GRAVITATIONAL WAVE RADIATION FROM BINARY COMPACT OBJECTS IN THE SCHWARZSCHILD DE-SITTER BACKGROUND

By

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To My Family

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Abstract

The success of general theory of relativity in testing deflection of light, radar echo delay, precession of planetary motion and gravitational redshift by gravity are the manifestation of progress in astronomy and astrophysical studies. The discovery of the expanding universe at an accelerating phase is another astounding progress in astronomy and astrophysics. Nowadays, the end products of stellar evolution called compact objects (White Dwarf (WD), Neutron Star (NS) and Black Hole (BH)) act as laboratory for the Theory of General Relativity tests over a wide range including origins and future determinations. These objects provide important information about the age of astrophysical objects; constrain models of galactic and cosmological evolutionary history from small scale to large scale structure. Currently, the development of astronomy has led an expansion of human knowledge reaching out, ever farther from our home where the observational tools were solely dependent on the information carried by electromagnetic waves (EMWs). However, due to EMWs interaction with matter there are limitations where these waves unable to penetrate a great deal of compact objects. However, it is hoped that, the transparency of media to GWs a laboratory for general relativity and a window to energetic astrophysical phenomena. Although no conclusive evidence for the direct detection of gravitational waves exists at present, as literatures point out a great hope that gravitational-wave astronomy may open a new window on the universe. Yet, the mechanisms of matching and testing theoretical models with observation need to be worked out for the completeness of the underlying physics. Motivated by this scientific background, we work on gravitational radiation emitted from binary compact objects like NS-NS or

NS-BH or BH-BH binaries that possibly support a mechanism to test the effect of cosmological constant at local level in Schwarzschild de-Sitter background. The project problem attack assumes a pure theoretical development that involves both analytical and numerical approaches.

Keywords: Compact objects -NSs-BHs, Gravitational Radiation

Introduction

The existence and ubiquity of gravitational waves is a clear prediction of Einstein's theory of general relativity. Although gravitational radiation has not yet been unambiguously and directly detected,there is already significant indirect evidence for its existence. In 1974,Russel Hulse and Joseph Taylor discovered the first binary pulsar,PSR 1913+16 which consists of two neutron stars with an orbital period of eight hours[8,10]. Today there is more than just hope in the existence of gravitational waves,which are one of the main predictions of Einstein theory of gravity through the measurement,performed by Hulse and Taylor,of the compact binary system.

The general theory of relativity predicts that the orbiting stars disturb spacetime around them,losing energy by emitting gravitational waves and therefore growth closer together[9]. Close binary stars consisting of two compact stellar remnants neutron stars (NS-NS),or black holes(BH-BH) are considered as primary targets of the forth coming field of gravitational waves(GWs) astronomy since their orbital evolution is entirely controlled by emission of gravitational waves and lead to ultimate merge of the components. For a given system its amount of gravitational radiation known exactly what is the amplitude and frequency of the gravitational waves in terms of the masses of the two bodies and their separation[2,4]. General Relativity also explains gravitation as a consequence of the curvature of spacetime,while in turn spacetime curvature is a consequence of the presence of matter. Spacetime curvature affects the movement of matter,which reciprocally determines the geometric properties and evolution of spacetime[3].

Recently,in 1998,a group of astronomers has claimed to have observed that our universe is currently undergoing accelerate expansion which is attributed to the existence of a positive cosmological constant. The idea that nature contains a cosmological constant stems from Newton. Newton being rather religious believed that the universe must be infinite in extend,must have existed at all times and must be static. However,gravity attracts causing such a space to be unstable. He therefore postulated that there must be some repulsive mechanism leading to a static universe. At that time and in centuries to follow nothing much was known about the universe and physicists ignored these ideas. Einstein too also believed that the universe must the static. However his theory of gravity led a dynamic universe,and he therefore in 1916 reintroduced the cosmological constant. At that time Einstein lived in Germany and could not because of World War I send his letters of correspondence to England and the USA. Still,he was able to send them to the Netherlands which was neutral at that time. The person to receive these letters was W.de Sitter who would then send them to whomever they were addressed to. In this way de Sitter became part of the cosmological debate that was held in those days and was therefore one of the first the hear of Einstein idea to reintroduce the cosmological constant. In 1917 de Sitter showed that for an empty space this new constant leads to a universe which undergoes accelerate expansion[13]. Since about far near decade,evidence has been accumulating that the expansion of universe is actually speeding up[3]. One explanation could be that there is a cosmological constant,Λ in the Einstein equations. Basically,Einstein introduced the cosmological constant because of the prevalent philosophical bias that the universe should be forever stationary,whereas in standard general relativity,it needs to either expand or contract. With $\Lambda < 0$,a stationary universe is possible,although we know that it could not be stable against small perturbations,again it leading to either expansion or construction[1,12]. Therefore,a non-stationary Universe is a prediction of general relativity,with or without cosmological constant,Λ. After Hubble's discovered the expansion of the universe,in 1929,Einstein called the introduction of the cosmological constant,"The biggest blunder of his life". If Einstein had put more trust in his equation,he could have predicted the expantion of the universe!However recently the idea has been resurrected. Observations currently favor a small,positive cosmological constant,Λ,which would make gravity slightly repulsive on large scales and give spacetime a natural tendency to expand. Often the cosmological constant is considered to be related to the vacuum energy density of some scalar field,$\rho_\Lambda = \Lambda[\frac{c^2}{8\pi G}]$. Whatever,the origin might be it will in this thesis be assumed to exist and its effects be discussed in Schwarzschild de-Sitter. A de Sitter spacetime describes an empty universe which has a positive cosmological constant,Λ.

In 1915 Einstein derived a wonderful set of formulas called the Einstein Field Equations. In these equations the Newtonian force of gravity is replaced by the curvature of space-time and related to the energy and momentum in the universe[10]. Unfortunately,these equations are very difficult to solve,even for simple energy-momentum configurations and we shall have to resort to approximations[7].

The success of the general theory of relativity in the deflection of light,the radar echo delay,precession of planetary motion and gravitational redshift by gravity are some of the success of astronomy and astrophysical studies. The discovery of the expanding universe at an accelerating phase is another astounding progress in astronomy and astrophysics. These past, present, and probably future astrophysical phenomena predications and observations all, more or less, rely on stars or where the stars evolve (like galaxies). Now days, the end products of stellar evolution (in particular star evolution) called compact objects serve as laboratory for the Theory of General Relativity tests that extends its application over the large scale structure phenomena in the universe for origins and future determinations. On the other hand, the development of astronomy has led an expansion of human knowledge reaching out, ever farther from our home. Up to date, the observational tools we use were solely dependent on the information carried away or towards by EMWs. However, due to EMWs interaction with matter there are limitations where these waves unable to penetrate a great deal of compact objects or some of the most interesting events in the universe where they lie hidden behind an impenetrable veil of dust and some scattered light. However, the transparency of media to GWs, gravitational waves provide a laboratory for general relativity and a window to energetic astrophysical phenomena invisible with electromagnetic radiation. The deep interior of neutron stars or the neighborhood of merging BHs can be probed as well as physical processes in the very early universe. With this motivational scientific background,We work on gravitational radiation emitted from binary compact object like NS-NS or NS-BH or BH-BH binaries that possibly support a mechanism to test the effect of cosmological constant at local level in Schwarzschild de-Sitter background.

The theoretical modeling of binary systems is the main topic of this thesis. As is well known,binaries can be modelled using various approximation schemes. The one approximation is the so-called quadrupole formula,in which the components of the binary are considered as pointlike particles following Newtonian orbits,and the emitted radiation is computed from the time variation of the quadrupole moment of the system. The main point of this thesis is to show that relevant information on the gravitational waves emitted by binary systems that can be gained using perturbation approaches. In this thesis we started with gravitation and spacetime geometry in general relatively. The Einstein field equations are related the matter with spacetime geometry. Secondly,we deal with gravitational radiation in Schwarzschild de-Sitter background. The weak gravitational field approximation for lineared field equation and quadrupole radiation. Thirdly,gravitational radiation from binary system in Schwarzschild de-Sitter background,the information carried by gravitational wave from binary system. Finally,the the observational data was analyzed.

Chapter 1

Space-Time Geometry and Gravitation

1.1 Tensor In General Relativity

1.1.1 Metric Tensor and Affine Connection

Affine connection is the field that determines the gravitational force and used as to represent the gravitational field. It also call as Christoffel second symbol which denoted as $\{\mu\nu, \lambda\}$ or $\{{\lambda \atop \mu\nu}\}$ or $\Gamma^{\lambda}_{\mu\nu}$. The metric tensor is use to determine the proper time interval between two events with a given infinitesimal coordinate separation and also the gravitational potential. Its derivatives helps to determine the field $\Gamma^{\lambda}_{\mu\nu}$ as well as denoted as $g_{\mu\nu}$. The mathematical definition of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ as,

$$\begin{aligned} g_{\mu\nu} &\equiv \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \eta^{\alpha\beta}, \\ \Gamma^{\lambda}_{\mu\nu} &\equiv \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \end{aligned} \tag{1.1.1}$$

Where ξ^{α} and ξ^{β} are local inertial coordinates. The infinitesimal line element and the motion of particle in a gravitational field can be written as,

$$\begin{aligned} d\tau^2 &= -g_{\mu\nu} dx^{\mu} dx^{\nu}, \\ \frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} &= 0 \end{aligned} \tag{1.1.2}$$

Now differentiating the metric tensor in a gravitational field with respect to the general coordinate system x^λ,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial}{\partial x^\lambda}\left[\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta^{\alpha\beta}\right]$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta} \tag{1.1.3}$$

Equation(1.1.3) can be written as,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \Gamma^\rho_{\lambda\nu}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\eta_{\alpha\beta} \tag{1.1.4}$$

Where,

$$\Gamma^\rho_{\lambda\mu} = \frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}$$

$$\Gamma^\rho_{\lambda\nu} = \frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\nu}$$

Equation(1.1.4) can be written as,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\rho\mu} \tag{1.1.5}$$

Where,

$$g_{\rho\nu} = \frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

$$g_{\rho\mu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\eta_{\alpha\beta}$$

The two $\Gamma^\rho_{\lambda\mu}$ and $\Gamma^\rho_{\lambda\nu}$ are the affine connections. If we considering freely falling particles affine connection is field that determines the gravitational force. Now using the symmetric property of affine connection with the exchange of lower indices,i.e $\Gamma^\rho_{\lambda\mu} = \Gamma^\rho_{\mu\lambda}$. To solve for the affine connection,It is a matter of adding to equation (1.1.5) the same equation with μ and λ interchange and subtract the same equation with ν and λ interchange.It shows,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\rho\mu} + \Gamma^\rho_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \Gamma^\rho_{\nu\mu} g_{\rho\lambda} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} \tag{1.1.6}$$

From the symmetric property of affine connection,$\Gamma^{\rho}_{\mu\nu}$ and metric tensor,$g_{\mu\nu}$,then

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = 2\Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} \tag{1.1.7}$$

.
Now let us define matrix $g^{\nu\sigma}$ as the inverse of $g_{\nu\sigma}$.

$$g^{\nu\sigma} g_{\rho\nu} = \delta^{\sigma}_{\rho}$$

Where,δ^{σ}_{ρ} is the kronecker delta define as $\delta^{\sigma}_{\rho} = 1$ for $\sigma = \rho$ and zero for else. Therefore,applying $(\sigma = \rho)$ to kronecker delta,thus

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right] \tag{1.1.8}$$

Equation (1.1.8) is the relation developed between the metric tensor and affine connection in a gravitational field. Here both of them represents the presence of gravitational effect.

1.1.2 Curvature Tensor

If we use only $g_{\mu\nu}$ and its first derivatives,then no new tensor can be contracted,for at any point we can find a coordinate system in which the first derivatives of the metric tensor vanish,so in this coordinate system the desired tensor must be equal to one of those that can constructed out of the metric tensor alone,and since this is an equality between tensors it must be true in all coordinate system[7]. The simplest possibility is to construct a tensor out of the metric tensor and its first and second derivatives. To do this it is possible to write the transformation rule of affine connection as,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

or it can be written as,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) \tag{1.1.9}$$

but,

$$\frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) = \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \right) + \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \left(\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \right)$$

Therefore the transformation of affine connection becomes,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\{\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}} + \frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau}\partial x^{\sigma}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\}$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\left(\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau}\partial x^{\sigma}}\right) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\right) \tag{1.1.10}$$

Using the relation given by affine connection and kronecker delta into equation (1.1.10) which are,

$$\Gamma^{\rho}_{\tau\sigma} = \frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau}\partial x^{\sigma}},$$

$$\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

Where,$\delta^{\rho}_{\sigma} = 1$ for $\rho = \sigma$ else zero.

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\left(\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}}\right) \tag{1.1.11}$$

According to the statement given by general coordinate transformation,equation (1.1.11) implies that $\Gamma^{\lambda}_{\mu\nu}$ is not a tensor. If $\Gamma^{\lambda}_{\mu\nu}$ is a tensor the expected term will be, $\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\tau\sigma}$.Now invert equation (1.1.11) as,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\Gamma'^{\tau}_{\rho\sigma} + \frac{\partial x^{\lambda}}{\partial x'^{\tau}}\frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}}$$

thus,

$$\frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}}\Gamma^{\lambda}_{\mu\nu} - \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\Gamma'^{\tau}_{\rho\sigma} \tag{1.1.12}$$

Differentiating this equation with respect to x^{κ} gives,

$$\frac{\partial^3 x'^{\tau}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}} = \frac{\partial^2 x'^{\tau}}{\partial x^{\kappa}\partial x^{\lambda}}\Gamma^{\lambda}_{\mu\nu} + \frac{\partial x'^{\tau}}{\partial x^{\lambda}}\frac{\partial}{\partial x^{\kappa}}\left(\Gamma^{\lambda}_{\mu\nu}\right) - \frac{\partial^2 x'^{\rho}}{\partial x^{\kappa}\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\Gamma'^{\tau}_{\rho\sigma}$$
$$+\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial^2 x'^{\sigma}}{\partial x^{\kappa}\partial x^{\nu}}\Gamma'^{\tau}_{\rho\sigma} - \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\frac{\partial}{\partial x^{\kappa}}\left(\Gamma'^{\tau}_{\rho\sigma}\right) \tag{1.1.13}$$

According to the relation given by equation(1.1.12) it is possible to write the following,

$$\frac{\partial^2 x'^\tau}{\partial x^\kappa \partial x^\lambda} = \frac{\partial x'^\tau}{\partial x^\eta}\Gamma^\eta_{\kappa\lambda} - \frac{\partial x'^\rho}{\partial x^\kappa}\frac{\partial x'^\sigma}{\partial x^\lambda}\Gamma'^\tau_{\rho\sigma}$$

,

$$\frac{\partial^2 x'^\rho}{\partial x^\kappa \partial x^\mu} = \frac{\partial x'^\rho}{\partial x^\eta}\Gamma^\eta_{\kappa\mu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\mu}\Gamma'^\rho_{\eta\xi}$$

,

$$\frac{\partial^2 x'^\sigma}{\partial x^\kappa \partial x^\nu} = \frac{\partial x'^\sigma}{\partial x^\eta}\Gamma^\eta_{\kappa\nu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\nu}\Gamma'^\sigma_{\eta\xi}$$

Substitute those equation into equation (1.1.13),we get,

$$\begin{aligned} \frac{\partial^3 x'^\tau}{\partial x^\kappa \partial x^\mu \partial x^\nu} &= \left(\frac{\partial x'^\tau}{\partial x^\eta}\Gamma^\eta_{\kappa\lambda} - \frac{\partial x'^\rho}{\partial x^\kappa}\frac{\partial x'^\sigma}{\partial x^\lambda}\Gamma'^\tau_{\rho\sigma}\right)\Gamma^\lambda_{\mu\nu} \\ &+\frac{\partial}{\partial x^\kappa}\Gamma^\lambda_{\mu\nu}\frac{\partial x'^\tau}{\partial x^\lambda} - \Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\sigma}{\partial x^\mu}\left(\frac{\partial x'^\rho}{\partial x^\eta}\Gamma^\eta_{\kappa\mu} - \frac{\partial'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\mu}\Gamma'^\rho_{\eta\xi}\right) \\ &-\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial}{\partial x^\kappa}\Gamma'^\tau_{\rho\sigma} - \Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\left(\frac{\partial x'^\sigma}{\partial x^\eta}\Gamma^\eta_{\kappa\nu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\nu}\Gamma^\sigma_{\eta\xi}\right) \end{aligned} \tag{1.1.14}$$

Now collect similar terms and juggle indices a bit gives,

$$\begin{aligned} \frac{\partial^3 x'^\tau}{\partial x^\kappa \partial x^\mu \partial x^\nu} = \frac{\partial x'^\tau}{\partial x^\lambda}\left(\frac{\partial}{\partial x^\kappa}\Gamma^\lambda_{\mu\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta}\right) - \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\kappa}\left(\frac{\partial}{\partial x'^\eta}\Gamma'^\tau_{\rho\sigma} - \Gamma'^\tau_{\rho\lambda}\Gamma'^\lambda_{\eta\sigma} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho}\right) \\ -\Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\sigma}{\partial x^\lambda}\left(\Gamma^\lambda_{\mu\nu}\frac{\partial x'^\rho}{\partial x^\kappa} + \Gamma^\lambda_{\kappa\nu}\frac{\partial x'^\rho}{\partial x^\mu} + \Gamma^\lambda_{\kappa\mu}\frac{\partial x'^\rho}{\partial x^\nu}\right) \end{aligned} \tag{1.1.15}$$

Then after subtract the same equation with interchanging $\nu \longleftrightarrow \kappa$ ta drop out the product of Γ and Γ',so that

$$\begin{aligned} 0 &= \frac{\partial x'^\tau}{\partial x^\lambda}\left(\frac{\partial}{\partial x^\kappa}\Gamma^\lambda_{\mu\nu} - \frac{\partial}{\partial x^\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}\right) \\ &-\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x^\eta}{\partial x^\kappa}\left(\frac{\partial}{\partial x'^\eta}\Gamma'^\lambda_{\rho\sigma} - \frac{\partial}{\partial x'^\sigma}\Gamma'^\tau_{\rho\eta} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho} + \Gamma'^\tau_{\lambda\eta}\Gamma'^\lambda_{\sigma\rho}\right) \end{aligned} \tag{1.1.16}$$

