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# ENTANGLED PHOTONIC MODES INSIDE COHERENTLY PUMPED OPTICAL CAVITY

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## Abstract

We analyze the quantum properties of the light generated by a non-degenerate three-level laser in which the three-level atoms available in an open cavity are pumped to the top level by means of strong coherent light. We carry out our analysis by putting the noise operators associated with a vacuum reservoir in normal order. It is found that the three-level laser generates squeezed light under certain conditions, with maximum intra-cavity squeezing being 55% below the coherent-state level. We have also established that the stimulated and spontaneous decay constants have directly and inversely proportional effect on the intensity of the light generated by the system, respectively. In addition, the light modes in the laser cavity are entangled at steady-state.

## 1. Introduction

One of the most fundamentally interesting and intriguing phenomena associated with a composite quantum system is entanglement. In recent years, the topic of continuous-variable entanglement has received a significant amount of attention as it plays an important role in all branches of quantum information processing [1]. The efficiency of quantum information schemes highly depends on the degree of

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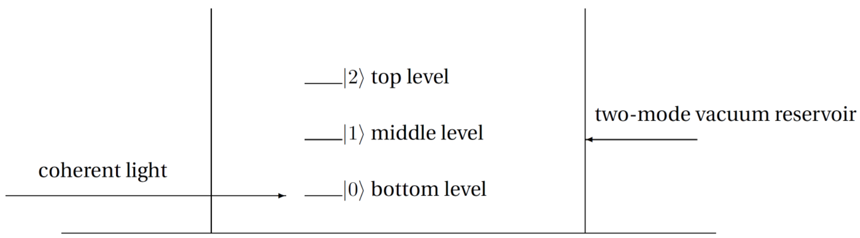
entanglement. A two-mode subharmonic generator at and above threshold has been theoretically predicted to be a source of light in an entangled state [2, 3]. Recently, the experimental realization of the entanglement in two-mode subharmonic generator has been demonstrated by Zhang et al. [4]. On the other hand, Xiong et al. [5] have recently proposed a scheme for an entanglement based on a non-degenerate three-level laser when the three level atoms are injected at the lower level and the top and bottom levels are coupled by a strong coherent light. They have found that a non-degenerate three-level laser can generate light in an entangled state employing the entanglement criteria for bipartite continuous-variable states [5].

Moreover, Tan et al. [6] have extended the work of Xiong et al. and examined the generation and evolution of the entangled light in the Wigner representation using the sufficient and necessary in separability criteria for a two-mode Gaussian state proposed by Dual et al. [5] and Simon [7]. Tesfa [8] has considered a similar system when the atomic coherence is induced by superposition of atomic states and analyzed the entanglement at steady-state. Furthermore, Ooi [9] has studied the steady-state entanglement in a two-mode $\Lambda$ laser.

More recently, Eyob [10] has studied continuous-variable entanglement in a non-degenerate three-level laser with a parametric amplifier. In this model the injected atomic coherence introduced by initially preparing the atoms in a coherent superposition of the top and bottom levels. In addition to exhibiting a two-mode squeezed light, this combined system produces light in an entangled state. In one model of such a laser, three-level atoms initially in the upper level are injected at a constant rate into the cavity and removed after they have decayed due to spontaneous emission. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the laser cavity. Besides, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity.

In order to avoid the aforementioned problems, Fesseha [11] have considered that $N$ two-level atoms available in a closed cavity are pumped to the top level by means of electron bombardment. He has shown that the light generated by this laser operating well above threshold is coherent and the light generated by the same laser operating below threshold is chaotic. In addition, Fesseha [12, 13] has studied the squeezing and the statistical properties of the light produced by a degenerate three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating far below threshold, is 50% below the coherent-state level. Alternatively, the three level atoms available in a closed cavity and pumped by coherent light also generated squeezed light under certain conditions, with the maximum quadrature squeezing being 43% below the coherent state level. In view of these results, better squeezing is found from the laser, in which the atoms are pumped by electron bombardment than by coherent light.

In this model, we seek to study CV entanglement for the light generated by a coherently pumped non-degenerate three-level laser coupled to a two-mode vacuum reservoirs via a single-port mirror whose open cavity contains $N$ non-degenerate three-level cascade atoms.




**Digram 1.** Coherently pumped non-degenerate three-level laser.

In order to carry out our analysis, we put the noise operators associated with the vacuum reservoir in the normal order and we consider the interaction of the three-level atoms with a twomode vacuum reservoir. We then first drive the quantum Langevin equations for the cavity mode operators. We next determine the equations of evolution of the expectation values of atomic operators employing the pertinent master equation. Applying the steady-state solution of equations of evolution, we analyze mean photon number, quadrature squeezing, and CV entanglement.

## 2. Dynamics of Atomic and Cavity Mode Operators

We consider a coherently pumped non-degenerate three level laser dynamics coupled to two-mode vacuum reservoir whose cavity contains $N$ three level atoms in cascade configuration as depicted in the schematic Diagram 1. For the sake of convenience, the bottom energy level is denoted by $|0\rangle_j$, the middle energy level by

$|1\rangle_j$, and the top energy level by $|2\rangle_j$. In order to expedite the cascading process, it is assumed that the parity of energy levels $|2\rangle_j$ and $|0\rangle_j$ is the same, where as that of $|1\rangle_j$ is different. This entails that direct transition between energy level $|2\rangle_j \leftrightarrow |0\rangle_j$ is electric dipole forbidden but due to parity difference, the transition between $|1\rangle_j \leftrightarrow |2\rangle_j$, and $|1\rangle_j \leftrightarrow |0\rangle_j$ are allowed.

