laser in which three-level atoms in a cascade configuration and initially prepared in coherent superposition of the top and bottom levels are injected at a constant rate and removed from the laser cavity after some time.We denote the top, intermediate, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$. We assume the cavity mode to be at resonance with two tranisition $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. We next drive the time evolution of reduced density operator for a cavity mode coupled to thermal resevoir via a single port-mirror. The Hamiltonian describing the interaction of a three-level

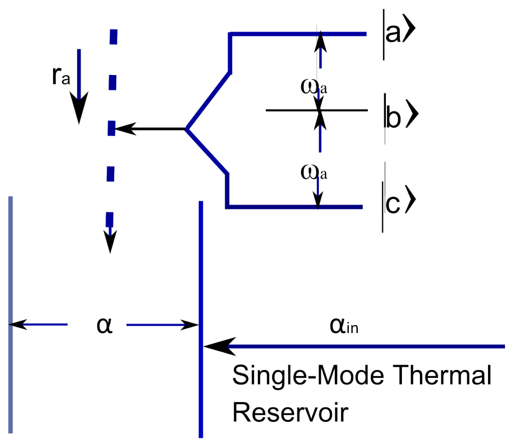

Figure 2.1: A schematic representation of a degenerate three level laser coupled to thermal reservoir.

atom with the cavity mode is expressible as

$$\hat{H}_I = \imath g\Big((|a\rangle\langle b| + |b\rangle\langle c|)\hat{a} - \hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|)\Big), \tag{2.1}$$

where g is the coupling constant and $\hat{a}$ is the annihilation operator for the cavity mode. We take the initial state of a single three-level atom to be

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle \tag{2.2}$$

and hence the initial density operator of a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.3}$$

where

$$\rho_{aa}^{(0)} = C_a^* C_a, \tag{2.4}$$

$$\rho_{ac}^{(0)} = C_a C_c^*, \tag{2.5}$$

$$\rho_{ca}^{(0)} = C_c C_a^*, \tag{2.6}$$

$$\rho_{cc}^{(0)} = C_c^* C_c. \tag{2.7}$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity mode at time t, with the atom injected at time t_j, such that $(t-\tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as $\hat{\rho}_{AR} = \sum_j N_j \hat{\rho}_{AR}(t, t_j)$. Then it follows

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j) \Delta t_j, \tag{2.8}$$

where $N = r_a \Delta t_j$, represents the number of atoms injected into the cavity in a time Δt_j and r_a is the rate at which atoms are injected into the cavity.

Now converting the summation into integration in the time $\Delta t_j \rightarrow 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t') dt' \tag{2.9}$$

and on differentiating with respect to t, there follows

$$\frac{d}{dt} \hat{\rho}_{AR}(t) = r_a [\hat{\rho}_{AR}(t) - \hat{\rho}_{AR}(t, t-\tau)] + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t} \hat{\rho}_{AR}(t, t') dt'. \tag{2.10}$$

We observe that $\hat{\rho}_{AR}(t, t')$ is the density operator for the cavity mode plus an atom injected at time t. This operator can thus be expressed as

$$\hat{\rho}_{AR}(t) = \hat{\rho}_A(t) \hat{\rho}(t), \tag{2.11}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone. We also note that $\hat{\rho}_{AR}(t, t-\tau)$ represents the density operator for an atom plus the cavity mode at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau) \hat{\rho}(t). \tag{2.12}$$

Now in view of Eqs. (2.11) and (2.12), one can rewrite Eq. (2.10) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau)]\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.13}$$

In the absence of damping of the cavity mode by a vaccum reservoir, the density operator $\hat{\rho}_{AR}(t,t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i[\hat{H}_I, \hat{\rho}_{AR}(t,t')], \tag{2.14}$$

so that using this relation and taking into account Eq. (2.9), one can put Eq. (2.13) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau)]\hat{\rho}(t) - i[\hat{H}_I, \hat{\rho}_{AR}(t)]. \tag{2.15}$$

Now tracing Eq. (2.15) over the atomic variables, we observe that

$$\frac{d}{dt}\hat{\rho}(t) = -iTr_A[\hat{H}_I, \hat{\rho}_{AR}(t)]. \tag{2.16}$$

Taking into account the damping of the cavity mode by a thermal reservoir, we found

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = &-\imath Tr_A[\hat{H}_I, \hat{\rho}_{AR}(t)] - hTr_R(\hat{H}_{SR}^2\hat{R})\hat{\rho}\\ &-h\hat{\rho}Tr_R(\hat{H}_{SR}^2\hat{R}) + 2hTr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}).\end{aligned} \tag{2.17}$$

The interaction of a cavity mode with a reservoir can be described by the Hamiltonian

$$\hat{H}_{SR} = \imath\lambda(a^\dagger\hat{a}_{in} - a_{in}^\dagger\hat{a}), \tag{2.18}$$

where λ is the coupling constant, $\hat{a}$ is the annihilation operator for the cavity mode and $\hat{a}_{in}$ is the annihilation operator for the thermal reservoir. Applying the fact $[\hat{a}_{in}, \hat{a}] = 0$, we observe that

$$Tr_R(\hat{H}_{SR}^2\hat{R}) = -\lambda^2\bigg[\hat{a}^{\dagger 2}\langle\hat{a}_{in}^2\rangle_R - \hat{a}^\dagger\hat{a}\langle\hat{a}_{in}\hat{a}_{in}^\dagger\rangle_R - \hat{a}\hat{a}^\dagger\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle_R + \hat{a}\hat{a}^\dagger\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle_R\bigg]. \tag{2.19}$$

We recall that density operator for a light in a chaotic state is given by

$$\hat{\rho} = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}|n\rangle\langle n|. \tag{2.20}$$

Employing this density operator, one can easily calculate the mean photon number of the thermal reservoir, we thus see that

$$\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}Tr\bigg(|n\rangle\langle n|\hat{a}_{in}^\dagger\hat{a}_{in}\bigg), \tag{2.21}$$

from which follows

$$\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle_R = \bar{n}. \tag{2.22}$$

Following the same proceder, we get

$$\langle\hat{a}_{in}^2\rangle_R = \langle\hat{a}_{in}^{\dagger 2}\rangle_R = 0, \tag{2.23}$$

$$\langle\hat{a}_{in}\hat{a}_{in}^\dagger\rangle_R = \bar{n}+1, \tag{2.24}$$

where the commutation relation $[\hat{a}_{in}, \hat{a}_{in}^\dagger] = 1$.

In view of Eq. (2.22), (2.23), and (2.24) along with (2.19), we have

$$Tr_R(\hat{H}_{SR}^2\hat{R}) = -\lambda^2\bigg[-(\bar{n}+1)\hat{a}^\dagger\hat{a} - \bar{n}\hat{a}\hat{a}^\dagger\bigg]. \tag{2.25}$$

Similarly, one can readily obtain

$$Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = -\lambda^2\bigg[-\bar{n}(\hat{a}^\dagger\hat{\rho}\hat{a}) - (\bar{n}+1)(\hat{a}\hat{\rho}\hat{a}^\dagger)\bigg]. \tag{2.26}$$

Now combination of Eqs. (2.25), (2.26), and (2.17) yields

$$\frac{d\hat{\rho}}{dt} = -\imath Tr_A[\hat{H}_I(t), \hat{\rho}_{AR}(t)] + \frac{1}{2}\kappa(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{1}{2}\kappa\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \tag{2.27}$$

in which

$$\kappa = 2h\lambda^2, \tag{2.28}$$

is the cavity dampying constant.

Moreover, employing Eq. (2.1), the master equation for the degenerate three-level atom in a cavity coupled to thermal reservoir, can be put in the form

$$\frac{d\hat{\rho}}{dt} = g(\rho_{ab}\hat{a}^\dagger - \hat{a}^\dagger\rho_{ab} + \rho_{bc}\hat{a}^\dagger - \hat{a}^\dagger\rho_{bc} + \hat{a}\rho_{ba} - \rho_{ba}\hat{a} + \hat{a}\rho_{cb} - \rho_{cb}\hat{a}) + \frac{1}{2}\kappa(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{1}{2}\kappa\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \tag{2.29}$$

in which the matrix element $\rho_{\alpha\beta}$ is defined by

$$\rho_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.30}$$

with $\alpha, \beta = a, b, c$.