This may be written as a transformation rule,

$$R'^{\tau}_{\rho\sigma\eta} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}}\frac{\partial x^{\mu}}{\partial x'^{\rho}}\frac{\partial x^{\nu}}{\partial x^{\sigma}}\frac{\partial x^{\kappa}}{\partial x'^{\eta}}R^{\lambda}_{\mu\nu\kappa} \tag{1.1.17}$$

Now defining the term in the first bracket of (1.18) using the curvature tensor notation as,

$$R^{\lambda}_{\mu\nu\kappa} = \frac{\partial}{\partial x^{\kappa}}\Gamma^{\lambda}_{\mu\nu} - \frac{\partial}{\partial x^{\nu}}\Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} \tag{1.1.18}$$

$R^{\lambda}_{\mu\nu\kappa}$ is called Riemann- Christoffel curvature tensor.
The Riemann-Christoffel curvature tensor plays an important role in specifying the geometrical properties of space-time. The space-time is considered flat,if the Riemann tensor vanishes every where. It is also possible to write the Riemann curvature tensor in its fully covariant form as,

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma}R^{\sigma}_{\mu\nu\kappa} \tag{1.1.19}$$

Riemann curvature tensor can also be written directly in terms of the space-time metric,using the definition of affine connection,

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}}\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right)$$

Thus,

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}g_{\lambda\sigma}\frac{\partial}{\partial x^{\kappa}}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right)$$
$$-\frac{1}{2}g_{\lambda\sigma}\frac{\partial}{\partial x^{\nu}}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\kappa}} + \frac{\partial g_{\rho\kappa}}{\partial x^{\mu}} - \frac{\partial g_{\mu\kappa}}{\partial x^{\rho}}\right) + g_{\lambda\sigma}\left[\Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\eta}\right] \tag{1.1.20}$$

Now define the kronecker delta $\delta^{\rho}_{\lambda} = 1$,where $\rho = \lambda$ and ,

$$g_{\lambda\sigma}\frac{\partial}{\partial x^{\kappa}}g^{\sigma\rho} = -g^{\lambda\rho}\frac{\partial}{\partial x^{\kappa}}g_{\lambda\sigma} = -g^{\sigma\rho}\left(\Gamma^{\eta}_{\kappa\lambda}g_{\eta\sigma} + \Gamma^{\eta}_{\kappa\sigma}g_{\eta\lambda}\right)$$

Therefore most of $\Gamma\Gamma$ terms cancel,then

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right] + g_{\eta\sigma}\left[\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right] \tag{1.1.21}$$

This is the covariant form of Riemann-Christoffel curvature tensor.The algebraic properties of the curvature tensor are,

1. symmetry

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \tag{1.1.22}$$

2. Antisymmetry ,

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\nu\kappa\lambda\mu} \tag{1.1.23}$$

3. Cyclicity

$$R_{\lambda\mu\nu\kappa} + R_{\mu\kappa\lambda\nu} + R_{\lambda\kappa\mu\nu} = 0 \tag{1.1.24}$$

Therefore,the Riemann tensor in 4-dimensional space-time has only 20 independent components because of symmetries. Thus the general rule for computing the number of independent components in a N-dimensional space-time is $\frac{N^2(N^2-1)}{12}$[7].

1.1.3 Ricci Tensor,Ricci Scalar and Einstein Field Tensor

Ricci Tensor: Obtained from the Riemann curvature tensor by contracting over two of the indices

$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa}$$

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa} \tag{1.1.25}$$

which can be also written as,

$$R_{\mu\kappa} = \frac{1}{2} g^{\lambda\nu} \left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right] + g^{\lambda\nu} g_{\eta\sigma} \left[\Gamma^{\eta}_{\nu\lambda} \Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda} \Gamma^{\sigma}_{\mu\nu} \right] \tag{1.1.26}$$

and also one can write the Ricci tensor as,

$$R_{\mu\kappa} = \frac{\partial}{\partial x^\kappa} \Gamma^{\lambda}_{\mu\lambda} - \frac{\partial}{\partial x^\lambda} \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\eta}_{\mu\lambda} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\lambda\eta} \tag{1.1.27}$$

Ricci tensor is symmetric. Therefore,it has at most ten independent components.

$$R_{\mu\kappa} = R_{\kappa\mu}$$

Ricci Scalar: By further contracting the Ricci tensor with the contravariant components of the metric,one obtains the curvature scalar,

$$R = g^{\mu\kappa} R_{\mu\kappa} = g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa} \tag{1.1.28}$$

or

$$R = R^{\mu}_{\mu} \tag{1.1.29}$$

Einstien Field Tensor: Einstien field tensor $G_{\mu\kappa}$ is constructed only from the Riemann tensor and the metric and is automatically divergence free as an identity.thus,

$$G_{\mu\kappa} = R_{\mu\kappa} - \frac{1}{2} g_{\mu\kappa} R \tag{1.1.30}$$

Where,$G_{\mu\kappa}$ is a linear combination of the metric tensor and its first and second derivatives. Since the Ricci tensor and metric tensor are symmetric,so Einstien field tensor also symmetric,thus

$$G_{\mu\kappa} = G_{\kappa\mu} \tag{1.1.31}$$

1.1.4 Bianchi Identities

The Riemann curvature tensor obeys important differential identities. These can be derived at a given point,x by adopting a locally inertial coordinate system in which $\Gamma^{\lambda}_{\mu\nu}$ vanishes at x thus at x,

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2} \frac{\partial}{\partial x^{\eta}} \left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa} \partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu} \partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu} \partial x^{\lambda}} \right] \tag{1.1.32}$$

By permuting ν,κ and η cyclically,we obtain the Bianchi identities,

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{1.1.33}$$

Now recalling that the covariant derivatives of $g^{\lambda\nu}$ vanish,then we find on contraction of λ with ν that,

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0 \tag{1.1.34}$$

Again contracting gives,

$$R_{;\eta} - R^{\mu}_{\eta;\mu} - R^{\nu}_{\eta;\nu} = 0 \tag{1.1.35}$$

or

$$\left(R^{\mu}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta}R\right)_{;\mu} = 0 \tag{1.1.36}$$

An equivalent but more familiar form is,

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0 \tag{1.1.37}$$

or

$$G^{\mu\nu}_{;\mu} = 0 \tag{1.1.38}$$

1.1.5 Energy-Momentum Tensor

Some time Energy Momentum called as stress-energy tensor. It describes the density and flows of the 4 momentum$(-E, P_1, P_2, P_3)$. The four velocity,U^{μ} is define as

$$U^{\mu} = \frac{dx^{\mu}}{d\tau} \tag{1.1.39}$$

From line element,

$$d\tau^2 = -\eta_{\mu\upsilon}dx^{\mu}dx^{\upsilon} \tag{1.1.40}$$

If we rearranging the above equation(1.1.40),

$$-1 = \eta_{\mu\upsilon}\frac{dx^{\mu}}{d\tau}\frac{dx^{\upsilon}}{d\tau} \tag{1.1.41}$$

Comparing equation(1.1.41) to equation(1.1.39),we get,

$$-1 = \eta_{\mu\upsilon}U^{\mu}U^{\upsilon} \tag{1.1.42}$$

In the rest frame of a particle,It's four-velocity has components,

$$U^{\mu} = (1, 0, 0, 0) \tag{1.1.43}$$

A related vector is the energy-momentum four-vector as,

$$P^{\mu} = mU^{\mu} \tag{1.1.44}$$

where m =mass of the particle U^μ =four-velocity P^μ =four-momentum

The energy of the particle is simply P^o,the time-like component of its energy-momentum vector.

$$P^o = mc^2 = E \tag{1.1.45}$$

where $dx^o = dt$ is time-like and $c = 1$. The time-like component is not invariant under lorentz transformation,since the particle's at rest frame is not the same as that of the same particle in motion. In a moving frame we can find the components of P^μ be performing a lorentz transformation;for a particle moving with velocity v along the x axis is,

$$P^\mu = (\gamma m, v\gamma m, 0, 0) \tag{1.1.46}$$

where $\gamma = (1 - v^2)^{\frac{-1}{2}}$. For the case of $v <<< c$ or small v. The energy of particle is the sum of its mass and the kinetic energy,

$$P^o = m + \frac{1}{2}mv^2 \tag{1.1.47}$$

As well as,

$$P^1 = mv \tag{1.1.48}$$

In the Newtonian mechanics,the Newton's second law is,

$$\vec{f} = m\vec{a} = m\frac{d^2x}{dt^2} = \frac{d\vec{P}}{dt} \tag{1.1.49}$$

Analogous equation should hold in special relativity and it can be tensorial force four-vector f^μ,

$$f^\mu = m\frac{d^2x^\mu}{d\tau^2} = \frac{dP^\mu}{d\tau} \tag{1.1.50}$$

The simplest example of a force in Newtonian physics is the force due to gravity. In relativity,however,gravity is not described by a force,but rather by the curvature of spacetime itself. The P^μ provides a complete description of the energy-momentum of a particle,but now let us extend to go further to define the energy-momentum tensor(sometime called as stress-energy tensor) $T^{\mu v}$[7]. This is a symmetric second rank $(2, 0)$ tensor which tells us all we need to know about the energy like aspects of a system energy density,pressure,stress and so on. To make the idea more concrete,let us consider the very general category of matter as a fluid. Fluid is a continuum of matter described by macroscopic quantity such as temperature,pressure,entropy,viscosity,etc. In general relativity essentially all interesting types of matter can be thought of as perfect fluids,one with no heat conduction and no viscosity(schurt definition) or which looks isotropic in its rest frame(Wienberg definition). Generally,a perfect fluid as one which may be completely characterized by its pressure and density. For more understand of perfect fluids,let's start with simpler example of dust,that is a collection of particles at rest with respect to each other,or as a perfect fluid with zero pressure ($P = 0$). Since the particles all have an equal velocity in any fixed inertial frame,we can imagine a "$four - velocityfields$" U^μ define all over spacetime.then the number-flux four-vector N^μ can be,

$$N^\mu = nU^\mu \tag{1.1.51}$$

where,$n =$ is number density of the particles as measured in their rest frame. N^o is the number density of particles as measured in any other frame. N^i is flux of particles in the x^i direction. Assuming that each particles have the same mass m,then in the rest frame the energy density of the dust is,

$$\rho = nm \tag{1.1.52}$$

By the definition,the energy density completely specifies the dust. Where as n and m are 0-component of four-vector in their rest frame. Specifically,

$$N^\mu = (n, 0, 0, 0)$$

$$P^\mu = (m, 0, 0, 0) \tag{1.1.53}$$

Therefore,ρ is the component of $\mu = 0$ and $v = 0$ tensor $P \bigotimes N$ as measured in its rest frame. The energy-momentum tensor for dust is the,

$$T^{\mu v} = P^\mu N^v = \rho U^\mu U^v \tag{1.1.54}$$

where,$N^v = nU^v$,$P^\mu = mU^\mu$ and $\rho = nm$

We also remember that "perfect" take to mean "isotropic in its rest frame." This in turn means that $T^{\mu v}$ is diagonal. Furthermore the nonzero space like components must all be equal,means,

$$T^{11} = T^{22} = T^{33} \tag{1.1.55}$$

The only two independent numbers are therefore T^{00} and T^{ii}. The first is energy density ρ and the second is pressure P. The energy-momentum tensor of a perfect fluid therefore takes the following form in its rest frame.

$$T^{\mu v} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1.1.56}$$

One thing is left,we would like,ofcourse,a formula which is good in any frame. For dust,$T^{\mu v} = \rho U^\mu U^v$,we might guess $(\rho + P)U^\mu V^v$ which gives,

$$(\rho + P)U^\mu V^v = \begin{pmatrix} \rho + P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.1.57)$$

To get the answer we want we must add the following equation to the above,

$$P\eta^{\mu v} = \begin{pmatrix} -P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \quad (1.1.58)$$

Finally,the general form of the energy-momentum tensor for a perfect fluid is,

$$T^{\mu v} = (\rho + P)U^\mu U^v + P\eta^{\mu v} \quad (1.1.59)$$

This is an important formula for applications such as stellar structure and cosmology[7,11]. In fact,one way to define $T^{\mu v}$ would be " a(2,0) tensor with units of energy per volume,which is conserved."

1.2 Einstein Field Equations

In Newtonian theory, gravity can only exist where there exists matter. However Einstein showed that matter and energy are only different faces of the same coin. This encouraged him to make the conclusion that gravity is not only created by the presence of matter, it is in fact the product of the presence of energy. General relativity must present appropriate analogues of the two parts of the dynamics,one how particles move in the response to gravity,and secondly,how particles generate gravitational effects[7]. The analogue of the poisson equation of the second idea can be,

$$\nabla^2\phi(\vec{x}) = 4\pi G\rho(\vec{x}) \quad (1.2.1)$$

Consider the case where a particle is moving slowly in a weak stationary gravitational field. For sufficiently slow motion of a particle,the equation of motion of a particle can be written as,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \quad (1.2.2)$$

This is from the equation of motion of the particle in a gravitational field which can be given by,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

For which $\lambda = \nu = 0$ and $dx^0 = dt$. Recall the relation given by,

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2}g^{\nu\sigma}\{\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\}$$

Since the field is stationary,all time derivatives of $g_{\mu\nu}$ vanish,so that

$$\Gamma^\lambda_{00} = \frac{1}{2}g^{\lambda\nu}\frac{\partial g_{00}}{\partial x^\nu} \tag{1.2.3}$$

For a weak static field produced by non-relativistic mass density ρ,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where,$\|h_{\alpha\beta}\| << 1$ and $\eta_{\alpha\beta}$ is minkowski metric tensor. For $\alpha = \beta = 0$ and applying the relation $\eta_{00} = -1$.

$$g_{00} = -1 + h_{00} \tag{1.2.4}$$

Therefore we have,

$$\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta} \tag{1.2.5}$$

Now the equation of motion has take the form of,

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\alpha\beta}\{\frac{\partial h_{00}}{\partial x^\beta}\}\{\frac{dt}{d\tau}\}^2 \tag{1.2.6}$$

For $\alpha = \beta = 1, 2, 3$ the minkowski metric tensor,$\eta_{\alpha\beta} = \eta^{\alpha\beta}$,then the above equation can be written as,

$$\frac{d^2x}{d\tau^2} = \frac{1}{2}\{\frac{dt}{d\tau}\}^2\{\frac{\partial h_{00}}{\partial x}\}$$

It is possible to write as,

$$\frac{d^2x}{d\tau^2} = \frac{1}{2}\{\frac{dt}{d\tau}\}^2\nabla h_{00} \tag{1.2.7}$$

Where, $\frac{\partial h_{00}}{\partial x} = \nabla h_{00}$. Once rearranging the equation that gives,

$$\frac{d^2x}{d\tau^2} = \frac{1}{2}\nabla h_{00} \tag{1.2.8}$$

Now the corresponding Newtonian result is,

$$\frac{d^2x}{d\tau^2} = -\nabla\phi \tag{1.2.9}$$

Where, ϕ is the Newtonian potential. The comparison of equations result,

$$\frac{1}{2}\nabla h_{00} = -\nabla\phi$$

$$\nabla h_{00} = -2\nabla\phi$$

$$h_{00} = -2\phi + constant \tag{1.2.10}$$

Furthermore,the coordinates system must become Minkowskian at great distance so h_{00} vanish at infinity. Then if $\phi|$ defined to vanish at infinity (where $\phi = -\frac{GM}{r}$,r is the distance from the center of a spherical body of mass M. By recall the relation for a weak static field given by,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

Therefore,

$$g_{00} = -1 + h_{00} \tag{1.2.11}$$

Using the value of h_{00} for zero constant and

$$g_{00} = -(1 + 2\phi) \tag{1.2.12}$$

Now we start to derive Einstein field equation under the approximation of a weak static field produced by a non-relativistic mass density ρ[7,11]. Therefore,the energy density for non-relativistic matter is,

$$T_{00} = \rho = T_0^0 \tag{1.2.13}$$

One can write the poisson equation as,

$$\nabla^2\phi = 4\pi G T_{00} \tag{1.2.14}$$

From equation,

$$g_{00} = -1 + h_{00}$$

And we get,

$$\nabla^2\phi = -\frac{1}{2}\nabla^2 g_{00} \tag{1.2.15}$$

Therefore the poisson equation result,

$$-\frac{1}{2}\nabla^2 g_{00} = 4\pi G T_{00}$$

$$\nabla^2 g_{00} = 8\pi G T_{00} \tag{1.2.16}$$

From this fact the weak field equation for a general distribution of energy and momentum $T_{\alpha\beta}$ will take the form,

$$G_{\alpha\beta} = -8\pi G T_{\alpha\beta} \tag{1.2.17}$$

Where, $G_{\alpha\beta}$ is a linear combination of the metric tensor and its first and second derivatives. The principle of equivalence that the equation which govern gravitational fields of arbitrary strength must take the form,

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{1.2.18}$$

Therefore,equation (1.2.17) is the approximated form of equation (1.2.18) in a weak static gravitational field as equivalence principle states. Here $G_{\mu\nu}$ is a tensor which reduce to $G_{\alpha\beta}$ for a weak fields and since $T_{\mu\nu}$ is symmetric $G_{\mu\nu}$ also.

To go further consider the nature of $G_{\mu\nu}$;

1. By definition $G_{\mu\nu}$ is a tensor

2. By assumption $G_{\mu\nu}$ contain terms that are linear in the second derivative of the metric tensor or quadratic in the first derivative of the metric.