While the atom undergoes a direct transition from energy level $|2\rangle_j$ to $|1\rangle_j$, suppose it emits a photon represented by $a_2$. In principle, it can still undergo a direct transition and go over to the lower energy level $|0\rangle_j$, in the process emitting a photon described by $a_1$. And the two atomic transitions are resonant with two different modes of the cavity. The interaction of a three-level atom with cavity modes can be described at resonance by the Hamiltonian

$$\hat{H}_1(t) = ig[\hat{\sigma}_1^{+j}(t)\hat{a}_1(t) - \hat{a}_1^\dagger(t)\hat{\sigma}_1^j(t) + \hat{\sigma}_2^{+j}(t)\hat{a}_2(t) - \hat{a}_2^\dagger(t)\hat{\sigma}_2^j(t)], \tag{1}$$

where

$$\hat{\sigma}_1^j(t) = |0\rangle_{jj}\langle 1| \tag{2}$$

and

$$\hat{\sigma}_2^j(t) = |1\rangle_{jj}\langle 2| \tag{3}$$

are lowering atomic operators, $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are the annihilation operators for the light modes $a_1$ and $a_2$, respectively, and $g$ is the coupling constant between the atom and the cavity modes.

The top and bottom level of the three-level atoms are coupled by a strong driving coherent light. The coupling of the top and bottom levels of a three-level atom by coherent light can be described by the Hamiltonian

$$\hat{H}_2(t) = \frac{i\Omega}{2}[\hat{\sigma}_0^{+j}(t) - \hat{\sigma}_0^j(t)] \tag{4}$$

in which

$$\hat{\sigma}_0^j(t) = |0\rangle_{jj}\langle 2|, \tag{5}$$

$$\Omega = 2\mu_0\lambda_0. \tag{6}$$

Here, $\mu_0$ is the amplitude of the coherent light and $\lambda_0$ is the coupling constant between the coherent light and the three-level atom. Thus combining Eqs. (1) and (4), the interaction of a three-level atoms with the cavity modes and the driving coherent light can be described by the Hamiltonian

$$\hat{H}_S(t) = ig[\hat{\sigma}_1^{+j}(t)\hat{a}_1(t) - \hat{a}_1^\dagger(t)\hat{\sigma}_1^j(t) + \hat{\sigma}_2^{+j}(t)\hat{a}_2(t) - \hat{a}_2^\dagger(t)\hat{\sigma}_2^j(t)]$$

$$+ \frac{i\Omega}{2}[\hat{\sigma}_0^{+j}(t) - \hat{\sigma}_0^j(t)]. \tag{7}$$

The master equation for a three-level atom coupled to a two-mode vacuum reservoir has the form

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}_S, \hat{\rho}] + \frac{\beta}{2}[2\hat{\sigma}_0^j\hat{\rho}\hat{\sigma}_0^{+j} - \hat{\sigma}_0^{\dagger j}\hat{\sigma}_0^j\hat{\rho} - \hat{\rho}\hat{\sigma}_0^{+j}\hat{\sigma}_0^j]$$

$$+ \frac{\beta}{2}[2\hat{\sigma}_1^j\hat{\rho}\hat{\sigma}_1^{+j} - \hat{\sigma}_1^{\dagger j}\hat{\sigma}_1^j\hat{\rho} - \hat{\rho}\hat{\sigma}_1^{+j}\hat{\sigma}_1^j]$$

$$+ \frac{\beta}{2}[2\hat{\sigma}_2^j\hat{\rho}\hat{\sigma}_2^{+j} - \hat{\sigma}_2^{\dagger j}\hat{\sigma}_2^j\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{+j}\hat{\sigma}_2^j], \tag{8}$$

where $\beta = 2hg'^2$ is spontaneous emission decay constant. Now with the aid of Eq. (7), one can put (8) in the form

$$\frac{d}{dt}\hat{\rho}(t) = g[\hat{\sigma}_1^{\dagger j}\hat{a}_1\hat{\rho} - \hat{\rho}\hat{\sigma}_1^{\dagger j}\hat{a}_1 + \hat{\rho}\hat{a}_1^\dagger\hat{\sigma}_1^j - \hat{a}_1^\dagger\hat{\sigma}_1^j\hat{\rho} + \hat{\sigma}_2^{\dagger j}\hat{a}_2\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{\dagger j}\hat{a}_2 + \hat{\rho}\hat{a}_2^\dagger\hat{\sigma}_2^j - \hat{a}_2^\dagger\hat{\sigma}_2^j\hat{\rho}]$$

$$+ \frac{\Omega}{2}[\hat{\sigma}_0^{\dagger j}\hat{\rho} - \hat{\rho}\hat{\sigma}_0^{\dagger j} + \hat{\rho}\hat{\sigma}_0^j - \hat{\sigma}_0^j\hat{\rho}] + \frac{\beta}{2}[2\hat{\sigma}_0^j\hat{\rho}\hat{\sigma}_0^{+j} - \hat{\sigma}_0^{\dagger j}\hat{\sigma}_0^j\hat{\rho} - \hat{\rho}\hat{\sigma}_0^{+j}\hat{\sigma}_0^j]$$

$$+ \frac{\beta}{2}[2\hat{\sigma}_1^j\hat{\rho}\hat{\sigma}_1^{+j} - \hat{\sigma}_1^{\dagger j}\hat{\sigma}_1^j\hat{\rho} - \hat{\rho}\hat{\sigma}_1^{+j}\hat{\sigma}_1^j] + \frac{\beta}{2}[2\hat{\sigma}_2^j\hat{\rho}\hat{\sigma}_2^{+j} - \hat{\sigma}_2^{\dagger j}\hat{\sigma}_2^j\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{+j}\hat{\sigma}_2^j]. \tag{9}$$