On the other hand , we see from Eq. (2.15) that

$$\frac{d\rho_{\alpha\beta}}{dt} = [r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - r_a\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle]\hat{\rho}(t) - i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}_I|\beta\rangle] - \gamma\rho_{\alpha\beta}, \tag{2.31}$$

where the last term is included to account for the decay of atoms due to spontaneous emission. Here γ, considerered to be the same for all the three-levels, is the atomic decay constant. We assume that the atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle] = 0 \tag{2.32}$$

and hence Eq. (2.31) reduces to

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}_I|\beta\rangle] - \gamma\rho_{\alpha\beta}. \tag{2.33}$$

Applying this equation and taking into acount Eqs. (2.1) and (2.3), we can readily obtain

$$\frac{d\rho_{ab}}{dt} = g(\rho_{ac}^{(0)}\hat{a}^\dagger + \hat{a}\rho_{bb}^{(0)} - \rho_{aa}^{(0)}\hat{a}) - \gamma\rho_{ab}, \tag{2.34}$$

$$\frac{d\rho_{bc}}{dt} = g(\hat{a}\rho_{cc}^{(0)} - \rho_{bb}^{(0)}\hat{a} - \hat{a}^\dagger\rho_{ac}^{(0)}) - \gamma\rho_{bc}, \tag{2.35}$$

$$\frac{d\rho_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho} + g(\rho_{ab}^{(0)}\hat{a}^\dagger + \hat{a}\rho_{ba}^{(0)}) - \gamma\rho_{aa}, \tag{2.36}$$

$$\frac{d\rho_{bb}}{dt} = g(\rho_{bc}^{(0)}\hat{a}^\dagger + \hat{a}\rho_{cb}^{(0)} - \hat{a}^\dagger\rho_{ab}^{(0)} - \rho_{ba}^{(0)}\hat{a}) - \gamma\rho_{bb}, \tag{2.37}$$

$$\frac{d\rho_{ac}}{dt} = r_a\rho_{ac}^{(0)}\hat{\rho} + g(\hat{a}\rho_{bc}^{(0)} - \rho_{ab}^{(0)}\hat{a}) - \gamma\rho_{ac}, \tag{2.38}$$

$$\frac{d\rho_{cc}}{dt} = r_a\rho_{cc}^{(0)}\hat{\rho} - g(\hat{a}^\dagger\rho_{ba}^{(0)} + \rho_{cb}^{(0)}\hat{a}) - \gamma\rho_{cc}. \tag{2.39}$$

Thus upon dropping the g-terms and applying the large-time approximation scheme, we get from Eqs. (2.36) - (2.39) that

$$\rho_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}(t), \tag{2.40}$$

$$\rho_{bb} = 0, \tag{2.41}$$

$$\rho_{ac} = \frac{r_a \rho_{ac}^{(0)}}{\gamma} \hat{\rho}(t), \tag{2.42}$$

$$\rho_{cc} = \frac{r_a \rho_{cc}^{(0)}}{\gamma} \hat{\rho}(t). \tag{2.43}$$

Now combination of Eqs. (2.34), (2.35), (2.40), (2.41), (2.42), and (2.43) leads to

$$\frac{d\rho_{ab}}{dt} = \frac{g r_a}{\gamma} (\rho_{ac}^{(0)} \hat{\rho} \hat{a}^{\dagger} - \rho_{aa}^{(0)} \hat{\rho} \hat{a}) - \gamma \rho_{ab}, \tag{2.44}$$

$$\frac{d\rho_{bc}}{dt} = \frac{g r_a}{\gamma} (\rho_{cc}^{(0)} \hat{a} \hat{\rho} - \rho_{ac}^{(0)} \hat{a}^{\dagger} \hat{\rho}) - \gamma \rho_{bc}. \tag{2.45}$$

Using once more the large -time approximation scheme, we easily find

$$\rho_{ab} = \frac{g r_a}{\gamma^2} (\rho_{ac}^{(0)} \hat{\rho} \hat{a}^{\dagger} - \rho_{aa}^{(0)} \hat{\rho} \hat{a}), \tag{2.46}$$

$$\rho_{bc} = \frac{g r_a}{\gamma^2} (\rho_{cc}^{(0)} \hat{a} \hat{\rho} - \rho_{ac}^{(0)} \hat{a}^{\dagger} \hat{\rho}). \tag{2.47}$$

Finally, on account of Eqs. (2.44) and (2.45), the master equation for the cavity mode produced by degenerate three-level laser coupled to thermal reservoir takes the form

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = &\frac{1}{2}(A\hat{\rho}_{aa}^{(0)} + \kappa\bar{n})(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}) \\ &+\frac{1}{2}(A\hat{\rho}_{cc}^{(0)} + \kappa(\bar{n}+1))(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) \\ &+\frac{1}{2}A\hat{\rho}_{ac}^{(0)}(\hat{a}^{\dagger 2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger 2} - 2\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}) \\ &+\frac{1}{2}A\hat{\rho}_{ca}^{(0)}(\hat{a}^{2}\hat{\rho} + \hat{\rho}\hat{a}^{2} - 2\hat{a}\hat{\rho}\hat{a}), \end{aligned} \tag{2.48}$$

where

$$A = \frac{2 r_a g^2}{\gamma^2}, \tag{2.49}$$

is linear gain coefficient.

2.2 Stochastic differential equations

In this section, we obtain the stochastic differential equations with the aid of master equation described by Eq. (2.48). Then empolying the relation

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\bigg(\frac{d\hat{\rho}}{dt}\hat{a}\bigg), \tag{2.50}$$

one can write

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle = &\frac{1}{2}\bigg(A\hat{\rho}^{(0)}_{aa} + \kappa\bar{n}\bigg)Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a})\\ &+\frac{1}{2}\bigg(A\hat{\rho}^{(0)}_{cc} + \kappa(\bar{n}+1)\bigg)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a})\\ &+\frac{1}{2}A\hat{\rho}^{(0)}_{ac}Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a} + \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a})\\ &+\frac{1}{2}A\hat{\rho}^{(0)}_{ca}Tr(\hat{\rho}\hat{a}^3 + \hat{a}^2\hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^2).\end{aligned} \tag{2.51}$$

Applying the cyclic property of the trace operation and the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2.52}$$

we get

$$\frac{d}{dt}\langle\hat{a}\rangle = \frac{1}{2}\bigg(A\rho^{(0)}_{aa} - A\rho^{(0)}_{cc} - \kappa\bigg)\langle\hat{a}\rangle. \tag{2.53}$$

Following the same procedure, it can also be verfied that

$$\frac{d}{dt}\langle\hat{a}^2\rangle = \bigg(A\rho^{(0)}_{aa} - A\rho^{(0)}_{cc} - \kappa\bigg)\langle\hat{a}^2\rangle + A\rho^{(0)}_{ac}, \tag{2.54}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = \bigg(A\rho^{(0)}_{aa} - A\rho^{(0)}_{cc} - \kappa\bigg)\langle\hat{a}^\dagger\hat{a}\rangle + A\rho^{(0)}_{aa} + \kappa\bar{n}. \tag{2.55}$$

We note that the c-number equations corresponding to Eqs. (2.53), (2.54), and (2.55) in the normal order are

$$\frac{d}{dt}\langle\alpha(t)\rangle = \frac{1}{2}\bigg(A\rho^{(0)}_{aa} - A\rho^{(0)}_{cc} - \kappa\bigg)\langle\alpha(t)\rangle, \tag{2.56}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = \left(A\rho_{aa}^{(0)} - A\rho_{cc}^{(0)} - \kappa\right)\langle\alpha^2(t)\rangle + A\rho_{ac}^{(0)}, \tag{2.57}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \left(A\rho_{aa}^{(0)} - A\rho_{cc}^{(0)} - \kappa\right)\langle\alpha^*(t)\alpha(t)\rangle + A\rho_{aa}^{(0)} + \kappa\bar{n}. \tag{2.58}$$