3. Since $T_{\mu\nu}$ is symmetric so does $G_{\mu\nu}$.

4. Since $T_{\mu}\nu$ is conserved in the absence of external forces,so does $G_{\mu\nu}$.

5. For a weak stationary field produced by non-relativistic matter ,the 00 component must satisfy.

$$G_{00} \cong \nabla 2 g_{00} \tag{1.2.19}$$

Hence (1) and (2) require $G_{\mu\nu}$ to take the form

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R \tag{1.2.20}$$

Where,C_1 and C_2 are constants.
Since this is symmetric condition(3) is automatically satisfied. It follows from the above relation that.

$$g^{\sigma\mu} G_{\mu\nu} = C_1 g^{\sigma\mu} R_{\mu\nu} + C_2 g^{\sigma\mu} g_{\mu\nu} R \tag{1.2.21}$$

Equivalent to,

$$G^{\sigma}_{\nu} = C_1 R^{\sigma}_{\nu} + C_2 \delta^{\sigma}_{\nu} R \tag{1.2.22}$$

this follows as

$$G^{\sigma}_{\nu;\sigma} = C_1 R^{\sigma}_{\nu;\sigma} + C_2 \delta^{\sigma}_{\nu} R_{;\sigma} \tag{1.2.23}$$

Using the result,$R^{\sigma}_{\nu;\sigma} = \frac{1}{2}\delta^{\sigma}_{\nu} R_{;\sigma}$ in to the above equation and it follows,

$$G^{\sigma}_{\nu;\sigma} = \frac{1}{2} C_1 \delta^{\sigma}_{\nu} R^{\sigma}_{;\sigma} + C_2 \delta^{\sigma}_{\nu} R_{;\sigma}$$

If $\nu = \sigma$

$$G^{\sigma}_{\nu;\sigma} = \left(\frac{C_1}{2} + C_2\right) R_{;\sigma} \tag{1.2.24}$$

By the conservation of $G_{\mu\nu}$ we have $G^{\sigma}_{\nu;\sigma} = 0$ and this yields the relation,

$$(\frac{C_1}{2} + C_2) R_{;\sigma} = 0$$

$$\frac{C_1}{2} = -C_2 \tag{1.2.25}$$

Therefore we can rewrite $G_{\mu\nu}$ as,

$$G_{\mu\nu} = C_1 R_{\mu\nu} - \frac{C_1}{2} g_{\mu\nu} R$$

$$G_{\mu\nu} = C_1 (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \tag{1.2.26}$$

To fix the constant C_1,use the property[5]. A non-relativistic system always has $|T_{ij}| << |T_{00}|$ and here look the case where $|G_{ij}| << |G_{00}|$ thus,

$$G_{ij} \cong 0 \tag{1.2.27}$$

From equation(1.2.20) we can be write as,

$$R_{ij} - \frac{1}{2} g_{ij} R = 0$$

$$R_{ij} = \frac{1}{2} g_{ij} R \tag{1.2.28}$$

Since we deal here with a weak field approximation(i.e $g_{\alpha\beta} \cong \eta_{\alpha\beta}$) as well as $g_{ij} \cong \eta_{ij}$. Therefore,this lead to write as,

$$R_{ij} \cong \frac{1}{2} \eta_{ij} R \tag{1.2.29}$$

By applying the property of metric tensor $\eta_{ij} = 1$,for $i = j = 1, 2, 3$ and taking the sum over each indices,

$$R_{ij} = \sum_{i,j=1}^{3} \frac{1}{2} \eta_{ij} R \cong \frac{3}{2} R$$

$$R_{kk} = \frac{3}{2} R \tag{1.2.30}$$

The curvature scalar is therefore given by,

$$R \cong R_{kk} - R_{00} = \frac{3}{2} R - R_{00}$$

$$R \cong 2R_{00} \tag{1.2.31}$$

Thus in the weak field approximation we have the following information,

$$R \cong 2R_{00}$$

$$g_{\alpha\beta} \cong \eta_{\alpha\beta}$$

$$G_{\mu\nu} = C_1\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \tag{1.2.32}$$

For the 00 component of $G_{\mu\nu}$ equals to,

$$G_{00} = C_1\left(R_{00} - \frac{1}{2}g_{00}R\right) = C_1\left(R_{00} - \frac{1}{2}\eta_{00}\right)$$

,

$$G_{00} = C_1 R = 2C_1 R_{00} \tag{1.2.33}$$

Now the task is to calculate R_{00}. Recall the expression given by the Riemann curvature tensor $R_{\lambda\mu\nu\kappa}$ that is,

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right]$$

Since we are looking for a weak field approximation,it is better to use the linear part of $R_{\lambda\mu\nu\kappa}$,given by

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right] \tag{1.2.34}$$

When the field is static all the time derivatives vanish,and the components that we need are,

$$R_{0000} \cong 0$$

$$R_{i0j0} \cong \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = \frac{1}{2}\nabla^2 g_{00} \tag{1.2.35}$$

Where $\frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = \nabla^2 g_{00}$ From the contraction of curvature tensor over the two indices,

$$R_{00} = g^{\lambda\nu}R_{\lambda 0\nu 0} = \eta^{\lambda\nu}R_{\lambda 0\nu 0}$$

$$R_{00} = R_{i0j0} - R_{0000} \tag{1.2.36}$$

By using this relation in to equation(1.2.33)for $G_{\mu\nu}$,

$$G_{00} = 2C_1\left(R_{i0j0} - R_{0000}\right)$$

$$G_{00} = 2C_1\left(\frac{1}{2}\nabla^2 g_{00} - 0\right) = C_1\nabla^2 g_{00} \tag{1.2.37}$$

Comparing equation (1.2.19) to equation (1.2.37),

$$G_{00} = C_1\nabla^2 g_{00} = \nabla^2 g_{00} \tag{1.2.38}$$

This gives the value of $C_1 = 1$,and therefore we can write the equation for $G_{\mu\nu}$ as,

$$G_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -8\pi G T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{1.2.39}$$

Equation (1.2.39) is Einstein field equation. This shows that the metric of spacetime is dependent upon the matter present in that spacetime.

Chapter 2

Gravitational Radiation In Schwarzschild de-Sitter Background

The Theory of General Relativity is a relativistic theory of gravitation and predicts the existence of gravitational radiation[7,11,12]. Unfortunately,it is extremely difficult to explore gravitational radiation from the full Einstein field equations,not only in mathematics but even conceptually. This could be explained qualitatively as follow;In electromagnetic radiation,it is the electric and magnetic fields that propagate as waves with the speed of light. What propagates in gravitational radiation? The answer unfortunately is not as clear as the electromagnetic waves. The gravitational effects in relativity are intimately related to the geometric structure of space-time[9]. Hence we expect the structural changes in space-time to propagate as gravitational waves. In practice,it is very difficult to single out any particular quantity that relates to such changes of space-time structure and that we can claim to be propagating as waves. The difficulty lies partly in the coordinate description of space and time. Einstein field equations have the beautiful property that they have the same formal structure,whatever the coordinate frame of reference used. But every observer uses a coordinate system to describe the geometric properties of space-time. The above-mentioned property gets in the way of deciding whether a particular solution does represent a gravitational wave or it is a result of the choice of a particular frame of reference. When gravitational fields are strong and the geometric properties of space-time are very different from Euclidians,the problem of interpreting a disturbance as a gravitational wave becomes very difficult. But in the case of weak gravitational fields it is simpler to identify certain disturbances as gravitational waves[11]. For examples,massive bodies undergoing acceleration and two (binary) stars going around each other emit gravitational waves. It is best to regard the weak field solutions not as approximate solutions to the full equations,but as solutions to give an idea of the behavior expected in the full theory. In this chapter we will reformulate the Einstein equations so that they can be expressed as a wave equation. This is done by linearizing the equations by assuming that the gravitational field is weak,and by choosing appropriate coordinates in which to express the equations. This chapter will be divided into two sections.In the first section the linearization of the Einstein equations with cosmological constant will be discussed. In the second section it will be proved that one always can choose a set of coordinates, such that the linearized equations take on the form of an inhomogeneous wave equation.

2.1 Linearization Of Einstein Field Equations In Schwarzschild de-Sitter Background

The spacetime metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and treating $h_{\mu\nu}$ as a small perturbation. For most of the astrophysical applications,such an approximation is adequate if we are interested in the propagation of gravitational wave disturbances in a(nearly) flat background spacetime. There are,however,situations in which one would like to study the gravitational waves propagating in a curved spacetime like Schwarzschild de-Sitter. In this section we shall see such spacetime. It is certainly possible formally to separate any spacetime metric as,

$$g_{\mu\nu} = g_{\mu\nu}^{(B)} + h_{\mu\nu} \tag{2.1.1}$$

Where,$h_{\mu\nu}$ is treat as a perturbation and $g_{\mu\nu}^{(B)}$ is treat as background[11]. But since a given metric $g_{\mu\nu}$ could be separated into a background and a perturbation in many different ways,it is not possible to treat $h_{\mu\nu}$ as a well-defined gravitational wave without further physical input.
The line element in the Schwarzschild de-Sitter has a form of,

$$ds^2 = V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.1.2}$$

Where,

$$V(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2$$

and Λ is cosmological constant with $\Lambda > 0$.
To linearize the field equation with Schwarzschild de-sitter background,we start with the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(w)} + h_{\mu\nu}^{(\Lambda)} \tag{2.1.3}$$

where,$h^{(w)}_{\mu\nu}$ is Newtonian perturbation and $h^{(\Lambda)}_{\mu\nu}$ is cosmological perturbation. The contravariant of the metric can be as,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{(w)\mu\nu} - h^{(\Lambda)\mu\nu} \tag{2.1.4}$$

2.1.1 Linearization of Affine Connection

The affine connection is given by,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\Bigg(g_{\sigma\mu;\nu} + g_{\sigma\nu;\mu} - g_{\mu\nu;\sigma}\Bigg) \tag{2.1.5}$$

Then the first order linearize affine connection where the higher orders are omitted in Schwarzschild de-Sitter has the form of,

$$\begin{aligned}\Gamma^{\lambda(1)}_{\mu\nu} = & \frac{1}{2}\Bigg(h^{\lambda(w)}_{\mu;\nu} + h^{\lambda(w)}_{\nu;\mu} - h^{;\lambda(w)}_{\mu\nu}\Bigg)\\ & -\frac{1}{2}\Bigg(h^{\lambda(\Lambda)}_{\mu;\nu} + h^{\lambda(\Lambda)}_{\nu;\mu} - h^{;\lambda(\Lambda)}_{\mu\nu}\Bigg)\end{aligned} \tag{2.1.6}$$

Here we are adopting the convenient convention tha t indices on $h_{lmu\nu}$,$\Gamma^{\lambda}_{\mu\nu}$,$R_{\mu\nu}$ and $\frac{\partial}{\partial x^{\lambda}}$ are raised and lowered with $\eta's$
Examples,

$$h^{\lambda}_{\lambda} \equiv \eta^{\lambda\nu}h_{\lambda\nu} \equiv h$$

and

$$\frac{\partial}{\partial x_{\lambda}} \equiv \eta^{\lambda\nu}\frac{\partial}{\partial x^{\nu}}$$

2.1.2 Linearization Of Ricci Tensor

We know that the Ricci curvature tensor is obtained from the Riemann curvature tensor by,

$$R_{\mu\nu} = g^{\lambda\kappa}R_{\lambda\mu\kappa\nu}$$

$$R_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}\Bigg[\frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}}\Bigg] + g^{\lambda\kappa}g_{\eta\sigma}\Bigg[\Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa}\Bigg] \tag{2.1.7}$$

To Linearize the Ricci Tensor to the first and second orders,we use,

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu}$$

and

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu(w)} - h^{\mu\nu(\Lambda)}$$

As well as,$\eta's$ are raised and lowered the indices on $h_{\mu\nu}$,$R^{(1)}_{\mu\nu}$,$R^{(2)}_{\mu\nu}$ and $\frac{\partial}{\partial x^{\lambda}}$ as before,then the product of,

$$g^{\lambda\kappa}g_{\eta\sigma} = \eta^{\lambda\kappa}\eta_{\eta\sigma}+\eta^{\lambda\kappa}h^{(w)}_{\eta\sigma}+\eta^{\lambda\kappa}h^{(\Lambda)}_{\eta\sigma}-\eta_{\eta\sigma}h^{\lambda\kappa(w)}-\eta_{\eta\sigma}h^{\lambda\kappa(\Lambda)}-\left[\left(h^{\lambda\kappa(w)}+h^{\lambda\kappa(\Lambda)}\right)\left(h^{(w)}_{\eta\sigma}+h^{(\Lambda)}_{\eta\sigma}\right)\right] \tag{2.1.8}$$

Therefore by ignoring the higher order than second,the Ricci tensor linearize to,

$$R_{\mu\nu}=\frac{1}{2}\left[\frac{\partial^2 h_{\lambda}^{\lambda(w)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu}^{\lambda(w)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\nu}^{\kappa(w)}}{\partial x^{\kappa}\partial x^{\mu}}+\frac{\partial^2 h_{\mu\nu}^{(w)}}{\partial x_{\lambda}\partial x^{\lambda}}\right]+\frac{1}{2}\left[\frac{\partial^2 h_{\lambda}^{\lambda(\Lambda)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu}^{\lambda(\Lambda)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\nu}^{\kappa(\Lambda)}}{\partial x^{\kappa}\partial x^{\mu}}+\frac{\partial^2 h_{\mu\nu}^{(\Lambda)}}{\partial x_{\lambda}\partial x^{\lambda}}\right]$$
$$-\frac{1}{2}h^{\lambda\kappa}_{(w)}\left[\frac{\partial^2 h_{\lambda\kappa}^{(w)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu\kappa}^{(w)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\lambda\nu}^{(w)}}{\partial x^{\kappa}\partial x^{\lambda}}+\frac{\partial^2 h_{\mu\nu}^{(w)}}{\partial x^{\kappa}\partial x^{\lambda}}\right]-\frac{1}{2}h^{\lambda\kappa}_{(w)}\left[\frac{\partial^2 h_{\lambda\kappa}^{(\Lambda)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu\kappa}^{(\Lambda)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\lambda\nu}^{(\Lambda)}}{\partial x^{\kappa}\partial x^{\lambda}}+\frac{\partial^2 h_{\mu\nu}^{(\Lambda)}}{\partial x^{\kappa}\partial x^{\lambda}}\right]$$
$$-\frac{1}{2}h^{\lambda\kappa}_{(\Lambda)}\left[\frac{\partial^2 h_{\lambda\kappa}^{(w)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu\kappa}^{(w)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\lambda\nu}^{(w)}}{\partial x^{\kappa}\partial x^{\lambda}}+\frac{\partial^2 h_{\mu\nu}^{(w)}}{\partial x^{\kappa}\partial x^{\lambda}}\right]-\frac{1}{2}h^{\lambda\kappa}_{(\Lambda)}\left[\frac{\partial^2 h_{\lambda\kappa}^{(\Lambda)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu\kappa}^{(\Lambda)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\lambda\nu}^{(\Lambda)}}{\partial x^{\kappa}\partial x^{\lambda}}+\frac{\partial^2 h_{\mu\nu}^{(\Lambda)}}{\partial x^{\kappa}\partial x^{\lambda}}\right]$$
$$+\frac{1}{4}\left[2\frac{\partial h_{\sigma}^{\kappa(w)}}{\partial x^{\kappa}}-\frac{\partial h_{\kappa}^{\kappa(w)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(w)}}{\partial x^{\nu}}+\frac{\partial h_{\nu}^{\sigma(w)}}{\partial x^{\mu}}-\frac{\partial h_{\mu\nu}^{(w)}}{\partial x_{\sigma}}\right]+\frac{1}{4}\left[2\frac{\partial h_{\sigma}^{\kappa(w)}}{\partial x^{\kappa}}-\frac{\partial h_{\kappa}^{\kappa(w)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(\Lambda)}}{\partial x^{\nu}}+\frac{\partial h_{\nu}^{\sigma(\Lambda)}}{\partial x^{\mu}}-\frac{\partial h_{\mu\nu}^{(\Lambda)}}{\partial x_{\sigma}}\right]$$
$$+\frac{1}{4}\left[2\frac{\partial h_{\sigma}^{\kappa(\Lambda)}}{\partial x^{\kappa}}-\frac{\partial h_{\kappa}^{\kappa(\Lambda)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(w)}}{\partial x^{\nu}}+\frac{\partial h_{\nu}^{\sigma(w)}}{\partial x^{\mu}}-\frac{\partial h_{\mu\nu}^{(w)}}{\partial x_{\sigma}}\right]+\frac{1}{4}\left[2\frac{\partial h_{\sigma}^{\kappa(\Lambda)}}{\partial x^{\kappa}}-\frac{\partial h_{\kappa}^{\kappa(\Lambda)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(\Lambda)}}{\partial x^{\nu}}+\frac{\partial h_{\nu}^{\sigma(\Lambda)}}{\partial x^{\mu}}-\frac{\partial h_{\mu\nu}^{(\Lambda)}}{\partial x_{\sigma}}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h_{\sigma\nu}^{(w)}}{\partial x^{\lambda}}+\frac{\partial h_{\sigma\lambda}^{(w)}}{\partial x^{\nu}}-\frac{\partial h_{\mu\lambda}^{(w)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(w)}}{\partial x_{\lambda}}+\frac{\partial h^{\sigma\lambda}_{(w)}}{\partial x^{\mu}}-\frac{\partial h_{\mu}^{\lambda(w)}}{\partial x_{\sigma}}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h_{\sigma\nu}^{(w)}}{\partial x^{\lambda}}+\frac{\partial h_{\sigma\lambda}^{(w)}}{\partial x^{\nu}}-\frac{\partial h_{\mu\lambda}^{(w)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(\Lambda)}}{\partial x_{\lambda}}+\frac{\partial h^{\sigma\lambda}_{(\Lambda)}}{\partial x^{\mu}}-\frac{\partial h_{\mu}^{\lambda(\Lambda)}}{\partial x_{\sigma}}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h_{\sigma\nu}^{(\Lambda)}}{\partial x^{\lambda}}+\frac{\partial h_{\sigma\lambda}^{(\Lambda)}}{\partial x^{\nu}}-\frac{\partial h_{\mu\lambda}^{(\Lambda)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(w)}}{\partial x_{\lambda}}+\frac{\partial h^{\sigma\lambda}_{(w)}}{\partial x^{\mu}}-\frac{\partial h_{\mu}^{\lambda(w)}}{\partial x_{\sigma}}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h_{\sigma\nu}^{(\Lambda)}}{\partial x^{\lambda}}+\frac{\partial h_{\sigma\lambda}^{(\Lambda)}}{\partial x^{\nu}}-\frac{\partial h_{\mu\lambda}^{(\Lambda)}}{\partial x^{\sigma}}\right]\left[\frac{\partial h_{\mu}^{\sigma(\Lambda)}}{\partial x_{\lambda}}+\frac{\partial h^{\sigma\lambda}_{(\Lambda)}}{\partial x^{\mu}}-\frac{\partial h_{\mu}^{\lambda(\Lambda)}}{\partial x_{\sigma}}\right] \tag{2.1.9}$$

Now if we ignore the cosmological perturbation $h_{\mu\nu}^{(\Lambda)}$ of the order above the first,finally