We model that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror. In addition, we carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators. We can therefore drop the noise operators and write the quantum Langevin equation for the operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$ as

$$\frac{d}{dt}\hat{a}_1(t) = -i[\hat{a}_1(t),\ \hat{H}_S(t)] - \frac{1}{2}\kappa\,\hat{a}_1(t) \tag{10}$$

and

$$\frac{d}{dt}\hat{a}_2(t) = -i[\hat{a}_2(t),\ \hat{H}_S(t)] - \frac{1}{2}\kappa\hat{a}_2(t), \tag{11}$$

in which $\kappa$ is assumed to be the cavity damping constant for the light modes $a_1$ and $a_2$. Then with the aid of Eqs. (7), (10), and (11) together with the commutation relation

$$[\hat{a}_i,\ \hat{a}_j^{\dagger}] = \delta_{ij}, \tag{12}$$

we easily find

$$\frac{d}{dt}\hat{a}_1(t) = -g\hat{\sigma}_1^j - \frac{1}{2}\kappa\hat{a}_1(t), \tag{13}$$

$$\frac{d}{dt}\hat{a}_2(t) = -g\hat{\sigma}_2^j - \frac{1}{2}\kappa\hat{a}_2(t). \tag{14}$$

Making use of the pertinent master equation and the fact that

$$\frac{d}{dt}\langle A(t)\rangle = Tr\left(\frac{d\hat{\rho}(0)}{dt}\hat{A}(t)\right), \tag{15}$$

where $\hat{A}(t)$ is an operator in the Heisenberg picture, it is not difficult to verify that

$$\frac{d}{dt}\langle\hat{\sigma}_1^j\rangle = g[\langle\hat{n}_0^j\hat{a}_1\rangle - \langle\hat{n}_1^j\hat{a}_1\rangle - \langle\hat{a}_2^{\dagger}\hat{\sigma}_0^j\rangle] - \frac{\Omega}{2}\langle\sigma_2^{\dagger j}\rangle - \frac{\beta}{2}\langle\sigma_1^j\rangle, \tag{16}$$

$$\frac{d}{dt}\langle\hat{\sigma}_2^j\rangle = g[\langle\hat{n}_1^j\hat{a}_2\rangle - \langle\hat{\eta}_2^j\hat{a}_2\rangle + \langle\hat{a}_1^{\dagger}\hat{\sigma}_0^j\rangle] + \frac{\Omega}{2}\langle\sigma_1^{\dagger j}\rangle - \beta\langle\sigma_2^j\rangle, \tag{17}$$

$$\frac{d}{dt}\langle\hat{\sigma}_0^j\rangle = g[\langle\hat{\sigma}_1^j\hat{a}_2\rangle - \langle\hat{\sigma}_2^j\hat{a}_1\rangle] + \frac{\Omega}{2}[\langle n_0^j\rangle - \langle n_2^j\rangle] - \frac{\beta}{2}\langle\sigma_0^j\rangle, \tag{18}$$

$$\frac{d}{dt}\langle\hat{n}_1^j\rangle = g[\langle\hat{\sigma}_1^{\dagger j}\hat{a}_1\rangle + \langle\hat{a}_1^{\dagger}\hat{\sigma}_1^j\rangle - \langle\hat{\sigma}_2^{\dagger j}\hat{a}_2\rangle - \langle\hat{a}_2^{\dagger}\hat{\sigma}_2^j\rangle] + \beta(\langle n_2^j\rangle - \langle n_1^j\rangle), \tag{19}$$

$$\frac{d}{dt}\langle \hat{n}_2^j \rangle = g[\langle \hat{\sigma}_2^{\dagger j} \hat{a}_2 \rangle + \langle \hat{a}_2^{\dagger} \hat{\sigma}_2^j \rangle] + \frac{\Omega}{2}[\langle \hat{\sigma}_0^{\dagger j} \rangle + \langle \hat{\sigma}_0^j \rangle] - 2\beta\langle n_2^j \rangle, \tag{20}$$

$$\frac{d}{dt}\langle \hat{n}_0^j \rangle = -g[\langle \hat{\sigma}_1^{\dagger j} \hat{a}_1 \rangle + \langle \hat{a}_1^{\dagger} \hat{\sigma}_1^j \rangle] - \frac{\Omega}{2}[\langle \hat{\sigma}_0^{\dagger j} \rangle + \langle \hat{\sigma}_0^j \rangle + \beta(\langle n_1^j \rangle + \langle n_2^j \rangle), \tag{21}$$

where

$$\hat{n}_0^j(t) = |0\rangle_{jj}\langle 0|, \tag{22}$$

$$\hat{n}_1^j(t) = |1\rangle_{jj}\langle 1|, \tag{23}$$

$$\hat{n}_2^j(t) = |2\rangle_{jj}\langle 2|. \tag{24}$$

The three-level atoms available in the cavity are pumped from the bottom level to the top level by strong coherent light. The pumping process must surely affect the dynamics of $\langle \hat{n}_0^j \rangle$ and $\langle \hat{n}_2^j \rangle$. We see that Eqs. (16)-(21) are coupled nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation. Then employing this approximation scheme, we get from Eqs. (13) and (14), the approximately valid relation

$$\hat{a}_1 = -\frac{2g}{\kappa}\hat{\sigma}_1^j(t) \tag{25}$$

and

$$\hat{a}_2 = -\frac{2g}{\kappa}\hat{\sigma}_2^j. \tag{26}$$

Evidently, these turn out to be exact relations at steady-state. Solving these expressions simultaneously one easily verify that

$$\hat{a}_1 = -\frac{2g\kappa}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}_1^j \tag{27}$$

and

$$\hat{a}_2 = -\frac{2g\kappa}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}_2^j. \tag{28}$$