On the basis of Eq. (2.56), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu\alpha(t) + f_\alpha(t), \tag{2.59}$$

where $\mu = A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)} + \kappa$ and $f_\alpha(t)$ is a noise force whose correlation property remains to be determined. We now proceed to determine the correlation properties of the noise force. Taking the expectation value of Eq. (2.59), we observe that

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu\langle\alpha(t)\rangle + \langle f_\alpha(t)\rangle. \tag{2.60}$$

We note that Eqs. (2.56) and (2.60), will have the same form if

$$\langle f_\alpha(t)\rangle = 0. \tag{2.61}$$

Applying the relation

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = \langle\alpha(t)(\frac{d}{dt}\alpha(t))\rangle + \langle(\frac{d}{dt}\alpha(t))\alpha(t)\rangle, \tag{2.62}$$

along with Eq. (2.59), we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + \langle\alpha(t)f_\alpha(t)\rangle + \langle f_\alpha(t)\alpha(t)\rangle. \tag{2.63}$$

Comparison of Eqs. (2.57) and (2.63) shows that

$$\langle\alpha(t)f_\alpha(t)\rangle = \frac{1}{2}A\rho_{ac}^{(0)}. \tag{2.64}$$

The formal solution of Eq. (2.59) can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}f_\alpha(t')dt'. \tag{2.65}$$

Thus employing this formal solution, we can express Eq. (2.64) as

$$\langle\alpha(0)f_\alpha(t)\rangle e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle dt' = \frac{1}{2}A\rho_{ac}^{(0)}. \tag{2.66}$$

Taking into account Eq. (2.61) and the fact that a noise force at some time does not affects the cavity mode variables at earlier time, we see that

$$\langle\alpha(0)f_\alpha(t)\rangle = \langle\alpha(0)\rangle\langle f_\alpha(t)\rangle = 0, \tag{2.67}$$

in view of which Eq. (2.66) becomes

$$\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle dt' = \frac{1}{2}A\rho_{ac}^{(0)}. \tag{2.68}$$

Applying the relation

$$\int_0^t e^{-\frac{1}{2}a(t-t')}\langle f(t)g(t')\rangle dt' = D, \tag{2.69}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'). \tag{2.70}$$

We therefore, see from Eq. (2.68) that

$$\langle f_\alpha(t')f_\alpha(t)\rangle = A\rho_{ac}^{(0)}\delta(t-t'). \tag{2.71}$$

Futhermore, applying the relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = (\langle\alpha(t)\frac{d}{dt}\alpha^*(t)\rangle) + \langle\alpha^*(t)\frac{d}{dt}\alpha(t)\rangle), \tag{2.72}$$

along with Eq. (2.59) and its complex conjugate, we have

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle + \langle\alpha(t)f_\alpha^*(t)\rangle. \tag{2.73}$$

Comparison of this equation with Eq. (2.58) indicates that

$$\langle \alpha^*(t) f_\alpha(t) \rangle + \langle \alpha(t) f^*(t) \rangle = A\rho_{aa}^{(0)} + \kappa\bar{n}. \tag{2.74}$$

Now applying Eq. (2.65) and its complex conjugate in Eq. (2.74), we get

$$\begin{aligned} A\rho_{aa}^{(0)} + \kappa\bar{n} = {} & (\langle \alpha^*(0) f_\alpha(t) \rangle + \langle \alpha(0) f_\alpha^*(t) \rangle) e^{-\frac{1}{2}\mu(t-t')} \\ & + \int_0^t e^{-\frac{1}{2}\mu(t-t')} [\langle f_\alpha^*(t') f_\alpha(t) \rangle + \langle f_\alpha^*(t') f_\alpha(t) \rangle] dt'. \end{aligned} \tag{2.75}$$

In view the relation $\langle \alpha^*(0) f_\alpha(t) \rangle = \langle \alpha(0) f_\alpha^*(t) \rangle = 0$, we can put Eq. (2.75) in the form

$$\int_0^t e^{-\frac{1}{2}\mu(t-t')} [\langle f_\alpha^*(t') f_\alpha(t) \rangle + \langle f_\alpha^*(t) f_\alpha(t') \rangle] dt' = A\rho_{aa}^{(0)} + \kappa\bar{n} \tag{2.76}$$

and assuming that

$$\langle f_\alpha^*(t') f_\alpha(t) \rangle = \langle f(t') f^*(t) \rangle, \tag{2.77}$$

we have

$$\int_0^t e^{-\frac{1}{2}\mu(t-t')} \langle f_\alpha^*(t') f_\alpha(t) \rangle] dt' = \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa\bar{n}). \tag{2.78}$$

It then follows that

$$\langle f^*(t') f(t) \rangle = \left(A\rho_{aa}^{(0)} + \kappa\bar{n} \right) \delta(t-t'). \tag{2.79}$$

In view of Eq. (2.77), we note that

$$\langle f_\alpha(t') f_\alpha^*(t) \rangle = \left(A\rho_{aa}^{(0)} + \kappa\bar{n} \right) \delta(t-t'). \tag{2.80}$$

The results described by Eqs. (2.61), (2.71), and (2.79) represents the correlation properties of noise forces associated with normal ordering.

We next proceed to determine the solution Eq. (2.59). We then define a new variable

$$\alpha_{\pm}(t)=\alpha^{*}(t)\pm\alpha(t) \tag{2.81}$$

and using Eq. (2.59) along with its complex conjugate, we find

$$\frac{d}{dt}\alpha_{\pm}=-\frac{1}{2}\mu\alpha_{\pm}+f_{\alpha}^{*}(t)\pm f_{\alpha}(t), \tag{2.82}$$

where we have used

$$\mu=A(\rho_{cc}^{(0)}-\rho_{aa}^{(0)})+\kappa. \tag{2.83}$$

The solution Eq. (2.82) can be written in the form

$$\alpha_{\pm}(t)=\alpha_{\pm}(0)e^{-\frac{1}{2}\mu t}+\int_{0}^{t}e^{-\frac{1}{2}\mu(t-t')}[f_{\alpha}^{*}(t')\pm f_{\alpha}(t')]dt', \tag{2.84}$$

so that with the aid of Eqs. (2.81) and (2.84), we readily find

$$\alpha(t)=E_{+}(t)\alpha(0)+E_{-}(t)\alpha^{*}(0)+F(t), \tag{2.85}$$

where

$$E_{\pm}(t)=\frac{1}{2}(e^{-\frac{1}{2}\mu(t-t')}\pm(e^{-\frac{1}{2}\mu(t-t')}), \tag{2.86}$$

$$F(t)=\int_{0}^{t}[E_{+}(t)f(t')+E_{-}(t)f^{*}(t')]dt'. \tag{2.87}$$

2.3 The Q function

We now proceed to obtain the Q function for the cavity mode coupled to thermal reservoir. The Q function for a single-mode light can be expressed in terms of the antinormally ordered characteristic function as

$$Q(\alpha^{*},\alpha,t)=\frac{1}{\pi}\int\frac{d^{2}z}{\pi}\Phi_{a}(z^{*},z,t)e^{z^{*}\alpha-z\alpha^{*}}, \tag{2.88}$$

where $\Phi_a(z^*, z, t)$ is defined in the Heisenberge picture by

$$\Phi_a(z^*, z, t) = Tr(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}). \tag{2.89}$$

Apply the relation

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{2.90}$$

which holds for $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, Eq. (2.89) can be rewritten as

$$\Phi_a(z^*, z, t) = e^{-z^*z}Tr(\rho(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}). \tag{2.91}$$

We note that Eq. (2.91) can be expressed in terms of c-number variables associated with the normal ordering as

$$\Phi_a(z^*, z, t) = e^{-z^*z}\langle e^{z\alpha^*(t)}e^{-z^*\alpha(t)}\rangle. \tag{2.92}$$

On the basis of Eq. (2.85) and assuming that the cavity mode is initially in vaccum state thus we observe that $\alpha(t)$ is a Gaussian variable with zero mean. On account of this, Eq. (2.92) can be put in the form [2].

$$\Phi_a(z^*, z, t) = e^{-z^*z}exp\langle\frac{1}{2}(z\alpha^*(t) - z^*\alpha(t))^2\rangle. \tag{2.93}$$