1. The Ricci tensor is linearize to the first order is,

$$\begin{aligned}R^{(1)}_{\mu\nu}=&\frac{1}{2}\left[\frac{\partial^2 h_{\lambda}^{\lambda(w)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu}^{\lambda(w)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\nu}^{\kappa(w)}}{\partial x^{\kappa}\partial x^{\mu}}+\frac{\partial^2 h_{\mu\nu}^{(w)}}{\partial x_{\lambda}\partial x^{\lambda}}\right]\\&+\frac{1}{2}\left[\frac{\partial^2 h_{\lambda}^{\lambda(\Lambda)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu}^{\lambda(\Lambda)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\nu}^{\kappa(\Lambda)}}{\partial x^{\kappa}\partial x^{\mu}}+\frac{\partial^2 h_{\mu\nu}^{(\Lambda)}}{\partial x_{\lambda}\partial x^{\lambda}}\right]\end{aligned} \tag{2.1.10}$$

2. The Ricci tensor that linearize to the second order is,

$$R^{(2)}_{\mu\nu}=-\frac{1}{2}h^{\lambda\kappa(w)}\left[\frac{\partial^2 h_{\lambda\kappa}^{(w)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu\kappa}^{(w)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\lambda\nu}^{(w)}}{\partial x^{\kappa}\partial x^{\lambda}}+\frac{\partial^2 h_{\mu\nu}^{(w)}}{\partial x^{\kappa}\partial x^{\lambda}}\right]$$

$$-\frac{1}{2}h^{\lambda\kappa(w)}\left[\frac{\partial^2 h^{(\Lambda)}_{\lambda\kappa}}{\partial x^\nu \partial x^\mu}-\frac{\partial^2 h^{(\Lambda)}_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}-\frac{\partial^2 h^{(\Lambda)}_{\lambda\nu}}{\partial x^\kappa \partial x^\lambda}+\frac{\partial^2 h^{(\Lambda)}_{\mu\nu}}{\partial x^\kappa \partial x^\lambda}\right]$$
$$-\frac{1}{2}h^{\lambda\kappa(\Lambda)}\left[\frac{\partial^2 h^{(w)}_{\lambda\kappa}}{\partial x^\nu \partial x^\mu}-\frac{\partial^2 h^{(w)}_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}-\frac{\partial^2 h^{(w)}_{\lambda\nu}}{\partial x^\kappa \partial x^\lambda}+\frac{\partial^2 h^{(w)}_{\mu\nu}}{\partial x^\kappa \partial x^\lambda}\right]$$
$$+\frac{1}{4}\left[2\frac{\partial h^{\kappa(w)}_{\sigma}}{\partial x^\kappa}-\frac{\partial h^{\kappa(w)}_{\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x^\nu}+\frac{\partial h^{\sigma(w)}_{\nu}}{\partial x^\mu}-\frac{\partial h^{(w)}_{\mu\nu}}{\partial x_\sigma}\right]$$
$$+\frac{1}{4}\left[2\frac{\partial h^{\kappa(w)}_{\sigma}}{\partial x^\kappa}-\frac{\partial h^{\kappa(w)}_{\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(\Lambda)}_{\mu}}{\partial x^\nu}+\frac{\partial h^{\sigma(\Lambda)}_{\nu}}{\partial x^\mu}-\frac{\partial h^{(\Lambda)}_{\mu\nu}}{\partial x_\sigma}\right]$$
$$+\frac{1}{4}\left[2\frac{\partial h^{\kappa(\Lambda)}_{\sigma}}{\partial x^\kappa}-\frac{\partial h^{\kappa(\Lambda)}_{\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x^\nu}+\frac{\partial h^{\sigma(w)}_{\nu}}{\partial x^\mu}-\frac{\partial h^{(w)}_{\mu\nu}}{\partial x_\sigma}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h^{(w)}_{\sigma\nu}}{\partial x^\lambda}+\frac{\partial h^{(w)}_{\sigma\lambda}}{\partial x^\nu}-\frac{\partial h^{(w)}_{\mu\lambda}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x_\lambda}+\frac{\partial h^{\sigma\lambda(w)}}{\partial x^\mu}-\frac{\partial h^{\lambda(w)}_{\mu}}{\partial x_\sigma}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h^{(w)}_{\sigma\nu}}{\partial x^\lambda}+\frac{\partial h^{(w)}_{\sigma\lambda}}{\partial x^\nu}-\frac{\partial h^{(w)}_{\mu\lambda}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(\Lambda)}_{\mu}}{\partial x_\lambda}+\frac{\partial h^{\sigma\lambda(\Lambda)}}{\partial x^\mu}-\frac{\partial h^{\lambda(\Lambda)}_{\mu}}{\partial x_\sigma}\right]$$
$$-\frac{1}{4}\left[\frac{\partial h^{(\Lambda)}_{\sigma\nu}}{\partial x^\lambda}+\frac{\partial h^{(\Lambda)}_{\sigma\lambda}}{\partial x^\nu}-\frac{\partial h^{(\Lambda)}_{\mu\lambda}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x_\lambda}+\frac{\partial h^{\sigma\lambda(w)}}{\partial x^\mu}-\frac{\partial h^{\lambda(w)}_{\mu}}{\partial x_\sigma}\right] \tag{2.1.11}$$

2.1.3 Coordinate Transformations

Suppose that the coordinates are varied at the same time that the metric tensor is varied. Thus,the coordinates become,

$$x^{'\mu} = x^\mu + \xi^\mu(x^\nu) \tag{2.1.12}$$

This new coordinate is generated by a vector ξ^μ,where the components are function of position. If we demand that ξ^μ be small in the sense that

$$\|\xi^\mu_{;\nu}\| << 1$$

Therefore,the metric in new coordinate,$x^{'\mu}$ is,

$$g^{'\mu\nu} = \frac{\partial x^{'\mu}}{\partial x^\lambda} g^{\lambda\rho} \frac{\partial x^{'\nu}}{\partial x^\rho} \tag{2.1.13}$$

Generally,the new metric is differ from that of flat background spacetime,now in the Schwarzschild de-Sitter background it has a form of,

$$g'^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu(w)} - h^{\mu\nu(\Lambda)} + \frac{\partial\xi^{\mu}}{\partial x^{\lambda}}\eta^{\lambda\nu} + \frac{\partial\xi^{\nu}}{\partial x^{\rho}}\eta^{\mu\rho} \tag{2.1.14}$$

This implies that,

$$g'^{\mu\nu} = \eta^{\mu\nu} - h'^{\mu\nu(w)} - h'^{\mu\nu(\Lambda)} \tag{2.1.15}$$

Where,

$$h'^{\mu\nu(w)} + h'^{\mu\nu(\Lambda)} = h^{\mu\nu(w)} + h^{\mu\nu(\Lambda)} - \frac{\partial\xi^{\mu}}{\partial x^{\lambda}}\eta^{\lambda\nu} - \frac{\partial\xi^{\nu}}{\partial x^{\rho}}\eta^{\mu\rho}$$

or

$$h'^{(w)}_{\mu\nu} + h'^{(\Lambda)}_{\mu\nu} = h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu} - \frac{\partial\xi_{\mu}}{\partial x^{\nu}} - \frac{\partial\xi_{\nu}}{\partial x^{\mu}} \tag{2.1.16}$$

Now back to the metric variation,in the new coordinate,x'^{μ} is,

$$\delta g_{\mu\nu} = h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu} - \frac{\partial\xi_{\mu}}{\partial x^{\nu}} - \frac{\partial\xi_{\nu}}{\partial x^{\mu}} \tag{2.1.17}$$

This is differ from the variation,

$$\delta g_{\mu\nu} = h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu} \tag{2.1.18}$$

Therefore,they represents the same spacetime geometry as a change of the form,

$$h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu} \longrightarrow h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu} - \frac{\partial\xi_{\mu}}{\partial x^{\nu}} - \frac{\partial\xi_{\nu}}{\partial x^{\mu}} \tag{2.1.19}$$

This is called a gauge transformation because it should have no effect on the physical law (gauge invariance).

2.1.4 Harmonic Coordinate Conditions

The gauge invariance is not easy when it comes to actually solving the field equations. However,the difficulty can be removed by choosing some particular gauge. i.e coordinate system. The most convenient choice is to work in a harmonic coordinate system,

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \tag{2.1.20}$$

Using the affine connection of the first order and definition of metric in Schwarzschild de-Sitter,yield that

$$\frac{\partial}{\partial x^{\mu}}h_{\nu}^{\mu(w)}+\frac{\partial}{\partial x^{\mu}}h_{\nu}^{\mu(\Lambda)}=\frac{1}{2}\frac{\partial}{\partial x^{\nu}}h_{\mu}^{\mu(w)}+\frac{1}{2}\frac{\partial}{\partial x^{\nu}}h_{\mu}^{\mu(\Lambda)} \tag{2.1.21}$$

If $h_{\mu\nu}$ does not satisfy the above equation,then we can find an $h'_{\mu\nu}$ that does,by performing the coordinate transformation,therefore

$$\Box^2\xi_{\nu}=\frac{\partial}{\partial x^{\mu}}h_{\nu}^{\mu(w)}+\frac{\partial}{\partial x^{\mu}}h_{\nu}^{\mu(\Lambda)}-\frac{1}{2}\frac{\partial}{\partial x^{\nu}}h_{\mu}^{\mu(w)}-\frac{1}{2}\frac{\partial}{\partial x^{\nu}}h_{\mu}^{\mu(\Lambda)} \tag{2.1.22}$$

2.1.5 Linearization The Einstein Field Equations

We are looking for the forms of field equations that can be tested observationally in the context of quadrupole formalism. The simplest assumption to start with is,that at least the theory to be developed must be in agreement with Newtonian approximation of gravitational whenever the later gives a successful account of observations. So in a region where we are going to implement the linearzation theory with the Schwarzschild metric be subjected to this assumption. But this is possible in a region where the effect of ,Λ is so small. Recall that the metric of the Schwarzschild de-Sitter,

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}^{(w)}+h_{\mu\nu}^{(\Lambda)} \tag{2.1.23}$$

We have the first order affine connection and Ricci tensor are given by

$$\Gamma_{\mu\nu}^{\lambda(1)}=\frac{1}{2}\left[\frac{\partial h_{\mu}^{\lambda(w)}}{\partial x^{\nu}}+\frac{\partial h_{\nu}^{\lambda(w)}}{\partial x^{\mu}}-\frac{h_{\mu\nu}^{(w)}}{\partial x_{\lambda}}\right]-\frac{1}{2}\left[\frac{\partial h_{\mu}^{\lambda(\Lambda)}}{\partial x^{\nu}}+\frac{h_{\nu}^{\lambda(\Lambda)}}{\partial x^{\mu}}-\frac{h_{\mu\nu}^{(\Lambda)}}{\partial x_{\lambda}}\right] \tag{2.1.24}$$

$$R_{\mu\nu}^{(1)}=\frac{1}{2}\left[\frac{\partial^2 h_{\lambda}^{\lambda(w)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu}^{\lambda(w)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\nu}^{\kappa(w)}}{\partial x^{\kappa}\partial x^{\mu}}+\frac{\partial^2 h_{\mu\nu}^{(w)}}{\partial x_{\lambda}\partial x^{\lambda}}\right]+\frac{1}{2}\left[\frac{\partial^2 h_{\lambda}^{\lambda(\Lambda)}}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial^2 h_{\mu}^{\lambda(\Lambda)}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2 h_{\nu}^{\kappa(\Lambda)}}{\partial x^{\kappa}\partial x^{\mu}}+\frac{\partial^2 h_{\mu\nu}^{(\Lambda)}}{\partial x_{\lambda}\partial x^{\lambda}}\right] \tag{2.1.25}$$

Further also applied the D'Alembertial operator of the form,

$$\eta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}=-\frac{\partial^2}{\partial t^2}+\nabla^2=\Box^2 \tag{2.1.26}$$

The field equation with cosmological constant can be written as,

$$R_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}R-\Lambda g_{\mu\nu}=-8\pi G T_{\mu\nu} \tag{2.1.27}$$

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = -8\pi G(T_{\mu\nu} - \frac{\Lambda}{8\pi G}g_{\mu\nu})$$
$$= -8\pi G(T_{\mu\nu} - T^{(vacuum)}_{\mu\nu}) \quad (2.1.28)$$

Where $T^{(vacuum)}_{\mu\nu} = \frac{\Lambda}{8\pi G}g_{\mu\nu}$ and also known as energy-momentum tensor of vacuum. We may reduced the energy-momentum tensor as follow,

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = -8\pi G\tilde{T}_{\mu\nu} \quad (2.1.29)$$

where $\tilde{T}_{\mu\nu} = T_{\mu\nu} - T^{(vacuum)}_{\mu\nu}$ as energy-momentum of matter and vacuum.

Therefore the Einstien field equations to first order can be written as,

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda} = -8\pi G\tilde{T}_{\mu\nu} \quad (2.1.30)$$

$$R^{(1)}_{\mu\nu} = -8\pi G\tilde{T}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda}$$
$$= -8\pi G[\tilde{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{T}^{\lambda}_{\lambda}] \quad (2.1.31)$$

.

Note that the vacuum energy-momentum $T^{(vacuum)}$ to the first order leave with a cosmological constant since it take as perturbation,meaning $T^{(vacuum)}_{\mu\nu} = \frac{\Lambda}{8\pi G}\eta_{\mu\nu}$. Therefore one can write the linearized field equation as,

$$R^{(1)}_{\mu\nu} = -8\pi G\tilde{\tilde{T}}_{\mu\nu} \quad (2.1.32)$$

Where $\tilde{\tilde{T}}_{\mu\nu}$ is the source define as,

$$\tilde{\tilde{T}}_{\mu\nu} = \tilde{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{T}^{\lambda}_{\lambda}$$

Here we rewrite the field equation in the first order with the approximation we mentioned above for cosmological constant and also the harmonic coordinate(Lorentz gauge)and ignoring the higher order of cosmological than first order the above equation reduce to,

$$\Box^2 h^{(w)}_{\mu\nu} = -16\pi G\tilde{\tilde{T}}_{\mu\nu} \quad (2.1.33)$$

The well known one solution of equation is retarded potential given by the following relation,

$$h^{(w)}_{\mu\nu}(x,t) = 4G\int \frac{d^3x'}{|x-x'|}\tilde{\tilde{T}}_{\mu\nu}(x', t-|x-x'|) \tag{2.1.34}$$

In the above equation the time argument $t-|x-x'|$ shows that gravitational radiation propagate with a unit velocity,i.e,with speed of light. If we concerned with the conservational law for $\tilde{\tilde{T}}_{\mu\nu}$ is equivalent to

$$\frac{\partial}{\partial x^{\mu}}\tilde{\tilde{T}}^{\mu}_{\nu} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\tilde{\tilde{T}}^{\mu}_{\mu} \tag{2.1.35}$$

As a result the solution of retarded potential for a source $\tilde{\tilde{T}}_{\mu\nu}$ confined to a finite volume will be satisfies the harmonic coordinate condition.

In a vacuum,the only energy-momentum tensor is the vacuum energy-momentum tensor $T^{(vacuum)}_{\mu\nu}$. Now the linearize Einstein field equation in a vacuum become to,

$$\Box^2 h^{(w)}_{\mu\nu} = 16\pi G T^{(vacuum)}_{\mu\nu} \tag{2.1.36}$$

Where $T^{(vacuum)}_{\mu\nu} = \frac{\Lambda}{8\pi G}\eta_{\mu\nu}$ finally,the field equation can be written as,

$$\Box^2 h^{(w)}_{\mu\nu} = 2\Lambda\eta_{\mu\nu} \tag{2.1.37}$$

2.2 Gravitational Wave

The general approach to solving inhomogeneous wave equations like, $\Box^2 h^{(w)}_{\mu\nu} - 2\Lambda\eta_{\mu\nu} = 0$ is to use the technique of Green's function. Now,we are interested in obtaining the solutions to the inhomogeneous linearized field equations in schwarzschild de-sitter spacetime.

$$\Box^2 h^{(w)}_{\mu\nu}(x,t) - 2\Lambda\eta_{\mu\nu}(x,t) = 0$$

where $h^{(w)}_{\mu\nu}$ and $\eta_{\mu\nu}$ are function of space and coordinates. The retarded solution is then

$$h^{(w)}_{\mu\nu}(x,t) = \frac{1}{2\pi}\int 2\Lambda d^3x' \frac{\eta_{\mu\nu}(x', t-|x-x'|)}{|x-x'|} \tag{2.2.1}$$

where $t-|x-x'|$ is retarded time shows that effects propagate with unit velocity that is,with speed of light. This also written as Fourier integral,

$$h^{(w)}_{\mu\nu}(x,t) = \frac{\Lambda}{\pi}\int \frac{d^3x'}{|x-x'|}\eta_{\mu\nu}(x',\omega)e^{-i\omega t + i\omega|x-x'|} \tag{2.2.2}$$

Having in mind that this radiation is observed in the wave zone,that is at distance $r \equiv |x|$ much larger than the dimension $R = |x|_{max}$,then the denominator $|x - x'|$ can be replaced by r,while in the exponent we may approximate,

$$|x - x'| \simeq r - x' \cdot \hat{x} \quad and \quad \hat{x} \equiv \frac{x}{r}$$

The field become,

$$h^{(w)}_{\mu\nu}(x,t) = \frac{\Lambda}{\pi r} e^{i\omega r - i\omega t} \int \frac{d^3x'}{|x - x'|} \eta_{\mu\nu}(x', \omega) e^{-i\omega x' \cdot \hat{x}} \tag{2.2.3}$$

Since $r\omega$ is assumed large,the above equation just looks like a plane wave of,

$$h^{(w)}_{\mu\nu}(x,t) = e_{\mu\nu}(x, \omega) exp(i\mathbf{k}_\mu x^\mu) \tag{2.2.4}$$

where "wave vector " and "polarization tensor" are given by,

$$\mathbf{k} \equiv \omega\hat{x} \quad and \quad k^o = \omega$$

and

$$e_{\mu\nu}(x, \omega) = \frac{\Lambda}{\pi r} \int d^3x' \eta_{\mu\nu}(x, \omega) e^{-i\mathbf{k}\cdot x'} \tag{2.2.5}$$

From Fourier transformation,

$$\eta_{\mu\nu}(\mathbf{k}, \omega) = \int \eta_{\mu\nu}(x, \omega) e^{-i\mathbf{k}\cdot x'}$$

Then the polarization tensor has the form of

$$e_{\mu\nu}(x, \omega) = \frac{\Lambda}{\pi r} \eta_{\mu\nu}(\mathbf{k}, \omega) \tag{2.2.6}$$

Therefore the field can be written as,

$$h^{(w)}_{\mu\nu}(x,t) = \frac{\Lambda}{\pi r} \eta_{\mu\nu}(\mathbf{k}, \omega) e^{i\omega r - i\omega t} \tag{2.2.7}$$

Where,$|x-x'|^3$ and $|x-x'|^2$ are much larger than $|x-x'|$ We approximate to firs order,then

$$k_\mu k^\mu = 2\pi r e^{i\omega t - i\omega r}$$

,
The harmonic coordinates condition in the weak field approximation$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$ that is equivalently,

$$\frac{\partial}{\partial x^\mu} h^{\mu(w)}_\nu - \frac{1}{2}\frac{\partial}{\partial x^\nu} h^{\mu(w)}_\mu = 0 \tag{2.2.8}$$

This equation lead to,

$$k_\mu e^\mu_\nu = k_\nu e^\mu_\mu \tag{2.2.9}$$

This reduces the number of independent components of the symmetric tensor $e_{\mu\nu}$ to *two*. And it can be shown that the only non-zero components are e_{11}, e_{12}, e_{21} and e_{22}.

