Entanglement Amplification of Nondegenerate Three-Level Laser With Driven by Coherent Light and Coupled to a Vacuum Reservoir

In Partial Fulfilment of

the Requirement of the Degree of

Master of Science in Physics

(**Quantum Optics and Information)**

by

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Jimma, Ethiopia

June 2019

<u>DECLARATION</u>

I hereby declare that this MSc thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the thesis have been duly acknowledged.

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Place and date of submission:

Jimma University Department of Physics

June 2019

Jimma University
Department of Physics

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Acknowledgement

First of all, I would like to thank the almighty God for letting me to accomplish this study. Secondly, I would like to express my deepest gratitude and respect to my advisor and instructor, Dr. Tamirat Abebe, for continued follow up and unreserved help for the completion of this thesis. Finally, I am greatly indebted to my family and friends for their encouragement and indirect involvement of this work.

Selamu Didilo

June 2019

Abstract

In this thesis we study the squeezing entanglement and statistical properties of the cavity light beams produced by a coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, we have calculated the mean , variance ,correlation and entanglement of the photon number as well as the quadrature squeezing of the cavity light. We also obtain the anti-normally ordered characteristic function defined in the Heisenberg picture. With the aid of the resulting characteristic function, we determine the Q function which is then used to calculate the photon number variance of two-mode. The maximum quadrature squeezing is the same for different values of spontaneous emission decay constant and occurs at different values of the amplitude of the driving coherent light. We have also noticed that the maximum quadrature squeezing is 43.42% below the vacuum state-level.

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1

Introduction

Quantum Optics is an area of atomic, molecular, and optical physics. Its main characteristics, however, is that it deals with lasers, i.e. sources of coherent electromagnetic radiation. The quantum properties of light are largely determined by the state of the light mode. Accordingly, the most important quantum states of light are the number state, the chaotic state, the coherent, and squeezed states. In particular the interaction of three-level atoms, in different configurations, with radiation has attracted a great deal of interest for the last 20 years. In recent years, the subject of squeezing of light has also received a great deal of attention by several authors [1-7]. These non classical states of light (squeezed states) are characterized by a reduction of quantum fluctuations (noise) in one quadrature component of the light below the vacuum level, or below that achievable in a coherent state, at the expense of increased fluctuations in the other component such that the product of these fluctuations still obeys the uncertainty relation [7],[8], and [11]. Squeezed light has potential applications in low-noise optical communications and weak signal detection [7],[11], [17]. Also, entanglement is one of the fundamental tools for quantum information processing and communication protocols. The generation and manipulation of entanglement has attracted a great deal of interest with wide applications in quantum teleportation, quantum dense coding, quantum computation, quantum error correction, and quantum cryptography[12-18].

There has been a considerable interest in the analysis of the squeezing, and statistical properties of the light generated by three-level lasers [14-24]. In a cascade three-level laser, three-level atoms in a cascade configuration are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. When a three-level atom the top, intermediate, and the bottom levels are denoted by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$ in which the transitions between levels $|a\rangle_k$ to $|b\rangle_k$ and $|b\rangle_k$ to $|c\rangle_k$ are assumed to be dipole allowed, with direct transition between levels $|a\rangle_k$ and $|c\rangle_k$ to be dipole forbidden. When the atom makes a transition from the top to the intermediate level and then from the intermediate to the bottom level, two photons are emitted. If the two photons have different frequencies, then the three-level atom is called a non-degenerate three-level atom otherwise it is called degenerate.

Some of the authors have studied the squeezing and statistical properties of the light produced by three-level lasers when the atoms are initially prepared in a coherent superposition of the top and bottom levels or when these levels are coupled by a strong coherent light [16-34]. These authors have found that these quantum optical systems can generate squeezed light under certain conditions. Moreover, Fesseha[17] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, is found to be 50% below the vacuum-state level. In addition, Sintayehu Tesfa[23] studied the squeezing properties and entanglement amplification of the cavity radiation. The authors have calculated the correlation of the photon numbers and the fluctuation of the intensity difference. The study has shown that the generated light exhibits a two-mode squeezing and entanglement when initially they are more atoms in the lower level. Moreover, a strong correlation between photon numbers along with a significant fluctuation in the intensity difference is found .

In this thesis, we study the squeezing , entanglement and statistical properties of the light generated by a coherently driven non-degenerate three-level laser with an open cavity coupled to a two-mode vacuum reservoir via a single-port mirror. In order to carry out our calculation, we put the noise operators associated with the vacuum reservoir in normal order. We thus first determine the master equation and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation and the large-time approximation scheme, we obtain equations of evolution of the expectation values of atomic operators. Moreover, we determine the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators. Furthermore, applying the same solutions, we obtain the photon number entanglement.

2

Operator Dynamics

In this chapter we consider a non degenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode vacuum reservoir via a single-port mirror as shown in Fig. (2.1). Here we first derive the master equation and the quantum Langevin equations for the cavity mode operators. moreover, employing the master equation and the large-time approximation scheme, we drive the equations of evolution of the expectation values of the atomic oprators. Finally, we determine the steady-state solutions of the resulting equations of evolution.

2.1 Master equation

In this section, we consider here a system of N nondegenerate three-level atoms in cascade configuration are available in an open cavity and interacting with the two non-degenerate cavity modes. The top, intermediate, and bottom levels of the three-level atom by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. As shown in Fig. (2.1) when the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and from levels $|b\rangle_k$ to $|c\rangle_k$ two photons with different frequencies are emitted. We wish to represent the light emitted from the top level by a and the emitted from the intermediate level by b. In addition, we assume that the cavity

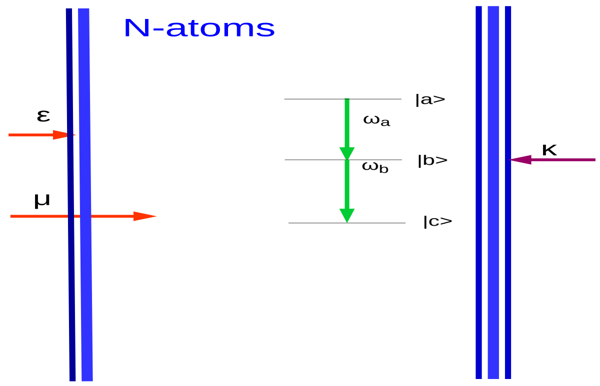

Figure 2.1: Schematic representation of a coherently driven nondegenerate three-level laser coupled to a two-mode vacuum reservoir.

mode a and b to be at resonance with the two transitions $|a\rangle_k \rightarrow |b\rangle_k$ and$|b\rangle_k \rightarrow |c\rangle_k$ with top and bottom levels of the three-level atom coupled by coherent light. The interaction of the three-level atoms with light mode a and b can be described by the Hamiltonian[17]

$$\hat{H}' = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^{\dagger}\hat{\sigma}_b^k\big], \tag{2.1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_k\,{}_k\langle a|, \tag{2.2}$$

$$\hat{\sigma}_b^k = |c\rangle_k\,{}_k\langle b|, \tag{2.3}$$

are lowering atomic operators, $\hat{a}$ and $\hat{b}$ are the annihilation operators for light mode a and b,and g is the coupling constant between the atoms and the light mode a or light mode b. And the coupling of the top and the bottom levels by coherent light can be described by the Hamiltonian

$$\hat{H}'' = \frac{i\Omega}{2}[\hat{\sigma}_c^{+k} - \hat{\sigma}_c^k], \tag{2.4}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{k\ k}\langle a| \tag{2.5}$$

is lowering atomic operator and

$$\Omega = 2\mu\lambda. \tag{2.6}$$

Here μ, considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the atoms and coherent light. In addition, the interaction of the cavity modes and coherent driving modes can be described by the Hamiltonian

$$\hat{H}''' = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \tag{2.7}$$

where ε, considered to be real and constant, is proportional to the amplitude of the driving coherent light modes. Thus up on combining eqs. (2.1), (2.4) and(2.7). The interaction of the three- level atom with the driving coherent light and cavity mode $\hat{a}$ and $\hat{b}$ is described by the Hamiltonian as

$$\hat{H}_S(t) = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big] + \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big] + i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \tag{2.8}$$

where $\hat{H}_s$ is the Hamiltonian of the system. We next seek to obtain the time evolution of the density operator for a two-mode cavity radiation coupled to a two-mode squeezed vacuum reservoir via a single-port mirror. In general, the time evolution of the reduced density operator for a cavity radiation coupled to a reservoir has the form [4]

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i[\hat{H}_{SR}(t),\hat{\rho}(t)] - i[\langle\hat{H}_{SR}(t)\rangle_R,\hat{\rho}(0)] \\ &- \int_0^t [\langle\hat{\rho}_{SR}(t)\rangle_R,[\hat{H}_s(t')]]dt' \\ &- \int_0^t Tr_R[\hat{H}_{SR}(t'),[\hat{H}_{SR}(t')]]dt', \end{aligned} \tag{2.9}$$

where S and H refers to the system and reservoir variables. Furthermore, the interaction of a two-mode cavity radiation with a two-mode reservoir can be described the Hamiltonian

$$\begin{aligned}\hat{H}_{SR}(t) &= i\sum_k g_k\big(\hat{\sigma}_a^{\dagger k}\hat{a}_k\exp[i(\omega_0-\omega_k)t] - \hat{a}_k^{\dagger}\hat{\sigma}_a^{k}\exp[-i(\omega_0-\omega_k)t] \\ &+ \hat{\sigma}_b^{\dagger k}\hat{b}_k\exp[i(\omega_0-\omega_k)t] - \hat{b}_k^{\dagger}\hat{\sigma}_b^{k}\exp[-i(\omega_0-\omega_k)t]\big),\end{aligned} \tag{2.10}$$

where $\hat{a}_k$ and $\hat{b}_k$ is the annihilation operator for a reservoir sub-mode characterized by wave vector 'k' and

$$g_k = [\frac{\omega_k}{2\varepsilon_0 v}]^{\frac{1}{2}} d_{ab}.U_k. \tag{2.11}$$

In addition, $\omega_0 = \frac{\omega_a+\omega_b}{2}$, with ω_a and ω_b representing frequencies and $(\hat{a},\hat{b})$ being the annihilation operators for the cavity modes, ω_k is frequency, and g_k is the coupling constant. In view of Eq.(2.10), we can write

$$\begin{aligned}\langle\hat{H}_S(t)\rangle_{\hat{R}} &= i\sum_k g_k\big(\hat{\sigma}_a^{\dagger k}\langle\hat{a}_k\rangle_{\hat{R}}\exp[i(\omega_0-\omega_k)t - \langle\hat{a}_k^{\dagger}\rangle_{\hat{R}}\hat{\sigma}_a^{k}\exp[-i(\omega_0-\omega_k)t] \\ &+ \hat{\sigma}_b^{\dagger k}\langle\hat{b}_k\rangle_{\hat{R}}\exp[i(\omega_0-\omega_k)t] - \langle\hat{b}_k^{\dagger}\rangle_{\hat{R}}\hat{\sigma}_b^{k}\exp[-i(\omega_0-\omega_k)t]\big).\end{aligned} \tag{2.12}$$

Since for a two-mode reservoir, we find

$$\langle\hat{a}_k\rangle_{\hat{R}} = \langle\hat{b}_k\rangle_{\hat{R}} = 0. \tag{2.13}$$

Hence, one can easily see that

$$\langle \hat{H}_{SR} \rangle_{\hat{R}} = 0. \tag{2.14}$$

We therefore see that

$$[\langle \hat{H}_{SR} \rangle_{\hat{R}}, \hat{\rho}(0)] = 0, \tag{2.15}$$

$$[\langle \hat{H}_{SR} \rangle_{\hat{R}}, [\hat{H}(t'), \hat{\rho}(t')]] = 0. \tag{2.16}$$

On account of Eqs. (2.15) and (2.16), Eq. (2.10) can be put in the form

$$\begin{aligned} \frac{d}{dt}\hat{\rho} &= -i[\hat{H}_S(t), \hat{\rho}] - \int_0^t Tr[\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t')]dt' \\ &- \int_0^t \hat{\rho}(t')Tr[\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t)]dt' \\ &+ \int_0^t Tr[\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t')]dt'. \end{aligned} \tag{2.17}$$

Making use of Eq. (2.10) and the cyclic property of the trace operation, one can easily verify that

$$\begin{aligned} Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t')) &= -[-\Gamma_1\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^{k'} - \Gamma_2\hat{\sigma}_a^{k}\hat{\rho}\hat{\sigma}_a^{\dagger k'} + \Gamma_3\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^{\dagger k'} + \Gamma_4\hat{\sigma}_a^{k}\hat{\rho}\hat{\sigma}_a^{k'} \\ &- \Gamma_5\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^{k'} - \Gamma_6\hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_b^{\dagger k'} + \Gamma_7\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^{\dagger k'} + \Gamma_8\hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_b^{k'} \\ &+ \Gamma_9(\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_b^{\dagger k'} + \hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_a^{\dagger k'}) + \Gamma_{10}(\hat{\sigma}_a^{k}\hat{\rho}\hat{\sigma}_b^{k'} + \hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_a^{k'}) \\ &- \Gamma_{11}(\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_b^{k'} + \hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_a^{\dagger k'}) - \Gamma_{12}(\hat{\sigma}_a^{k}\hat{\rho}\hat{\sigma}_b^{\dagger k'} + \hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_a^{k'})], \end{aligned} \tag{2.18}$$

where

$$\Gamma_1 = \sum_{k,k'} g_k g_{k'} \langle \hat{a}_k^\dagger \hat{a}_{k'} \rangle_R \exp[-i(\omega_0 - \omega_k)t + i(\omega_0 - \omega_{k'})t'], \tag{2.19}$$

$$\Gamma_2 = \sum_{k,k'} g_k g_{k'} \langle \hat{a}_k \hat{a}_{k'}^\dagger \rangle_R \exp[i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.20}$$

$$\Gamma_3 = -\sum_{k,k'} g_k g_{k'} \langle \hat{a}_k \hat{a}_{k'} \rangle_R \exp[i(\omega_0 - \omega_k)t + i(\omega_0 - \omega_{k'})t'], \tag{2.21}$$

$$\Gamma_4 = -\sum_{k,k'} g_k g_{k'} \langle \hat{a}_k^\dagger \hat{a}_{k'}^\dagger \rangle_R \exp[-i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.22}$$

$$\Gamma_5 = \sum_{k,k'} g_k g_{k'} \langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle_R \exp[-i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.23}$$

$$\Gamma_6 = \sum_{k,k'} g_k g_{k'} \langle \hat{b}_k \hat{b}_{k'}^\dagger \rangle_R \exp[i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.24}$$

$$\Gamma_7 = \sum_{k,k'} g_k g_{k'} \langle \hat{b}_k \hat{b}_{k'} \rangle_R \exp[i(\omega_0 - \omega_k)t + i(\omega_0 - \omega_{k'})t'], \tag{2.25}$$

$$\Gamma_8 = \sum_{k,k'} g_k g_{k'} \langle \hat{b}_k^\dagger \hat{b}_{k'}^\dagger \rangle_R \exp[-i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.26}$$

$$\Gamma_9 = -\sum_{k,k'} g_k g_{k'} \langle \hat{a}_k \hat{b}_{k'} \rangle_R \exp[i(\omega_0 - \omega_k)t + i(\omega_0 - \omega_{k'})t'], \tag{2.27}$$

$$\Gamma_{10} = -\sum_{k,k'} g_k g_{k'} \langle \hat{a}_k^\dagger \hat{b}_{k'}^\dagger \rangle_R \exp[-i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.28}$$

$$\Gamma_{11} = \sum_{k,k'} g_k g_{k'} \langle \hat{a}_k \hat{b}_{k'}^\dagger \rangle_R \exp[i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t'], \tag{2.29}$$

$$\Gamma_{12} = \sum_{k,k'} g_k g_{k'} \langle \hat{a}_k^\dagger \hat{b}_{k'} \rangle_R \exp[-i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t']. \tag{2.30}$$