Then leads to

$$\Phi_a(z^*, z, t) = e^{-^*z}exp\Big[- z^*z\langle\alpha^*(t)\alpha(t)\rangle + \frac{1}{2}z^2\langle\alpha^{*2}(t)\rangle + \frac{1}{2}z^{*2}\langle\alpha^2(t)\rangle\Big]. \tag{2.94}$$

We now proceed to obtain the expectation value of c-number variables appearing in Eq. (2.94). Taking into account the fact that a noise force at a given instant does not affect the cavity mode variable at earlier time, we find

$$\langle\alpha^*(t)\alpha(t)\rangle = \langle F^*(t)F(t)\rangle, \tag{2.95}$$

so that using Eq. (2.86), we have

$$\langle\alpha^*(t)\alpha(t)\rangle = \int_0^t dt'dt''\Bigg[(E_+(t-t')E_+(t-t'')\langle f^*(t')f(t'')\rangle$$
$$+E_-(t-t')E_-(t-t'')\langle f^*(t'')f(t')\rangle$$
$$+E_+(t-t')E_-(t-t'')\langle f^*(t')f^*(t'')\rangle$$
$$+E_-(t-t')E_+(t-t'')\langle f(t')f(t'')\rangle\Bigg]. \tag{2.96}$$

With the aids of Eqs. (2.71) and (2.79), we get

$$\langle\alpha^*(t)\alpha(t)\rangle = \int_0^t dt'\Bigg[(E_+^2(t-t')+E_-^2(t-t')[A\rho_{aa}^{(0)}+\kappa\bar{n}]$$
$$+E_+(t-t')E_-(t-t')[A(\rho_{ac}^{(0)}+\rho_{ac}^{*(0)})\Bigg] \tag{2.97}$$

and employing Eq. (2.86), we have

$$\langle\alpha^*(t)\alpha(t)\rangle = \frac{1}{4}\int_0^t dt'\Bigg[(A[\rho_{ac}^{(0)}+\rho_{ac}^{*(0)}+2\rho_{aa}^{(0)}]+2\kappa\bar{n})e^{-\mu_-(t-t')}$$
$$-(A[\rho_{ac}^{(0)}+\rho_{ac}^{*(0)}-2\rho_{aa}^{(0)}]-2\kappa\bar{n})e^{-\mu_+(t-t')}\Bigg]. \tag{2.98}$$

It then follows that

$$\langle\alpha^*(t)\alpha(t)\rangle = \frac{1}{4\mu_-}\bigg(A[\rho_{ac}^{(0)}+\rho_{ac}^{*(0)}+2\rho_{aa}^{(0)}]+2\kappa\bar{n})(1-e^{-\mu_-t}\bigg)$$
$$-\frac{1}{4\mu_+}\bigg(A([\rho_{ac}^{(0)}+\rho_{ac}^{*(0)}-2\rho_{aa}^{(0)}]-2\kappa\bar{n}\bigg)(1-e^{-\mu_+t})\bigg). \tag{2.99}$$

Following a similar proceeder, we easily find

$$\langle\alpha^2(t)\rangle = \frac{1}{4\mu_-}\bigg(A[\rho_{ac}^{(0)}+\rho_{ac}^{*(0)}+2\rho_{aa}^{(0)}]+2\kappa\bar{n}\bigg)(1-e^{-\mu_-t}\bigg)$$
$$+\frac{1}{4\mu_+}\bigg(A([\rho_{ac}^{(0)}+\rho_{ac}^{*(0)}-2\rho_{aa}^{(0)}]-2\kappa\bar{n}\bigg)(1-e^{-\mu_+t}\bigg)$$
$$+\frac{A}{\mu_-+\mu_+}\bigg(\rho_{ac}^{(0)}-\rho_{ac}^{*(0)}\bigg)(1-e^{-(\mu_-+\mu_+)\frac{t}{2}}\bigg) \tag{2.100}$$

and

$$\langle\alpha^{*2}(t)\rangle = \frac{1}{4\mu_-}\bigg(A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} + 2\rho_{aa}^{(0)}] + 2\kappa\bar{n}\bigg)(1 - e^{-\mu_- t}\bigg)$$
$$+\frac{1}{4\mu_+}\bigg(A([\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} - 2\rho_{aa}^{(0)}] - 2\kappa\bar{n}\bigg)(1 - e^{-\mu_+ t}\bigg)$$
$$-\frac{A}{\mu_- + \mu_+}\bigg(\rho_{ac}^{(0)} - \rho_{ac}^{*(0)}\bigg)(1 - e^{-(\mu_- + \mu_+)\frac{t}{2}}\bigg). \tag{2.101}$$

In view of Eq. (2.94), we see that

$$\Phi_a(z^*, z, t) = exp\bigg[- az^*z + \frac{1}{2}(b^*z^2 + bz^{*2})\bigg], \tag{2.102}$$

in which

$$a = 1 + \langle\alpha^*(t)\alpha(t)\rangle, \tag{2.103}$$

$$b = \langle\alpha^2(t)\rangle, \tag{2.104}$$

$$b^* = \langle\alpha^{*2}(t)\rangle. \tag{2.105}$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1 - \eta}{2}, \tag{2.106}$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1 \tag{2.107}$$

and

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{2.108}$$

one easily finds

$$\rho_{cc}^{(0)} = \frac{1 + \eta}{2}, \tag{2.109}$$

$$|\rho_{ac}^{(0)}| = \frac{1}{2}(1-\eta^2)^{\frac{1}{2}}, \tag{2.110}$$

and

$$\rho_{cc}^{(0)} - \rho_{aa}^{(0)} = \eta. \tag{2.111}$$

With the aid of Eqs. (2.111) and (2.83), we see that

$$\mu = A\eta + \kappa. \tag{2.112}$$

Upon setting

$$\rho_{ac}^{(0)} = |\rho_{ac}^{(0)}|e^{i\theta}, \tag{2.113}$$

we have

$$\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} = \sqrt{1-\eta^2}\cos\theta. \tag{2.114}$$

On account of Eqs. (2.99), (2.100), and (2.101) along with Eqs.(2.103), (2.104), and (2.105), we observe that

$$\begin{aligned} a = 1 &+ \frac{\left((A[1-\eta+\sqrt{1-\eta^2}\cos\theta] + 2\kappa\bar{n})(1-e^{-(A\eta+\kappa)t})\right)}{4(A\eta+\kappa)} \\ &+ \frac{\left((A[1-\eta-\sqrt{1-\eta^2}\cos\theta] + 2\kappa\bar{n})(1-e^{-(A\eta+\kappa)t})\right)}{4(A\eta+\kappa)}, \end{aligned} \tag{2.115}$$

$$\begin{aligned} b = &\frac{\left((A[1-\eta+\sqrt{1-\eta^2}\cos\theta] + 2\kappa\bar{n})(1-e^{-(A\eta+\kappa)t})\right)}{4(A\eta+\kappa)} \\ &- \frac{\left((A[1-\eta-\sqrt{1-\eta^2}\cos\theta] + 2\kappa\bar{n})(1-e^{-(A\eta+\kappa)t})\right)}{4(A\eta+\kappa)} \\ &+ \frac{iA\sqrt{1-\eta^2}\sin\theta\left(1-e^{-(A\eta+\kappa)t}\right)}{2(A\eta+\kappa)}, \end{aligned} \tag{2.116}$$

$$b^* = \frac{\left((A[\sqrt{1-\eta^2}\cos\theta + (1-\eta)] + 2\kappa\bar{n})(1-e^{-(A\eta+\kappa)t})\right)}{4(A\eta+\kappa)} - \frac{\left((A[\sqrt{1-\eta^2}\cos\theta - (1-\eta)] + 2\kappa\bar{n})(1-e^{-(A\eta+\kappa)t})\right)}{4(A\eta+\kappa)} - \frac{iA\sqrt{1-\eta^2}\sin\theta\left(1-e^{-(A\eta+\kappa)t}\right)}{2(A\eta+\kappa)}. \tag{2.117}$$

Hence applying Eq. (2.102) in Eq. (2.88) and upon carrying out the integration with the aid of the relation

$$\int \frac{d^2\alpha}{\pi} e^{-a\alpha^*\alpha + b\alpha + c\alpha^* + B\alpha^2 + C\alpha^{*2}} = \left[\frac{1}{a^2-4BC}\right]^{\frac{1}{2}} exp\left[\frac{abc + Bc^2 + Cb^2}{a^2-4BC}\right], a > 0 \tag{2.118}$$

the Q function is found to be

$$Q(\alpha^*, \alpha, t) = \frac{(u^2 - v^*v)^{\frac{1}{2}}}{\pi} exp\left[-u\alpha^*\alpha + \frac{1}{2}(v^*\alpha^2 + v\alpha^{*2})\right], \tag{2.119}$$

where

$$u = \frac{a}{(a^2 - b^*b)}, \tag{2.120}$$

$$v = \frac{b}{(a^2 - b^*b)} \tag{2.121}$$

and

$$v^* = \frac{b^*}{(a^2 - b^*b)}. \tag{2.122}$$

This represents the Q function for the degenerate three level laser coupled to thermal reservoir.

