MASS-RADIUS RELATION OF COMPACT OBJECTS

By

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The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a thesis entitled "**Mass-Radius Relation of Compact Objects**" by **Seman Abaraya** in partial fulfillment of the requirements for the degree of **Master of Science in Physics(Astrophysics)**.

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Abstract

Since the birth of General Relativity, (GR) the presence of compact objects (COs) as the stellar evolutionary end points were remained with great debates for several decades. However, in the course of new observational techniques and capabilities our current understanding is no more at debate at least in the presence of the COs. Today, the compact objects are among the most astrophysical objects used to study other systems. Thus, there active research interests in various respects of these objects. So, in this thesis we addressed some issues related to the compact objects mass limit in relation to their masses. Methodologically, we employed Einstein field equations to derive their equation of state. With the assumption of high densities and low temperatures characteristics of the compacts polytropic the derived equation of state is reduced to polytropic kind. Working out the polytropic equations we have obtained similar physical implications, in agreement to earlier works by others. But, we also noticed from our numerical results where the latest version Mathematica11 is used, slight differences in accuracy.

Keywords: COs, CO-mass-limit, GR, EOS, Mass-radius-relationship.

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General Introduction

I. Background of The Study

In Astronomy, the term " Compact star" (or " Compact Object") refers collectively to White Dwarfs, Neutron stars, and Black holes. It would grow to include exotic stars if such hypothetical dense bodies are confirmed. Most compact stars are the end points of stellar evolution, and thus often referred to as stellar remnants, the form of the remnant depending primarily on the mass of the star when it formed. All of these objects have a mass relative to their radius, giving them very high density. The term compact star is often used when the exact nature of the star is not known, but evidence suggests that it is very massive and has a small radius. A compact star that is not a black hole may be called a degenerate star.

A compact star can become a white dwarf, a neutron star, or a black hole depending upon its initial mass.The gravitational collapse of compact stars like white dwarf and neutron stars is halted by the degeneracy pressure of fermions - a quantum mechanical phenomenon.The theoretical analysis of the relation between the nature of a compact star and its mass was done by S. Chandrasekhar. This led him to predict a limiting mass for white dwarf stars.

The nature of the remains of a star after death depends on its mass. The mass of a star plays a crucial role in its evolution and determines its luminosity. Similarly, depending upon its mass, a dying star can turn into any one of the three kinds of compact stars, namely a white dwarf, neutron star or black hole.

Compact stars are simply the end products of ordinary stars and are characterised by smaller sizes and higher densities[2].

The final fate of a star depends mainly on its mass. Stars with a mass less than $4-8M_{\odot}$ finish the nuclear burning in their center when the nuclei in the core are carbon or oxygen (the lightest stars perhaps finish with helium and more heavy ones with silicon). Most of their outer mass is ejected in strong stellar wind, leading to the formation of a so-called planetary nebula; the cause of this instability is not well understood, however. The remnant of the star becomes a white dwarf of typically $0.5-1.4M_{\odot}$. The neutrons are degenerate at nuclear matter densities which induces a strong pressure. The increase in pressure stops the collapse of the iron core; it expands slightly (like an elastic ball which has been squeezed together), and sends a shock wave through the outer part of the star. The shock wave and the copious emission of neutrinos from the collapsed core apparently provide enough energy (originating from the released gravitational binding energy) to eject the outer mass of the star ($> 6M_{\odot}$) in a supernova explosion. In the center of the star there remains a compact object with $M \sim 1.4M_{\odot}$ and $R \sim 10km$. This compact remnant is called a neutron star. If the original star was heavier than about $20M_{\odot}$, its inner core has too much mass for the collapse to be stopped by the degeneracy pressure of the neutrons. In this case the collapse continues until a black hole is formed [2].

II.Literature Review

Compact stars are the final stage of the evolution of an ordinary star, and they also constitute a laboratory of tests in general relativity[4][3]. The compact stars are formed of matter in high densities, in the case of the white dwarfs they are constituted of degenerate electron gas or in the case of neutron star that are formed by neutron degenerate gas[4][3]. The mass of a white dwarf is typically 0.5 to $1M_{\odot}$($M_{\odot}$ represents the solar mass) and they have a radii of the order of 10000 km, however the mass of the neutron stars is about 1.4 to $3M_{\odot}$ and the radii is typically 10 km[4][3].