To evaluate the trace over the reservoir state,we need the trace of the various possible combination of the reservoir operators coming from $\hat{H}_{SR}$. These will involve the pairs of the operators $\hat{a}_k \hat{a}_{k'}^\dagger$. Using cyclic permutation; these trace terms can be written as thermal average $\langle X \rangle = Tr_R(\hat{\rho}_R X)$. A squeezed vacuum reservoir [4], the relevant expectations are

$$\langle\hat{a}_k\hat{a}_{k'}\rangle_R = \langle\hat{a}^\dagger_k\hat{a}^\dagger_{k'}\rangle_R = 0, \tag{2.31}$$

$$\langle\hat{a}_k\hat{a}_{k'}\rangle_R = N\delta_{kk'}, \tag{2.32}$$

$$\langle\hat{a}_k\hat{a}^\dagger_{k'}\rangle_R = (N+1)\delta_{kk'}, \tag{2.33}$$

$$\langle\hat{a}_{k'}\hat{a}_k\rangle_R = \langle\hat{a}^\dagger_{k'}\hat{a}^\dagger_k\rangle_R = -M\delta_{k',2k_{0-k}}. \tag{2.34}$$

Furthermore, for vacuum reservoir we have

$$N = M = 0. \tag{2.35}$$

On account of Eqs. (2.31), (2.32), (2.33) and (2.34), we see that

$$\Gamma_1 = \Gamma_3 = \Gamma_4 = \Gamma_5 = \Gamma_7 = \Gamma_8 = \Gamma_9 = \Gamma_{10} = \Gamma_{12} = 0. \tag{2.36}$$

On the other hand, using Eqs.(2.32) and (2.34) one can easily see that

$$\Gamma_2 = \Gamma_6 = \sum_k g_k^2 \exp[i(\omega_0 - \omega_k)(t - t')]. \tag{2.37}$$

In order to evaluate the dot product involved in Eq. (2.11), we adopt spherical coordinates in k-space, with the electric dipole matrix element d_{ab} taken to be along z-axis. In addition, we take the unit vector U_k to be in the plane formed by the vectors d_{ab} and k. Since U_k is normal to k, i.e, $k.U_K = 0$, the angle between U_k and d_{ab} is $(\frac{\pi}{2} - \theta)$[4]. We then see that

$$d_{ab}.U_k = d_{ab}\cos(\frac{\pi}{2} - \theta) = d_{ab}\sin\theta. \tag{2.38}$$

Now employing the transformation

$$\sum_k \rightarrow \frac{V}{(2\pi)^3}\int d_k^3 = \frac{V}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} k^2 d_k, \tag{2.39}$$

and taking in to account Eq. (2.10) along with (2.38), we can put Eq.(2.37) in the form

$$\begin{aligned}\Gamma_2 = \Gamma_6 &= \frac{d_{ab}^2}{2(2\pi)^3\varepsilon_0 c^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta(1-\cos^2\theta)d\theta \\ &\times \int_0^{\infty} \omega^3 \exp[i(\omega_0-\omega_k)(t-t')]d_\omega, \end{aligned} \quad (2.40)$$

where we have made use of the relation $k=\frac{\omega}{c}, \sin^2\theta = 1-\cos^2\theta$. Furthermore, carrying out the integration over ϕ as well as θ, we get

$$\Gamma_2 = \Gamma_6 = \frac{d_{ab}^2}{6(\pi)^2\varepsilon_0 c^3}\int_0^{\infty} \omega^3 \exp[i(\omega_0-\omega_k)(t-t')]d_\omega. \quad (2.41)$$

We assume that ω varies very little around ω_0. In view of this , we can replace ω^3 by ω_0^3 and extending the lower limit of the integration to $-\infty$ consequently, we have

$$\Gamma_2 = \Gamma_6 = \frac{d_{ab}^2}{6(\pi)^2)\varepsilon_0 c^3}\int_{-\infty}^{\infty} \exp[-i(\omega_0-\omega_k)(t-t')]d_\omega. \quad (2.42)$$

Moreover, up on setting $\omega^{'}=\omega-\omega_0$, we notice that

$$\Gamma_2 = \Gamma_6 = \frac{d_{ab}^2}{6(\pi)^2)\varepsilon_0 c^3}\int_{-\infty}^{\infty} \exp[i(\omega_0-\omega_k)(t-t')]d_{\omega'}. \quad (2.43)$$

It the follows that

$$\Gamma_2 = \Gamma_6 = \gamma\delta(t-t'), \quad (2.44)$$

in which

$$\gamma = \frac{d_{ab}^2}{3(\pi)^2)\varepsilon_0 c^3}, \quad (2.45)$$

where γ is the atomic decay rate. Furthermore, applying Eqs.(2.36) and (2.44), we can express Eq. (2.18) as

$$Tr(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')) = \gamma[\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k]\delta(t-t'). \quad (2.46)$$

We easily see that

$$\int_0^t Tr(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t')dt' = \frac{\gamma}{2}[\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho}], \tag{2.47}$$

with $\hat{\rho} = \hat{\rho}(t)$, we also see that

$$\int_0^t \hat{\rho}(t')Tr_R\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')dt' = \frac{\gamma}{2}[\hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho}]. \tag{2.48}$$

Hence in view of eq. (2.36) and (2.44), we also see that

$$Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t') = \gamma[\hat{\sigma}_a^k\hat{\rho}(t')\hat{\sigma}_a^{\dagger k} + \hat{\sigma}_b^k\hat{\rho}(t')\hat{\sigma}_b^{\dagger k}]\delta(t-t'), \tag{2.49}$$

from which follows

$$\int_0^t Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t')dt' = \frac{\gamma}{2}[\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k}], \tag{2.50}$$

and

$$\int_0^t Tr_R(\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)dt' = \frac{\gamma}{2}[\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k}]. \tag{2.51}$$

On substituting Eqs. (2.47), (2.48), (2.50) and (2.51) into Eq. (2.17) the master equation for a nondegenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode vacuum reservoir is found to be

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) &= -i[H_s(t),\hat{\rho}(t)] + \frac{\Omega}{2}\left[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\sigma}_c^k\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} + \hat{\rho}\hat{\sigma}_c^k\right] \\
&+ \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\right] \\
&+ \frac{\gamma}{2}\left[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\right].
\end{aligned} \tag{2.52}$$

where γ is the spontaneous emission decay constant associated with the two-modes $\hat{a}$ and $\hat{b}$. Therefore, on account of eqs.(2.1) and (2.7), the master equation for coherently driven three-level laser and coupled to a two- mode vacuum reservoir takes the form

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}(t) &= g\big[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\big] \\
&+ \varepsilon(\hat{\rho}\hat{a} - \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho} + \hat{\rho}\hat{b} - \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}) \\
&+ \frac{\Omega}{2}\big[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\sigma}_c^k\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} + \hat{\rho}\hat{\sigma}_c^k\big] + \frac{\gamma}{2}\big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\big] \\
&+ \frac{\gamma}{2}\big[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\big]. \qquad (2.53)
\end{aligned}$$

This is the master equation for a coherently driven non-degenerate three-level atom with a two-mode cavity in an open cavity and coupled to a two-mode vacuum reservoir.

2.2 Quantum Langevin Equations

We assume that the cavity modes are coupled to a vacuum reservoir via single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [8,17]. We can therefore drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}], \qquad (2.54)$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}], \qquad (2.55)$$

where κ is the cavity damping constant. By applying the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \qquad (2.56)$$

$$[\hat{a}, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = [\hat{a}, \hat{b}^\dagger] = 0, \qquad (2.57)$$

$$[\hat{a}^2, \hat{a}^\dagger] = 2\hat{a}. \qquad (2.58)$$

In view of Eqs. (2.56), (2.57) and (2.58), the quantum Langevin equations for cavity mode operators $\hat{a}$ and $\hat{b}$ turns out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \varepsilon, \tag{2.59}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k + \varepsilon. \tag{2.60}$$

2.3 Stochastic differential Equations

Next we proceed to derive the stochastic differential equations of the atomic operators by applying the master equation and the large-time approximation scheme. Moreover, we find the steady-state solutions of the equations of evolution of the atomic operators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{2.61}$$

along with the master equation eq. (2.53), one can readily establish that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle &= \varepsilon[\langle\hat{a}\hat{\sigma}_a\rangle - \langle\hat{\sigma}_a\hat{a}\rangle + \langle\hat{\sigma}_a\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\hat{b}\hat{\sigma}_a\rangle - \langle\hat{\sigma}_a\hat{b}\rangle\langle\hat{\sigma}_a\hat{b}^\dagger\rangle \\ &- \langle\hat{b}^\dagger\hat{\sigma}_a\rangle] + g[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle - \gamma\langle\hat{\sigma}_a^k\rangle,\end{aligned} \tag{2.62}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle &= \varepsilon[\langle\hat{a}\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_b^k\hat{a}\rangle + \langle\hat{\sigma}_b^k\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{b}\hat{\sigma}_b^k\rangle\langle-\hat{\sigma}_b^k\hat{b}\rangle + \langle\hat{\sigma}_b^k\hat{b}^\dagger\rangle \\ &- \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle] + g[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle] - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle - \frac{\gamma}{2}\langle\hat{\sigma}_b^k\rangle,\end{aligned} \tag{2.63}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle &= \varepsilon[\langle\hat{a}\hat{\sigma}_c^k\rangle - \langle\hat{\sigma}_c^k\hat{a}\rangle + \langle\hat{\sigma}_c^k\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle + \langle\hat{b}\hat{\sigma}_c^k\rangle - \langle\hat{\sigma}_c^k\hat{b}\rangle + \langle\hat{\sigma}_c^k\hat{b}^\dagger\rangle \\ &- \langle\hat{b}^\dagger\hat{\sigma}_a^k\rangle] + g[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle] + \frac{\Omega}{2}\left[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\right] - \frac{\gamma}{2}\langle\hat{\sigma}_c^k\rangle,\end{aligned} \tag{2.64}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_a^k\rangle &= \varepsilon[\langle\hat{a}\hat{\eta}_a^k\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{\eta}_a^k\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{\eta}_a^k\rangle + \langle\hat{b}\hat{\eta}_a^k\rangle - \langle\hat{\eta}_a^k\hat{b}\rangle + \langle\hat{\eta}_a^k\hat{b}^\dagger\rangle \\ &- \langle\hat{b}^\dagger\hat{\eta}_a^k\rangle] + g[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] + \frac{\Omega}{2}\left[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\right] - \gamma\langle\hat{\eta}_a^k\rangle,\end{aligned} \tag{2.65}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b^k\rangle &= \varepsilon[\langle\hat{a}\hat{\eta}_b^k\rangle - \langle\hat{\eta}_b^k\hat{a}\rangle + \langle\hat{\eta}_b^k\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{\eta}_a^k\rangle + \langle\hat{b}\hat{\eta}_b^k\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle + \langle\hat{\eta}_b^k\hat{b}^\dagger\rangle \\ &- \langle\hat{b}^\dagger\hat{\eta}_b^k\rangle] + g[\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle] + \gamma[\langle\hat{\eta}_b^k\rangle - \langle\hat{\eta}_a^k\rangle], \qquad (2.66)\end{aligned}$$

where

$$\hat{\eta}_a^k = |a\rangle_{k\ k}\langle a|, \qquad (2.67)$$

$$\hat{\eta}_b^k = |b\rangle_{k\ k}\langle b|, \qquad (2.68)$$

$$\hat{\eta}_c^k = |c\rangle_{k\ k}\langle c|. \qquad (2.69)$$

We see that Eqs. (2.62) - (2.66) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [8,17]. Therefore, employing this approximation scheme, we get from Eqs. (2.59) and (2.60) the approximately valid relations are

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2\varepsilon}{\kappa}, \qquad (2.70)$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2\varepsilon}{\kappa}. \qquad (2.71)$$

Evidently, these turn out to be exact relations at steady-state. Moreover, the conjugate of (2.70) and (2.71), we get

$$\hat{a}^\dagger = -\frac{2g}{\kappa}\hat{\sigma}_a^{\dagger k} + \frac{2\varepsilon}{\kappa}, \qquad (2.72)$$

$$\hat{b}^\dagger = -\frac{2g}{\kappa}\hat{\sigma}_b^{\dagger k} + \frac{2\varepsilon}{\kappa}. \qquad (2.73)$$

We next substituting Eqs. (2.70) - (2.73) into Eqs. (2.62)-2.66, the equations of evolution of the atomic operators take the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -(\gamma+\gamma_c)\langle\hat{\sigma}_a^k\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{2.74}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{\sigma}_b^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{2.75}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{\sigma}_c^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{2.76}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -(\gamma+\gamma_c)\langle\hat{\eta}_a^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.77}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = (\gamma+\gamma_c)[\langle\hat{\eta}_b^k\rangle - \langle\hat{\eta}_a^k\rangle], \tag{2.78}$$

where

$$\gamma_c = \frac{4g^2}{\kappa}. \tag{2.79}$$

We prefer to call the parameter defined by Eq. (2.79) is the stimulated emission decay constant. We next sum Eqs. (2.74) - (2.78) over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -(\gamma+\gamma_c)\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^{\dagger}\rangle, \tag{2.80}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^{\dagger}\rangle, \tag{2.81}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{1}{2}(\gamma+\gamma_c)\langle\hat{m}_c\rangle + \frac{\Omega}{2}\big[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\big], \tag{2.82}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma+\gamma_c)\langle\hat{N}_a\rangle + \frac{\Omega}{2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^{\dagger}\rangle\big], \tag{2.83}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = (\gamma+\gamma_c)[\langle\hat{N}_b\rangle - \langle\hat{N}_a\rangle], \tag{2.84}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N} \hat{\sigma}_a^k, \tag{2.85}$$

$$\hat{m}_b = \sum_{k=1}^{N} \hat{\sigma}_b^k, \tag{2.86}$$

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k, \tag{2.87}$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \tag{2.88}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{2.89}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k, \tag{2.90}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels, respectively. Employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{2.91}$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.92}$$