2.4 The densty operator

We seek to determine the densty operator for the cavity mode. Suppose $\hat{\rho}(\hat{a}^{\dagger}, \hat{a})$ is density operator for a certain lihgt beam. The normally-ordered density operator can be expressed as

$$\hat{\rho}^{l}(t) = \sum_{kl} C_{kl} \hat{a}^{\dagger k} \hat{a}^{l}. \tag{2.123}$$

Now we recall the completeness relation for coherent state as [2]

$$\frac{1}{\pi} \int d^2 |\alpha\rangle\langle\alpha| = \hat{I}. \tag{2.124}$$

On the other hand, the expectation value of an operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a}, t)$ can be put in the form

$$\langle \hat{A}(\hat{a}^{\dagger}, \hat{a}, t)\rangle = Tr(\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)\hat{A}(0)). \tag{2.125}$$

To this end, using then completeness relation given by Eq. (2.123) twice; we have

$$\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t) = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} |\alpha\rangle\langle\alpha|\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)|\beta\rangle\langle\beta|. \tag{2.126}$$

This can be rewriten as in the form

$$\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t) = \frac{1}{\pi} \int d^2\alpha d^2\beta Q(\alpha^*, \beta, t)\langle\alpha|\beta\rangle|\alpha\rangle\langle\beta| \tag{2.127}$$

in which

$$Q(\alpha^*, \beta, t) = \frac{1}{\pi} \langle\alpha|\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)|\beta\rangle. \tag{2.128}$$

Therefore, in view of Eq. (2.125) and (2.127), the expectation value of a given operator function $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ is expressible as

$$\langle \hat{A}(\hat{a}^\dagger, \hat{a}, t)\rangle = \frac{1}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta, t) exp\Big[-\alpha^*\alpha - \beta^*\beta + \beta^*\alpha - \alpha^*\beta\Big] A_n(\alpha, \beta^*), \tag{2.129}$$

where $A_n(\alpha^*, \beta)$ is c-number function corresponding to the $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ in the normal order. Then the expectation value of a given operator function $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ can be rewritten as

$$\begin{aligned}\langle \hat{A}(\hat{a}^\dagger, \hat{a}, t)\rangle = [u^2 - v^*v]^{\frac{1}{2}} \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} exp\Big[- u\alpha^*\beta \\ + \frac{1}{2}(v\alpha^{*2} + v^*\beta^2) - \alpha^*\alpha - \beta^*\beta + \beta^*\alpha + \alpha^*\beta\Big] A_n(\alpha, \beta^*).\end{aligned} \tag{2.130}$$

This is the expectation value of operator $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ for the degenerate three-level coupled to thermal light in which $A_n(\alpha, \beta^*)$ is c-number function corresponding to the operator variables in the normal order.

3

Photon Statistics

It would be helpful to classify the photon statistics of light modes based on the relation between the variance and mean of the photon number. Hence the photon statistics of a light mode for which $\Delta n^2 = \bar{n}$ is referred to as Poissonian and the photon statistics of light mode for which $\Delta n^2 > \bar{n}$ is called super-Poissonian. Otherwise, the photon statistics is said to be sup-Poissonian. Here we wish to calculate the mean photon number, variance of photon number, power spectrum, and the photon number distribution.

3.1 The mean photon number

The mean photon number can be expressed as

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \int d^2\alpha Q(\alpha^*, \alpha, t) A_a(\alpha^*, \alpha). \tag{3.1}$$

It then follows that

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \int d^2\alpha Q(\alpha^*, \alpha, t)(\alpha^*\alpha - 1), \tag{3.2}$$

where

$$A_a(\alpha^*, \alpha) = \alpha^*\alpha - 1, \tag{3.3}$$

is the c-number function corresponding to operator function $\hat{a}^\dagger(t)\hat{a}(t)$ in the antinormally order.

Taking in to account Eq. (2.119), the mean photon number can be written as

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2 - v^*v)^{\frac{1}{2}} \int \frac{d^2\alpha}{\pi} exp\Big[- u\alpha^*\alpha + \frac{1}{2}(v^*\alpha^2 + v\alpha^{*2})\Big](\alpha^*\alpha - 1), \tag{3.4}$$

this can be put in the form

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2 - v^*v)^{\frac{1}{2}} \frac{d^2}{dndm} \int \frac{d^2}{\pi} exp\Big[- u\alpha^*\alpha + n\alpha + m\alpha^* + \frac{1}{2}(v^*\alpha^2 + v\alpha^{*2})\Big]_{n=m=0} - 1, \tag{3.5}$$

so that upon carrying out the integration using the relation described by Eq. (2.118), and applying the condition n=m=o, we get

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{u}{u^2 - v^*v} - 1. \tag{3.6}$$

Now in view of Eqs. (2.120), (2.121), and (2.122) along with Eq. (2.115), we can write

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = a - 1, \tag{3.7}$$

then follows that

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{(A[1-\eta+\sqrt{1-\eta^2}\cos\theta] + 2\kappa\bar{n})}{4(A\eta+\kappa)}(1 - e^{-(A\eta+\kappa)t}) + \frac{(A[1-\eta-\sqrt{1-\eta^2}\cos\theta] + 2\kappa\bar{n})}{4(A\eta+\kappa)}(1 - e^{-(A\eta+\kappa)t}) \tag{3.8}$$

and this can be rewritten as

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{(2A[1-\eta] + 4\kappa\bar{n})}{4(A\eta+\kappa)}\Big(1 - e^{-(A\eta+\kappa)t}\Big). \tag{3.9}$$

This represents the mean photon number for the degenerate three-level laser

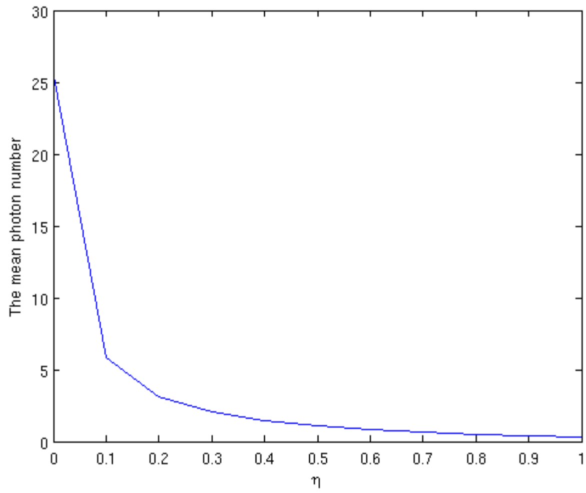

Fig. 3.1: plot of the mean photon number $\bar{n}$ versus η for $A = 25$, $\bar{n} = 20$, and $\kappa = 0.8$

coupled to themal reservoir. The result described by Eq. (3.9) at steady state can be put in the form

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} = \frac{A(1-\eta)}{2(A\eta+\kappa)} + \frac{\kappa\bar{n}}{A\eta+\kappa}. \tag{3.10}$$

We immediatly see that the mean photon number of the system under consideration does not happens to be the sum of the mean photon number of the laser and the thermal light. We clearly observe from fig. 3.1 that the mean photon number decreases with η.

Moreover, upon setting $\bar{n} = 0$, we easily get

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} = \frac{A(1-\eta)}{2(A\eta+\kappa)}. \tag{3.11}$$

This is the mean photon number of the the degenerate three level laser coupled to vacuum reservoir.