2.3 Energy and Momentum of Gravitational Wave

We can now interpret Einstein field equation as having two sources,one being the conventional matter source given by $T_{\mu\nu}$ and the other arising out of the energy-momentum tensor of the gravitational wave perturbations $t_{\mu\nu}$[7]. The solution of plane wave leads to calculating the energy and momentum it carries. The energy momentum tensor of gravitation given by the second order of $h_{\mu\nu}$.

$$t_{\mu\nu} \equiv \frac{1}{8\pi G}\left[-\frac{1}{2}h_{\mu\nu}\eta^{\lambda\rho}R^{(1)}_{\lambda\rho} + \frac{1}{2}\eta\mu\nu h^{\lambda\rho}R^{(1)} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho} + \Lambda\eta_{\mu\nu} + \Lambda h_{\mu\nu}\right] \tag{2.3.1}$$

where $R^{(n)}_{\mu\nu}$ is the term in Ricci tensor of order n in $h_{\mu\nu}$,the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies the first-order Einstien equation,$R^{(1)}_{\mu\nu} = \Lambda\eta_{\mu\nu}$,thus we can write the above equation as,

$$t_{\mu\nu} \simeq \frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho} + \Lambda\eta_{\mu\nu} - \frac{1}{2}\Lambda h_{\mu\nu}\right] \tag{2.3.2}$$

The approximation of cosmological constant to the first order yield that,

$$t_{\mu\nu} \simeq \frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho} + \Lambda\eta_{\mu\nu}\right] \tag{2.3.3}$$

The second order Ricci is given by,

$$R^{(2)}_{\mu\nu} = -\frac{1}{2}h^{\lambda\kappa(w)}\left[\frac{\partial^2 h^{(w)}_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h^{(w)}_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^{(w)}_{\lambda\nu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 h^{(w)}_{\mu\nu}}{\partial x^\kappa \partial x^\lambda}\right]$$

$$
\begin{aligned}
&-\frac{1}{2}h^{\lambda\kappa(w)}\left[\frac{\partial^2 h^{(\Lambda)}_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h^{(\Lambda)}_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^{(\Lambda)}_{\lambda\nu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 h^{(\Lambda)}_{\mu\nu}}{\partial x^\kappa \partial x^\lambda}\right] \\
&-\frac{1}{2}h^{\lambda\kappa(\Lambda)}\left[\frac{\partial^2 h^{(w)}_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h^{(w)}_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^{(w)}_{\lambda\nu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 h^{(w)}_{\mu\nu}}{\partial x^\kappa \partial x^\lambda}\right] \\
&+\frac{1}{4}\left[2\frac{\partial h^{\kappa(w)}_{\sigma}}{\partial x^\kappa} - \frac{\partial h^{\kappa(w)}_{\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x^\nu} + \frac{\partial h^{\sigma(w)}_{\nu}}{\partial x^\mu} - \frac{\partial h^{(w)}_{\mu\nu}}{\partial x_\sigma}\right] \\
&+\frac{1}{4}\left[2\frac{\partial h^{\kappa(w)}_{\sigma}}{\partial x^\kappa} - \frac{\partial h^{\kappa(w)}_{\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(\Lambda)}_{\mu}}{\partial x^\nu} + \frac{\partial h^{\sigma(\Lambda)}_{\nu}}{\partial x^\mu} - \frac{\partial h^{(\Lambda)}_{\mu\nu}}{\partial x_\sigma}\right] \\
&+\frac{1}{4}\left[2\frac{\partial h^{\kappa(\Lambda)}_{\sigma}}{\partial x^\kappa} - \frac{\partial h^{\kappa(\Lambda)}_{\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x^\nu} + \frac{\partial h^{\sigma(w)}_{\nu}}{\partial x^\mu} - \frac{\partial h^{(w)}_{\mu\nu}}{\partial x_\sigma}\right] \\
&-\frac{1}{4}\left[\frac{\partial h^{(w)}_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial h^{(w)}_{\sigma\lambda}}{\partial x^\nu} - \frac{\partial h^{(w)}_{\mu\lambda}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda(w)}}{\partial x^\mu} - \frac{\partial h^{\lambda(w)}_{\mu}}{\partial x_\sigma}\right] \\
&-\frac{1}{4}\left[\frac{\partial h^{(w)}_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial h^{(w)}_{\sigma\lambda}}{\partial x^\nu} - \frac{\partial h^{(w)}_{\mu\lambda}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(\Lambda)}_{\mu}}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda(\Lambda)}}{\partial x^\mu} - \frac{\partial h^{\lambda(\Lambda)}_{\mu}}{\partial x_\sigma}\right] \\
&-\frac{1}{4}\left[\frac{\partial h^{(\Lambda)}_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial h^{(\Lambda)}_{\sigma\lambda}}{\partial x^\nu} - \frac{\partial h^{(\Lambda)}_{\mu\lambda}}{\partial x^\sigma}\right]\left[\frac{\partial h^{\sigma(w)}_{\mu}}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda(w)}}{\partial x^\mu} - \frac{\partial h^{\lambda(w)}_{\mu}}{\partial x_\sigma}\right]
\end{aligned}
\tag{2.3.4}
$$

To find out $R^{(2)}_{\mu\nu}$,the result is extremely complicated,however to simplify,it is better to take the average of $t_{\mu\nu}$ over a region of space and time much larger than $|k|^{-1}$. From this approximation the averaging kills all terms proportional to $exp(\pm 2ik_\lambda x^\lambda)$ and left with only the x^μ-independent cross-terms. That mean,

1. $$[e_{\mu\nu}exp(ik_\lambda x^\lambda) + c.c][e_{\sigma\kappa}exp(ik_\lambda x^\lambda) + c.c] = e_{\mu\nu}e\sigma\kappa[exp(2ik_\lambda x^\lambda)]$$

2. $$\left\langle [e_{\mu\nu}exp(ik_\lambda x^\lambda) + c.c][e_{\sigma\kappa}exp(ik_\lambda x^\lambda) + c.c] \right\rangle = 2Re^*_{\mu\nu}e_{\sigma\kappa}$$

   Where R is real part and $\langle .. \rangle$ denotes time-average as well as c.c is complex part.

3. For plane wave solutions derivateves correspond to multiplication with k.

$$\frac{\partial h_{\mu\nu}}{\partial x^\lambda} = ik_\lambda h_{\mu\nu}$$

Now the second order of Ricci tensor of average can be reduced to,

$$\langle R^{(2)}_{\mu\nu}\rangle = Re\Big\{e^{\lambda\rho*}\Big[k_\mu k_\nu e_{\lambda\rho} - k_\mu k_\lambda e_{\nu\rho} - k_\nu k_\rho e_{\mu\lambda} + k_\lambda k_\rho e_{\mu\nu}\Big] + \Big[e^\lambda_\rho k_\lambda - \frac{1}{2}e^\lambda_\lambda\Big] * \Big[k_\mu e^\rho_\nu + k_\nu e^\rho_m u - k^\rho e_{\mu\nu}\Big] - \frac{1}{2}\Big[k_\lambda e_{\rho\nu} + k_\nu e_{\rho\lambda} - k_\rho e_{\lambda\nu}\Big] * \Big[k^\lambda e^\rho_\mu + k_\mu e^{\rho\lambda} - k^\rho e^\lambda_\mu\Big]\Big\} \tag{2.3.5}$$

Now applying the harmonic coordinate conditions $k_\mu k^\mu = 2\pi r e^{i\omega t - i\omega r}$ and $k_\mu e^\mu_\nu = \frac{1}{2}k_\nu e^\mu_\mu$ then we find that,

$$\langle R^{(2)}_{\mu\nu}\rangle \equiv \frac{k_\mu k_\nu}{2}[e^{\lambda\rho*}e_{\lambda\rho} - \frac{1}{2}|e^\lambda_\lambda|^2] + \frac{\Lambda^2}{2\pi r}\eta^2(k,\omega)e^{i\omega r - i\omega t} \tag{2.3.6}$$

and Similarly,

$$\eta^{\mu\nu}\langle R^{(2)}_{\mu\nu}\rangle \equiv \frac{k^\nu k_\nu}{2}[e^{\lambda\rho*}e_{\lambda\rho} - \frac{1}{2}|e^\lambda_\lambda|^2] + \frac{\Lambda^2}{2\pi r}\eta^2(k,\omega)e^{i\omega r - i\omega t} \tag{2.3.7}$$

Thus the time average of energy-momentum tensor of the gravitational wave reads,

$$\langle t_{\mu\nu}\rangle \equiv -\frac{k_\mu k_\nu}{16\pi G}[e^{\lambda\rho*}e_{\lambda\rho} - \frac{1}{2}|e^\lambda_\lambda|^2] + \frac{\Lambda^2}{16\pi^2 G r}\eta^2(k,\omega)e^{i\omega r - i\omega t} + \frac{\Lambda\eta_{\mu\nu}}{8\pi G} \tag{2.3.8}$$

We know that our approximation of cosmological constant to first order,thus

$$\langle t_{\mu\nu}\rangle \equiv -\frac{k_\mu k_\nu}{16\pi G}[e^{\lambda\rho*}e_{\lambda\rho} - \frac{1}{2}|e^\lambda_\lambda|^2] + \frac{\Lambda\eta_{\mu\nu}}{8\pi G} \tag{2.3.9}$$

In particular,for a wave traveling in the z-direction,with wave vector,

$$k^1 = k^2 = 0 \quad and \quad k^3 = k^0 = k > 0$$

And polarization tensor

$$e_{01} = -e_{31}, e_{02} = e_{32}, -\frac{1}{2}(e_{33} + e_{00}) \quad and \quad e_{22} = -e_{11}$$

,Therefore,the energy momentum tensor can be reduced to

$$\langle t_{\mu\nu}\rangle \equiv -\frac{k_\mu k_\nu}{8\pi G}[|e_{11}|^2 + |e_{12}|^2] + \frac{\Lambda\eta_{\mu\nu}}{8\pi G} \tag{2.3.10}$$

where e_{11} and e_{12} are couplled with cosmological constant,Λ.

2.4 Gravitational Quadrupole Radiation

Long-wavelength gravitational waves far from a non-relativistic source (wavelengths much larger than the characteristic source size imply low velocities for mass in the source relative to that of light)[5,7]. In the Schwarzschild de-Sitter spacetime the retarded potential where matter energy-momentum tensor included can be,

$$h_{\mu\nu}^{(w)}(x,t) = 4G\int \frac{d^3x'}{|x-x'|}\tilde{\tilde{T}}_{\mu\nu}(x', t-|x-x'|) \tag{2.4.1}$$

If we use the relation

$$-i\omega t_r = i\omega[t-|x-x'|] = -i\omega t + i\omega|x-x'|$$

Then the field emitted by source is

$$h_{\mu\nu}^{(w)}(x,t) = 4G\int \frac{d^3x'}{|x-x'|}\tilde{\tilde{T}}_{\mu\nu}(x', \omega)e^{-i\omega t + i\omega|x-x'|} \tag{2.4.2}$$

Assuming that we observe the radiation in wave zone,that is,at a distance $r \simeq |x|$ much larger than the dimension $R = |x'|_{max}$ of the source and also much larger than ωR^2 and $\frac{1}{\omega}$. Therefore the denominator $|x-x'|$ approximate to r and the exponent may approximate to

$$|x-x'| \simeq r - x'\cdot\hat{x} \quad and \quad \hat{x} \equiv \frac{x}{r}$$

Then

$$h_{\mu\nu}^{(w)}(x,t) = \frac{4G}{r}exp(i\omega r - i\omega t)\int d^3x'\tilde{\tilde{T}}_{\mu\nu}(x',\omega)e^{-i\omega\hat{x}\cdot x'} \tag{2.4.3}$$

Since $r\omega$ is assumed large,this looks just like a plane wave

$$h_{\mu\nu}(x,t) = e_{\mu\nu}(x,\omega)exp(ik_\lambda x^\lambda) + c.c \tag{2.4.4}$$

Where *wave vector*,

$$k = \omega\hat{x} \quad and \quad k^0 = \omega$$

And *polarization tensor* ,

$$e_{\mu\nu}(x,\omega) \simeq \frac{4G}{r}\int d^3x'\tilde{\tilde{T}}_{\mu\nu}(x,\omega)e^{-ik\cdot x'}$$

The polarization tensor may written explicitly in terms of the Fourier transform of $\tilde{T}_{\mu\nu}$,

$$e_{\mu\nu}(x,\omega) = \frac{4G}{r}[\tilde{T}_{\mu\nu}(k,\omega) - \frac{1}{2}\eta_{\mu\nu}\tilde{T}^\lambda_\lambda(k,\omega)] \tag{2.4.5}$$

The conservational law for $\tilde{T}_{\mu\nu}(x,t)$ is,

$$\frac{\partial}{\partial x^{\mu}}\tilde{T}^{\mu}_{\nu}(x,t) = 0 \tag{2.4.6}$$

Applying the Fourier transformation,gives,

$$\frac{\partial}{\partial x^{i}}\tilde{T}^{i}_{\nu}(x,\omega) - i\omega\tilde{T}^{0}_{\nu}(x,\omega) = 0 \tag{2.4.7}$$

where we multiply both sides with $e^{-ik\cdot x}$ and integrating over x,

$$\int \frac{\partial}{\partial x^{i}}\tilde{T}^{i}_{\nu}(x,\omega)e^{-ik\cdot x}d^3x - i\omega\int \tilde{T}^{0}_{\nu}(x,\omega)e^{ik\cdot x}d^3x = 0 \tag{2.4.8}$$

This also equivalent to,

$$ik_{\mu}\int \tilde{T}^{\mu}_{\nu}(x,\omega)e^{-ik\cdot x}d^3x = ik_{\mu}\tilde{T}^{\mu}_{\nu}(x,\omega) = 0 \tag{2.4.9}$$

where k^{μ} is given by,

$$k = \omega\hat{x} \quad and \quad k^0 = \omega$$

Therefore one can read,

$$k_{\mu}\tilde{T}^{\mu}_{\nu} = 0 \tag{2.4.10}$$

This equation verifies that (2.2.5) obeys the harmonic coordinate condition.

We now stand to move on to found out the quadrupole moment,we know that in the linear approximation,the $\tilde{T}^{\mu\nu}$ satisfies the conservational law as well as harmonic condition. That is,

$$\frac{\partial}{\partial x^{\mu}}\tilde{T}^{\mu\nu} = 0 \tag{2.4.11}$$

To simplify our calculations,let we separate this equations in to space and time components like,

$$\frac{\partial}{\partial x^{0}}\tilde{T}^{\mu 0} + \frac{\partial}{\partial x^{k}}\tilde{T}^{\mu k} = 0 \quad where \quad k = 1,2,3 \tag{2.4.12}$$

Integrating over the volume V,

$$\int_V \frac{\partial}{\partial t}\tilde{T}^{\mu 0}d^3x = \int_V \frac{\partial}{\partial x^{k}}\tilde{T}^{\mu k}d^3x \tag{2.4.13}$$

When we apply the Gauss's theorem to the R.H.S

$$\int_V \frac{\partial}{\partial t}\tilde{T}^{\mu 0} d^3x = \int_S \tilde{T}^{\mu k} dS_k \tag{2.4.14}$$

Where S is the surface that encloses,V. We assume that on the surface S,far $\tilde{T}^{\mu\nu} \to 0$, Therefore,

$$\int_V \frac{\partial}{\partial t}\tilde{T}^{\mu 0} d^3x = \frac{\partial}{\partial t}\Big(\int_V \tilde{T}^{\mu 0} d^3x\Big) = 0 \tag{2.4.15}$$

This equation indicate that, the the term in the bracket is constant,
i.e

$$\int_V \tilde{T}^{\mu 0} d^3x = constant \quad \Rightarrow \quad h^{\mu 0} = constant \tag{2.4.16}$$

Since we are interested in the time-dependent part of the field,we put,

$$h^{\mu 0}(x,t) = h_{\mu 0}(x,t) = 0 \tag{2.4.17}$$

Now let us consider the space component and time component conservational law separatively,
1. Space component of the conservational law has the form of,

$$\frac{\partial}{\partial x^0}\tilde{T}^{k0} + \frac{\partial}{\partial x^i}\tilde{T}^{ki} = 0 \quad where \quad k = 1,2,3 \tag{2.4.18}$$

Multiplying both side equation by x^j then integrating over all space,and leaving the surface terms on the assumption that goes to zero sufficiently rapidly at spatial infinity,

$$\int_V \frac{\partial}{\partial t}\tilde{T}^{k0} x^j d^3x = -\int_V \frac{\partial}{\partial x^i}\tilde{T}^{ki} d^3x$$
$$= -\Big[\int_V \frac{\partial}{\partial x^i}(\tilde{T}^{ki} x^j) d^3x - \int_V \tilde{T}^{ki}\frac{\partial x^j}{\partial x^i} d^3x\Big] \tag{2.4.19}$$

$$where \quad \frac{\partial x^j}{\partial xi} = \delta_i^j = 1$$

This may written as

$$\frac{\partial}{\partial t}\int_V \tilde{T}^{k0}x^j d^3x = -\int_S \tilde{T}^{ki}x^j dS_i + \int_V \tilde{T}^{kj}d^3x \tag{2.4.20}$$

As before $\int_S \tilde{T}^{ki}x^j dS_i = 0$ then

$$\frac{\partial}{\partial t}\int_V \tilde{T}^{k0}x^j d^3x = \int_V \tilde{T}^{kj}d^3x \tag{2.4.21}$$