Furthermore, using the definition given by Eq. (2.85) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{2.93}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{2.94}$$

Following the same procedure, one can also easily establish that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{2.95}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{2.96}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{2.97}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.98}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{2.99}$$

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b, \tag{2.100}$$

we see that

$$\hat{m}^\dagger = \hat{m}_a^\dagger + \hat{m}_b^\dagger, \tag{2.101}$$

where

$$\hat{m}_a^\dagger = N|a\rangle\langle b|, \tag{2.102}$$

$$\hat{m}_b^\dagger = N|b\rangle\langle a|. \tag{2.103}$$

Combination of Eqs. (2.100) and (2.101) yields

$$\hat{m}^\dagger\hat{m} = (\hat{m}_a^\dagger + \hat{m}_b^\dagger)(\hat{m}_a + \hat{m}_b). \tag{2.104}$$

Up on evaluating the terms on the right side of the above equation, we get

$$\hat{m}_a^\dagger\hat{m}_a = N\hat{N}_a, \tag{2.105}$$

$$\hat{m}_a^\dagger\hat{m}_b = \hat{m}_b^\dagger\hat{m}_a = 0, \tag{2.106}$$

$$\hat{m}_b^\dagger\hat{m}_b = N\hat{N}_b. \tag{2.107}$$

Substitution of Eqs.(2.105)-(2.107) into (2.104), we see that

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b). \tag{2.108}$$

In similar manner,one can also establish that

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{2.109}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{2.110}$$

In the presence of N three-level atoms, we rewrite Eqs. (2.59) and (2.60) as [17]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda\hat{m}_a + \varepsilon, \tag{2.111}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \beta\hat{m}_b + \varepsilon, \tag{2.112}$$

in which λ and β are constants whose values remain to be fixed. We note that the steady-state solutions of Eqs. (2.59) and (2.60)are

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2\varepsilon}{\kappa}, \tag{2.113}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2\varepsilon}{\kappa}. \tag{2.114}$$

Now employing Eqs. (2.113) and (2.114), the commutation relations for the cavity mode operators are found to be

$$[\hat{a}, \hat{a}^\dagger]_k = \frac{\gamma_c}{\kappa}\big[\hat{\eta}_b^k - \hat{\eta}_a^k\big], \tag{2.115}$$

$$[\hat{b}, \hat{b}^\dagger]_k = \frac{\gamma_c}{\kappa}\big[\hat{\eta}_c^k - \hat{\eta}_b^k\big], \tag{2.116}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_a\big], \tag{2.117}$$

$$[\hat{b}, \hat{b}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_b\big]. \tag{2.118}$$

where

$$[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N} [\hat{a}, \hat{a}^\dagger]_k, \tag{2.119}$$

$$[\hat{b}, \hat{b}^\dagger] = \sum_{k=1}^{N} [\hat{b}, \hat{b}^\dagger]_k. \tag{2.120}$$

We note that Eqs. (2.119) and (2.120) stand for the commutators $\hat{a}$ and $\hat{a}^\dagger$, and for $\hat{b}$ and $\hat{b}^\dagger$ when the light modes a and b are interacting with all the N three-level atoms. On the other hand, using the steady-state solutions of Eqs. (2.111) and (2.112), one can easily verify that

$$[\hat{a}, \hat{a}^\dagger] = N\left(\frac{2\lambda}{\kappa}\right)^2 \left(\hat{N}_b - \hat{N}_a\right), \tag{2.121}$$

$$[\hat{b}, \hat{b}^\dagger] = N\left(\frac{2\beta}{\kappa}\right)^2 \left(\hat{N}_c - \hat{N}_b\right). \tag{2.122}$$

Thus on account of Eqs. (2.117) and (2.121), we see that

$$\lambda = \pm \frac{g}{\sqrt{N}}. \tag{2.123}$$

Similarly, inspection of Eqs. (2.118) and (2.122) shows that

$$\beta = \pm \frac{g}{\sqrt{N}}. \tag{2.124}$$

Hence in view of these two results, the equations of evolution of the light modes a and b operators given by Eqs. (2.111) and (2.112) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \varepsilon, \tag{2.125}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \varepsilon. \tag{2.126}$$

Now adding Eqs. (2.117) and (2.118) as well as Eqs. (2.125) and (2.126), we get

$$[\hat{c}, \hat{c}^\dagger] = \frac{\gamma_c}{\kappa}\left[\hat{N}_c - \hat{N}_a\right], \tag{2.127}$$

and

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m} + 2\varepsilon, \tag{2.128}$$

in which

$$\hat{c} = \hat{a} + \hat{b}. \tag{2.129}$$

We next proceed to obtain the expectation value of the cavity mode operators. One can rewrite Eq. (2.80) and the adjoint of (2.81) as

$$\frac{d}{dt}\langle\hat{m}_a(t)\rangle = -(\gamma + \gamma_c)\langle\hat{m}_a(t)\rangle + \frac{\Omega}{2}\langle\hat{m}_b^\dagger(t)\rangle, \tag{2.130}$$

$$\frac{d}{dt}\langle\hat{m}_b^\dagger(t)\rangle = -\frac{\Omega}{2}\langle\hat{m}_a(t)\rangle - \frac{1}{2}(\gamma + \gamma_c)\langle\hat{m}_b^\dagger(t)\rangle. \tag{2.131}$$

On the basis of eqs.2.130 and 2.131, we observe that

$$\frac{d}{dt}\hat{m}_a(t) = -(\gamma + \gamma_c)\hat{m}_a(t) + \frac{\Omega}{2}\hat{m}_b^\dagger(t) + \hat{F}_a(t), \tag{2.132}$$

$$\frac{d}{dt}\hat{m}_b^\dagger(t) = -\frac{\Omega}{2}\hat{m}_a(t) - \frac{1}{2}(\gamma + \gamma_c)\hat{m}_b^\dagger(t) + \hat{F}_b(t). \tag{2.133}$$

where $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise force operators and the properties of which remain to be determined. To solve the coupled differential equations (2.132) and (2.133), we can write the single-matrix equation as

$$\frac{d}{dt}\begin{pmatrix} \hat{m}_a(t) \\ \hat{m}_b^\dagger(t) \end{pmatrix} = M\begin{pmatrix} \hat{m}_a(t) \\ \hat{m}_b^\dagger(t) \end{pmatrix} + \begin{pmatrix} \hat{F}_a(t) \\ \hat{F}_b^\dagger(t) \end{pmatrix}, \tag{2.134}$$

$$\frac{d}{dt}\hat{J}(t) = M\hat{J}(t) + \hat{F}(t), \tag{2.135}$$

where

$$\hat{J}(t) = \begin{pmatrix} \hat{m}_a(t) \\ \hat{m}_b^\dagger(t) \end{pmatrix}, \tag{2.136}$$

$$M = \begin{pmatrix} -(\gamma+\gamma_c) & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & -\frac{1}{2}(\gamma+\gamma_c) \end{pmatrix}, \tag{2.137}$$

$$\hat{E}(t) \;=\; \begin{pmatrix} \langle \hat{F}_a(t)\rangle \\ \langle \hat{F}_b^\dagger(t)\rangle \end{pmatrix}. \tag{2.138}$$

In order to solve Eq. (2.135), we need the eigenvalues and eigenvectors of M such that

$$MV_i = \lambda_i V_i, \tag{2.139}$$

where,

$$V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \tag{2.140}$$

is the eigenvectors. with $i = 1, 2$, the normalization condition

$$x_i^2 + y_i^2 = 1. \tag{2.141}$$

The eigenvalue equation (2.139) can be written as

$$(M - \lambda_i I)V_i, = 0 \tag{2.142}$$

where I is an identity operator, Eq. (2.142) has nontrivial solution provided that

$$det(M - \lambda I) = 0, \tag{2.143}$$

so that applying Eq. (2.137), the eigenvalues are found to be

$$\lambda_1 = -\frac{3}{4}(\gamma+\gamma_c) + \frac{1}{2}p, \tag{2.144}$$

$$\lambda_2 = -\frac{3}{4}(\gamma+\gamma_c) - \frac{1}{2}p, \tag{2.145}$$

where

$$p = \sqrt{\frac{1}{4}(\gamma+\gamma_c)^2 - \Omega^2}. \tag{2.146}$$

We next seek to obtain the eigenvectors of M. To this end, the eigenvector corresponding to λ_1 is expressible as

$$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.147}$$

Then employing Eqs. (2.137) and (2.139), we write the matrix equation

$$\begin{pmatrix} -(\gamma+\gamma_c) & \frac{\Omega}{2} \\ -\frac{\Omega}{2} & -\frac{1}{2}(\gamma+\gamma_c) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.148}$$

Taking into account this equation and the normalization condition

$$x_1^2 + y_1^2 = 1, \tag{2.149}$$

we get

$$V_1 = \frac{1}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1+\gamma+\gamma_c)^2}} \begin{pmatrix} \frac{\Omega}{2} \\ \lambda_1+\gamma+\gamma_c \end{pmatrix}. \tag{2.150}$$

The eigenvector corresponding to λ_2 can also be established following a similar procedure that

$$V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2+\gamma+\gamma_c)^2}} \begin{pmatrix} \frac{\Omega}{2} \\ \lambda_2+\gamma+\gamma_c \end{pmatrix}. \tag{2.151}$$

Finally, we construct a matrix V consisting of the eigenvectors of the matrix M as column matrices

$$V = \begin{pmatrix} \frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4}+(\lambda_1+\gamma+\gamma_c)^2}} & \frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4}+(\lambda_2+\gamma+\gamma_c)^2}} \\ \frac{\lambda_1+\gamma+\gamma_c}{\sqrt{\frac{\Omega^2}{4}+(\lambda_1+\gamma+\gamma_c)^2}} & \frac{\lambda_2+\gamma+\gamma_c}{\sqrt{\frac{\Omega^2}{4}+(\lambda_2+\gamma+\gamma_c)^2}} \end{pmatrix}. \tag{2.152}$$

We next proceed to determine the inverse of the matrix V . To this end, it can be readily verified that the characteristic equation

$$det(V - \lambda I) = 0 \tag{2.153}$$

has explicit form

$$\lambda^2 - \left[\frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}} + \frac{\lambda_2 + \gamma + \gamma_c}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}}\right]\lambda$$

$$- \frac{\frac{\Omega}{2}(\lambda_1 - \lambda_2)}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}} I = 0. \tag{2.154}$$

Thus applying the Cayley-Hamilton theorem that a matrix satisfies its own characteristic equation, we have

$$V^2 - \left[\frac{\frac{\Omega}{2}}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}} + \frac{\lambda_2 + \gamma + \gamma_c}{\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}}\right]V$$

$$- \frac{\frac{\Omega}{2}(\lambda_1 - \lambda_2)}{\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2}} I = 0. \tag{2.155}$$

In view of this, we obtain

$$V^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \frac{-(\lambda_2+\gamma+\gamma_c)}{\frac{\Omega}{2}}\sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2} & \sqrt{\frac{\Omega^2}{4} + (\lambda_1 + \gamma + \gamma_c)^2} \\ \frac{(\lambda_1+\gamma+\gamma_c)}{\frac{\Omega}{2}}\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2} & -\sqrt{\frac{\Omega^2}{4} + (\lambda_2 + \gamma + \gamma_c)^2} \end{pmatrix}. \tag{2.156}$$

Furthermore, using the fact that

$$VV^{-1} = V^{-1}V = I, \tag{2.157}$$

We can write Eq.(2.135) as

$$\frac{d}{dt}\hat{J}(t) = VV^{-1}MVV^{-1}\hat{J}(t) + \hat{F}(t), \tag{2.158}$$

Up on multiplying Eq. (2.158) by V^{-1} from the left, we get

$$\frac{d}{dt}(V^{-1}\hat{J}(t)) = DV^{-1}\hat{J}(t) + V^{-1}\hat{F}(t), \tag{2.159}$$

where

$$D = V^{-1}MV = \begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}, \tag{2.160}$$

in which $\beta = \gamma + \gamma_c$. The formal solution of Eq. (2.159) can be written as

$$\hat{J}(t+\tau) = Ve^{Dt}V^{-1}\hat{J}(t) + \int_0^\tau Ve^{D(\tau-\tau')}V^{-1}\hat{F}(t+\tau')d\tau'. \tag{2.161}$$

In view of the fact that D is diagonal, we have

$$e^{Dt} = \begin{pmatrix} e^{-\beta\tau} & 0 \\ 0 & e^{-\beta\tau} \end{pmatrix}, \tag{2.162}$$

and

$$e^{D(\tau-\tau')} = \begin{pmatrix} e^{-\beta(\tau-\tau'} & 0 \\ 0 & e^{-\beta(\tau-\tau')} \end{pmatrix}. \tag{2.163}$$

It then follows that

$$Ve^{D\tau}V^{-1}\hat{J}(t) = \begin{bmatrix} S(\tau)\langle\hat{m}_a(\tau)\rangle & 0 \\ 0 & S(\tau)\langle\hat{m}_b^\dagger(\tau)\rangle \end{bmatrix}, \tag{2.164}$$

and

$$\int_0^\tau Ve^{D(\tau-\tau'}V^{-1}\hat{F}(t+\tau')d\tau' = \begin{bmatrix} \int_0^\tau S(\tau-\tau')\hat{F}_a(\tau+\tau') & 0 \\ 0 & \int_0^\tau S(\tau-\tau')\hat{F}_b^\dagger(t+\tau) \end{bmatrix} \tag{2.165}$$

where

$$S(\tau) = e^{\beta\tau}, \tag{2.166}$$

$$S(\tau-\tau') = e^{\beta(\tau-\tau')}, \tag{2.167}$$

Finally, applications of these results yield

$$\hat{m}_a(t+\tau) = S(\tau)\hat{m}_a(t) + \int_0^\tau S(\tau-\tau')\hat{F}_b^\dagger(t+\tau')d\tau', \tag{2.168}$$

$$\hat{m}_b^\dagger(t+\tau) = S(\tau)\hat{m}_b^\dagger(t) + \int_0^\tau S(\tau-\tau')\hat{F}_a(t+\tau')d\tau'. \tag{2.169}$$

Now the expectation value of Eq. (2.168) and the adjoint of (2.169) can be expressed as

$$\langle\hat{m}_a(t+\tau)\rangle = S(\tau)\langle\hat{m}_a(t)\rangle + \int_0^\tau S(\tau-\tau')\langle\hat{F}_b^\dagger(t+\tau')\rangle d\tau', \tag{2.170}$$

$$\langle\hat{m}_b(t+\tau)\rangle = S(\tau)\langle\hat{m}_b(t)\rangle + \int_0^\tau S(\tau-\tau')\langle\hat{F}_a^\dagger(t+\tau')\rangle d\tau'. \tag{2.171}$$