3.2 The variance of the photon number

The variance of the photon number can be expressed as

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{3.12}$$

The photon number variance for the cavity mode can be rewritten as

$$(\Delta n)^2 = \langle \hat{a}^2(t)\hat{a}^{\dagger 2}(t) \rangle - 3\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle - \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^2 - 2, \tag{3.13}$$

where $\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle$ is the meam photon number for the laser plus the thermal light. Thus employing the Q function, we see that

$$\langle \hat{a}^2(t)\hat{a}^{\dagger 2}(t) \rangle = \int d^2\alpha Q(\alpha^*, \alpha, t)\alpha^{*2}\alpha^2. \tag{3.14}$$

This can be put in the form

$$\langle \hat{a}^2(t)\hat{a}^{\dagger 2}(t) \rangle = (u^2 - v^* v)^{\frac{1}{2}} \frac{d^4}{d\eta^2 dz^2} \int \frac{d^2\alpha}{\pi} exp\bigg[- u\alpha^*\alpha + \eta\alpha + z\alpha^* + \frac{1}{2}(v^*\alpha^2 + v\alpha^{*2}) \bigg]_{\eta = z = 0}. \tag{3.15}$$

Upon carrying out the intgration using Eq. (2.118), we get

$$\langle \hat{a}^2(t)\hat{a}^{\dagger 2}(t) \rangle = \frac{d^4}{d\eta^2 dz^2} exp\bigg[uz\eta + \frac{1}{2}vz^2 + \frac{1}{2}v^*\eta^2 \bigg]_{\eta = z = 0}, \tag{3.16}$$

so that carrying out the differentiation and applying the condition $z = \eta = 0$, we readily find

$$\langle \hat{a}^2(t)\hat{a}^{\dagger 2}(t) \rangle = \frac{2u^2 + v^* v}{(u^2 - v^* v)^2}. \tag{3.17}$$

On account of Eqs. (2.120), (2.121), and (2.122), we easily obtain

$$\langle \hat{a}^2(t)\hat{a}^{\dagger 2}(t) \rangle = 2a^2 + b^* b. \tag{3.18}$$

Now with the aid of Eqs. (3.7) and (3.18), Eq. (3.13) can be put in the form

$$(\Delta n)^2 = a^2 + b^* b - a. \tag{3.19}$$

Finally, in view of Eqs. (2.115), (2.116), and (2.117), the variance of the photon number for the degenerate three level laser coupled to thermal reservoir has the form

$$\begin{aligned}(\Delta n)^2 = {} & 2\left[\frac{A[1+\sqrt{1-\eta^2}\cos\theta]+\kappa\bar{n}}{4(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t})\right]^2 \\ & +2\left[\frac{A[1-\sqrt{1-\eta^2}\cos\theta]+\kappa\bar{n}}{4(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t})\right]^2 \\ & +\frac{A^2(1-\eta^2)\sin^2\theta}{4(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t})^2 \\ & -\left[\frac{A[1+\sqrt{1-\eta^2}\cos\theta]+2\kappa\bar{n}}{4(A\eta+\kappa)}(1-e^{(A\eta+\kappa)}) \right.\\ & \left. +\frac{A[1-\sqrt{1-\eta^2}\cos\theta]+2\kappa\bar{n}}{4(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t})\right].\end{aligned} \tag{3.20}$$

Then the variance of the photon number at steady state is expressible as

$$\begin{aligned}(\Delta n)^2 = {} & 2\left[\frac{A[1+\sqrt{1-\eta^2}\cos\theta]+\kappa\bar{n}}{4(A\eta+\kappa)}\right]^2 + 2\left[\frac{A[1-\sqrt{1-\eta^2}\cos\theta]+\kappa\bar{n}}{4(A\eta+\kappa)}\right]^2 \\ & +\left[\frac{A^2(1-\eta^2)\sin^2\theta}{4(A\eta+\kappa)}\right] - \left[\frac{A[1+\sqrt{1-\eta^2}\cos\theta]+2\kappa\bar{n}}{4(A\eta+\kappa)}\right] \\ & +\left[\frac{A[1-\sqrt{1-\eta^2}\cos\theta]+2\kappa\bar{n}}{4(A\eta+\kappa)}\right].\end{aligned} \tag{3.21}$$

On comparing Eq. (3.10) and Eq. (3.21) that the photon statistics is super-Poissonian.

For the case in which $\bar{n}$=0

$$\begin{aligned}(\Delta n)^2 = {} & 2\left[\frac{A[1+\sqrt{1-\eta^2}\cos\theta]}{4(A\eta+\kappa)}\right]^2 + 2\left[\frac{A[1-\sqrt{1-\eta^2}\cos\theta]}{4(A\eta+\kappa)}\right]^2 \\ & +\left[\frac{A^2(1-\eta^2)\sin^2\theta}{4(A\eta+\kappa)}\right] - \left[\frac{A[1+\sqrt{1-\eta^2}\cos\theta]}{4(A\eta+\kappa)}\right] \\ & +\left[\frac{A[1-\sqrt{1-\eta^2}\cos\theta]}{4(A\eta+\kappa)}\right].\end{aligned} \tag{3.22}$$

This is the variance of the photon number for the degenerate three-level laser coupled to vacuum reservoir.

3.3 The power spectrum

It is now interesting to consider the power spectrum of the cavity light. The power spectrum of a single-mode light with centeral frequency ω_0 is expressible as [2]

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}. \tag{3.23}$$

We now proceed to calculate the two-time correlation function that appears in Eq. (3.23) for the cavity light. To this end, we realize that the formal solution of Eq. (2.65) can be written as

$$\alpha(t+\tau) = \alpha(t)e^{\frac{-\mu\tau}{2}} + \int_0^\tau e^{\frac{-\mu}{2}\frac{(\tau-\tau')}{2}} f(t+\tau')d\tau'. \tag{3.24}$$

Then multiplying both sides on the left by $\alpha^*(t)$ and taking the expectation value of the resulting equation, we have

$$\langle\alpha^*(t)\alpha(t+\tau)\rangle = \langle\alpha^*(t)\alpha(t)\rangle e^{\frac{-\mu\tau}{2}} + \int_0^\tau e^{\frac{-\mu}{2}\frac{(\tau-\tau')}{2}} \langle\alpha^*(t)f(t+\tau')\rangle d\tau', \tag{3.25}$$

so that in view of the fact that

$$\langle\alpha^*(t)f(t+\tau')\rangle = \langle\alpha^*(t)\rangle\langle f(t+\tau')\rangle = 0, \tag{3.26}$$

it then follows

$$\langle\alpha^*(t)\alpha(t+\tau)\rangle_{ss} = \langle\alpha^*(t)\alpha(t)\rangle_{ss} e^{\frac{-1}{2}\mu\tau}. \tag{3.27}$$

Substitution of Eq. (3.27) into (3.23) and carryingout the integration employing the relation

$$\frac{1}{\pi} Re \int_0^\infty d\tau e^{-[\frac{\Gamma}{2} - i(\eta-\eta_0)]\tau} = \frac{\frac{\Gamma}{2\pi}}{(\eta-\eta_0)^2 + (\frac{\Gamma}{2})^2}, \tag{3.28}$$

we arive at

$$P(\omega) = \langle\alpha^*(t)\alpha(t)\rangle_{ss}\frac{\frac{\mu}{2\pi}}{(\omega-\omega_0)^2+(\frac{\mu}{2})^2}. \tag{3.29}$$

Upon integrating both sides of Eq. (3.29) over ω, we readily get

$$\int_{-\infty}^{+\infty} P(\omega)d\omega = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss}, \tag{3.30}$$

in which

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} = \frac{A(1-\eta)}{2A(\eta+\kappa)} + \frac{\kappa\bar{n}}{A\eta+\kappa}, \tag{3.31}$$

is the steady-state mean photon number of the cavity light produced by degenerate three-level laser coupled to thermal reservoir. From this result, we observe that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$.