Recently new massive White dwarfs was discovered[7] and also Supernovae type Ia [7] that suggest the possibility that highly massive White dwarf with masses exceed the Chandrasekhar limit in which have many studies about them[7].

The first to propose and infer the General relativity effects for White dwarf were Kaplan in 1949[7], these effects were calculated by S. Chandrasekhar and R. F.Topper in 1964[7]. They showed that General relativity breaks the dynamical instability for densities lower than for Newtonian calculations. They also showed that the maximum stable mass due to radial oscillations reduces in a small quantity if considering General relativity effects.

White dwarfs are stars of about one solar mass with a characteristic radius of 5000 km, corresponding to a mean density of $10^6 g/cm^3$[1].

Actual models of white dwarf stars, taking into account the special relativistic effects in the degenerate electron equation of state were then constructed in 1930 by Chandrasekhar[1]. He made the fundamental discovery of a maximum mass of $1.4M_{\odot}$ for white dwarfs the exact value somewhat depends on the chemical composition.

There is a unique relationship between the EoS of cold dense matter and the sequence of compact star configurations in the mass-radius (M-R) diagram provided by the Tolman-Oppenheimer- Volkof equations. It can be used to quantify the likelihood of EoS models by Bayesian analysis using a selection of mass and radius measurements as priors[6]. Neutron stars are relativistic compact objects formed by the collapsing cores of massive stars at the end of their evolution[8]. The energy released by the collapsing core launches a shock that ejects the outer layers of the progenitor star in a so-called supernova explosion[8].

The masses of neutron stars are in the range $M \simeq 1-3M_{\bigodot}$[8]. Accurately measured masses in binary pulsars are clustered near$M \simeq 1.4M_{\bigodot}$[8]. The highest measured masses are $M_{max} \simeq 2M_{\bigodot}$[8]. There is a firm theoretical upper limit to the mass of neutron stars $M_{max} \simeq 3.2M_{\bigodot}$[8]. Lowered this so-called Oppenheimer-Volkoff limit slightly[8]. Statistical analysis suggests [8] the existence of neutron stars up to $M \simeq 2.5M_{\bigodot}$ without a sharp cut-off, implying that this value is set by astrophysical processes rather than the theoretical upper limit. The radii of neutron stars are in the range of $R \simeq 9-15km$[8].

III.Statement of the problem

Since the birth of General Theory of Relativity the presence of compact objects as the stellar evolutionary end points were remained with great debates for several decades. However, in the course of new observational techniques capabilities our current understanding is no more at debate at least in the presence of the compact objects. There are a lot of noble works that boost General Theory of Relativity for the observational tests of the theory. However, still the physics of the compact objects is unfinished both theoretically and observationally. The limiting masses of the types of the compacts is not fully addressed. Though there is known that the compact objects mass limit is also related to their progenitor stars, the accurate correlation is still at debates.

Research questions

- What is the fate of the end point of stellar evolution?
- What parameters do determine the type of the end product of stellar evolution?
- How does the progenitor stellar object mass enter in the analysis of formation of its

end?

- In what way the radius of a progenitor stellar end product is incorporated to characterize the type of the end?

IV.Objectives

a. General objective

- To study the Mass-Radius Relation of Compact Objects

b.Specific objectives

- To describe the fate of the end point of stellar evolution.
- To derive the parameters that determine the type of the end product of stellar evolution.
- To derive the progenitor stellar object mass enter in the analysis of formation of its end.
- To describe the way of the radius of a progenitor stellar end product is incorporated to characterize the type of the end.

V. Methodology

General Theory of Relativity is used to derive the appropriate TOV-equations. With simplifying boundary conditions, the resulting TOV-equations be used to develop equation of state. Since, compact objects are characterized with high densities and low temperatures, we further impose conditions to derive polytropic kind of equations. Then, the resulting equations are being used to analyze our work. Moreover, the latest version Mathematica11 is used for numerical integration of the differential equations to derive the mass-radius relationship of the resulting compact objects. Finally, our numerical result is being compared to data produced by others theoretically and observationally.

Chapter 1

Astrophysical compact objects

Astrophysical compact objects represent the final stages of stellar evolution: white dwarfs, neutron stars, and black holes. They differ from normal stars in two basic ways.[9]

First, since they do not burn nuclear fuel, they cannot support themselves against gravitational collapse by generating thermal pressure. Instead, either they are prevented from collapsing by the degeneracy pressure (white dwarfs and neutron stars) or they are completely collapsed (Black Holes). With the exception of the spontaneously radiating mini black holes with masses M less than $10^{15}g$ and radii smaller than a fermi, all three compact objects are essentially static over the lifetime of the Universe.