Upon setting $t = 0$ and $\tau = t$, we see that

$$\langle\hat{m}_a(t)\rangle = S(t)\langle\hat{m}_a(0)\rangle + \int_0^t S(t-t')\langle\hat{F}_b^\dagger(t')\rangle dt', \tag{2.172}$$

$$\langle\hat{m}_b(t)\rangle = S(t)\langle\hat{m}_b(0)\rangle + \int_0^t S(t-t')\langle\hat{F}_a^\dagger(t')\rangle dt'. \tag{2.173}$$

Furthermore, the expectation value of Eqs. (2.132) and (2.133) are

$$\frac{d}{dt}\langle\hat{m}_a(t)\rangle = -(\gamma+\gamma_c)\langle\hat{m}_a(t)\rangle + \frac{\Omega}{2}\langle\hat{m}_b^\dagger(t)\rangle + \langle\hat{F}_a(t)\rangle, \tag{2.174}$$

$$\frac{d}{dt}\langle\hat{m}_b^\dagger(t)\rangle = -\frac{\Omega}{2}\langle\hat{m}_a(t)\rangle - \frac{1}{2}(\gamma+\gamma_c)\langle\hat{m}_b^\dagger(t)\rangle + \langle\hat{F}_b(t)\rangle. \tag{2.175}$$

we note that Eqs. (2.132) and (2.174) as well as (2.133) and (2.175) will have identical forms if

$$\langle\hat{F}_a^\dagger(t)\rangle = \langle\hat{F}_b^\dagger(t')\rangle = 0 \tag{2.176}$$

With the atoms considered to be initially in the bottom level, Eqs. (2.172) and (2.173) reduce to

$$\langle\hat{m}_a(t)\rangle = 0, \tag{2.177}$$

$$\langle\hat{m}_b(t)\rangle = 0. \tag{2.178}$$

Now to obtain the expectation value of the cavity mode operators, according to the quantum Langevin equation given by Eq. (2.125) is expressible as

$$\langle \hat{a}(t)\rangle = \langle \hat{a}(0)\rangle e^{-\kappa t/2} + \frac{g}{\sqrt{N}}\int_0^t e^{\kappa t'/2}\langle \hat{m}_a(t')\rangle dt' + \varepsilon e^{-\kappa t/2}\int_0^t e^{\kappa t'/2} dt'. \tag{2.179}$$

With the help of Eq. (2.177) and the assumption that the cavity light is initially in a vacuum state and carrying out the integration of Eq. (2.179), at steady state goes over into

$$\langle \hat{a}(t)\rangle_{ss} = \frac{2\varepsilon}{\kappa} \tag{2.180}$$

In view of Eq. (2.126) and the result given by Eq. (2.178), one can readily obtain

$$\langle \hat{b}(t)\rangle_{ss} = \frac{2\varepsilon}{\kappa}. \tag{2.181}$$

Then on account of Eq. (2.126) , Eq. (2.180), and (2.181) together with(2.129), we have

$$\langle \hat{c}(t)\rangle_{ss} = \frac{4\varepsilon}{\kappa}. \tag{2.182}$$

Finally, we seek to determine the steady-state solutions of the expectation values of the atomic operators. We note that the steady-state solutions of Eqs. (2.82), (2.83), and (2.84) are given by

$$\langle \hat{m}_c\rangle = \left(\frac{\Omega}{\gamma+\gamma_c}\right)\left[\langle \hat{N}_c\rangle - \langle \hat{N}_a\rangle\right], \tag{2.183}$$

$$\langle \hat{N}_a\rangle = \frac{1}{2}\left(\frac{\Omega}{\gamma+\gamma_c}\right)\left[\langle \hat{m}_c\rangle + \langle \hat{m}_c^{\dagger}\rangle\right], \tag{2.184}$$

$$\langle \hat{N}_b\rangle = \langle \hat{N}_a\rangle. \tag{2.185}$$

Furthermore, with the help of Eq. (2.92) together with (2.185), we see that

$$\langle \hat{N}_c\rangle = N - 2\langle \hat{N}_a\rangle. \tag{2.186}$$

With the aid of Eq. (2.186), Eq. (2.183) can be written as

$$\langle \hat{m}_c \rangle = \left(\frac{\Omega}{\gamma + \gamma_c} \right) \left[N - 3\langle \hat{N}_a \rangle \right], \tag{2.187}$$

and in view of Eq. (2.187), we observe that

$$\langle \hat{m}_c \rangle = \langle \hat{m}_c^{\dagger} \rangle. \tag{2.188}$$

Employing , Eq. (2.188) and (2.184) can be put in the form

$$\langle \hat{N}_a \rangle = \left(\frac{\Omega}{\gamma + \gamma_c} \right) \langle \hat{m}_c \rangle. \tag{2.189}$$

Using Eqs. (2.187) and (2.189), one readily gets

$$\langle \hat{N}_a \rangle = \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N. \tag{2.190}$$

Substitution of Eq. (2.190) into Eqs. (2.185), (2.186), and (2.187) results in

$$\langle \hat{N}_b \rangle = \left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N, \tag{2.191}$$

$$\langle \hat{N}_c \rangle = \left[\frac{(\gamma_c + \gamma)^2 + \Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N, \tag{2.192}$$

$$\langle \hat{m}_c \rangle = \left[\frac{\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right] N. \tag{2.193}$$

These equations represent the steady-state solutions of the equations of evolution of the atomic operators. Furthermore, upon setting ($\gamma = 0$), for the case in which spontaneous emission is absent and for ($\Omega \gg \gamma_c$), the steady-state solutions described by Eqs. (2.190)-(2.93) take the form

$$\langle \hat{N}_a \rangle = \frac{1}{3} N, \tag{2.194}$$

$$\langle \hat{N}_b \rangle = \frac{1}{3} N, \tag{2.195}$$

$$\langle \hat{N}_c \rangle = \frac{1}{3} N, \tag{2.196}$$

$$\langle \hat{m}_c \rangle = 0. \tag{2.197}$$

The results described by Eqs. (2.194)-(2.197) are exactly the same as those obtained by Fesseha [17].

3

Photon statistics

In this chapter we seek to study the statistical properties of light mode a and b is described in terms of the mean and variance of photon number as well as the statistical properties of the two-modes cavity light applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators. In addition, employing the Q-function obtained using antinormally ordered characteristics function defined in the Heisenberg picture we determine photon number variance for the cavity modes.

3.1 Single-mode photon statistics

Here we wish to calculate the mean and variance of the photon numbers for light mode a and b.

3.1.1 Mean photon number

we proceed to calculate the mean photon numbers of light mode a and b. The mean photon number of light mode a, represented by the operators a and $a^\dagger$ is defined by

$$\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle. \tag{3.1}$$

We note that the steady state solution of Eq.(2.125) and Eq.(2.126) can be written as

$$\hat{a} = \frac{2g}{k\sqrt{N}}\hat{m}_a + \frac{2\varepsilon}{k}, \tag{3.2}$$

and

$$\hat{b} = \frac{2g}{k\sqrt{N}}\hat{m}_b + \frac{2\varepsilon}{k}. \tag{3.3}$$

So introducing Eq.(3.2) and its adjoint into (3.1), we see that

$$\bar{n}_a = \frac{2g}{k\sqrt{N}}\langle\hat{m}_a^\dagger\hat{m}_a\rangle + \frac{4\varepsilon^2}{k^2}. \tag{3.4}$$

On account of Eq.(2.105) , Eq. (3.4) can be expressed as

$$\bar{n}_a = \langle\hat{a}^\dagger\hat{a}\rangle = \frac{\gamma_c}{k}\langle N_a\rangle + \frac{4\varepsilon^2}{k^2}. \tag{3.5}$$

also with the aid of Eq.(3.2), one can easily establish that

$$\langle\hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c}{k}\langle N_b\rangle + \frac{4\varepsilon^2}{k^2}, \tag{3.6}$$

$$\langle\hat{a}\hat{a}\rangle = \langle\hat{a}^\dagger\hat{a}^\dagger\rangle = \frac{4\varepsilon^2}{k^2}. \tag{3.7}$$

Therefore,in view of Eq.(2.190), there follows

$$\bar{n}_a = \frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]N + \frac{4\varepsilon^2}{k^2}. \tag{3.8}$$

Moreover, we consider the case in which spontaneous emission is absent($\gamma = 0$). Then the mean photon number of light mode a for this case has the form

$$\bar{n}_a = \frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c)^2+3\Omega^2}\right]N + \frac{4\varepsilon^2}{k^2}. \tag{3.9}$$

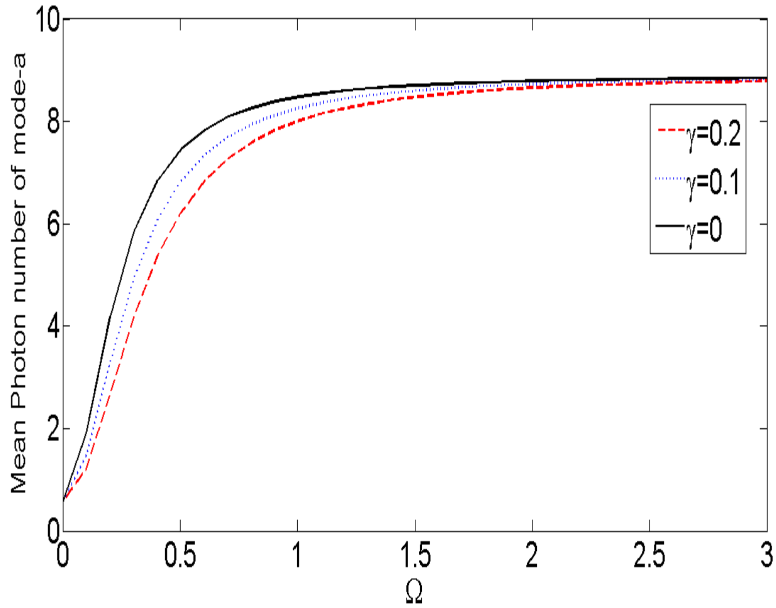

Figure 3.1: Plots of the mean photon number of light mode a [Eq. (3.8)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$, and for different values of γ.

In addition,we note that for the case in which there is no driving light radiation $(\varepsilon = 0)$ and $(\Omega >> \gamma_c)$ Eq.(3.9) the mean photon numbers reduces to

$$\bar{n}_a = \frac{\gamma_c}{3k} N. \tag{3.10}$$

The plots in Fig. (3.4) show that the mean photon number of the single-mode light increases with Ω and the steady-state mean photon number of light mode a in the absence of spontaneous emission when $(\gamma = 0)$ is greater than in the presence of spontaneous emission $(\gamma \neq 0)$. Therefore, the effect of spontaneous emission decreases the mean photon number.

Fig.(3.5) represents the plots of the mean photon number of light mode a [Eq.3.8] versus γ for $\Omega = 0.5$(solid curve), $\Omega = 0.4$(dotted curve) and $\Omega = 0.2$(dash curve) .We see

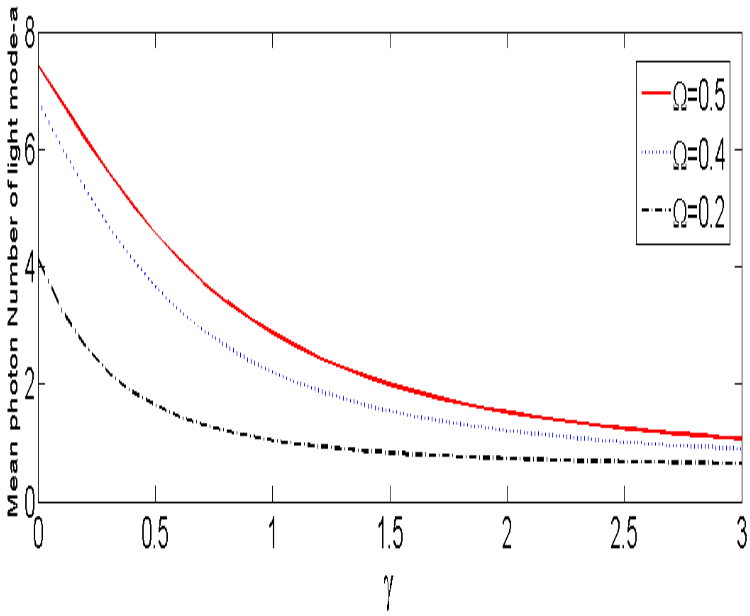

Figure 3.2: Plots of the mean photon number of light mode a [Eq. (3.8)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$, and for different values of Ω.

from the figure the steady-state mean photon number of light mode a decreases as the spontaneous emission increases.

From fig.(3.3) we see that the mean of the photon number of light mode$a$ with the driving light $(\varepsilon \neq 0)$ is greater than with out the driving light $(\varepsilon = 0)$. In other words, the driving light increases the mean of the photon number.

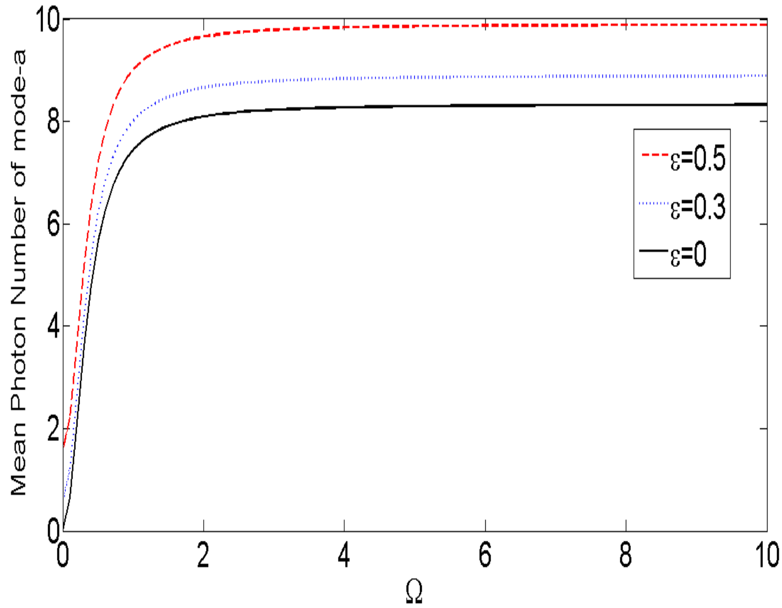

Figure 3.3: Plots of the mean photon number of light mode a [Eq. (3.8)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$, and for different values of ε.

Following a similar procedure, the mean photon number of light mode b is found to be

$$\bar{n}_b = \frac{\gamma_c}{k}\langle N_b\rangle + \frac{4\varepsilon^2}{k^2}. \tag{3.11}$$

Now on substituting Eq.(2.191) into Eq.(3.11), we have

$$\bar{n}_b = \frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]N + \frac{4\varepsilon^2}{k^2}. \tag{3.12}$$

we note that, the case in which there is no driving light radiation $(\varepsilon = 0)$ and $(\Omega >> \gamma_c)$, the mean photon number of mode b reduces to

$$\langle\hat{b}^\dagger\hat{b}\rangle = \bar{n}_b = \frac{\gamma_c}{3k}N. \tag{3.13}$$

In view of eq.(3.2) and eq.(3.13), we see that

$$\bar{n}_a = \bar{n}_b. \tag{3.14}$$

Therefore, we also observe that the mean photon number of light mode a is equal to light mode b.