Now introducing Eqs. (27) and (28) into Eqs. (16)-(21), we find

$$\frac{d}{dt}\langle\hat{\sigma}_1^j\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_1^j\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_2^{\dagger j}\rangle - \frac{\beta}{2}\langle\sigma_1^j\rangle, \tag{29}$$

$$\frac{d}{dt}\langle\hat{\sigma}_2^j\rangle = -\gamma_c\langle\hat{\sigma}_2^j\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_1^{\dagger j}\rangle - \beta\langle\sigma_2^j\rangle, \tag{30}$$

$$\frac{d}{dt}\langle\hat{\sigma}_0^j\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_0^j\rangle + \frac{\Omega}{2}[\langle\hat{n}_0^j\rangle - \langle\hat{n}_2^j\rangle] - \frac{\beta}{2}\langle\sigma_0^j\rangle, \tag{31}$$

$$\frac{d}{dt}\langle\hat{n}_1^j\rangle = -\gamma_c[\langle\hat{n}_1^j\rangle - \langle\hat{n}_2^j\rangle] + \beta(\langle n_2^j\rangle - \langle n_1^j\rangle), \tag{32}$$

$$\frac{d}{dt}\langle\hat{n}_2^j\rangle = -\gamma_c\langle\hat{n}_2^j\rangle + \frac{\Omega}{2}[\langle\hat{\sigma}_0^{\dagger j}\rangle + \langle\hat{\sigma}_0^j\rangle] - 2\beta\langle n_2^j\rangle, \tag{33}$$

$$\frac{d}{dt}\langle\hat{n}_0^j\rangle = \gamma_c\langle\hat{n}_1^j\rangle - \frac{\Omega}{2}[\langle\hat{\sigma}_0^{\dagger j}\rangle + \langle\hat{\sigma}_0^j\rangle] + \beta(\langle n_1^j\rangle + \langle n_2^j\rangle, \tag{34}$$

where

$$\gamma_c = \frac{4g^2}{\kappa}. \tag{35}$$

We prefer to call the parameter defined by Eq. (35) the stimulated emission decay constant. Then summing Eqs. (29)-(34) over the $N$ three-level atoms, we see that

$$\frac{d}{dt}\langle\hat{\Sigma}_1\rangle = -\frac{\gamma_c}{2}\langle\hat{\Sigma}_1\rangle - \frac{\Omega}{2}\langle\hat{\Sigma}_2^\dagger\rangle - \frac{\beta}{2}\langle\Sigma_1\rangle, \tag{36}$$

$$\frac{d}{dt}\langle\hat{\Sigma}_2\rangle = -\gamma_c\langle\hat{\Sigma}_2\rangle + \frac{\Omega}{2}\langle\hat{\Sigma}_1^\dagger\rangle - \beta\langle\Sigma_2\rangle, \tag{37}$$

$$\frac{d}{dt}\langle\hat{\Sigma}_0\rangle = -\frac{\gamma_c}{2}\langle\hat{\Sigma}_0\rangle + \frac{\Omega}{2}[\langle\hat{N}_0\rangle - \langle\hat{N}_2\rangle] - \frac{\beta}{2}\langle\Sigma_0\rangle, \tag{38}$$

$$\frac{d}{dt}\langle\hat{N}_1\rangle = -\gamma_c[\langle\hat{N}_1\rangle - \langle\hat{N}_2\rangle] + \beta[\langle N_2\rangle - \langle N_1\rangle], \tag{39}$$

$$\frac{d}{dt}\langle\hat{N}_2\rangle = -\gamma_c\langle\hat{N}_2\rangle + \frac{\Omega}{2}[\langle\hat{\Sigma}_0^\dagger\rangle + \langle\hat{\Sigma}_0\rangle] - 2\beta\langle N_2\rangle, \tag{40}$$

$$\frac{d}{dt}\langle\hat{N}_0\rangle = \gamma_c\langle\hat{N}_1\rangle - \frac{\Omega}{2}[\langle\hat{\Sigma}_0^\dagger\rangle + \langle\hat{\Sigma}_0\rangle] + \beta[\langle N_1\rangle + \langle N_2\rangle], \tag{41}$$

where

$$\hat{\Sigma}_1 = \sum_{j=1}^{N} \hat{\sigma}_1^j, \tag{42}$$

$$\hat{\Sigma}_2 = \sum_{j=1}^{N} \hat{\sigma}_2^j, \tag{43}$$

$$\hat{\Sigma}_0 = \sum_{j=1}^{N} \hat{\sigma}_0^j, \tag{44}$$

$$\hat{N}_0 = \sum_{j=1}^{N} \hat{n}_0^j, \tag{45}$$

$$\hat{N}_1 = \sum_{j=1}^{N} \hat{n}_1^j, \tag{46}$$

$$\hat{N}_2 = \sum_{j=1}^{N} \hat{n}_2^j \tag{47}$$

with the operators $\hat{N}_2$, $\hat{N}_1$, and $\hat{N}_0$ representing the number of atoms in the top, middle, and bottom levels. In addition, employing the completeness relation

$$\hat{I} = \hat{n}_0^j + \hat{n}_1^j + \hat{n}_2^j, \tag{48}$$

we easily arrive at

$$N = \langle \hat{N}_0 \rangle + \langle \hat{N}_1 \rangle + \langle \hat{N}_2 \rangle. \tag{49}$$