The second characteristic distinguishing compact objects from normal stars is their exceedingly small size. Relative to normal stars of comparable mass, compact objects have much smaller radii and hence, much stronger surface gravitational fields.

Compact stars -white and neutron stars -are the ashes of luminous stars. A black hole is the fate of the most massive stars - an inaccessible region of space-time in to which the star falls at the end of its luminous phase. White Dwarf stars are the size of the Earth but have mass comparable to that of the Sun. Neutron stars has density comparable to that of nuclei. Most stars will eventually come to a point in their evolution when the outward radiation

Table 1.1: Distinguishing Traits of Compact Objects

Objects	$Mass^a$ (M)	$Radius^b$ (R)	Mean density(gcm^{-3})	Surface potential(GM/Rc^2)
Sun	$M_\odot$	$R_\odot$	1	10^{-6}
White Dwarf	$\leq M_\odot$	$\sim 10^{-2} R_\odot$	$\leq 10^7$	$\sim 10^{-4}$
Neutron Star	$\sim 1 - 3M_\odot$	$\sim 10^{-5} R_\odot$	$\leq 10^{15}$	$\sim 10^{-1}$
Black Hole	Arbitrary	$2GM/C^2$	$\sim M/R^3$	~ 1

$$M_\odot = 1.989 \times 10^{33} g$$

$$R_\odot = 6.9599 \times 10^{10} cm$$

pressure from the nuclear fusions in its interior can no longer resist the ever-present gravitational forces. When this happens, the star collapses under its own weight and undergoes the process of stellar death. For most stars, this will result in the formation of a very dense and compact stellar remnant, also known as a compact star. Compact stars have no internal energy production, but will- with the exception of black holes- usually radiate for millions of years with excess heat left from the collapse itself.[1]

1.1 Classes of Compact Objects

The study of compact stars begins with the discovery of white dwarfs and the successful description of their properties by the Fermi-Dirac statistics, assuming that they are held up against gravitational collapse by the degeneracy pressure of the electrons, an idea first proposed by Fowler in 1926[1]. A maximum mass for white dwarfs was found to exist in 1930 by the seminal work of Chandrasekhar due to relativistic effects[1]. In 1932 Chadwick discovered the neutron. Immediately, the ideas formulated by Fowler for the electrons were generalized to neutrons. The existence of a new class of compact stars, with a large core of degenerate neutrons, was predicted-the neutron stars . The first neutron star model calculations were achieved by Oppenheimer and Volkoff[1] and Tolman[1] in 1939 ,describing the matter in such a star as an ideal degenerate neutron gas. Their calculations also showed the existence of a maximum mass, like in the case of white dwarfs, above which the star is not stable and collapses into a black hole. They found a maximum stable mass of $0.75M_{\odot}$[1].

1.1.1 White Dwarf

White dwarfs are stars of about one solar mass with a characteristic radius of 5000 km, corresponding to a mean density of $10^6 g/cm^{-3}$[1][9]. They are no longer burning nuclear fuel, but are steadily cooling away their internal heat.

There are several ways to observe white dwarf stars. The first white dwarf ever to be discovered was found because it is a companion star to Sirius, a bright star near the constellation Canis Major. In 1844, astronomer Friedrich Bessel noticed that Sirius had a slight back and forth motion, as if it were being orbited by an unseen object. In 1863, this mysterious object was finally resolved by optician Alvan Clark and it was found to be a white dwarf. This pair is now referred to as Sirius A and B, B being the white dwarf. The orbital period of this system is about 50 years. Since white dwarfs are very small and thus very hard to detect, binary systems are a helpful way to locate them. As with the Sirius system, if a star seems to have some sort of unexplained motion, we may find that the single star is really a multiple system. Upon close inspection we may find that it has a white dwarf companion.

The black-body spectrum of Sirius B peaks at 110 nm, corresponding to a temperature of 27,000 K [1]. From the known absolute magnitude (the distance of the system is 8.6 lightyears), the radius is calculated as 4200 km, smaller than the Earth, but as massive as the Sun.