3.2 The Q function

Here we wish to obtain, using anti-normally ordered characteristic function defined in the Heisenberg picture, the Q function for the two-mode laser light beam. Then employing the resulting Q function, we determine the photon number variance of single-mode and two-mode laser light beam.

Consider a two-mode laser light beam represented by the operators $\hat{c}$ and $\hat{c}^\dagger$ subjected to the commutation relation

$$[\hat{c}, \hat{c}^\dagger] \;=\; \lambda \tag{3.15}$$

where,

$$\lambda = \frac{\gamma_c}{\kappa}[\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle]. \tag{3.16}$$

is a constant c-number. The Q function for the two mode laser light beam can be expressed interms of the anti-normally ordered characteristics function as

$$Q(\gamma^*, \gamma) = \frac{\lambda}{\pi^2} \int d^2 z \phi_a(z) \exp[z^* \gamma - z\gamma^*], \tag{3.17}$$

where $\gamma = \alpha + \beta$ is the c-number variable corresponding to the operator $\hat{c}$ given by Eq.(2.129) the anti-normally ordered characteristics function is defined by

$$\phi_a(z) = Tr(\hat{\rho} \exp^{-z^* \hat{c}} \exp^{z\hat{c}^\dagger}). \tag{3.18}$$

with the aid of the completeness relation [17]

$$\hat{c}|\gamma\rangle = \lambda\gamma|\gamma\rangle, \tag{3.19}$$

we obtain

$$\phi_a(z) = \int d^2\gamma \lambda Q(\lambda\gamma^*, \lambda\gamma) \exp[z\lambda\gamma^* - z^*\lambda\gamma]. \tag{3.20}$$

where $Q(\lambda\gamma^*, \lambda\gamma)$ is the Q function for the two-mode laser light beam. Next introducing the variable $\xi = \lambda\gamma$, the anti-normally ordered function can be put in the form

$$\phi_a(z) = \int d^2\xi \frac{Q(\xi^*, \xi)}{\lambda} \exp[z\xi^* - z^*\xi]. \tag{3.21}$$

since $\frac{Q(\xi^*,\xi)}{\lambda}$ is the inverse fourier transform of the characteristics function,we see that

$$Q(\xi^*, \xi) = \frac{\lambda}{\pi^2} \int d^2z \phi_a(z) \exp[z^*\xi - z\xi^*]. \tag{3.22}$$

Up on integrating both sides of Eq.(3.22) over ξ, and taking into account the fact that

$$\frac{1}{\pi^2} \int d^2\xi \exp[z^*\xi - z\xi^*] = \delta^{(2)}, \tag{3.23}$$

we arrive at

$$\int d^2\xi Q(\xi^*, \xi) = \lambda \int d^2z Tr(\hat{\rho} \exp^{-z^*\hat{c}} \exp^{z\hat{c}^\dagger}). \tag{3.24}$$

from which follows

$$\int d^2\xi Q(\xi^*, \xi) = \lambda. \tag{3.25}$$

This shows that the normalized to λ. We next proceed to obtain the explicit form of the anti normally-ordered characteristic function for the two-mode laser light beam. Thus applying the Baker-Hausdorff identity[29]

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} - \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]}, \tag{3.26}$$

which holds for $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$ along with Eq.(3.15) the anti-normally ordered characteristic function given by Eq. (3.18) can be put in the form

$$\phi_a(z,t) = \exp^{\frac{-\lambda}{2}z^*z}\left\langle \exp[z\hat{c}^\dagger(t) - z^*\hat{c}(t)]\right\rangle . \tag{3.27}$$

In view of Eq.(2.129) and the assumption that the cavity modes are initially in a vacuum state,we can re-write Eq.(3.27) as

$$\phi_a(z,t) = \exp^{\frac{-\lambda}{2}z^*z}\exp\left[\langle\frac{1}{2}(z\hat{c}^\dagger(t) - z^*\hat{c}(t))^2\rangle\right]. \tag{3.28}$$

It then shows that

$$\phi_a(z,t) = \exp\left[\frac{-\lambda}{2}z^*z - \frac{z^*z}{2}\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle - \frac{z^*z}{2}\langle\hat{c}(t)\hat{c}^\dagger(t)\rangle + \frac{z^2}{2}\langle\hat{c}^{\dagger 2}(t)\rangle + \frac{z^{\dagger 2}}{2}\langle\hat{c}^2(t)\rangle\right]. \tag{3.29}$$

since $\langle\hat{m}_c\rangle$ is real, we see that

$$\langle\hat{c}^2\rangle = \langle\hat{c}^{\dagger 2}\rangle = \frac{\gamma_c}{k}\langle\hat{m}_c\rangle + \frac{16\varepsilon^2}{k^2}, \tag{3.30}$$

and with the aid of Eq.(2.129), we get

$$\langle\hat{c}^\dagger\hat{c}\rangle = \frac{\gamma_c}{k}[\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle] + \frac{16\varepsilon^2}{k^2}, \tag{3.31}$$

$$\langle\hat{c}\hat{c}^\dagger\rangle = \frac{\gamma_c}{k}[\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle] + \frac{16\varepsilon^2}{k^2}. \tag{3.32}$$

So that on account of EqS.(3.16), (3.30),(3.31) and (3.32) the anti-normally characteristics function can be put in the form

$$\phi_a(z,t) = \exp[-z^*z[\frac{\gamma_c}{k}\left[\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle\right] + \frac{16\varepsilon^2}{k^2}] + (\frac{\gamma_c}{k}\langle\hat{m}_c\rangle + \frac{16\varepsilon^2}{k^2})(\frac{z^{*2}}{2} + \frac{z^2}{2}). \tag{3.33}$$

This can be written as

$$\phi_a(z,t) = \exp[-Rz^*z + S(\frac{z^{*2}}{2} + \frac{z^2}{2})]. \tag{3.34}$$

in which

$$R = \frac{\gamma_c}{k}[\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle] + \frac{16\varepsilon^2}{k^2}, \tag{3.35}$$

$$S = \frac{\gamma_c}{k}\langle \hat{m}_c \rangle + \frac{16\varepsilon^2}{k^2}. \tag{3.36}$$

Now introducing Eq.(3.29) into Eq.(3.17) the Q function for two-mode laser light beam can be written as

$$Q(\gamma^*, \gamma) = \frac{\lambda}{\pi^2}\int d^2z \exp[-Rz^*z + S(\frac{z^{*2}}{2} + \frac{z^2}{2})]. \tag{3.37}$$

Furthermore, using the relation given [17]

$$\int \frac{d^2z}{\pi} e^{-az^*z+bz+cz^*+Az^2+Bz^{*2}} = \frac{1}{\sqrt{a^2-4BC}} e^{\frac{abc+Ab^2+Bb^2}{a^2-4BC}}, \quad a > 0, \tag{3.38}$$

and performing the integration, one readily obtain

$$Q(\gamma^*, \gamma) = \frac{\lambda}{\pi}\left[\frac{1}{u^2 - v^2}\right]^{\frac{1}{2}} \exp[-u\gamma^*\gamma + v(\frac{\gamma^{*2}}{2} + \frac{\gamma^2}{2})]. \tag{3.39}$$

in which

$$u = \frac{R}{R^2 - S^2} \tag{3.40}$$

$$v = \frac{S}{R^2 - S^2}. \tag{3.41}$$

We note that Eq.(3.39) is the Q function for the two-mode laser light beam. Moreover, following a similar procedure,the Q function for the single-mode a can be written as

$$Q(\alpha^*, \alpha) = \frac{\lambda}{\pi}\left[\frac{1}{a^2 - b^2}\right]^{\frac{1}{2}} \exp\left[-\frac{a}{a^2 - b^2}\alpha^*\alpha + \frac{b}{a^2 - b^2}(\frac{\alpha^{*2}}{2} + \frac{\alpha^2}{2})\right]. \tag{3.42}$$

where

$$a = \frac{\gamma_c}{\kappa}\langle N_b \rangle + \frac{4\varepsilon^2}{\kappa^2}, \tag{3.43}$$

$$b = \frac{4\varepsilon^2}{\kappa^2}. \tag{3.44}$$

It then follows that

$$Q(\alpha^*,\alpha) = \frac{\lambda}{\pi}\left[\frac{1}{u'^2-v'^2}\right]^{\frac{1}{2}} \exp\left[-u'\alpha^*\alpha + v'(\frac{\alpha^{*2}}{2}+\frac{\alpha^2}{2})\right]. \tag{3.45}$$

in which $u' = \frac{a}{a^2-b^2}$, and $v' = \frac{b}{a^2-b^2}$.

3.3 Photon number variance

We next proceed to obtain the variance of the light mode a and b. The photon number variance of light mode a and b defined by

$$(\Delta n_a)^2 = \langle (a^\dagger a)^2\rangle - \bar{n}^2, \tag{3.46}$$

$$(\Delta n_b)^2 = \langle (b^\dagger b)^2\rangle - \bar{n}^2. \tag{3.47}$$

using the commutation relation $[\hat{a}^\dagger, \hat{a}] = 1$, we can write of the first on the right hand sides terms of Eq.(3.46) and Eq.(3.47)as

$$\langle (a^\dagger a)^2\rangle = \langle a^2 a^{\dagger 2}\rangle - 3\bar{n}_a - 2, \tag{3.48}$$

$$\langle (b^\dagger b)^2\rangle = \langle b^2 b^{\dagger 2}\rangle - 3\bar{n}_b - 2. \tag{3.49}$$

So that expression (3.46), can be written as

$$(\triangle\ n_a)^2 = \langle (a^2 a^{\dagger 2})\rangle - \bar{n}^2 - 3\bar{n}_a - 2. \tag{3.50}$$

Thus employing Eq.(3.45), we have

$$\langle a^2 a^{\dagger 2}\rangle = \int d^2\alpha Q(\alpha)\alpha^{\dagger 2}\alpha^2. \tag{3.51}$$

This can be put in the form

$$\langle a^2 a^{\dagger 2}\rangle = [u'^2 - v'^2]^{\frac{1}{2}} \frac{d^4}{d\eta^2 dz^2}\int \frac{d^2\alpha}{\pi} e^{-u'\alpha^*\alpha + \eta\alpha + z\alpha^* + v'(\frac{\alpha^{*2}}{2}+\frac{\alpha^2}{2})}, \quad \eta = z = 0 \tag{3.52}$$

On carrying out the integration using Eq.(3.38), we obtain

$$\langle a^2 a^{\dagger 2}\rangle = \frac{d^4}{d\eta^2 dz^2}\exp[\frac{(u'z\eta + \frac{v'z^2}{2} + \frac{v'\eta^2}{2})}{u'^2 - v'^2}], \quad \eta = z = 0. \tag{3.53}$$

Performing the derivative and applying the condition $z = \eta = 0$, we get

$$\langle a^2 a^{\dagger 2}\rangle = 2(\bar{n}_a + 1)^2. \tag{3.54}$$

Using Eq.(3.54) into (3.50), we get

$$(\Delta n_a)^2 = \bar{n}_a^2 + \bar{n}_a. \tag{3.55}$$

Therefore, substitution of Eq.(3.8),into Eq.(3.55), we have

$$(\triangle\ n_a)^2 = \left[\frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]N + \frac{4\varepsilon^2}{k^2}\right]^2 + \frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]N + \frac{4\varepsilon^2}{k^2}. \tag{3.56}$$

It can also be shown in similar manner

$$(\triangle\ n_b)^2 = \bar{n}_b^2 + \bar{n}_b. \tag{3.57}$$

We see from eqs.(3.55) and (3.57) that the light mode a and b are separately in a chaotic light. Furthermore, inspection of these equation indicates that $(\Delta n)_a^2 > \bar{n}_a$ and $(\Delta n)_b^2 > \bar{n}_b$, and hence the photon statistics of each light mode is super-poissonian statistics.

3.4 Two-mode photon statistics

In this section, applying the steady-state solutions of the equations of evolution of the expectation value of the atomic operators and quantum langevin equations for the cavity mode operators, we seek to obtain the mean and variance of the photon numbers for two-mode light beam.

3.4.1 Two-mode mean photon number

Here we seek to calculate the steady state mean photon number of the two-mode cavity light beam. The mean photon number of the two-mode light beam, represented by the operators $\hat{c}$ and $\hat{c}^{\dagger}$ is defined by

$$\bar{n} = \langle \hat{c}^{\dagger}\hat{c}\rangle. \tag{3.58}$$

The steady-state solution of Eq.(2.129) is found to be

$$\langle \hat{c}\rangle = \frac{2g}{k\sqrt{N}}\hat{m} + \frac{4\varepsilon}{k}. \tag{3.59}$$

Hence at steady state the mean photon numbers goes over into

$$\bar{n} = \frac{\gamma_c}{k}\left[\langle N_a\rangle + \langle N_b\rangle\right] + \frac{16\varepsilon^2}{k^2}. \tag{3.60}$$

We see from Eq.(3.60) that the mean photon number of the two-mode light beam is the sum of photon numbers of the separate single mode light beams given by Eq.(3.5) and (3.11). Therefore, on account of Eqs.(3.8) and (3.12),into Eq.(3.60), we have

$$\bar{n} = \frac{\gamma_c}{k}\left[\frac{2\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]N + \frac{16\varepsilon^2}{k^2}. \tag{3.61}$$

We next proceed to consider for the case in which spontaneous emission is absent($\gamma = 0$) and $\Omega \gg \gamma_c$. Then the mean photon number for this case takes the form

$$\bar{n} = \frac{2\gamma_c}{3k}N + \frac{16\varepsilon^2}{k^2}. \tag{3.62}$$

Furthermore, we note that for $(\varepsilon = 0)$ Eq.3.62reduces to

$$\bar{n} = \frac{2\gamma_c}{3k}N. \tag{3.63}$$

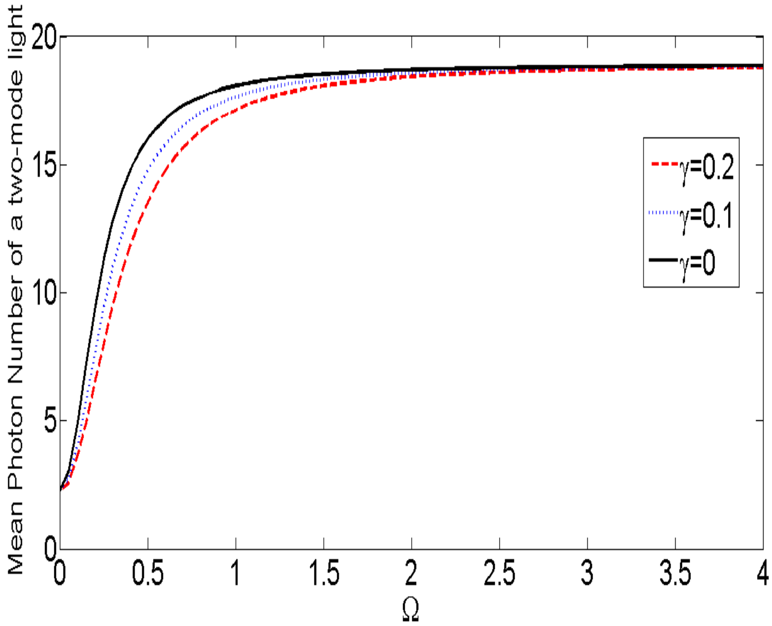

Figure 3.4: Plots of the mean photon number of two-mode light [Eq. (3.60)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$, and for different values of γ.