Since $\tilde{T}^{jk}$ is symmetric in j and k,

$$\frac{\partial}{\partial t}\int_V \tilde{T}^{k0}x^j d^3x = \int_V \tilde{T}^{jk}d^3x \tag{2.4.22}$$

Adding equations (2.5.21) and (2.5.22), We get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V [\tilde{T}^{k0}x^j + \tilde{T}^{j0}x^k]d^3x = \int_V \tilde{T}^{jk}d^3x \tag{2.4.23}$$

2. Similarly use the time-component of the conservational law,

$$\frac{\partial}{\partial x^0}\tilde{T}^{00} + \frac{\partial}{\partial x^i}\tilde{T}^{0i} = 0 \quad where \quad i = 1,2,3 \tag{2.4.24}$$

Multiplying both side by $x^j x^k$ and integrate over the volume,d^3x,

$$\begin{aligned} \int_V \frac{\partial}{\partial t}\tilde{T}^{00}x^j x^k d^3x &= -\int_V \frac{\partial}{\partial x^i}\tilde{T}^{0i}x^j x^k d^3x \\ = -\Big[\int_V \frac{\partial}{\partial x^i}(\tilde{T}^{0i}x^j x^k)d^3x &- \int_V (\tilde{T}^{0i}\frac{\partial x^j}{\partial x^i}x^k + \tilde{T}^{0i}\frac{\partial x^k}{\partial x^i}x^j)d^3x\Big] \\ &= -\int_S \tilde{T}^{0i}x^i x^k dS_i + \int_V [\tilde{T}^{0j}x^k + \tilde{T}^{0k}x^j]d^3x \end{aligned} \tag{2.4.25}$$

Now also as before $\int_S \tilde{T}^{0i}x^i x^k dS_i = 0$,then,

$$\frac{\partial}{\partial t}\int_V \tilde{T}^{00}x^j x^k d^3x = \int_V [\tilde{T}^{0j}x^k + \tilde{T}^{0k}x^j]d^3x \tag{2.4.26}$$

Differentiate this equation with respect to $x^0 = ct$

$$\frac{\partial^2}{\partial t^2}\int_V \tilde{T}^{00}x^j x^k d^3x = \frac{\partial}{\partial t}\int_V [\tilde{T}^{0j}x^k + \tilde{T}^{0k}x^j]d^3x \tag{2.4.27}$$

Comparing equations (2.5.23) and (2.5.27) we get,

$$\frac{\partial^2}{\partial t^2}\int_V \tilde{T}^{00}x^j x^k d^3x = 2\int_V \tilde{T}^{jk}d^3x \tag{2.4.28}$$

The quantity,$\int_V \tilde{T}^{00}x^j x^k d^3x$ is the ***Quadrupole Moment Tensor*** of the system,

$$D^{jk}(t) = \int_V \tilde{T}^{00}(x,t)x^j x^k d^3x \tag{2.4.29}$$

Where,

$$\tilde{T}^{00} = T^{00} - T^{00}_{vacuum} = T^{00} - \frac{\Lambda}{8\pi G}$$

For non-relativistic matter,($\lambda \gg R$) and $v \gg c$we can approximate T^{00} as an energy density that call as rest mass density,ρ

$$\tilde{T}^{00} = \rho c^2 - \frac{\Lambda}{8\pi G}c^2$$

The term T^{00}_{vacuum} may assumed as the cosmological density in vacuum as ρ_Λ in Schwarzschild de-Sitter space. Then the Quadrupole moment tensor has two parts as $D^{(w)jk}(t)$ and $D^{(\Lambda)jk}(t)$ that explicitly write as,

$$\begin{aligned} D^{jk}(t) &= D^{(w)jk}(t) + D^{(\Lambda)jk}(t) \\ &= \int_V T^{00}(x,t)x^j x^k d^3x - \int_V T^{00}_{vacuum}x^j x^k d^3x \\ &= \int_V \rho_{(w)}(x,t)x^j x^k d^3x - \int_V \rho_\Lambda x^j x^k d^3x \end{aligned} \tag{2.4.30}$$

Where $D^{(w)jk}(t)$ is the quadrupole moment tensor to Newtonian as well as $D^{(\Lambda)jk}(t)$ is the quadrupole moment tensor to cosmological constant,

To make a sense we return to the Transversa-Traceless(TT-gauge),or trace-reversed,as

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.4.31}$$

$$and \quad \tilde{h}^\mu_\mu = -h^\mu_\mu$$

By using the above form of perturbation,we can rewrite the Einstein Field Equations under the harmonic coordinates condition as,

$$\Box^2\tilde{h}_{\mu\nu} = -16\pi G\tilde{T}_{\mu\nu} \tag{2.4.32}$$

Now using the fact that in the $TT-gauge$,only the spatial components of $\tilde{h}_{\mu\nu}$ are non-zero (hence $\tilde{h}_{\mu 0}=0$),which means that the wave is transverse to its own direction of propagation,and, additionally,the sum of the diagonal components is zero(traceless). Due to this property and equation(2.5.31),in this gauge there is no deference between $h_{\mu\nu}$(the perturbation of the metric) and $\tilde{h}_{\mu\nu}$(the gravitational field),therefore

$$\Box^2 h^{TT}_{\mu\nu}(x,t) = [-16\pi G\tilde{T}_{\mu\nu}(x,t)]^{TT} \tag{2.4.33}$$

Where $\tilde{T}_{\mu\nu}$ is the energy-momentum tensor of matter and cosmological constant.

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + T^{(vacuum)}_{\mu\nu}$$

From equation (2.5.28),we have

$$\begin{aligned} h^{TT}_{jk}(x,t) &= \left[\frac{4G}{r}\int d^3x'\tilde{T}_{jk}(x',t-r)\right]^{TT} \\ &= \left[\frac{2G}{r}\frac{\partial^2}{\partial t^2}\int \tilde{T}_{00}(x',t-r)x'^j x'^k d^3x'\right]^{TT} \\ &= \frac{2G}{r}\left[\frac{\partial^2}{\partial t^2}D_{jk}(t-r)\right]^{TT} \end{aligned} \tag{2.4.34}$$

we also write the above equation as,

$$h^{TT}_{jk}(x,t) = \frac{2G}{r}[\ddot{D}_{jk}(t-r)]^{TT} \tag{2.4.35}$$

Where,$\ddot{D}_{jk}(t-r)$ is the second moment of the source's mass distribution and cosmological constant and the dotes denotes time derivatives.We can define the quadrupole moments in a traceless form as,

$$Q_{jk} = \int d^3x'\tilde{T}^{00}(t-r)[x'^j x'^k - \frac{1}{3}\delta^j_k r'^2] \tag{2.4.36}$$

Note that D_{jk} and Q_{jk} are only differ by a trace which is logically removed in the TT-gauge. Thus one can write the gravitational and cosmological as,

$$h^{TT}_{jk} = \frac{2G}{r}[\ddot{Q}_{jk}(t-r)]^{TT} \tag{2.4.37}$$

2.5 Energy Carried by Gravitational Wave

The energy density in gravitational wave is defined as,

$$\tilde{T}^{00} = \frac{1}{16\pi G}\langle h^{TT}_{jk,0} h^{TT}_{jk,0}\rangle$$
$$= \frac{1}{16\pi G}\langle \dot{h}^{TT}_{jk} \dot{h}^{TT}_{jk}\rangle \tag{2.5.1}$$

Where dote denotes the time components of this can be also written as

$$\tilde{T}^{00} = \frac{1}{16\pi G}\langle h_+^2 + h_\times^2\rangle \tag{2.5.2}$$

The associated radial flux $\tilde{T}^{0r}$ is given by,

$$\tilde{T}^{0r} = \frac{1}{16\pi G}\langle h^{TT}_{jk,0} h^{TT}_{jk,r}\rangle \tag{2.5.3}$$

The energy flow across the surface that has radius of r and the area element of $r^2 d\Omega$ with in a time at a rate,

$$\int r^2 d\Omega \tilde{T}^{0r}$$

The rate of energy decrease is then given by,

$$\frac{dE}{dt} = -\int r^2 d\Omega \tilde{T}^{0r} \tag{2.5.4}$$

Then after,the energy radiated by the system per unit solid angle and unit time in the radial direction is then,

$$\frac{d^2E}{dt d\Omega} = -r^2 \tilde{T}^{0r} \quad = \quad -r^2 \langle h^{TT}_{jk,0} h^{TT}_{jk,0}\rangle \tag{2.5.5}$$

$$h^{TT}_{jk} = \frac{2G}{r}[\ddot{Q}_{jk}(t-r)]^{TT}$$

As a result,

$$h^{TT}_{jk,0} = \frac{2G}{r}[\dddot{Q}_{jk}(t-r)]^{TT} \tag{2.5.6}$$

$$h^{TT}_{jk,r} = -\frac{2G}{r}[\dddot{Q}_{jk}(t-r)]^{TT} \tag{2.5.7}$$

Put equations (2.6.6) and (2.6.7) into equation (2.6.3),then the radial flux is,

$$\tilde{T}^{0r} = \frac{1}{8\pi G r^2}\langle[\dddot{Q}_{jk}(t-r)]^{TT}[\dddot{Q}_{jk}(t-r)]^{TT}\rangle \tag{2.5.8}$$

Here also the dot denotes the time derivatives.

Chapter 3

Gravitational Radiation From Massive Compact Binary Stars In Schwarzschild de-Sitter Background

Sources of gravitational waves include collapsing stars,exploding stars,stars in orbit around one another,and the big bang. Neither electromagnetic waves nor gravity waves result from a spherically symmetric distribution of charge(for electromagnetic waves)or matter(for gravitational waves),even when that spherical distribution pulses symmetrically in and out[3,9]. Therefore,symmetric collapses or explosions emit no waves,either electromagnetic or gravitational. The most efficient source of electromagnetic radiation is oscillating pairs of electric charges of opposite sign,the result technically called dipole radiation. But mass has only one "polarity;" there is no gravity dipole radiation from masses that oscillate back and forth along a line. Emission of gravity waves requires asymmetric movement or oscillation;the technical name for the result is quadrupole radiation[4]. Happily,most collapses and explosions are asymmetric;even the motion in a binary system is sufficiently asymmetric to emit gravitational waves. We study here gravity waves emitted by a binary system consisting of two black hole stars or-neutron stars-or a neutron star and a black hole orbiting about one another. All such pairs that we have detected are too far away to see directly;As the two objects orbit,they also emit gravity waves that cause the binary system to lose energy,so that the orbiting objects gradually spiral in toward one another.

3.1 The Newtonian Two-Body Problem

We shall begin studying the two-body problem in the case of Newtonian mechanics. This is necessary,because a lot of the techniques used in this chapter will have their analogies in later ,the Newtonian theory will serve as a limiting case of the general theory of relativity. More explicitly,the Newtonian theory should reappear when velocities are negligible($v << 1$) and the fields are weak ($\Phi << 1$). We will start by deriving the radial equation,which describes the change of the distance between the two bodies. This expression can be written in the form $\dot{r} = E + V$,where V is the effective potential. A plot of this potential gives a visual representation of the types of orbits that are possible in our system (circular, elliptical, parabolic and hyperbolic orbits). Finally we shall calculate the paths of the elliptical orbits explicitly.

We shall now apply some of the results obtained to study Newtonian mechanics. This can be modeled as a two body system of a neutron star or black hole. Since the pulsar is orbiting near a strongly gravitating compact remnant, its orbit will be an ellipse (unlike in the case of Newtonian gravity) but will precess. We shall work out the effect of the emission of gravitational radiation on the orbital parameters of the system and will then describe how such a system can be used to test the predictions of general relativity. Let us consider two bodies of masses m_1 and m_2 which are orbiting around the common centre of mass in a Newtonian elliptical orbit. We know from the standard Newtonian analysis,valid to the lowest order,that the orbital parameters are completely determined by the energy ($E < 0$) and the angular momentum J. The radial equation of motion is given by,

$$\dot{r}^2 = \frac{2}{\mu}\left(E - V - \frac{J^2}{2\mu r^2}\right) \tag{3.1.1}$$

Where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is reduced mass,V potential and J angular momentum,the potential is written as,

$$V = \frac{G(m_1 + m_2)}{r} = \frac{M}{r},$$

where $M = m_1 + m_2$.Thus,

$$\dot{r}^2 = \frac{2}{\mu}\left(E - \frac{M}{r} - \frac{J^2}{2\mu r^2}\right) \tag{3.1.2}$$

The above equation may written as,

$$\dot{r}^2 = \frac{2}{\mu}(E - V_{eff}) \tag{3.1.3}$$

Where,

$$V_{eff} = \frac{M}{r} - \frac{J^2}{2\mu r^2}$$

We derive the shape of the orbits by looking over how the angle θ and radius r are related,

$$d\theta = d\dot{\theta}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \quad but \quad J = \mu r^2\dot{\theta}$$

$$d\theta = \frac{J}{\mu r^2 \dot{r}}dr \quad \Rightarrow \quad \dot{r} = \frac{dr}{d\theta}\frac{J}{\mu r^2} \tag{3.1.4}$$

From equations (3.2.1) and (3.2.2),we get

$$\theta(r) = \pm\int \frac{J/\mu r^2}{\frac{2}{\mu}(E + \frac{M}{r} - \frac{J^2}{2\mu r})} \tag{3.1.5}$$

Then the solution to this equation is,

$$r = \frac{(J/\mu)^2}{M(1 + e\cos\theta)}, \quad and \quad e = \sqrt{1 + \frac{2EJ^2}{\mu^3 M^2}} \tag{3.1.6}$$

where e is the eccentricity. For elliptical orbits we have $0 < e < 1$. In this case the elliptical orbits we can rewrite equation (3.2.6) in terms of the semi-major axis a(the longest radius of the ellipse). This is the famous orbital equation for the ellipse:

$$r = \frac{a(1 - e^2)}{1 + ecos\theta} \quad , \quad a = \frac{J^2/\mu^2}{M(1 - e^2)} \tag{3.1.7}$$

It also enable us to write the energy and the angular momentum interms of the semi-major axis and the eccentricity as,

$$E = -\frac{m_1 m_2}{2a} \quad , \quad J^2 = -\frac{(m_1 m_2)^2}{M}a(1 - e^2) \tag{3.1.8}$$

Now let us look how the radius is dependent on time. The radial equation can be written as,

$$\frac{dr}{dt} = \pm\left(\frac{2}{\mu}(E + \frac{m_1 m_2}{r}) - \frac{J^2}{\mu^2 r^2}\right)^{\frac{1}{2}} \tag{3.1.9}$$

Thus,

$$t = \sqrt{\frac{\mu}{2}}\int_0^r \frac{dr}{\left(\frac{2}{\mu}(E + \frac{m_1 m_2}{r}) - \frac{J^2}{2\mu r^2}\right)^{\frac{1}{2}}} \tag{3.1.10}$$

In case of elliptical motion this is most conveniently integrated through an auxiliary variable $u(t)$,called the eccentric anomaly. Therefore,

$$r = a(1 - e\cos u) \tag{3.1.11}$$

Inserting eqn (3.2.11) into the orbital eqns (3.2.7),We get,

$$\begin{aligned} \cos\theta &= \frac{\cos u - e}{1 - e\cos u} \\ 1 + e\cos\theta &= \frac{1 - e^2}{1 - e\cos u} \end{aligned} \tag{3.1.12}$$

We can rewrite the above equations,using trigonometric identities as,

$$\theta = A_e(u), A_e \equiv 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right]$$

Now we can rewrite the time integral eqn (3.2.10) in terms of u,a and e.

$$t = \sqrt{\frac{\mu a^3}{m_1 m_2}}\int_0^u (1 - e\cos u)du \tag{3.1.13}$$

Integrating this from 0 to 2π we find Kepler's Law;

$$\omega_0 = \frac{2\pi}{T} \quad = \quad \sqrt{\frac{m_1 + m_2}{a^3}} \tag{3.1.14}$$

Integrating with out fixing u in the integral we arrive at the Kepler equation,

$$\omega_0 t = u - e\sin u \tag{3.1.15}$$

3.2 Gravitational Radiation From Binaries

Among the most interesting source of gravitational waves are system containing binary compact object(BH-BH,BH-NS,NS-NS). Now we want to calculate the power radiated as gravitational wave in Schwarzschild de-Sitter space. To do this we assume in the first approximation is that the motion is Keplerian.Therefore we take the parametric representation of the motion,

$$\begin{aligned} r &= a(1 - e\cos u), \qquad x = a(\cos u - e) \\ y &= a(1 - e^2)^{\frac{1}{2}}\sin u, \qquad \omega t = u - e\sin u \end{aligned} \tag{3.2.1}$$

Where a is semi-major axis,e is eccentricity and (x, y) are (x^1, x^2) respectively,as assumption we will take the orbital plane to be rectangular plane with the centre of mass at the origin. To compute the quadrupole moment,We found that,

$$D_{kl}(t) = \frac{1}{c^2}\int_V \tilde{T}^{00}(x', t)x'^k x'^l d^3x' \tag{3.2.2}$$

where,$\tilde{T}^{00}(x, t) = T^{00}(x, t) - T^{00}_{\Lambda}(x, t)$ we may use $T^{\mu\nu}_{vacuum}$ as simply $T^{\mu\nu}_{\Lambda}$. Using the reduced mass of the system of two bodies,$\mu = \frac{m_1 m_2}{m_1+m_2}$,We will get the following quadrupoles,

$$\begin{aligned} D_{11}(t) = \mu(x^1(t))^2 &= \mu a^2(\cos u - e)^2 \\ D_{22}(t) = \mu(x^2(t))^2 &= \mu a^2(1 - e^2)\sin^2 u \\ D_{12}(t) = D_{21}(t) = \mu x^1(t)x^2(t) &= \mu a^2(1 - e^2)^{\frac{1}{2}}\sin u(\cos u - e) \end{aligned} \tag{3.2.3}$$

It is better to take the following conversion on the expression for the time-average of function. So that its integration on du,

$$\langle f(t)\rangle \equiv \frac{1}{T}\int_0^T dt f(t) \tag{3.2.4}$$

From eqn (3.2.16) we have,

$$dt = \frac{T}{2\pi}(1 - e\cos u)du \tag{3.2.5}$$

Where $\omega = \frac{2\pi}{T}$ and if we let $f(t) = g(u(t))$,Then eqn (3.2.18) rewrite as,

$$\langle f(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} g(u)(1 - e\cos u)du \tag{3.2.6}$$