Furthermore, applying the definition given by Eq. (2) and setting for any $j$

$$\hat{\sigma}_1^j = |0\rangle\langle 1|, \tag{50}$$

we have

$$\hat{\Sigma}_1 = N|0\rangle\langle 1|. \tag{51}$$

Following the same procedure, one can also check that

$$\hat{\Sigma}_2 = N|1\rangle\langle 2|, \tag{52}$$

$$\hat{\Sigma}_0 = N|0\rangle\langle 2|, \tag{53}$$

$$\hat{N}_0 = N|0\rangle\langle 0|, \tag{54}$$

$$\hat{N}_1 = N|1\rangle\langle 1|, \tag{55}$$

$$\hat{N}_2 = N|2\rangle\langle 2|. \tag{56}$$

Moreover, using the definition

$$\hat{\Sigma} = \hat{\Sigma}_1 + \hat{\Sigma}_2 \tag{57}$$

and taking into account Eqs. (51)- (57), it can be readily established that

$$\hat{\Sigma}^{\dagger}\hat{\Sigma} = N(\hat{N}_1 + \hat{N}_2), \tag{58}$$

$$\hat{\Sigma}\hat{\Sigma}^{\dagger} = N(\hat{N}_1 + \hat{N}_0), \tag{59}$$

$$\hat{\Sigma}^2 = N\hat{\Sigma}_0 . \tag{60}$$

Upon adding Eqs. (13) and (14), we have

$$\frac{d}{dt}\hat{a}(t) = -g\hat{\sigma}^j - \frac{1}{2}\kappa\hat{a}(t), \tag{61}$$

in which

$$\hat{a}(t) = \hat{a}_1(t) + \hat{a}_2(t), \tag{62}$$

$$\hat{\sigma}^j = \hat{\sigma}_1^j + \hat{\sigma}_2^j. \tag{63}$$

The steady-state solution of Eq. (61) is expressible as

$$\hat{a} = \frac{2g}{\kappa}\hat{\sigma}^j. \tag{64}$$

Taking into account of Eq. (64) and its complex conjugate, the commutation relation of the cavity mode operator is

$$[\hat{a}, \hat{a}^{\dagger}]_j = \frac{\gamma_c}{\kappa}(\hat{n}_0^j - \hat{n}_2^j) \tag{65}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}(\hat{N}_0 - \hat{N}_2), \tag{66}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{j=1}^{N}[\hat{a}, \hat{a}^\dagger]_j \tag{67}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^\dagger$ when the cavity mode is interacting with all the $N$ three-level atoms.

In the presence of $N$ three-level atoms, we rewrite Eq. (61) as

$$\frac{d}{dt}\hat{a}(t) = -\frac{1}{2}\kappa\hat{a}(t) + \lambda\hat{\Sigma} \tag{68}$$

in which $\lambda$ is a constant whose value remains to be fixed. The steady-state solution of Eq. (68) is

$$\hat{a} = \frac{2\lambda\hat{\Sigma}}{\kappa}. \tag{69}$$

In view of (69) and its complex conjugate, the commutation relation for the cavity mode operator is

$$[\hat{a}, \hat{a}^\dagger] = \frac{4\lambda^2}{\kappa^2}N(\hat{N}_0 - \hat{N}_2). \tag{70}$$

Comparing Eqs. (66) and (70), shows that

$$\lambda = \frac{g}{\sqrt{N}}. \tag{71}$$

Then Eqs. (68) and (69) can be rewritten as

$$\frac{d}{dt}\hat{a}(t) = -\frac{1}{2}\kappa\hat{a}(t) + \frac{g}{\sqrt{N}}\hat{\Sigma} \tag{72}$$

and

$$\hat{a} = \frac{2g}{\kappa\sqrt{N}}\hat{\Sigma}. \tag{73}$$

### 3. The Mean Photon Number

$W$ next seek to calculate the mean photon number at steady-state. The mean photon number is given by

$$\bar{n} = \langle \hat{a}^{\dagger} \hat{a} \rangle. \tag{74}$$

To this end, we note that the steady-state solutions of Eqs. (38), (39), and (40), are

$$\langle \hat{\Sigma}_0 \rangle = \frac{\Omega}{(\gamma_c + \beta)} [\langle \hat{N}_0 \rangle - \langle \hat{N}_2 \rangle], \tag{75}$$

$$\langle \hat{N}_2 \rangle = \frac{\Omega}{2(\gamma_c + 2\beta)} [\langle \hat{\Sigma}_0^{\dagger} \rangle + \langle \hat{\Sigma}_0 \rangle], \tag{76}$$

$$\langle \hat{N}_1 \rangle = \langle \hat{N}_2 \rangle. \tag{77}$$

Now employing Eqs. (49), (75), (76), and (77), we readily find

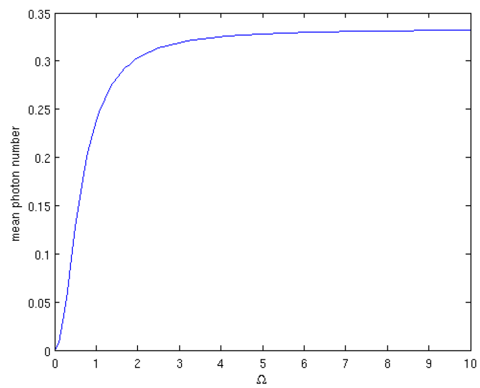



**Figure 1.** A plot of $\bar{n}$ Eq. (81) versus $\Omega$ for $\beta = 0.5, \gamma_c = 0.5, \kappa = 0.8,$ and $N = 1$.

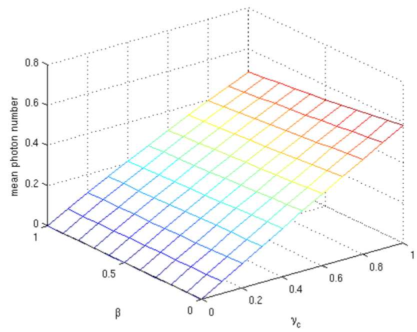



**Figure 2.** A plot of $\bar{n}$ Eq. (81) versus $\beta$ and $\gamma_c$ for $\Omega = 5$, $\kappa = 0.8$, and $N = 1$.

$$\langle \hat{N}_2 \rangle = \frac{\Omega^2 N}{(\gamma_c + \beta)(\gamma_c + 2\beta) + 3\Omega^2} \tag{78}$$