The plots in Fig. (3.4) we observe that the steady-state mean photon number of two-mode light in the absence of spontaneous emission when $(\gamma = 0)$ is greater than in the presence of spontaneous emission $(\gamma \neq 0)$. Therefore, the effect of spontaneous emission decreases the mean photon number two-mode light. also, from fig.(3.5) we see that the mean of the photon number of two-mode light with the driving light $(\varepsilon \neq 0)$ is greater than with out the driving light $(\varepsilon = 0)$. In other words, the driving light increases the mean of the photon number.

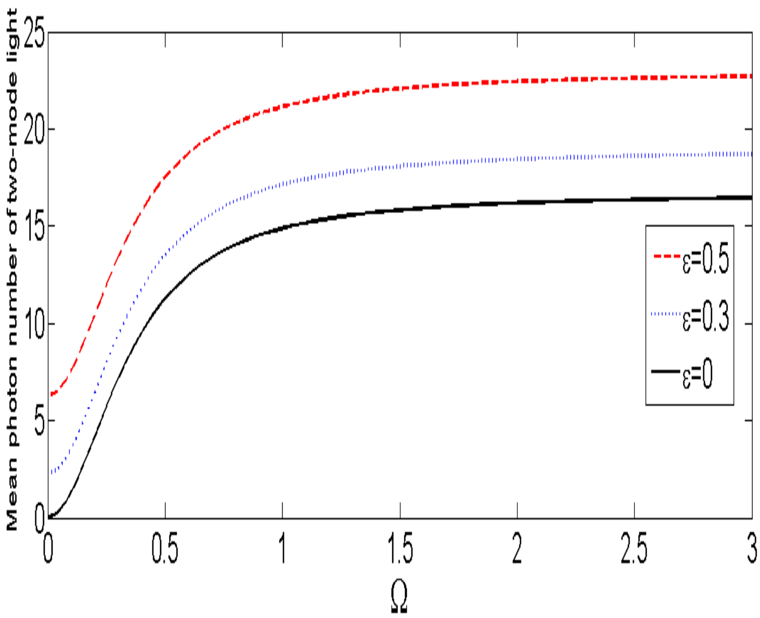

Figure 3.5: Plots of the mean photon number of light mode a [Eq. (3.60)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0$ $N = 50$, and for different values of ε.

3.4.2 Two-mode photon number variance

Here we proceed to study the steady state photon number variance for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{3.64}$$

where $\hat{n} = \hat{c}^\dagger \hat{c}$. Using the usual boson commutation relation, it is possible to rewrite Eq. (3.64) in terms of the c-number variables associated with the normal ordering the photon number variance can be written as

$$(\Delta n)^2 = \langle (\gamma^2 \gamma^{\dagger 2}) \rangle - \bar{n}^2 - 3\bar{n} - 2. \tag{3.65}$$

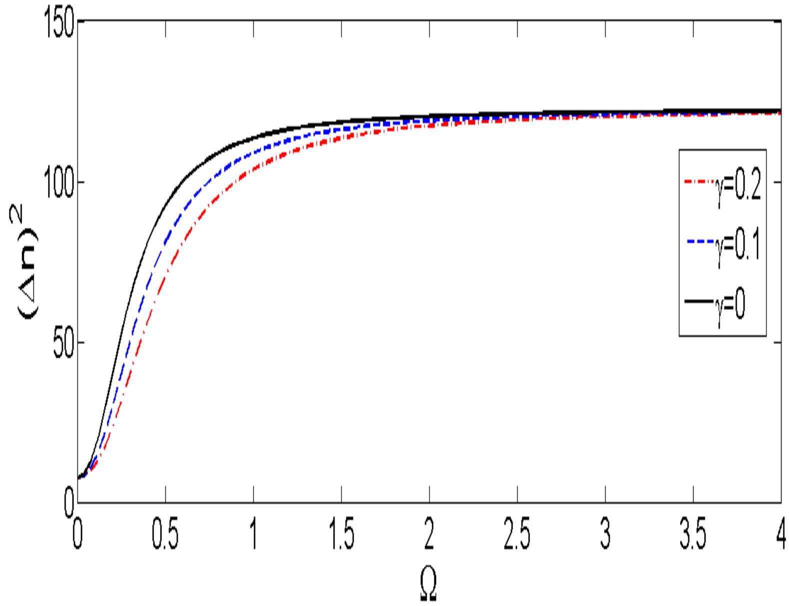

Figure 3.6: Plots of the photon number variance of two-mode light [Eq. (3.69)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$ $N = 50$, and for different values of γ.

The term on the right side of Eq.(3.65) can be expressed in terms of Q function as

$$\langle \gamma^2 \gamma^{\dagger 2} \rangle = \frac{2\lambda}{\pi}[u^2 - v^2]^{\frac{1}{2}} \frac{d^4}{dm^2 dn^2} \int d^2\alpha e^{-u\gamma^*\gamma + m\gamma + n\gamma^* + v(\frac{\gamma^{*2}}{2} + \frac{\gamma^2}{2})}, \quad m = n = 0. \tag{3.66}$$

so that up on carrying out integration, differentiation and setting the condition $m = n = o$, we get

$$\langle \gamma^2 \gamma^{\dagger 2} \rangle = 2(\bar{n} + 1)^2. \tag{3.67}$$

substitution of Eq.(3.67) into (3.65) yields

$$(\Delta n)^2 = \bar{n}^2 + \bar{n}. \tag{3.68}$$

Therefor, with the aid Eq.(3.61), we can write

$$(\Delta n)^2 = \left[\frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right] N + \frac{16\varepsilon^2}{k^2}\right]^2 + \frac{\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right] N + \frac{16\varepsilon^2}{k^2}, \tag{3.69}$$

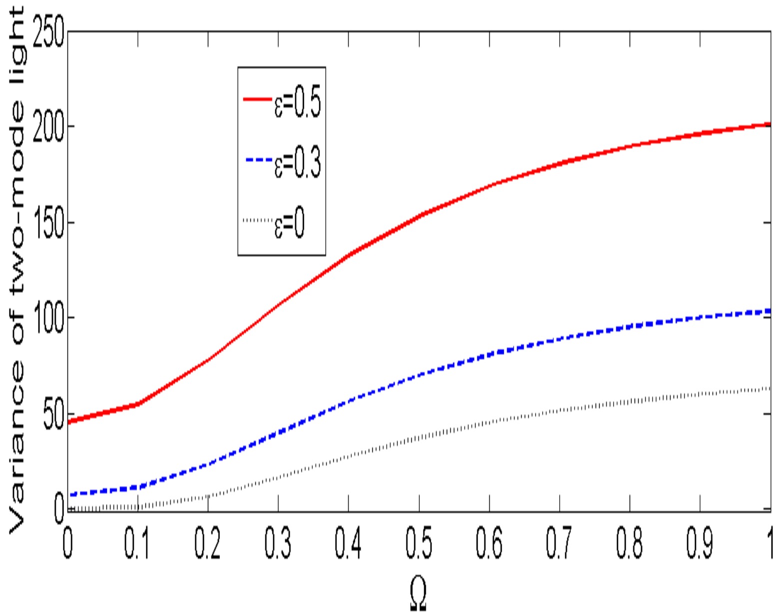

Figure 3.7: Plots of the variance of photon number of two-mode light [Eq. (3.69)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$ $N = 50$, and for different values of ε.

we note that for the case in which there is no driving light radiation($\varepsilon = 0$), ($\Omega \gg \gamma_c$) and the spontaneous emission is absent($\gamma = 0$).The variance of the photon for this case takes the form

$$(\Delta n)^2 = \left[\frac{2\gamma_c}{3k}N\right]^2 + \frac{2\gamma_c}{3k}. \tag{3.70}$$

Plots on fig.(3.6) we observe that the steady-state variance photon number of two-mode light is greater than the mean photon number of two-mode light. Furthermore, we have also observe that the normally ordered variance of the photon number for chaotic light and the photon statistics of the two-mode light is super-poissonian.

From fig.(3.7) we observe that the variance of the photon number of two-mode light with the driving light ($\varepsilon \neq 0$) is greater than with out the driving light ($\varepsilon = 0$). In other words, the driving light increases the variance of the photon number.

3.5 Photon number correlation

In order to determine whether the photon numbers of mode a and mode b are correlated or not, we must examine the normalized photon numbers correlation. Thus the photon number correlation of a two-mode light can be defined by [35]

$$g_{ab}^{(2)}(0) = \frac{\langle \hat{n}_a \hat{n}_b \rangle}{\bar{n}_a \bar{n}_b} \tag{3.71}$$

where,$\hat{n}_a = \hat{a}^\dagger \hat{a}$ and $\hat{n}_b = \hat{b}^\dagger \hat{b}$ are the photon number operators for cavity modes a and b, respectively. If $g_{ab}^{(2)} = 1$ the photon numbers of the cavity modes are uncorrelated. If on the other hand $g_{ab}^{(2)} \neq 1$, the photon numbers of the cavity modes are correlated. Therefore, one can express eq.(3.71) in the form

$$g_{ab}^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle}{\bar{n}_a \bar{n}_b}. \tag{3.72}$$

Since Eqs.(2.125) and (2.126) are linear differential equation, one can express Eq. (3.72) in the form

$$g_{ab}^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle \langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle \langle \hat{a} \hat{b}^\dagger \rangle}{\bar{n}_a \bar{n}_b}. \tag{3.73}$$

It then follows that;

$$g_{ab}^{(2)}(0) = 1 + \frac{\langle \hat{a}^\dagger \hat{b}^\dagger \rangle \langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle \langle \hat{a} \hat{b}^\dagger \rangle}{\bar{n}_a \bar{n}_b}, \tag{3.74}$$

with the aid of Eqs.(3.2) and (3.3) the steady-state second order correlation function takes the form

$$g_{ab}^{(2)}(0) = 1 + \frac{4\frac{\gamma_c}{\kappa^3}\varepsilon^2 \langle m_c \rangle + \frac{32\varepsilon^4}{\kappa^4}}{\bar{n}_a \bar{n}_b}. \tag{3.75}$$

The result in Eq. (3.75) shows that the second order correlation function of the two-mode light depends on the number of atom and the chaotic light has no effect on the photon number correlation. It also fore the case; we note that upon setting absence of driving light($\varepsilon = 0$) the above equation reduces to the photon number correlation for the two-mode chaotic light.

Therefore, this result shows that the photon numbers for the cavity modes a and b are correlated.

3.6 Intensity Difference Fluctuation

On the other hand, the variance of the intensity difference can be defined as

$$I_D^2 = \langle \hat{I}_D^2 \rangle - \langle \hat{I}_D \rangle^2, \tag{3.76}$$

where the difference of intensity is

$$\hat{I}_D = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}. \tag{3.77}$$

Hence making use of Eq. (3.77), it is possible to express

$$\begin{aligned} I_D^2 &= \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle^2 + \langle \hat{b}^{\dagger 2} \rangle \langle \hat{b}^2 \rangle \\ &+ 2\langle \hat{b}^\dagger \hat{b} \rangle^2 - 2\langle \hat{a}\hat{b} \rangle^2 - 2\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle \\ &- 2\langle \hat{a}^\dagger \hat{b} \rangle^2 + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle. \end{aligned} \tag{3.78}$$

This can be rewritten as

$$\begin{aligned} I_D^2 &= \langle \hat{a}^\dagger \hat{a} \rangle [1 + \langle \hat{a}^\dagger \hat{a} \rangle] + \langle \hat{b}^\dagger \hat{b} \rangle \\ &\times [1 + \langle \hat{b}^\dagger \hat{b} \rangle] - 2\langle \hat{a}\hat{b} \rangle^2, \end{aligned} \tag{3.79}$$

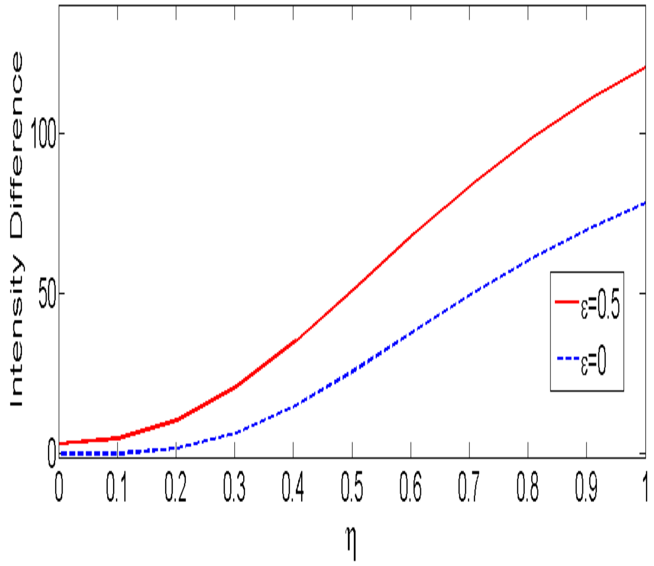

Figure 3.8: Plot of the variance of the intensity difference ΔI^2versus η occurs at steady state for $N = 50$, $(\varepsilon = 0, 0.5)$,$\kappa = 0.8$, and $\gamma_c = 0.4$

On account of Eqs. (3.2), (3.3), (3.4) and (3.10) in to Eq. (3.79) one can readily finds

$$I_D^2 = 2\left[\frac{\gamma_c}{k}\langle N_a\rangle\right]^2 + 2\left[\frac{\gamma_c}{k}\langle N_a\rangle\right] + \frac{16\gamma_c\varepsilon^2}{k^3}\langle N_a\rangle + \frac{8\varepsilon^2}{k^2}. \quad (3.80)$$

In addition, employing Eq. (2.190), we have

$$I_D^2 = 2\left[\frac{\gamma_c}{k}\left[\frac{N\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]\right]^2 + \frac{16\gamma_c\varepsilon^2}{k^3}\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]N$$
$$+\frac{2\gamma_c}{k}\left[\frac{\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]N + \frac{8\varepsilon^2}{k^2}. \quad (3.81)$$

Up on setting $\eta = \frac{\Omega}{\gamma_c}$, and $(\gamma = 0)$, Eq. (3.81) reduces to

$$I_D^2 = 2\left[\frac{\gamma_c}{k}\left[\frac{N\eta^2}{1+3\eta^2}\right]\right]^2 + \frac{16\gamma_c\varepsilon^2}{k^3}\left[\frac{\eta^2}{1+3\eta^2}\right]N + \frac{2\gamma_c}{k}\left[\frac{\eta^2}{1+3\eta^2}\right]N + \frac{8\varepsilon^2}{k^2}. \quad (3.82)$$

Eq. (3.82), describes the variance of the intensity difference of a coherently driven three-level laser and coupled to a two-mode vacuum reservoirs.