We next seek to calculate the mean photon number in a given frequency interval. We thus realize that the steady-state mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ can be written as

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{\pm\lambda} = \int_{-\lambda}^{+\lambda} P(\omega')d\omega', \tag{3.32}$$

wher $\omega' = \omega - \omega_0$. Therefore, using Eq. (3.29) and the fact that

$$\int_{-\lambda}^{+\lambda} \frac{d\omega'}{\omega^2+a^2} = \frac{2}{a}\tan^{-1}(\frac{\lambda}{a}), \tag{3.33}$$

we readily obtain

$$\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{\pm\lambda} = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss}z(\lambda), \tag{3.34}$$

where

$$z(\lambda) = \frac{2}{\pi}\tan^{-1}(\frac{2\lambda}{\mu}). \tag{3.35}$$

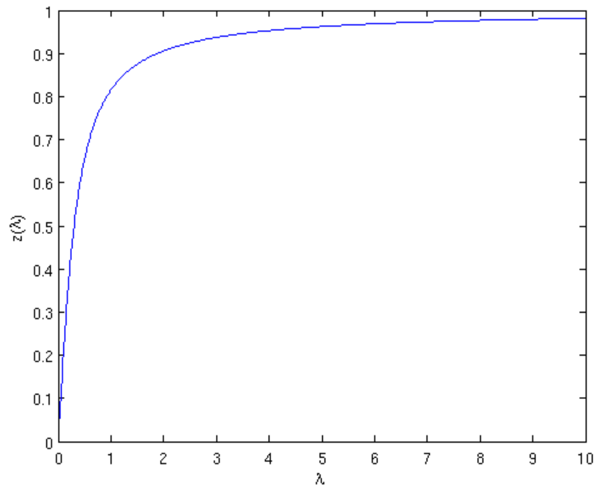

Fig. 3.2: Plot for z(λ) versus λ for $\eta = \kappa = 0.2$ and A=2.

One can easily get from Fig. 3.2 that z(0.15)=0.295, z(0.61)=0.709, z(1.36)=0.822, z(2.72)=0.93, z(6.76)=0.97. The combonation of this results with Eq. (3.34) yields

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{\pm 0.15} = 0.295\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle, \tag{3.36}$$

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{\pm 0.61} = 0.709\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle, \tag{3.37}$$

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{\pm 1.36} = 0.822\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle, \tag{3.38}$$

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{\pm 2.72} = 0.93\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle, \tag{3.39}$$

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{\pm 6.76} = 0.97\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle. \tag{3.40}$$

We immediatly see that a large part of the total mean photon number is confined in a relatively small frequency interval.

3.4 The photon number distribution

We wish here to obtain an explict expression of the photon number distribution employing the Q function for single-mode light.

The photon number distribution for a single-mode light defined by

$$P(n,t) = \langle n|\hat{\rho}(\hat{a}^{\dagger},\hat{a},t)|n\rangle. \tag{3.41}$$

Introducing Eq. (2.127) in (3.41), we see that

$$P(n) = \frac{1}{\pi^2}\int d^2z d^2\eta Q(z^*,\eta,t)\langle n|z\rangle\langle\eta|n\rangle\langle z|\eta\rangle. \tag{3.42}$$

Now using the Q function described by Eq. (2.119), Eq. (3.42), can be rewritten as

$$\begin{aligned} P(n) = \frac{[u^2-v^*v]^{\frac{1}{2}}}{n!\pi}\frac{\partial^{2n}}{\partial\alpha\partial\alpha^{*n}}\int\frac{d^2z}{\pi}exp\Big[-z^*z+\frac{v}{2}z^{*2}+\alpha^*z\Big] \\ \int\frac{d^2\eta}{\pi}exp\Big[-\eta^*\eta+\frac{v^*}{2}\eta^2-uz^*\eta+\alpha\eta^*+z^*\eta\Big]\Big|_{\alpha^*=\alpha=0}, \end{aligned} \tag{3.43}$$

where

$$\langle z|\eta\rangle = e^{-\frac{z^*z}{2}-\frac{\eta^*\eta}{2}+z^*\eta}, \tag{3.44}$$

$$\langle n|z\rangle = e^{\frac{-z^*z}{2}}\frac{z^n}{\sqrt{n!}}, \tag{3.45}$$

and

$$\langle\eta|n\rangle = e^{\frac{-\eta^*\eta}{2}}\frac{\eta^{*n}}{\sqrt{n!}}. \tag{3.46}$$

Upon carry out the integration, we readily obtain

$$P(n) = \frac{(u^2-v^*v)^{\frac{1}{2}}}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}exp\Big[(1-u)\alpha^*\alpha+\frac{v}{2}\alpha^{*2}+\frac{v^*}{2}\alpha^2\Big]_{\alpha^*=\alpha=0}. \tag{3.47}$$

Now expanding the exponential functions using of the power series, we have

$$P(n,t) = \frac{(u^2 - v^*v)^{\frac{1}{2}}}{n!} \sum_{l,k,p} \frac{(-1)^{(k+p)}(1-u)^l v^{*k} v^p}{2^{k+p} l! k! p!} \times \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}(\alpha^{*l+2k}\alpha^{l+2p})_{\alpha^*=\alpha=0}. \tag{3.48}$$

Thus performing the differentiation, employing the relation

$$\frac{\partial^m}{\partial\alpha^m} x^n = \sum_s^{\infty} \frac{n!}{(n-m)!} x^{n-m}, \tag{3.49}$$

we notice that

$$\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\alpha^{*l+2k}\alpha^{l+2p} = \frac{(l+2k)!\alpha^{*l+2k-n}}{(l+2k-n)!}\frac{(l+2p)!\alpha^{l+2p-n}}{(l+2p-n)!}. \tag{3.50}$$

Then the combination Eq. (3.47) and (3.50) leads to

$$P(n,t) = \frac{(u^2 - v^*v)^{\frac{1}{2}}}{n!} \sum_{l,k,p} \frac{(-1)^{(k+p)}(1-u)^l v^{*k} v^p (l+2k)!(l+2p)!}{2^{k+p} l! k! p! (l+2k-n)!(l+2p-n)!} \times (\alpha^{*(l+2k-n)}\alpha^{(l+2p-n)})_{\alpha^*=\alpha=0}. \tag{3.51}$$

Imposing the condition $\alpha^* = \alpha = 0$,we see that

$$P(n,t) = \frac{(u^2 - v^*v)^{\frac{1}{2}}}{n!} \sum_{l,k,p} \frac{(-1)^{(k+p)}(1-u)^l v^{*k} v^p (l+2k)!(l+2p)!}{2^{k+p} l! k! p! (l+2k-n)!(l+2p-n)!} \times \delta_{l+2k,n}\delta_{l+2p,n}. \tag{3.52}$$

Finally, in view of the fact that $p = k$ and $l = n - 2k$, the photon number distribution can put in form

$$P(n) = (u^2 - v^*v)^{\frac{1}{2}} \sum_{k=0}^{[n]} \frac{n!(1-u)^{n-2k}(v^*v)^k}{2^{2k}(k!)^2(n-2k)!}, \tag{3.53}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = (n - \frac{1}{2})$ for odd. From this result, we note that there is a finite probability to find number of photons inside the cavity.

4

Quadrature Fluctuations

In this chapter we seek to calculate the quadrature Fluctuations of the light produced by degenerate three-level laser coupled to thermal reservoir.

4.1 Quadrature variance

The quadratue variance of a single-mode light can be defined as

$$(\Delta a_{\pm})^2 = 1 \pm \langle : \hat{a}_{\pm}(t), \hat{a}_{\pm}(t) : \rangle, \tag{4.1}$$

where

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a} \tag{4.2}$$

and

$$\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a}), \tag{4.3}$$

with $\hat{a}_{+}$ and $\hat{a}_{-}$ are the plus and minus quadratue operators and :: stands for normal ordering.