We know that,

$$\begin{aligned} D^k_k = D^1_1 + D^2_2 = \mu\Big((x^1)^2 + (x^2)^2\Big) &= \mu r^2 \\ D^k_k = \mu a^2(1 - e\cos u)^2 & \end{aligned} \tag{3.2.7}$$

From the relation of,

$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = \frac{2\pi}{T}(1 - e\cos u)^{-1}\frac{d}{du} \tag{3.2.8}$$

Therefore,

$$\dot{D}^k_k = 2\mu a^2 e(\frac{2\pi}{T})\sin u$$
$$\ddot{D}^k_k = 2\mu a^2 e(\frac{2\pi}{T})^2(1-e\cos u)^{-1}\cos u$$
$$\dddot{D}^k_k = -2\mu a^2 e(\frac{2\pi}{T})^3(1-e\cos u)^{-3}\sin u \quad (3.2.9)$$

If we squared the third time derivative of eqn (3.2.24),

$$(\dddot{D}^k_k)^2 = 4\mu^2 a^4 e^2(\frac{2\pi}{T})^6(1-e\cos u)^{-6}\sin^2 u \quad (3.2.10)$$

The formulation of Keplerian period and areal relation is,

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{G(m_1+m_2)}{a^3}$$

Then eqn (3.2.25) as,

$$(\dddot{D}^k_k)^2 = \frac{4G^3}{a^5}(m_1 m_2)^2(m_1+m_2)e^2\frac{sin^2 u}{(1-e\cos u)^6} \quad (3.2.11)$$

Using eqn (3.2.21) for averaging,Thus

$$\langle\dddot{D}^k_k\rangle^2 = \frac{4G^3}{\pi a^5}(m_1 m_2)^2(m_1+m_2)e^2\int_0^T\frac{sin^2 u}{(1-e\cos u)^6}du \quad (3.2.12)$$

Where we have used also the symmetry of the integrand over the range of integration.

$$\int_0^{2\pi}\frac{\sin^2 u}{(1-e\cos u)^6} = \frac{\pi}{8}\frac{4+e^2}{(1-e^2)^{\frac{7}{2}}} \quad (3.2.13)$$

Therefore eqn(3.2.27) become

$$\langle\dddot{D}^k_k\rangle^2 = \frac{1}{2}\frac{4G^3}{a^5}(m_1 m_2)^2(m_1+m_2)e^2\frac{4+e^2}{(1-e^2)^{\frac{7}{2}}} \quad (3.2.14)$$

Now turn on to find, $\dddot{D}^{kl}\dddot{D}_{kl}$ by using eqn(3.2.18) and plane approximation,

$$\dddot{D}^{kl}\dddot{D}_{kl} = (\dddot{D}^{11})^2 + 2(\dddot{D}^{12})^2 + (\dddot{D}^{22})^2 \quad (3.2.15)$$

So that from eqn(3.2.18) and (3.2.23) we know that

$$D_{11}(t) = \mu a^2(\cos u - e)^2$$
$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = \frac{2\pi}{T}(1 - e\cos u)^{-1}\frac{d}{du}$$

Then,

$$\dot{D}^{11} = -2\mu a^2(\frac{2\pi}{T})\frac{\sin u(\cos u - e)}{1 - e\cos u}$$
$$\ddot{D}^{11} = -2\mu a^2(\frac{2\pi}{T})^2(1 - e\cos u)^{-3}(2\cos^2 u - e\cos u - e\cos^3 u + e^2 + 1)$$
$$\dddot{D}^{11} = -2\mu a^2(\frac{2\pi}{T})^3\frac{\sin u}{(1 - e\cos u)^5}(e\cos^2 u + 2e^2\cos u - 4\cos u - 3e^2 + 4e) \quad (3.2.16)$$

The second term of eqn(3.2.18) and (3.2.23) also give us,

$$D_{22}(t) = \mu a^2(1 - e^2)\sin^2 u$$
$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = \frac{2\pi}{T}(1 - e\cos u)^{-1}\frac{d}{du}$$

Then,

$$\dot{D}^{22} = 2\mu a^2(\frac{2\pi}{T})\frac{1 - e^2}{1 - e\cos u}\sin u\cos u$$
$$\ddot{D}^{22} = 2\mu a^2(\frac{2\pi}{T})^2\frac{1 - e^2}{(1 - e\cos u)^3}(\cos^2 u - \sin^2 u - e\cos^3 u)$$
$$\dddot{D}^{22} = 2\mu a^2(\frac{2\pi}{T})^3\frac{1 - e^2}{(1 - e\cos u)^5}\sin u(3e - 4\cos u + e\cos^2 u) \quad (3.2.17)$$

Similarly for

$$D_{12}(t) = D_{21}(t) = \mu a^2(1 - e^2)^{\frac{1}{2}}\sin u(\cos u - e)$$
$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = \frac{2\pi}{T}(1 - e\cos u)^{-1}\frac{d}{du}$$

Then,

$$\dot{D}^{12} = \mu a^2(\frac{2\pi}{T})\frac{(1-e)^{\frac{1}{2}}}{1-e\cos u}(2e\cos^2 u - e\cos u - 1)$$
$$\ddot{D}^{12} = \mu a^2(\frac{2\pi}{T})^2\frac{(1-e)^{\frac{1}{2}}}{(1-e\cos u)^3}\sin u(2e\cos^2 u - 2\cos u + 2e)$$
$$\dddot{D}^{12} = \mu a^2(\frac{2\pi}{T})^3\frac{(1-e)^{\frac{1}{2}}}{(1-e\cos u)^5}(e\cos^3 u + e^2\cos^2 u + 3e\cos u - 3e^2 - 4\cos^2 u + 2) \tag{3.2.18}$$

Now turn to the averaging of $\dddot{D}^{kl}\dddot{D}_{kl}$ by using equations (3.2.31),(3.2.32) and (3.2.33),in (3.2.30). Thus,

$$\begin{aligned}\langle\dddot{D}^{kl}\dddot{D}_{kl}\rangle &= \langle(\dddot{D}^{11})^2\rangle + 2\langle(\dddot{D}^{12})^2\rangle + \langle(\dddot{D})^2\rangle \\ &= \frac{4}{\pi}\mu^2(\frac{2\pi}{T})^6\int_0^{2\pi}\frac{8(1-e^2)+e^2\sin^2 u}{(1-e\cos u)^6}\end{aligned} \tag{3.2.19}$$

Using the relations of $\omega^2 = \frac{2\pi}{T} = \frac{G(m_1+m_2)}{a^3}$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$.The above eqn becomes,

$$\langle\dddot{D}^{kl}\dddot{D}_{kl}\rangle = \frac{4}{\pi}(m_1m_2)^2(m_1+m_2)\frac{G^3}{a^5}\left[8(1-e^2)\int_0^{2\pi}\frac{du}{(1-ecosu)^6}+e^2\int_0^{2\pi}\frac{sin^2u}{(1-ecosu)^6}du\right] \tag{3.2.20}$$

Using the relation that,

$$\int_0^{2\pi}\frac{du}{(1-e\cos u)^6} = \frac{\pi}{8}\frac{(3e^4+24e^2+8)}{(1-e^2)^{\frac{9}{2}}}$$
$$\int_0^{2\pi}\frac{sin^2u}{(1-e\cos u)^6}du = \frac{\pi}{8}\frac{4+e^2}{(1-e^2)^{\frac{7}{2}}} \tag{3.2.21}$$

From the property of eqn (3.2.36) into eqn (3.2.35) ,We have the average,

$$\langle\dddot{D}^{kl}\dddot{D}_{kl}\rangle = \frac{1}{2}\frac{G^3}{a^5}\frac{(25e^4+196e^2+64)}{(1-e^2)^{\frac{7}{2}}}(m_1m_2)^2(m_1+m_2) \tag{3.2.22}$$

The above eqn is for the Newtonian part of the quadrupole moment $D_{kl}^{(w)}$ and may written as,

$$\langle\dddot{D}^{kl}_{(w)}\dddot{D}^{(w)}_{kl}\rangle = \frac{1}{2}\frac{G^3}{a^5}\frac{(25e^4+196e^2+64)}{(1-e^2)^{\frac{7}{2}}}(m_1m_2)^2(m_1+m_2) \tag{3.2.23}$$

We know that the quadrupole moment has two parts since it is in Schwarzschild de-Sitter. The one found above and the cosmological part,Now let move on the cosmological part,by defining,as

$$\begin{aligned}
D^{(\Lambda)}_{11}(t) = \frac{\Lambda}{8\pi G}r^3(x^1(t))^2 &= \frac{\Lambda}{8\pi G}r^3a^2(\cos u - e)^2 \\
D^{(\Lambda)}_{22}(t) = \frac{\Lambda}{8\pi G}r^3(x^2(t))^2 &= \frac{\Lambda}{8\pi G}r^3a^2(1-e^2)\sin^2 u \\
D^{(\Lambda)}_{12}(t) = D_{21}(t) = \frac{\Lambda}{8\pi G}r^3x^1(t)x^2(t) &= \frac{\Lambda}{8\pi G}r^3a^2(1-e^2)^{\frac{1}{2}}\sin u(\cos u - e)
\end{aligned} \tag{3.2.24}$$

Here we can also follow the conversion function that expressed for average-time from eqns(3.2.19) to (3.2.22).

$$\begin{aligned}
D^{k(\Lambda)}_k = D^1_1 + D^2_2 = \frac{\Lambda}{8\pi G}r^3\left((x^1)^2 + (x^2)^2\right) &= \frac{\Lambda}{8\pi G}r^5 \\
D^{k(\Lambda)}_k &= \frac{\Lambda}{8\pi G}r^3a^2(1 - e\cos u)^2
\end{aligned} \tag{3.2.25}$$

We have the relation of,

$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = \frac{2\pi}{T}(1 - e\cos u)^{-1}\frac{d}{du} \tag{3.2.26}$$

Now the time derivative of D^k_k from the first to third is read as

$$\begin{aligned}
\dot{D}^{k(\Lambda)}_k &= \frac{\Lambda}{4\pi G}r^3a^2e(\frac{2\pi}{T})\sin u \\
\ddot{D}^{k(\Lambda)}_k &= \frac{\Lambda}{4\pi G}r^3a^2e(\frac{2\pi}{T})^2(1 - e\cos u)^{-1}\cos u \\
\dddot{D}^{k(\Lambda)}_k &= -\frac{\Lambda}{4\pi G}r^3a^2e(\frac{2\pi}{T})^3(1 - e\cos u)^{-3}\sin u
\end{aligned} \tag{3.2.27}$$

If we squared the third time derivative of eqn (3.2.43),

$$(\dddot{D}^{k(\Lambda)}_k)^2 = \frac{\Lambda^2}{16\pi^2G^2}r^6a^4e^2(\frac{2\pi}{T})^6(1 - e\cos u)^{-6}\sin^2 u \tag{3.2.28}$$

From the Keplerian relation of period and areal relation is,then eqn (3.2.44) written as,

$$(\dddot{D}^{k(\Lambda)}_k)^2 = \frac{\Lambda^2 G}{16\pi^2a^5}r^6(m_1 + m_2)^3e^2\frac{\sin^2 u}{(1 - e\cos u)^6} \tag{3.2.29}$$

The time-averaging also become,

$$\langle \dddot{D}_k^{k(\Lambda)} \rangle^2 = \frac{\Lambda^2 G}{16\pi^2 a^5} r^6 (m_1 + m_2)^3 e^2 \int_0^T \frac{\sin^2 u}{(1 - e\cos u)^6} du \tag{3.2.30}$$

Using the relation of eqn(3.2.29) we have,

$$\langle \dddot{D}_k^{k(\Lambda)} \rangle^2 = \frac{\Lambda^2 G}{128\pi a^5} r^6 (m_1 + m_2)^3 e^2 \frac{4 + e^2}{(1 - e^2)^{\frac{7}{2}}} \tag{3.2.31}$$

Now we come to findout, $\dddot{D}^{kl}_{(\Lambda)} \dddot{D}^{(\Lambda)}_{kl}$ by using eqn(3.2.39),

$$\dddot{D}^{kl}_{(\Lambda)} \dddot{D}^{(\Lambda)}_{kl} = (\dddot{D}^{11}_{(\Lambda)})^2 + 2(\dddot{D}^{12}_{(\Lambda)})^2 + (\dddot{D}^{22}_{(\Lambda)})^2 \tag{3.2.32}$$

From eqn(3.2.39) and the relation of (3.2.23) one can read the first to third time derivative of $D^{11}_{(\Lambda)}$,$D^{12}_{(\Lambda)}$ and $D^{22}_{(\Lambda)}$ as follows,

$$\begin{aligned}
\dot{D}^{11}_{(\Lambda)} &= -\frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T}) \frac{\sin u(\cos u - e)}{1 - e\cos u} \\
\ddot{D}^{11}_{(\Lambda)} &= -\frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T})^2 (1 - e\cos u)^{-3} (2cos^2 u - e\cos u - e\cos^3 u + e^2 + 1) \\
\dddot{D}^{11}_{(\Lambda)} &= -\frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T})^3 \frac{\sin u}{(1 - e\cos u)^5} (e\cos^2 u + 2e^2 \cos u - 4\cos u - 3e^2 + 4e)
\end{aligned} \tag{3.2.33}$$

$$\begin{aligned}
\dot{D}^{22}_{(\Lambda)} &= \frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T}) \frac{1 - e^2}{1 - e\cos u} \sin u \cos u \\
\ddot{D}^{22}_{(\Lambda)} &= \frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T})^2 \frac{1 - e^2}{(1 - e\cos u)^3} (\cos^2 u - \sin^2 u - e\cos^3 u) \\
\dddot{D}^{22}_{(\Lambda)} &= \frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T})^3 \frac{1 - e^2}{(1 - e\cos u)^5} \sin u(3e - 4\cos u + e\cos^2 u)
\end{aligned} \tag{3.2.34}$$

Similarly for

$$\begin{aligned}
\dot{D}^{12}_{(\Lambda)} &= \frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T}) \frac{(1 - e)^{\frac{1}{2}}}{1 - e\cos u} (2e\cos^2 u - e\cos u - 1) \\
\ddot{D}^{12}_{(\Lambda)} &= \frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T})^2 \frac{(1 - e)^{\frac{1}{2}}}{(1 - e\cos u)^3} \sin u(2e\cos^2 u - 2\cos u + 2e) \\
\dddot{D}^{12}_{(\Lambda)} &= \frac{\Lambda}{4\pi G} r^3 a^2 (\frac{2\pi}{T})^3 \frac{(1 - e)^{\frac{1}{2}}}{(1 - e\cos u)^5} (e\cos^3 u + e^2 \cos^2 u + 3e\cos u - 3e^2 - 4\cos^2 u + 2)
\end{aligned} \tag{3.2.35}$$

The average of $\dddot{D}^{kl}_{(\Lambda)}\dddot{D}^{(\Lambda)}_{kl}$ by using equations (3.2.49),(3.2.50) and (3.2.51),in to (3.2.48),

$$\begin{aligned}\langle\dddot{D}^{kl}_{(\Lambda)}\dddot{D}^{(\Lambda)}_{kl}\rangle &= \langle(\dddot{D}^{11}_{(\Lambda)})^2\rangle + 2\langle(\dddot{D}^{12}_{(\Lambda)})^2\rangle + \langle(\dddot{D})^2_{(\Lambda)}\rangle \\ &= \frac{\Lambda^2}{16\pi^2 G^2} r^6 a^4 (\frac{2\pi}{T})^6 \int_0^{2\pi} \frac{8(1-e^2) + e^2\sin^2 u}{(1-e\cos u)^6}\end{aligned} \tag{3.2.36}$$

Using the relations of $\omega^2 = \frac{2\pi}{T} = \frac{G(m_1+m_2)}{a^3}$,We get,

$$\langle\dddot{D}^{kl}_{(\Lambda)}\dddot{D}^{(\Lambda)}_{kl}\rangle = \frac{\Lambda^2 G}{16\pi^2 a^5} r^6 (m_1+m_2)^3 \left[8(1-e^2)\int_0^{2\pi}\frac{du}{(1-e\cos u)^6} + e^2\int_0^{2\pi}\frac{\sin^2 u}{(1-e\cos u)^6}du\right] \tag{3.2.37}$$

Using the properties of eq<u>n</u> (3.2.36) into eq<u>n</u> (3.2.53),We have the time-average,

$$\langle\dddot{D}^{kl}_{(\Lambda)}\dddot{D}^{(\Lambda)}_{kl}\rangle = \frac{\Lambda^2}{128\pi}\frac{G}{a^5} r^6 \frac{(25e^4 + 196e^2 + 64)}{(1-e^2)^{\frac{7}{2}}}(m_1+m_2)^3 \tag{3.2.38}$$

We know that the rate of Energy radiated is given by,

$$-\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\langle\dddot{Q}_{kl}\dddot{Q}_{kl}\rangle \tag{3.2.39}$$

Where the negative sign indicated energy loss of the system. we can write as follow interms of D^{kl},

$$-\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\langle 3(\dddot{D}_{kl})^2 - (\dddot{D}^k_k)^2\rangle \tag{3.2.40}$$

The total energy radiated in the Schwarzschild de-Sitter bacground has two parts which is Newtonian as well as cosmological constant part. The total energy radiated is the sum of the two. So using eq<u>ns</u> (3.2.29) and (3.2.38) into (3.2.56) and eq<u>ns</u> (3.2.47) and (3.2.55) into (3.2.56). Thus

$$\begin{aligned}-\left(\frac{dE}{dt}\right)^{(w)} &= \frac{32}{5}\frac{G^5}{a^5c^5}\frac{(1+\frac{37}{24}e^2+\frac{37}{96}e^4)}{(1-e^2)^{\frac{7}{2}}}(m_1m_2)^2(m_1+m_2) \\ &= \frac{32}{5}\frac{G^5}{a^5c^5}(m_1m_2)^2(m_1+m_2)f(e)\end{aligned} \tag{3.2.41}$$

$$\begin{aligned}-\left(\frac{dE}{dt}\right)^{(\Lambda)} &= \frac{\Lambda^2}{10}\frac{G^2}{a^5c^5}r^6\frac{(1+\frac{37}{24}e^2+\frac{37}{96}e^4)}{(1-e^2)^{\frac{7}{2}}}(m_1+m_2)^3 \\ &= \frac{\Lambda^2}{10}\frac{G^2}{a^5c^5}r^6(m_1+m_2)^3 f(e)\end{aligned} \tag{3.2.42}$$

where$(\frac{dE}{dt})^{(w)}$ is Newtonian part and $(\frac{dE}{dt})^{(\Lambda)}$ as cosmological part,we can also define the function $f(e)$ as follows,

$$f(e) = \frac{(1+\frac{37}{24}e^2+\frac{37}{96}e^4)}{(1-e^2)^{\frac{7}{2}}} \tag{3.2.43}$$

$$-\frac{dE}{dt} = -\left(\left(\frac{dE}{dt}\right)^{(w)} + \left(\frac{dE}{dt}\right)^{(\Lambda)}\right) \tag{3.2.44}$$

$$-\frac{dE}{dt} = \frac{32}{5}\frac{G^5}{a^5c^5}(m_1m_2)^2(m_1+m_2)f(e) + \frac{\Lambda^2}{10}\frac{G^2}{a^5c^5}r^6(m_1+m_2)^3f(e) \tag{3.2.45}$$