Figure (3.8) shows that the variance of the intensity difference for a coherently driven three-level laser with in the absence of the driving light and presence of driving light and coupled to a two-mode vacuum reservoirs versus η for the values $N = 50$, $\kappa = 0.8$, and $\gamma_c = 0.4$. The plot shows that variance of the intensity difference, ΔI^2, increases as η increases.

4

Quadrature Squeezing

In this chapter we seek to study the quadrature variance and the quadrature squeezing of the light produced by a non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir . Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the quadrature variances for light modes a and b. In addition, we determine the quadrature squeezing of the two-mode cavity light.

4.1 Single-mode quadrature variance

We now proceed to calculate the quadrature variance of light mode a in the entire frequency interval. The squeezing properties of light mode a are described by two quadrature operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{4.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.2}$$

where $\hat{a}_+$and $\hat{a}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures using Eq.(4.1) and (4.2) one can write

$$[\hat{a}_-, \hat{a}_+] = i\langle[\hat{a}^\dagger - \hat{a}, \hat{a} + \hat{a}^\dagger]\rangle, \tag{4.3}$$

with the aid of the identity

$$[\hat{A} + \hat{B}, \hat{C} + \hat{D}] = [\hat{A}, \hat{C}] + [\hat{A}, \hat{D}] + [\hat{B}, \hat{C}] + [\hat{B}, \hat{D}], \tag{4.4}$$

we have

$$[\hat{a}_-, \hat{a}_+] = i(\langle[\hat{a}^\dagger, \hat{a}] + [\hat{a}^\dagger, \hat{a}^\dagger] - [\hat{a}, \hat{a}] - [\hat{a}, \hat{a}^\dagger]\rangle). \tag{4.5}$$

So that in view of Eq.(2.117),one can express this commutation relation in the form

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}\left[\hat{N}_a - \hat{N}_b\right]. \tag{4.6}$$

In view of this result,the uncertainty relation for plus and minus quadrature operators of mode a is expressible as

$$\Delta\hat{a}_+\Delta\hat{a}_- \geq \frac{1}{2}\left|\langle[\hat{a}_+, \hat{a}_-]\rangle\right|, \tag{4.7}$$

$$\geq \left|\langle[\hat{a}, \hat{a}^\dagger]\rangle - \langle[\hat{a}^\dagger, \hat{a}]\rangle\right|. \tag{4.8}$$

so that using Eqs.(3.4) and (3.5) then follows

$$\Delta\hat{a}_+\Delta\hat{a}_- \geq \frac{\gamma_c}{k}\left|\langle N_a\rangle - \langle N_b\rangle\right|. \tag{4.9}$$

On account of Eq.(2.185), the uncertainty relation for the quadrature operators can be expressed as

$$\Delta\hat{a}_+\Delta\hat{a}_- \geq 0. \tag{4.10}$$

The quadrature variance for mode a is defined by

$$(\Delta\hat{a}_+)^2 = \langle(\hat{a}_+)^2\rangle - \langle(\hat{a}_+)\rangle^2, \tag{4.11}$$

and

$$(\Delta\hat{a}_-)^2 = \langle(\hat{a}_-)^2\rangle - \langle(\hat{a}_-)\rangle^2. \tag{4.12}$$

In view of Eqs.(4.1) and (4.11) can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle. \tag{4.13}$$

In addition on account of Eqs.(4.12) and (4.2), we get

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle. \tag{4.14}$$

So that combing of Eqs.(4.12) and (4.13)the quadrature variance can be put in the form

$$(\Delta\hat{a}_\pm)^2 = \pm(\pm(\langle\hat{a}\hat{a}^\dagger\rangle \pm \langle\hat{a}^\dagger\hat{a}\rangle) + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle) \mp (\langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 \pm 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle). \tag{4.15}$$

we observe on the basis of Eq.(2.180) one can write

$$\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle = \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle = \frac{4\varepsilon^2}{k^2} \tag{4.16}$$

Applying Eqs. (3.4), (3.5), (3.6), and (4.16) into (4.15), we arrive at

$$(\Delta\hat{a}\pm)^2 = \frac{\gamma_c}{\kappa}\big[\langle N_a\rangle + \langle N_b\rangle\big]. \tag{4.17}$$

On account of (Eq.2.185), we see that

$$(\Delta\hat{a}_\pm)^2 = \frac{2\gamma_c}{\kappa}\langle N_a\rangle. \tag{4.18}$$

Now substitution of Eq. (2.190) in to Eq. (4.18) results in

$$(\Delta\hat{a}_{\pm})^2 = \frac{\gamma_c}{k}\left[\frac{2\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]N. \tag{4.19}$$

Furthermore, the case in which spontaneous emission is absent $(\gamma = 0)$,and $\Omega \gg \gamma_c$, then the quadrature variance for this case takes the form

$$(\Delta\hat{a}_{\pm})^2 = \frac{2\gamma_c}{3k}N. \tag{4.20}$$

On account of Eq. (3.10), Eq. (4.20) can be put in the form

$$(\Delta\hat{a}_{\pm})^2 = 2\bar{n}_a. \tag{4.21}$$

Therefore, we see that the cavity mode a is in a chaotic state.

Next we seek to calculate the quadrature variance of light mode b. The squeezing properties of light mode b are described by two quadrature operators defined by

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b}, \tag{4.22}$$

and

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}). \tag{4.23}$$

where $\hat{b}_+$ and $\hat{b}_-$These operators are Hermitian which represents physical quantities called plus and minus quadratures,respectively. While $\hat{b}^\dagger$ and $\hat{b}$ are creation and annihilation operators for light mode b with the help of Eq.(4.22) and (4.23), we can show that the two quadratures satisfy the commutation relation

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}\left[\hat{N}_b - \hat{N}_c\right]. \tag{4.24}$$

In view of Eq. (4.24), the uncertainty relation for the plus and minus quadrature operators of mode b is expressible as

$$\Delta\hat{b}_+\Delta\hat{b}_- \geq \frac{1}{2}\left|\langle\hat{b}_+\hat{b}_-\rangle\right|,$$

$$\geq \left|\langle[\hat{b},\hat{b}^\dagger]\rangle - \langle[\hat{b}^\dagger,\hat{b}]\rangle\right|. \tag{4.25}$$

On account of Eqs. (4.24), (2.191) and (2.192), into Eq. (4.25), we can easily find that

$$\Delta\hat{b}_+\Delta\hat{b}_- \geq \frac{\gamma_c}{k}N\left|\frac{(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right|. \tag{4.26}$$

Now setting, spontaneous emission is absent $(\gamma = 0)$ and $\Omega = 0$, one can finds

$$\Delta\hat{b}_+\Delta\hat{b}_- \geq \frac{\gamma_c}{k}N, \tag{4.27}$$

We therefore notice that the product of the uncertainties in the two quadratures satisfies the minimum uncertainty relation.

Next we proceed to calculate the quadrature variance of light mode b. The variance of the plus and minus quadrature operators for light mode b are defined by

$$(\Delta\hat{b}_\pm)^2 = \langle\hat{b_\pm^2}\rangle - \langle\hat{b}_\pm\rangle^2. \tag{4.28}$$

On account of Eq. (4.22), Eq. (4.23) and Eq. (4.28) can be expressed in terms of the creation and annihilation operator as

$$(\Delta\hat{b}_\pm)^2 = \pm(\langle\hat{b}^2\rangle + \langle\hat{b}^{\dagger 2}\rangle \pm \langle\hat{b}\hat{b}^\dagger\rangle \pm \langle\hat{b}^\dagger\hat{b}\rangle) \mp (\langle\hat{b}\rangle^2 + \langle\hat{b}^\dagger\rangle^2 \pm 2\langle\hat{b}\rangle\langle\hat{b}^\dagger\rangle). \tag{4.29}$$

Moreover, with the aid of Eq. (3.3) , we get

$$(\Delta\hat{b}_\pm)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - 2\langle\hat{b}\rangle\langle\hat{b}^\dagger\rangle. \tag{4.30}$$

and in view of Eqs. (3.3) and (2.181) one can write

$$\langle\hat{b}\rangle\langle\hat{b}^{\dagger}\rangle = \langle\hat{b}^{\dagger}\rangle\langle\hat{b}\rangle = \frac{4\varepsilon^2}{\kappa^2}. \tag{4.31}$$

also with the help Eq. (4.31), we have

$$\langle\hat{b}^2\rangle = \langle\hat{b}^{\dagger 2}\rangle = \frac{4\varepsilon^2}{\kappa^2}. \tag{4.32}$$

moreover, using Eqs. (4.32), (4.31) and (3.12) into (4.30) and, the quadrature variance of light mode b takes, at steady-state, the form

$$(\Delta\hat{b}_{\pm})^2 = \frac{\gamma_c}{k}\left[\frac{(\gamma_c+\gamma)^2+2\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]N. \tag{4.33}$$

With the help of Eqs. (4.19) and (4.33), we see that

$$(\Delta\hat{b}_{\pm})^2 = (\Delta\hat{a}_{\pm})^2 + \frac{\gamma_c}{k}\left[\frac{(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right]N. \tag{4.34}$$

In view of this result, the quadrature variance of light mode b is greater than the quadrature variance of light mode a. This must be due to the fact that some of the atoms that make transition from the upper and intermediate level to the bottom level with spontaneous and stimulated emissions. Furthermore,we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the quadrature variance for this case has the form

$$(\Delta\hat{b}_{\pm})^2 = \frac{\gamma_c}{k}\left[\frac{(\gamma_c)^2+2\Omega^2}{(\gamma_c)^2+3\Omega^2}\right]N. \tag{4.35}$$

In addition, we note that $(\Omega \gg \gamma_c)$, and the quadrature variance of the single mode light is independent of the deriving coherent light Eq. (4.35), reduces to

$$(\Delta\hat{b}_{\pm})^2 = \frac{2\gamma_c}{3k} \tag{4.36}$$

In view of eq. (3.13) this can be expressed as

$$(\Delta\hat{b}_{\pm})^2 = 2\bar{n}_b, \tag{4.37}$$

which is the normal ordered quadrature variance for chaotic light.

4.2 Two-mode quadrature squeezing

we proceed to determine the quadrature variances of the two-mode light beam. The squeezing properties of the two-mode cavity light are described by two quadrature operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{4.38}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{4.39}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{c}^\dagger$ and $\hat{c}$ are the creation and annihilation operators of the two-mode cavity light. With the aid of Eqs. (4.38) and (4.39), we show that the two quadrature operators satisfy the commutation relation

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}\big[\hat{N}_a - \hat{N}_c\big]. \tag{4.40}$$

Using Eq. (4.40), the uncertainty relation for the plus and minus quadrature operators of the two-mode cavity light is expressible as

$$\Delta\hat{c}_+\Delta\hat{c}_- \geq \frac{1}{2}\left|\langle[\hat{c}_+, \hat{c}_-]\rangle\right|, \tag{4.41}$$

$$\geq \left|\langle\hat{c}, \hat{c}^\dagger\rangle - \langle\hat{c}^\dagger, \hat{c}\rangle\right|, \tag{4.42}$$

so that using Eqs. (3.31) and (3.32), there follows

$$\Delta\hat{c}_{+}\Delta\hat{c}_{-} \geq \frac{\gamma_c}{\kappa}\left|\langle N_a\rangle - \langle N_c\rangle\right|. \tag{4.43}$$

On account of Eqs. (2.190) and (2.191), the uncertainty relation for the plus and minus quadrature operators is found to be

$$\Delta\hat{c}_{+}\Delta\hat{c}_{-} \geq \frac{\gamma_c}{k}N\left|\frac{(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right|. \tag{4.44}$$

In addition, we consider the case in which spontaneous emission is absent $(\gamma = 0)$,the driving coherent light is absent and up on setting $(\Omega = 0)$. Then the uncertainty relation for this case takes the form

$$\Delta\hat{c}_{+}\Delta\hat{c}_{-} \geq \frac{\gamma_c}{k}N. \tag{4.45}$$

which is the minimum uncertainty relation for the two-mode cavity vacuum state. Next we proceed to calculate the quadrature variance of the two-mode cavity light. The variance of the plus and minus quadrature operators of the two-mode cavity light are defined by

$$(\Delta\hat{c}_{\pm})^2 = \langle\hat{c}_{\pm}^2\rangle - \langle\hat{c}_{\pm}\rangle^2. \tag{4.46}$$

On account of Eqs.In view of Eqs. (4.38), (4.39) and (4.46), the plus and minus quadrature variance can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{c}_{\pm})^2 = \langle\hat{c}\hat{c}^{\dagger}\rangle + \langle\hat{c}^{\dagger}\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \mp \langle\hat{c}\rangle^2 \mp \langle\hat{c}^{\dagger}\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^{\dagger}\rangle. \tag{4.47}$$

with the aid of Eqs. (3.30), (3.31), (3.32) and (3.56) into Eq.(4.47), expression goes over into

$$(\Delta\hat{c}_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\langle N_a\rangle + 2\langle N_b\rangle + \langle N_c\rangle \pm \langle\hat{m}_c\rangle \pm \langle\hat{m}_c^{\dagger}\rangle\right]. \tag{4.48}$$

Now using Eqs.(2.186) and (2.188) , the quadrature variance of the two-mode cavity light is found to be

$$(\Delta\hat{c}_{\pm})^2 = \frac{\gamma_c}{\kappa}\left[\langle N\rangle + \langle N_b\rangle \pm \langle\hat{m}_c\rangle\right]. \tag{4.49}$$

Finally, on account of Eqs.(2.191) and (2.193), the quadrature variance of the two-mode cavity light takes, at steady-state, the form

$$(\Delta\hat{c}_{+})^2 = \frac{\gamma_c}{\kappa}N\left[\frac{(\gamma_c+\gamma)^2+4\Omega^2+2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2+3\Omega^2}\right], \tag{4.50}$$

and

$$(\Delta\hat{c}_{-})^2 = \frac{\gamma_c}{\kappa}N\left[\frac{(\gamma_c+\gamma)^2+4\Omega^2-2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2+3\Omega^2}\right]. \tag{4.51}$$