We note that the c-number equation corresponding to Eq. (4.1) is

$$(\Delta a_{\pm})^2 = 1 \pm \langle : \alpha_{\pm}(t), \alpha_{\pm}(t) : \rangle. \tag{4.4}$$

Then Eq. (4.4) can be rewritten as

$$(\Delta a_{\pm})^2 = 1 \pm \langle : \alpha_{\pm}^2(t)\rangle \mp \langle \alpha_{\pm}(t) :\rangle^2, \tag{4.5}$$

where $\alpha_{\pm}(t) = \alpha^*(t) \pm \alpha(t)$. Using Eq. (2.84), one can write

$$\langle \alpha_{\pm}(t)\rangle = \langle \alpha_{\pm}(0)\rangle e^{-\mu\frac{t}{2}} + \int_0^t e^{-\mu\frac{t-t'}{2}}[\langle f^*(t')\pm\langle f(t')\rangle]dt'. \tag{4.6}$$

Taking into account Eq. (2.61) and assumming the cavity mode to be initially a vaccum state, we have

$$\langle \alpha_{\pm}(t)\rangle = 0. \tag{4.7}$$

Furthermore, employing Eq. (2.65) along with Eq. (2.61) and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, one can write

$$\begin{aligned}\langle \alpha_{\pm}^2\rangle = \int dt'dt'' \Big[\langle f^*(t')f^*(t'')\rangle + \langle f(t')f(t'')\rangle + \langle f^*(t')f(t'')\rangle \\ \pm\langle f(t')f^*(t'')\rangle\Big] e^{-\mu\frac{(2t-t'-t'')}{2}}.\end{aligned} \tag{4.8}$$

Applying Eq. (2.71) and (2.80) and carrying out the integration, we obtain

$$\langle \alpha_{\pm}^2\rangle = \frac{[A(\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}) \pm 2\kappa\bar{n}]}{\mu}(1-e^{-\mu t}). \tag{4.9}$$

Hence using Eq. (4.9), we finally obtain that

$$\langle : \alpha_{\pm}(t), \alpha_{\pm}(t) :\rangle = \frac{[A(\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}) \pm 2\kappa\bar{n}]}{\mu}(1-e^{-\mu t}). \tag{4.10}$$

Now employing Eqs. (2.106), (2.112), and (2.114), we put Eq. (4.10) in the form

$$\langle : \alpha_{\pm}(t), \alpha_{\pm}(t) :\rangle = \frac{\left(A[\sqrt{1-\eta^2}\cos\theta \pm (1-\eta)] \pm 2\kappa\bar{n}\right)}{A\eta+\kappa}(1-e^{-(A\eta+\kappa)t}), \tag{4.11}$$

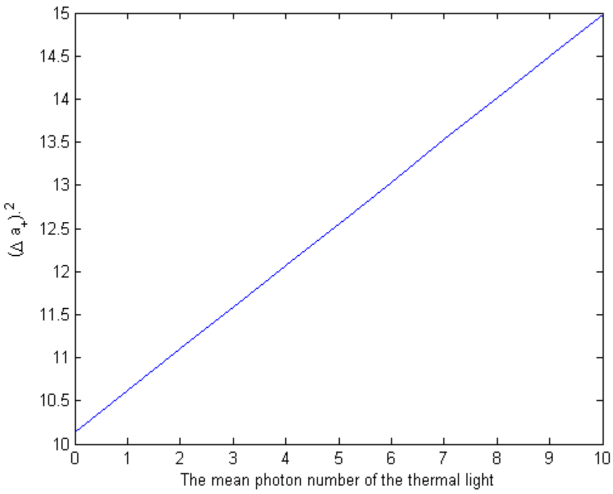

Figure 4.1: Plot for $(\Delta a_+)^2$ versus $\overline{n}$ (Eq. 4.13) for $\kappa = 0.8$, A=5, $\eta = 0.5$, and $\theta = 0^0$

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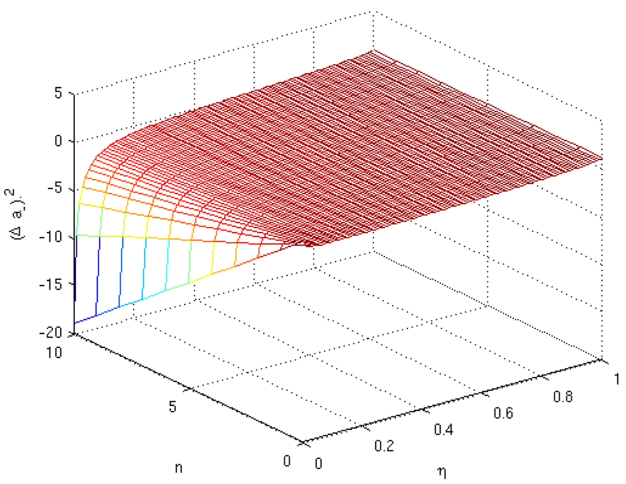

Figure 4.2: Plot for $(\Delta a_-)^2$ versus $\bar{n}$ and η (Eq. 4.14) for $\kappa = 0.8$, $A = 75$, and $\theta = 0^0$.

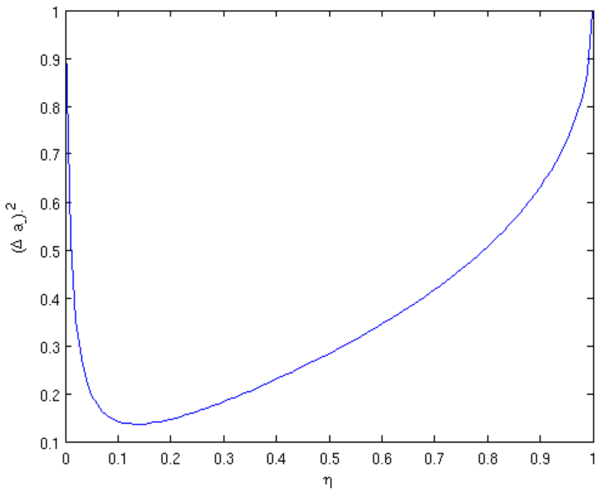

Figure 4.3: Plot of the quadratue variance $(\Delta a_{-})^2$ versus η [Eq. 4.15] for $\kappa = 0.8$, $A = 75$, and $\theta = 0^0$.

so that Eq. (4.4) becomes

$$(\Delta a_{\pm})^2 = 1 \pm \frac{\left[A\left(\sqrt{1-\eta^2}\cos\theta \pm (1-\eta)\right) \pm 2\kappa\bar{n}\right]}{A\eta + \kappa}(1 - e^{-(A\eta+\kappa)t}). \qquad (4.12)$$

Then the plus and minus quadrature variance of the cavity mode at the steady state can be put in the form

$$(\Delta a_{+})^2 = \frac{A[1 + \sqrt{1-\eta^2}\cos\theta] + \kappa(1+2\bar{n})}{A\eta + \kappa} \qquad (4.13)$$

and

$$(\Delta a_{-})^2 = \frac{A[1 - \sqrt{1-\eta^2}\cos\theta] + \kappa(1-2\bar{n})}{A\eta + \kappa}. \qquad (4.14)$$

We clearly observe from fig. 4.1 and 4.2 that the quadrature squeezing of the laser light indeed is affected by the thermal light and squeezing does not occur in all values of η. The quadratue variance of cavity mode coupled to a vaccum reservoir can be written as

$$(\Delta a_{\pm})^2 = \frac{A[1 \pm \sqrt{1-\eta^2}\cos\theta] + \kappa}{A\eta + \kappa}. \qquad (4.15)$$

Fig. 4.3 clearly indicates that the cavity mode is in the squeezed state for all values of η between zero and one.

5

Conclussion

In this thesis, we have studied the quadrature flactuations and photon statistics of the light produced by degenerate three level laser coupled to thermal reservoir. We have obtained the master equation together with stochastic differential equations. Applying the solutions of the resulting c-number Langevin equation, we have determined the Q function and the density operator. Then with the aid of the Q function along with the density operator, we have calculated the mean photon number, the variance of the photon, the power spectrum, the photon number distribution, and the quadrature variance. We have seen that the mean photon number of the system under consideration does not happens to be the sum of the mean photon number of the laser and the thermal light. We also clearly indicated that the mean photon number decreases with η but increases with $\overline{n}$. Furthermore, we have shown that a large part of the mean photon number is confined in a relatively small frequency interval. Moreover, the presence of the thermal light indeed affects the squeezing of the degenerate three level laser.

6

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DECLARATION

I hereby declare that this MSC thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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This MSC thesis has been submitted to for examination with our approval as University advisor and co-advisor.

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