We know that the gravitational wave radiation also carried the angular momentum from the system. We use the fact that the orbital parameters θ and r vary as,

$$\dot{\theta} = \frac{[Ma(1-e^2)]^{\frac{1}{2}}}{r^2}$$
$$\dot{r} = e\sin\theta\left[\frac{M}{a(1-e^2)}\right]^{\frac{1}{2}}$$

The classical approach to angular momentum as $J = \mu r^2\dot{\theta}$. The angular momentum is related to the quadrupole moment by,

$$\frac{dJ}{dt} = -\frac{2}{5}\epsilon^{ilk}(\ddot{D}_l^j\dddot{D}_{jk})$$
$$= -\frac{2}{5}[\ddot{D}_{12}(\dddot{D}_{22}-\dddot{D}_{11}) + \dddot{D}_{12}(\ddot{D}_{11}-\ddot{D}_{22})] \tag{3.2.46}$$

The rate of angular momentum radiated can be written interms of both Newtonian and cosmological as,

$$\left[\frac{dJ}{dt}\right]^{(w)} = -\frac{32}{5}\frac{\mu^2G^4}{c^5}\frac{M^{\frac{5}{2}}}{a^{\frac{7}{2}}}\frac{1+\frac{7}{8}e}{(1-e^2)^2}$$
$$\left[\frac{dJ}{dt}\right]^{(\Lambda)} = -\frac{\Lambda^2}{10}\frac{G}{c^5}\frac{M^{\frac{5}{2}}}{a^{\frac{7}{2}}}r^6\frac{1+\frac{7}{8}e}{(1-e^2)^2} \tag{3.2.47}$$

From eqn (3.2.8),we can determine how the orbital parameters of the binary system changes,that is $E = -\frac{m_1m_2}{2a}$,thus

$$\left\langle\frac{da}{dt}\right\rangle = \frac{2a^2}{m_1m_2}\left\langle\frac{dE}{dt}\right\rangle$$
$$= -\frac{64}{5}\frac{G^5}{a^3c^5}(m_1m_2)(m_1+m_2)f(e) - \frac{\Lambda^2}{5}\frac{G^2}{a^3c^5}r^6\frac{(m_1+m_2)^3}{m_1m_2}f(e) \tag{3.2.48}$$

$$\left\langle \frac{de}{dt} \right\rangle = \frac{(m_1+m_2)}{m_1 m_2 e}\left[\frac{a(1-e^2)}{m_1+m_2}\left\langle \frac{dE}{dt} \right\rangle - \frac{(1-e^2)^1 2}{a^{\frac{1}{2}}(m_1+m_2)^{\frac{1}{2}}}\left\langle \frac{dJ}{dt} \right\rangle\right] \tag{3.2.49}$$

The period of orbit $T \sim a^{\frac{3}{2}}$ decreases due to gravitational emission by amount of $\frac{\dot{T}}{T} = \frac{3\dot{a}}{2a} = \frac{\dot{E}}{2E}$,

$$\frac{1}{T}\frac{dT}{dt} = -\frac{96}{5}\frac{G^4}{a^4 c^5} m_1 m_2 (m_1+m_2) f(e) - \frac{3}{20}\frac{\Lambda^2 G}{a^4 c^5} r^6 \frac{m_1+m_2}{m_1 m_2} f(e) \tag{3.2.50}$$

Chapter 4

Result and Discussion

We are interested in solutions far from an isolated source,$r \gg R_{source}$ and whose size is much smaller than the emitted gravitational wave wavelength,$\lambda \gg R_{source}$. In the context of gravitational wave astronomy the limitation of being far from the source is not much of a hindrance. However,the limitation that the source is small does imply that it must not contain relativistic motions. This follows directly from $v_{source} \approx 2\pi R_{source}/\lambda$. The perturbation has part of Newtonian and cosmological constant,Λ that approximated to the first order. We defined this perturbation in Schwarzschild de-Sitter with the quadrupole formalism of binary system as the second time derivatives. $h_{jk} = \frac{2G}{c^4 r}[\ddot{D}_{jk}(t-r)]$. The quadrupole has the Newtonian and cosmological parts. The most famous example of the effects of gravitational radiation on an orbiting system is the HulseTaylor Binary Pulsar, PSR B1913+16. In this system, two neutron stars orbit in a close eccentric orbit.

Back to the gravitational field perturbation from eq$\underline{n}$ (2.4.35) has non-vanish components

$$
\begin{aligned}
h_{11}^{TT} = -h_{22}^{TT} = \frac{G}{c^4 r}[\ddot{D}_{11}(t-r) - \ddot{D}_{22}(t-r)]^{TT} \\
h_{12}^{TT} = -h_{21}^{TT} = \frac{2G}{c^4 r}[\ddot{D}_{12}(t-r)]^{TT}
\end{aligned}
\tag{4.0.1}
$$

The quadrupole moment,D_{jk} has the components of Newtonian and cosmological. Thus using eq$\underline{ns}$ (3.2.31), (3.2.32), (3.2.33), (3.2.48)(3.2.49) and (3.2.50),

$$
\begin{aligned}
\ddot{D}_{11}(t-r) &= -a^2(\frac{2\pi}{T})^2\left(2\mu - \frac{\Lambda}{3G}r^3\right)\beta \\
\ddot{D}_{22}(t-r) &= a^2(\frac{2\pi}{T})^2\left(2\mu - \frac{\Lambda}{3G}r^3\right)\alpha \\
\ddot{D}_{12}(t-r) &= a^2(\frac{2\pi}{T})^2\left(\mu - \frac{\Lambda}{6G}r^3\right)\gamma
\end{aligned}
\tag{4.0.2}
$$

Where,

$$\beta = \frac{2\cos^2 u - e\cos u - e\cos^3 u + e^2 + 1}{(1-e\cos u)^3}$$
$$\alpha = \frac{(1-e)^2(\cos^2 u - \sin^2 u - e\cos^3 u)}{(1-e\cos u)^3}$$
$$\gamma = \frac{(1-e)^{\frac{1}{2}}\sin u(2e\cos^2 u - 2\cos u + 2e)}{(1-e\cos u)^3} \tag{4.0.3}$$

Therefore our perturbation components become,

$$h_{11} = \frac{2G}{c^4 r}\left[a^2(\frac{2\pi}{T})^2\left(2\mu - \frac{\Lambda}{3G}r^3\right)\right](\beta - \alpha)$$
$$h_{12} = \frac{2G}{c^4 r}\left[a^2(\frac{2\pi}{T})^2\left(\mu - \frac{\Lambda}{6G}r^3\right)\right]\gamma \tag{4.0.4}$$

We are at turning point to test our derivation with the Observation. To do these we simplify our approximation to binary system on circular orbit ($e = 0$),we get

$$h \sim \frac{2G}{c^4 r}\left[a^2(\frac{2\pi}{T})^2\left(2\mu - \frac{\Lambda}{3G}r^3\right)\right]2$$
$$\sim \frac{G}{c^4 r}\left[a^2(2\omega_o)^2\left(2\mu - \frac{\Lambda}{3G}r^3\right)\right] \tag{4.0.5}$$

Where,ω_o is the orbital speed that related to orbital frequency (ν_o) by $\omega_o = 2\pi\nu_o$. From the Keplerian we have,$\omega_o^2 = (\frac{2\pi}{T})^2 = \frac{G(m_1+m_2)}{a^3}$. Clearly,the gravitational wave frequency is twice of orbital frequency in this case. $\Omega_{GW} = 2\omega_o$,Now

$$h \sim \frac{8G^2 M\mu}{ac^4 r} - \frac{4}{3}\frac{\Lambda}{ac^4}Gr^2 \tag{4.0.6}$$

Where,$M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$. From observation,the data of binary pulsar PSR1931+16(Taylor-Weisberg,1982)

$$m_1 \sim m_2 \sim 1.4M_{\odot}$$
$$T = 7hr45m7s$$
$$\nu_o = 3.58\times 10^{-5}Hz$$
$$a = 0.19\times 10^{12}cm \simeq 2R_{\odot}$$
$$r = 5Kpc, 1pc = 3.08\times 10^{18}cm$$
$$\Lambda = 10^{-49}cm^{-1}to10^{-56}cm^{-1}$$

Where m_1,m_2 are masses of pulsar,$M_{\odot}$ solar mass,$R_{\odot}$ solar radius,a daimeter of circular orbit,ν_o orbital frequency,T period and Λ is cosmological constant. The perturbation from this data result,

$$h \sim \frac{8G^2M\mu}{ac^4r} - \frac{4}{3}\frac{\Lambda}{a}Gr^3$$
$$= 1.19654 \times 10^{-22}$$

We know that the perturbation has newtonian and cosmological parts,

$$h^{(w)} \sim 1.19654 \times 10^{-22}$$
$$h^{(\Lambda)} \sim 1.30078 \times 10^{-72}$$

The frequency of orbit,$\nu_o = 3.70465 \times 10^{-5} Hz$,since the gravitational wave frequency is twice to orbital,we have $\nu_{GW} = 7.41 \times 10^{-5} Hz$. Therefore the potential perturbation h,and gravitational wave frequency are approximately comparable to observation.If the orbit is elliptic,waves are emitted at frequency multiple of the orbital frequency and the number of equal spaced spectral lines increases with ellipticity.

Finally,eqn(3.2.61) is the rate of total energy radiated in Schwarzschild de-Sitter background. From the data what we have,the gravitational power radiated from the pulsar PSR1931+16 for circular orbit($e = 0$) is

$$P = -\frac{dE}{dt} = \frac{32}{5}\frac{G^5}{a^5c^5}(m_1m_2)^2(m_1+m_2) + \frac{\Lambda^2}{10}\frac{G^2}{a^5c^5}r^6(m_1+m_2)^3$$
$$P \approx 7.0621 \times 10^{30} erg/s \qquad (4.0.7)$$

,where we write this explicitly,

$$P^{(w)} \approx 7.0621 \times 10^{30} erg/s$$
$$P^{(\Lambda)} \approx 1.456 \times 10^{-2} erg/s$$

From observational data we may compute the rate of angular momentum radiation as,

$$\frac{dJ}{dt} = -\frac{32}{5}\frac{\mu^2G^4}{c^5}\frac{M^{\frac{5}{2}}}{a^{\frac{7}{2}}}\frac{1+\frac{7}{8}e}{(1-e^2)^2} - \frac{\Lambda^2}{10}\frac{G}{c^5}\frac{M^{\frac{5}{2}}}{a^{\frac{7}{2}}}r^6\frac{1+\frac{7}{8}e}{(1-e^2)^2}$$
$$\frac{dJ}{dt} \approx -7.84 \times 10^{30} \qquad (4.0.8)$$

Separately the cosmological constant part can be,

$$\left[\frac{dJ}{dt}\right]^{(\Lambda)} = -\frac{\Lambda^2}{10}\frac{G}{c^5}\frac{M^{\frac{5}{2}}}{a^{\frac{7}{2}}}r^6\frac{1+\frac{7}{8}e}{(1-e^2)^2}$$
$$\left[\frac{dJ}{dt}\right]^{(\Lambda)} \approx -2.1259 \times 10^{-128} \qquad (4.0.9)$$

From eqn (3.2.8),we can determine how the orbital parameters of the binary system changes,that is $E = -\frac{m_1 m_2}{2a}$,thus

$$\left\langle \frac{da}{dt} \right\rangle = \frac{2a^2}{m_1 m_2} \left\langle \frac{dE}{dt} \right\rangle$$
$$= -\frac{64}{5} \frac{G^5}{a^3 c^5} (m_1 m_2)(m_1 + m_2) f(e) - \frac{\Lambda^2}{5} \frac{G^2}{a^3 c^5} r^6 \frac{(m_1 + m_2)^3}{m_1 m_2} f(e) \qquad (4.0.10)$$

$$\left\langle \frac{de}{dt} \right\rangle = \frac{(m_1 + m_2)}{m_1 m_2 e} \left[\frac{a(1 - e^2)}{m_1 + m_2} \left\langle \frac{dE}{dt} \right\rangle - \frac{(1 - e^2)^1 2}{a^{\frac{1}{2}} (m_1 + m_2)^{\frac{1}{2}}} \left\langle \frac{dJ}{dt} \right\rangle \right] \qquad (4.0.11)$$

From observational data the orbital parameters decay,

$$\frac{da}{dt} \approx -6.5757 \times 10^{-14} cm/s$$
$$\frac{de}{dt} \approx -6.4029 \times 10^{-25} cm/s \qquad (4.0.12)$$

The cosmological part read as,

$$\left[\frac{da}{dt} \right]^{(\Lambda)} \approx -1.1903 \times 10^{-179} cm/s$$
$$\left[\frac{de}{dt} \right]^{(\Lambda)} \approx -2.4037 \times 10^{-183} cm/s \qquad (4.0.13)$$

The period of orbit $T \sim a^{\frac{3}{2}}$ decreases due to gravitational emission by amount of $\frac{\dot{T}}{T} = \frac{3\dot{a}}{2a} = \frac{\dot{E}}{2E}$,

$$\frac{1}{T} \frac{dT}{dt} = -\frac{96}{5} \frac{G^4}{a^4 c^5} m_1 m_2 (m_1 + m_2) f(e) - \frac{3}{20} \frac{\Lambda^2 G}{a^4 c^5} r^6 \frac{m_1 + m_2}{m_1 m_2} f(e) \qquad (4.0.14)$$

From observational data the rate of period decay,and part of cosmological constant can be,

$$\frac{dT}{dt} \approx -2.171 \times 10^{-13} s^{-1}$$
$$\left[\frac{dT}{dt} \right]^{(\Lambda)} \approx -6.341 \times 10^{-113} s^{-1} \qquad (4.0.15)$$

Chapter 5

Summary and Conclusion

General theory of relativity is the theory of gravitation and geometry of spacetime. It generalizes the spacial theory of relativity and Newton's law of universal gravitation. The matter and geometry of spacetime are related by the Einstein field equations($G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}$),where $G_{\mu\nu}$ is Einstein's field tensor that tales geometry of spacetime and $T_{\mu\nu}$ is energy-momentum tensor that is matter. Techniques equivalent this energy-momentum tensor is deduce from perfect fluid that is important to stellar structure and cosmology($T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\eta_{\mu\nu}$). Generally, the spacetime geometry and gravitation are described by tensors specially second rank $(0, 2)$ tensors like Metric tensor,Reiman curvature tensor,Ricci tensor,Ricci scalar,Einstein field tensor and energy-momentum tensor in addition to Affine connections. The existence of gravitational wave predicted through general relativity,which have since been measured indirectly. It implies that the solutions to linearized Einstein equation suggests the existence of gravitational wave since these waves take away energy and angular momentum from the system. The weak Gravitational field radiative solution of Einstein field equations,which describe waves carrying not enough energy and momentum that affect their own propagation was studied with linearize Einstein equation in the Schwarzschild de-Sitter background. The approximation were done by keeping the cosmological constant to the first order and the metric also linearized to,$g_{\mu\nu} = \eta_{\mu\nu} + h^{(w)}_{\mu\nu} + h^{(\Lambda)}_{\mu\nu}$ .In vacuum,the linearize field equation has a form of($\Box^2 h_{\mu\nu} = 2\Lambda\eta_{\mu\nu}$). The gravitational plane wave solution of this inhomogeneous linearize field equation was solved by the technique of Green's function. It carried energy-momentum. The quadrupole formalism approximation is used for the system,in which energy density $\tilde{T}^{00}(x, t)$ is dominated by the rest-mass density of the system. Since the Energy and momentum are conserved,we are freely approximate $\tilde{T}^{00}(x, t)$ to the rest-mass density. In Schwarzschild de-Sitter,we have the rest-mass density and vacuum energy density related to (Λ). The potential perturbation($h_{\mu\nu}$) and the rate energy radiated($\frac{dE}{dt}$) from the system are related to quadrupole moment tensor($D_{\mu\nu}$) as the second and third time derivatives respectively.

The gravitational wave radiated from binary system in Schwarzschild de-Sitter background was approximated to Keplerian and quadrupole. We observed that from observational data for binary Pulsar PSR 1931+16 the perturbation,$h \sim 1.19654 \times 10^{-22}$,that fit. The perturbation has the cosmological part,$h^{\Lambda} \sim 1.30018 \times 10^{-73}$ that is too small. Since the binary emitted gravitational wave energy,the rate of energy(power) radiated,$\frac{dE}{dt} \sim 7.0621 \times 10^{30} erg/s$ that has cosmological part,$[\frac{dE}{dt}]^{\Lambda} \sim 1.456 \times 10^{-2}$,an also the rate of orbital period decay,$\frac{dT}{dt} \sim 2.17 \times 10^{-13}$ with cosmological part,$[\frac{dT}{dt}]^{\Lambda} \sim 6.341 \times 106{-}113$. We concluded that the cosmological constant contribute insignificant amount. The rate of angular momentum lost from the system,$\frac{dJ}{dt} \sim 7.84{\times}10^{30} erg/s$ that has cosmological part,$[\frac{dJ}{dt}]^{\Lambda} \sim 2.171{\times}10^{-128} erg/s$,and the orbital parameters also decay as time rate,$\frac{da}{dt} \sim 6.5757 \times 10^{-14}$ and $\frac{de}{dt} \sim 2.1259 \times 10^{-25}$ with their respective cosmological part,$[\frac{da}{dt}]^{\Lambda} \sim 1.1903 \times 10^{-179}$ and $[\frac{de}{dt}]^{\Lambda} \sim 2.4037 \times 10^{-183}$.

Generally,Gravitational wave radiation has effect on orbital parameters. We saw that where two masses are moving in an elliptic orbit with eccentricity e and semi-major axis a lose energy and angular momentum due to radiations wave leading to a change in orbital period.

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