Furthermore, we consider the case in which spontaneous emission is absent $(\gamma = 0)$,and for $(\Omega \gg \gamma_c)$. Thus the quadrature variance for this case has the form

$$(\Delta\hat{c}\pm)^2 = \frac{4\gamma_c}{3\kappa}N. \tag{4.52}$$

This result is exactly the same as the one obtained by Fesseha [17].also this can be written as

$$(\triangle\ \hat{c}\pm)^2 = 2\bar{n}, \tag{4.53}$$

where $\bar{n}$ is given by Eq. (3.59). We see that Eq. (4.53) represents the normally ordered quadrature variance for chaotic light. Thus upon setting $(\Omega = 0)$ in Eq. (4.51), we get

$$(\Delta\hat{c}_{+})^2 = (\Delta\hat{c}_{-})^2 = \frac{\gamma_c}{\kappa}N. \tag{4.54}$$

which is the normally ordered quadrature variance of the two-mode cavity vacuum state. We note that for $(\Omega = 0)$ the uncertainty in the plus and minus quadratures

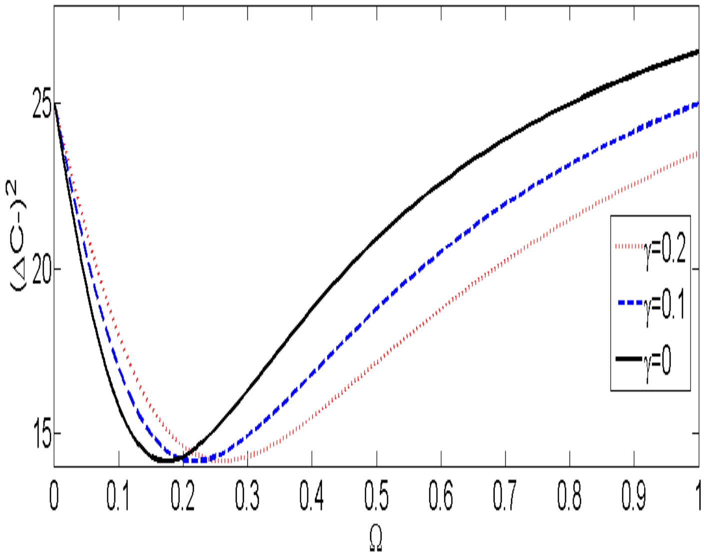

Figure 4.1: Plots of $(\Delta c_{-})^2$ [Eq. (4.51)] versus Ω for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$ and for different values of γ.

are equal and satisfy the minimum uncertainty relation.The plot in Fig. (4.1) clearly indicates that the minimum value of the quadrature variance for $\gamma = 0.2, \gamma = 0.1$, and $\gamma = 0$is $(\Delta\hat{c}_{-})^2 = 14.14$ and occur at $\Omega = 0.2626, \Omega = 0.2121$, and $\Omega = 0.1717$,respectively. Therefore, from fig.(4.1) we see that squeezed state for all values of Ω between 0 and 1 and squeezing occurs in the minus quadrature.

Next we proceed to calculate the global quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode vacuum state. We then define the quadrature squeezing of the two-mode cavity light by [17]

$$S = \frac{(\Delta\hat{c}_{-})^2_v - (\Delta\hat{c}_{-})^2}{(\Delta\hat{c}_{-})^2_v}, \tag{4.55}$$

it then follows that

$$S = 1 - \frac{(\Delta \hat{c}_{-})^2}{(\Delta \hat{c}_{-})^2_v}. \tag{4.56}$$

In view of Eqs.(4.50) and (4.54) , the quadrature squeezing of the two-mode cavity light takes, at steady-state, the form

$$S = \left[\frac{2\Omega(\gamma_c + \gamma) - \Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right]. \tag{4.57}$$

We observe from this equation that unlike the mean photon number, the photon number variance, and the quadrature variance, the quadrature squeezing does not depend on the number of three-level atoms in the cavity and the cavity damping constant. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons.

Applying Eqs. (3.2) and (3.3), we find

$$\langle \hat{b}\hat{a} \rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_c \rangle + \frac{4\varepsilon^2}{\kappa^2}. \tag{4.58}$$

Since $\langle \hat{b} \rangle = \langle \hat{a} \rangle$, we see that light modes a and b are correlated. The squeezing of the two-mode cavity light is due to this correlation. The two-mode light can be used in experiments involving entangled light modes [36]. In addition, we consider the case in which spontaneous emission is absent $(\gamma = 0)$and up on setting $(\varepsilon = 0)$. Then the quadrature squeezing for this case takes the form

$$S = \left[\frac{2\Omega\gamma_c - \Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{4.59}$$

Eq.4.54 result is exactly the same as the one obtained by Fesseha [8]. In Fig. (4.2) we plot the quadrature squeezing of Eq. (4.57) versus Ω for $\gamma = 0.2$(dot curve) and $\gamma = 0.1$(dash

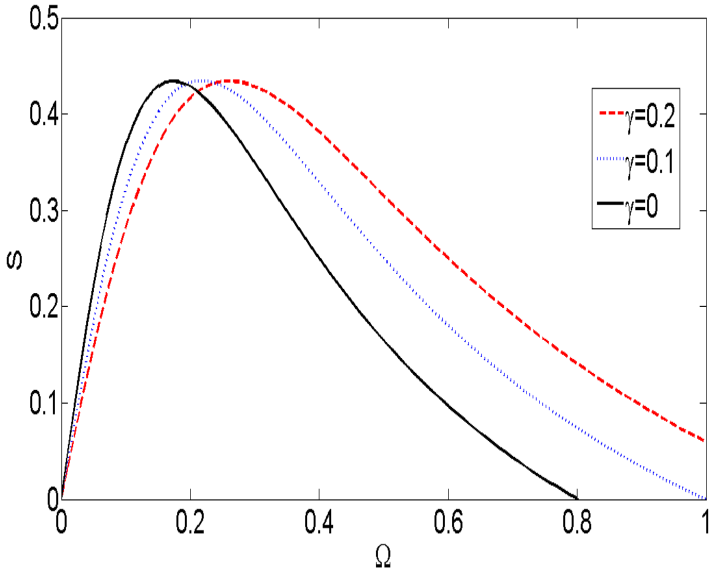

Figure 4.2: Plots of S [Eq. (4.57)] versus Ω for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$ and for different values of γ.

curve) $\gamma = 0$(solid curve). We see from the figure that can be obtained for small values of Ω, the degree of squeezing of the two-mode cavity light increases with Ω and for large values of Ω the quadrature squeezing decreases as Ω increases. In addition, we found that the maximum quadrature squeezing to be the same in the presence $(\gamma \neq 0)$ as well as in the absence of spontaneous emission $(\gamma = 0)$. Therefore, this result implies that the maximum intracavity squeezing for the above values is 43.42% below the vacuum level.

5

Entanglement Amplification

Here we seek to study the entanglement condition of the two modes in the cavity. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles. That is, if the density operator for the combined state cannot be described as a combination of the product of density operators of the constituents,

$$\hat{\rho} \neq \sum_j P_j \hat{\rho}_j^{(1)} \bigotimes \hat{\rho}_j^{(2)}. \tag{5.1}$$

$P_j \geq 0$ and $\sum_j P_j = 1$ is set to ensure normalization of the combined density of state. To study the properties of entanglement produced by this quantum optical system, we need an entanglement criterion for the system. According to the criteria set by Duan et al. [20], a quantum state of the system is entangled provided that the sum of the variances of the two EPR(Einstein-Podolsky-Rosen)-type operators (entanglement) $\hat{u}$ and $\hat{v}$ satisfies the condition;

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 < 2N, \tag{5.2}$$

where,

$$\hat{u} = \hat{x}_a - \hat{x}_b, \hat{v} = \hat{p}_a + \hat{p}_b, \tag{5.3}$$

with

$$\hat{x}_a = \frac{(\hat{a}^\dagger + \hat{a})}{\sqrt{2}}, \hat{x}_b = \frac{(\hat{b}^\dagger + \hat{b})}{\sqrt{2}}, \tag{5.4}$$

$$\hat{p}_a = \frac{i(\hat{a}^\dagger - \hat{a})}{\sqrt{2}}, \hat{p}_b = \frac{i(\hat{b}^\dagger - \hat{b})}{\sqrt{2}}. \tag{5.5}$$

being the quadrature operators for modes $\hat{a}$ and$\hat{b}$. The total variance of the operators $\hat{u}$ and $\hat{v}$ can be written as

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 < 2N. \tag{5.6}$$

This implies that

$$(\Delta\hat{u})^2 = \langle u^2 \rangle - \langle u \rangle^2. \tag{5.7}$$

On account of Eq. (5.3), we see that

$$(\Delta\hat{u})^2 = \langle (\frac{1}{2}(\hat{a} + \hat{a}^\dagger) - \langle \frac{1}{2}(\hat{b} + \hat{b}^\dagger))^2 \rangle, \tag{5.8}$$

from which follows

$$(\Delta\hat{u})^2 = \frac{1}{2}[1 + 2\langle \hat{a}^\dagger \hat{a} \rangle] - \frac{1}{2}[2\langle \hat{a}\hat{b} \rangle] - \frac{1}{2}[2\langle \hat{a}\hat{b} \rangle] + \frac{1}{2}[1 + 2\langle \hat{b}^\dagger \hat{b} \rangle]. \tag{5.9}$$

It then follows that

$$(\Delta\hat{u})^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + 2\langle \hat{b}^\dagger \hat{b} \rangle - 2\langle \hat{a}\hat{b} \rangle. \tag{5.10}$$

Following the same procedure , we easily obtain

$$(\Delta\hat{v})^2 = [1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + 2\langle \hat{b}^\dagger \hat{b} \rangle - 2\langle \hat{a}\hat{b} \rangle]. \tag{5.11}$$

Thus, the sum of the variances of u and v can be expressed as

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2(\Delta\hat{u})^2 = 2(\Delta\hat{c}_{\pm})^2. \tag{5.12}$$

We see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light. Therefore, we see that

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2\left[1 + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle - 2\langle\hat{b}\hat{a}\rangle\right]. \tag{5.13}$$

Now making use of Eqs.(3.5), (3.11) and (4.58), we see that

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = \frac{2\gamma_c}{\kappa}\left[N + \langle N_b\rangle_{ss} - 2\langle m_c\rangle\right]. \tag{5.14}$$

Thus in view of Eq.(5.12) to gather with Eq.(4.51), the sum of the variances of u and v can be expressed as

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2(\Delta\hat{c}_-)^2 = \frac{2\gamma_c N}{\kappa}\left[\frac{(\gamma_c+\gamma)^2 + 4\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]. \tag{5.15}$$

We next consider some special cases. We note that the spontaneous emission is absent $\gamma = 0$, one can readily verify that

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = 2(\Delta\hat{c}_-)^2 = \frac{2\gamma_c N}{\kappa}\left[\frac{(\gamma_c)^2 + 4\Omega^2 - 2\Omega(\gamma_c)}{(\gamma_c)^2 + 3\Omega^2}\right]. \tag{5.16}$$

This represents the photon entanglement of the cavity modes for a non degenerate three level laser coupled to a two-mode vacuum reservoir. The plot in Fig. (5.1) we see that the minimum value of the photon entanglement for $\gamma = 0.2, \gamma = 0.1,$ and, $\gamma = 0$ is $(\Delta\hat{u}^2 + \Delta\hat{v}^2) = 28.28$ and occur at $\Omega = 0.2626, \Omega = 0.2121,$ and $\Omega = 0.1717$,respectively.This result implies that the maximum intracavity photon entanglement for the above values is 56%below the coherent-state level.

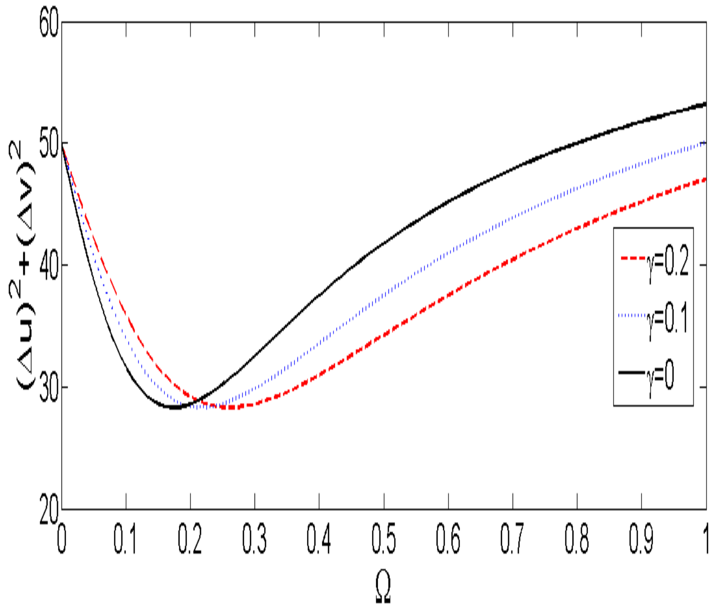

Figure 5.1: Plots of $\Delta\hat{u}^2 + \Delta\hat{v}^2$ [Eq. (5.15)] versus Ω for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$ and for different values of γ.

One can immediately notice that, this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It is noted that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing as given by Eq.(4.49). It also follows that like the mean photon number and quadrature variance the degree of entanglement depends on the number of atoms and the degree of entanglement does not depends on the external driving light.

Based on the criteria (5.2),that a significant entanglement between the states of the light generated in the cavity due to the strong correlation between the radiation emitted when the atoms decay from the upper energy level to the lower via the intermediate level. the sum of the variances of a pair of EPR-type operators $(\Delta\hat{u})^2 + (\Delta\hat{v})^2$ is plotted against the amplitude of the driving coherent light so that the available entanglement is clearly evident for various values of Ω between 0 and 1.

6

Conclusion

In this thesis we have considered a coherently driven non-degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir. We have carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. first we have derived the master equation and the quantum lagevin equations for the cavity light. Applying these equations, the equations of evolution of the cavity mode and the atomic operators are obtained. Making use of the solutions of atomic and cavity mode operators, the mean photon number, the variance photon number, the quadrature variance, the quadrature squeezing, and the photon number entanglement are determined.

We have found that the mean photon number of light mode a is equal to the mean photon number of light mode b. In addition, we have found that the mean photon number of the two-mode light beam is the sum of the mean photon numbers of the separate single-mode light beams. Moreover, we have observed that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beams.

We have found that the light generated by the three-level laser is in a squeezed state and the squeezing occurs in the minus quadrature. The maximum squeezing is found to be 43.42% below the vacuum state-level. we have also seen that the two-mode deriving light has no effect on the squeezing of the cavity modes. Unlike the quadrature squeezing, the deriving light affect the mean photon number, the photon number variance, the quadrature variance,and the photon number correlation. We have also found that increasing the amplitude of the deriving light increases the mean photon number, the photon number variance, the quadrature variance,and the photon number correlation. In addition like the mean photon number, the photon number variance, the quadrature variance,and the photon number correlation depends on the number of three-level atoms in the cavity and the cavity damping constant. But,the quadrature squeezing does not depend it. This implies that the quadrature squeezing of the two-mode light beam is independent of the number of photons.

Finally, we observe that the squeezing and entanglement in the two-mode light is directly related. As a result, an increase in the degree of squeezing directly implies an increase in the degree of entanglement. This shows that whenever there is squeezing in the two-mode light, there exists entanglement in the system.

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