and

$$\langle \hat{\Sigma}_0 \rangle = \frac{\Omega(\gamma_c + 2\beta)N}{(\gamma_c + \beta)(\gamma_c + 2\beta) + 3\Omega^2}. \tag{79}$$

Hence using Eqs. (73) along with (57), the mean photon number of the two-mode cavity light is expressible as

$$\bar{n} = \frac{\gamma_c}{\kappa}[\langle \hat{N}_1 \rangle + \langle \hat{N}_2 \rangle] \tag{80}$$

so that on account of Eqs. (73) and (57), we have

$$\bar{n} = \frac{2N\gamma_c\Omega^2}{\kappa(\gamma_c + \beta)(\gamma_c + 2\beta) + 3\Omega^2}. \tag{81}$$

We note from the plot in Figure 1 that maximum mean photon number can be observed when $\Omega \geq 5$ for $\beta = 0.5$, $\gamma_c = 0.5$, $\kappa = 0.8$. And also we see from the plot in Figure 2 that the stimulated and spontaneous decay constants have directly and inversely proportional effect on the intensity of the light generated by the laser cavity, respectively.

## 4. Quadrature Squeezing

In this section, we wish to calculate the quadrature squeezing of the cavity light in the entire frequency interval. The squeezing properties of the cavity light are described by two quadrature operators defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{82}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{83}$$

It can be readily established that

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}[\langle \hat{N}_2 \rangle - \langle \hat{N}_0 \rangle]. \tag{84}$$

It then follows that

$$\Delta a_+ \Delta a_- \geq \frac{\gamma_c}{\kappa}[\langle \hat{N}_2 \rangle - \langle \hat{N}_0 \rangle]. \tag{85}$$

The variance of the quadrature operators is expressible as

$$\Delta a_\pm = \pm\langle [\hat{a}^\dagger \pm \hat{a}]^2 \rangle \mp [\langle \hat{a}^\dagger \rangle \pm \langle \hat{a} \rangle]^2. \tag{86}$$

Next we wish to know the expectation value of the atomic operator $\hat{\Sigma}$. To this end, applying large time approximation scheme to Eq. (59), we easily get

$$\langle \hat{\Sigma}_2 \rangle = \frac{\Omega}{2(\gamma_c + \beta)}\langle \hat{\Sigma}_1^\dagger \rangle \tag{87}$$

and in view of this result, Eq. (36) takes the form

$$\frac{d}{dt}\langle \Sigma_1 \rangle = -\frac{1}{2}\eta_0 \langle \Sigma_1 \rangle, \tag{88}$$

where

$$\eta_0 = \beta + \gamma_c + \frac{\Omega^2}{2(\gamma_c + \beta)}. \tag{89}$$

We notice that the steady-state solution of (90) for $\eta_0$ different from zero is

$$\langle \Sigma_1 \rangle = 0 \tag{90}$$

from which follows

$$\langle \Sigma_2 \rangle = 0. \tag{91}$$

Then on account of (92) and (93) along with (57), we see that

$$\langle \Sigma \rangle = 0. \tag{92}$$

In addition, the expectation value of the solution of Eq. (72) is expressible as

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\frac{1}{2}\kappa t} + \frac{g}{\sqrt{N}} e^{-\frac{1}{2}\kappa t} \int_0^t e^{-\frac{1}{2}\kappa t'} \langle \hat{\Sigma}(t') \rangle dt'. \tag{93}$$

Now with the aid of (94) and the assumption that the cavity light is initially in a vacuum state, Eq. (95) goes over into

$$\langle \hat{a}(t) \rangle = 0. \tag{94}$$

We observe on the basis of Eqs. (72) and (96) that $\hat{a}$ is a Gaussian variable with zero mean. Hence employing Eq. (88) along with (97), the quadrature variance leads to

$$\Delta a_{\pm} = \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{a} \hat{a}^{\dagger} \rangle \pm [\langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^{2} \rangle]. \tag{95}$$

Now taking into account (58) and (59) together with (73) and (95), the quadrature variance turns out to be

$$\Delta a_{\pm} = \frac{\gamma_c}{\kappa} [1 + \frac{\Omega^2 \pm 2\Omega(\gamma_c + 2\beta)}{(\gamma_c + \beta)(\gamma_c + 2\beta) + 3\Omega^2} N. \tag{96}$$

We observe that the cavity mode is in a squeezed state and the squeezing occurs in the minus quadrature.

We recall that the light generated by a two-level laser operating well above thresh hold is coherent, the quadrature variance of which is given by [11]

$$\Delta a_{+} = \Delta a_{-} = \frac{\gamma_c}{\kappa} N. \tag{97}$$

This is just the first term in (96). We prefer to call this term the coherent level. We calculate the quadrature squeezing of the cavity light relative to the quadrature variance of the cavity coherent level. We then define the quadrature squeezing of the cavity light by

$$S = \frac{(\Delta a_{\pm})^2_{coh} - (\Delta a_{-})^2_{sys}}{(\Delta a_{\pm})^2_{coh}}, \tag{98}$$

so that on account of (96) and (97) together with (98), we see that

$$S = \frac{2\Omega(\gamma_c + 2\beta) - \Omega^2}{(\gamma_c + \beta)(\gamma_c + 2\beta) + 3\Omega^2}. \tag{99}$$

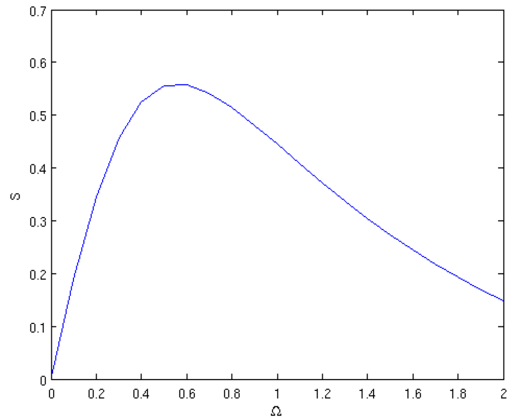



**Figure 3.** A plot of $S$ Eq. (99) versus $\Omega$ for $\gamma_c = \beta = 0.5$.

We note that unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light does not depend on the number of photons. The plot in Figure 3 shows that the the maximum quadrature squeezing is 55% below the coherent level, which is slightly greater than the result found so far.

## 4. Entanglement

To this end, we prefer to analyze the entanglement of photon-states in the laser cavity. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently - instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, polarization, etc. performed on entangled particles are found to be appropriately correlated. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have non-classical and non-local properties lead to a better understanding of the basic quantum principles. It is in this spirit that this section is devoted to the analysis of the entanglement of the two modes (photon-states). In other words, it is a wellknown fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product density operators of the constituents,

$$\hat{\rho} \neq \sum_k P_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}, \tag{100}$$

in which $P_k >> 0$ and $\sum_k P_k = 1$ to verify the normalization of the combined density states. On the other hand, an entangled continuous variable (CV) state can be expressed as a common eigenstate of a pair of EPR-type operators [14] such as $\hat{x}_2 - \hat{x}_1$ and $\hat{p}_2 - \hat{p}_1$. The total variance of these two operators reduces to zero for maximally entangled CV states. According to the criteria given by Duan et al [5], cavity photon-states of a system are entangled, if the sum of the variance of a pair of EPR-like operators,

$$\hat{s} = \hat{x}_2 - \hat{x}_1 \tag{101}$$

and

$$\hat{t} = \hat{p}_2 + \hat{p}_1, \tag{102}$$

where

$$\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_1^\dagger), \tag{103}$$

$$\hat{x}_2 = \frac{1}{\sqrt{2}}(\hat{a}_2 + \hat{a}_2^\dagger), \tag{104}$$

$$\hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1), \tag{105}$$

$$\hat{p}_2 = \frac{i}{\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_2) \tag{106}$$

are quadrature operators for modes $a_1$ and $a_2$, satisfy

$$(\Delta s)^2 + (\Delta t)^2 < 2N. \tag{107}$$

On the other hand, using Eqs. (13) and (14) together with (27) and (28), the equation of evolution of cavity mode operators $\hat{a}_1$ and $\hat{a}_2$ can be rewritten as

$$\frac{d}{dt}\hat{a}_1(t) = -\frac{1}{2}\kappa\hat{a}_1(t) - g\hat{\sigma}_1^j \tag{108}$$

and

$$\frac{d}{dt}\hat{a}_2(t) = -\frac{1}{2}\kappa\hat{a}_2(t) - g\hat{\sigma}_2^j. \tag{109}$$

Applying the steady-state solution of Eqs. (108) and (109), one can readily establish the commutation relation of the cavity mode operators $\hat{a}_1$ and $\hat{a}_1^\dagger$ as well as $\hat{a}_2$ and $\hat{a}_2^\dagger$. Hence we notice that

$$[\hat{a}_1, \hat{a}_1^\dagger] = \frac{\gamma_c}{\kappa}[\hat{n}_0^j - \hat{n}_1^j], \tag{110}$$

$$[\hat{a}_2, \hat{a}_2^\dagger] = \frac{\gamma_c}{\kappa}[\hat{n}_1^j - \hat{n}_2^j] \tag{111}$$

and on summing over all atoms, we obtain

$$[\hat{a}_1, \hat{a}_1^\dagger] = \frac{\gamma_c}{\kappa}[\hat{N}_0 - \hat{N}_1] \tag{112}$$

and

$$[\hat{a}_2, \hat{a}_2^\dagger] = \frac{\gamma_c}{\kappa}[\hat{N}_1 - \hat{N}_2], \tag{113}$$

where

$$[\hat{a}_i, \hat{a}_k^\dagger] = \delta_{ik}\sum_{j=1}^{N}[\hat{a}_i, \hat{a}_k^\dagger]_j \tag{114}$$

stands for the commutator of mode operators when the cavity light is interacting with all the *N* three-level atoms.

In the presence of *N* three-level atoms, we rewrite Eqs. (108) and (109) as

$$\frac{d}{dt}\hat{a}_1(t) = -\frac{1}{2}\kappa\hat{a}_1(t) + \lambda'_1\hat{\Sigma}_1 \tag{115}$$

and

$$\frac{d}{dt}\hat{a}_2(t) = -\frac{1}{2}\kappa\hat{a}_2(t) + \lambda'_2\,\hat{\Sigma}_2 \tag{116}$$

in which $\lambda_1^{'}$ and $\lambda_2^{'}$ are constants whose values remain to be fixed. The steady-state solution of Eqs. (115) and (116) is

$$\hat{a}_1 = \frac{2\lambda_1^{'}\kappa}{\kappa^2 - 4\varepsilon^2}\hat{\Sigma}_1 \tag{117}$$

and

$$\hat{a}_2 = \frac{2\lambda_2^{'}\kappa}{\kappa^2 - 4\varepsilon^2}\hat{\Sigma}_2\,. \tag{118}$$

In view of (117) and (118) as well as their complex conjugate, the commutation relation for the cavity mode operators is

$$[\hat{a}_1,\,\hat{a}_1^{\dagger}] = \frac{4N\lambda_1^{'2}}{\kappa^2}[\hat{N}_0 - \hat{N}_1] \tag{119}$$

and

$$[\hat{a}_2,\,\hat{a}_2^{\dagger}] = \frac{4N\lambda_2^{'2}}{\kappa^2}[\hat{N}_1 - \hat{N}_2]. \tag{120}$$

On comparing Eqs. (112) and (119) together with (113) and (120), shows that

$$\lambda_1^{'} = \lambda_2^{'} = \frac{g}{\sqrt{N}}\,. \tag{121}$$

Then Eqs. (117) and (118) can be rewritten as

$$\hat{a}_1 = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{\Sigma}_1\,. \tag{122}$$

$$\hat{a}_2 = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{\Sigma}_2\,. \tag{123}$$

Furthermore, the expectation value of the solution of Eqs. (115) and (116) together with (121) is expressible as

$$\langle\hat{a}_1(t)\rangle = \langle\hat{a}_1(0)\rangle e^{-\frac{1}{2}\kappa t} + \frac{g}{\sqrt{N}}e^{-\frac{1}{2}\kappa t}\int_0^t e^{-\frac{1}{2}\kappa t'}\langle\hat{\Sigma}_1(t')\rangle dt'. \tag{124}$$

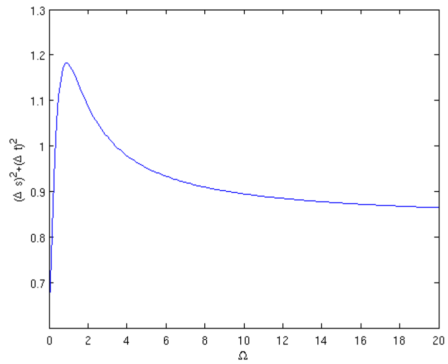



**Figure 4.** A plot of $(\Delta s)^2 + (\Delta t)^2$ Eq. (128) versus $\Omega$ for $\kappa = 0.8,$ $\gamma_c = 0.5,$ $\beta = 0.5,$ and $N = 1.$

$$\langle \hat{a}_2(t) \rangle = \langle \hat{a}_2(0) \rangle e^{-\frac{1}{2}\kappa t} + \frac{g}{\sqrt{N}} e^{-\frac{1}{2}\kappa t} \int_0^t e^{-\frac{1}{2}\kappa t'} \langle \hat{\Sigma}_2(t') \rangle dt'. \tag{125}$$

Now in view of Eqs. (90) and (91) with the assumption that the cavity light is initially in a vacuum state, Eqs. (124) and (125) goes over into

$$\langle \hat{a}_1(t) \rangle = \langle \hat{a}_2(t) \rangle = 0. \tag{126}$$

On account of this result as well as Eqs. (115) and (116) that $\hat{a}_1(t)$ and $\hat{a}_2(t)$ are Gaussian variables with zero mean. Employing Eqs. (101) and (102) along with (126), we readily get

$$(\Delta s)^2 + (\Delta t)^2 = [\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_2^\dagger \rangle]$$
$$- [\langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle + \langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle]. \tag{127}$$

Finally, in view of (78) and (79), Eq. (126) reduces to

$$(\Delta s)^2 + (\Delta t)^2 = \frac{\gamma_c}{\kappa} \left[ 1 + \frac{\Omega^2 + 2\Omega(\gamma_c + 2\beta)}{(\gamma_c + \beta)(\gamma_c + 2\beta) + 3\Omega^2} \right] N. \tag{128}$$

The plot in Figure 4 shows that the light modes in the laser cavity are entangled and the degree of entanglement decreases when the value of $\Omega$ taken to be $0 \le \Omega \le 0.9$ and increases when the value of $\Omega > 0.9.$

## 5. Conclusion

We have studied a coherently pumped non-degenerate three-level laser coupled to a two-mode vacuum reservoirs via a single-port mirror whose open cavity contains $N$ non-degenerate three-level cascade atoms. We carried out our analysis by putting the noise operators associated with a vacuum reservoir in normal order. We then first obtained the quantum Langevin equations for the cavity mode operators. We next determined the equations of evolution of the expectation values of atomic operators employing the pertinent master equation. Applying the steady-state solution of these equations, we have analyzed the mean photon number, the quadrature squeezing, and CV bipartite photon-state entanglement. It is found that the light modes in the laser cavity are entangled and the degree of entanglement decreases when the value of $\Omega$ taken to be $0 \leq \Omega \leq 0.9$ and increases when the value of $\Omega > 0.9$. In addition, we have established that maximum mean photon number can be observed when $\Omega \geq 5$ for $\beta = 0.5, \gamma_c = 0.5, \kappa = 0.8$. And we have also observed that the stimulated and spontaneous decay constants have directly and inversely proportional effect on the intensity of the light generated by the system, respectively. Moreover, we have shown that unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light does not depend on the number of photons. And also the maximum quadrature squeezing is 55% below the coherent level, under certain conditions, which is slightly greater than the result found so far.

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