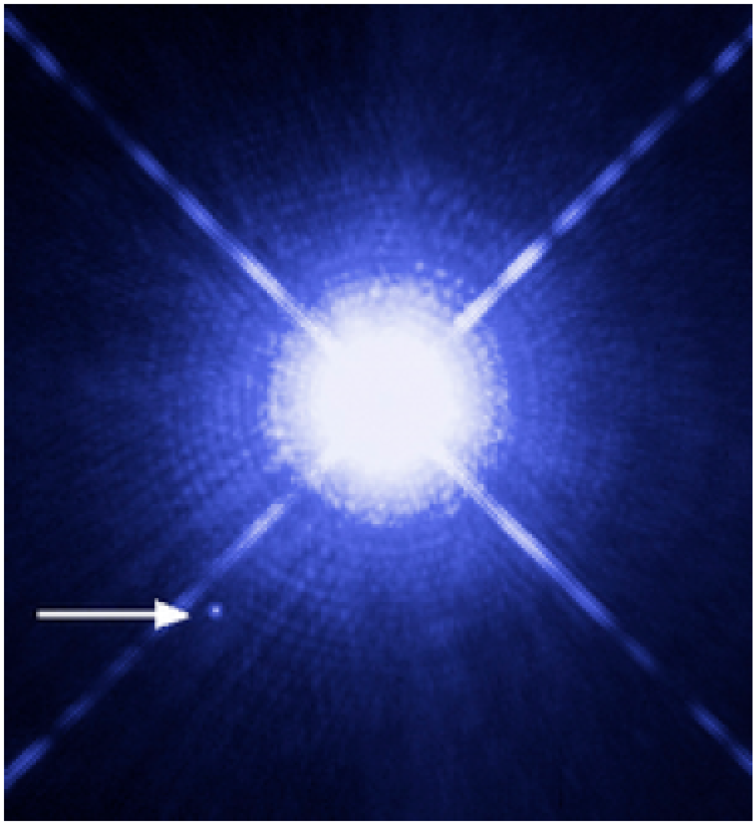

Figure 1.1: Image of Sirius A and Sirius B taken by the Hubble Space Telescope. Sirius B, which is a white dwarf, can be seen as a faint point of light to the lower left of the much brighter Sirius A

In 1926, only three white dwarfs were firmly detected. In that year, Dirac formulated the FermiDirac statistics, which was used by Fowler in the same year, in a pioneering paper on compact stars to explain the puzzling nature of white dwarf stars. He identified the pressure holding up the stars from gravitational collapse with the electron degeneracy pressure[1].

Actual models of white dwarf stars, taking into account the special relativistic effects in the degenerate electron equation of state were then constructed in 1930 by Chandrasekhar. He made the fundamental discovery of a maximum mass of 1.4 $M_\odot$ for white dwarfs the exact value somewhat depends on the chemical composition[1]. This maximum mass is called the Chandrasekhar limit in honor of its discoverer.

The role of general relativity in modifying the mass-radius relation for massive white dwarfs above about lM_O was first discussed by Kaplan (1949). He concluded that general relativity probably induces a dynamical instability when the radius becomes smaller than 1.1 $\times lo^3$ km. The general relativistic instability for white dwarfs was discovered independently by Chandrasekhar in 1964[9].

1.1.2 Neutron Stars

In 1932 Chadwick discovered the neutron. Immediately, the ideas formulated by Fowler for the electrons were generalized to neutrons. The existence of a new class of compact stars, with a large core of degenerate neutrons, was predicted the neutron stars (NS)[1].

In 1934 Baade and Zwicky proposed the idea of neutron stars, pointing out that they would be at very high density and small radius, and would be much more gravitationally bound than ordinary stars. They also made the remarkably prescient suggestion that neutron stars would be formed in supernova explosions[9].

The first NS model calculations were achieved by Oppenheimer and Volkoff and Tolman in 1939,[1] describing the matter in such a star as an ideal degenerate neutron gas. Their calculations also showed the existence of a maximum mass, like in the case of white dwarfs, above which the star is not stable and collapses into a black hole. They found a maximum stable mass of $0.75M_{\odot}$[1]. Only nearly 30 years later, in 1967, was the first neutron The prediction of the existence of neutron stars as a possible endpoint of stellar evolution was independent of observations. Following the discovery of the neutron by Chadwick, it was realized by many people that at very high densities electrons would react with protons to form neutrons via inverse beta decay. Neutron stars had been found at the end of 1960s as radio pulsars and in the beginning of 1970s as X-ray stars. A firm upper limit for the mass of neutron stars was then seen as evidence for the existence of even more exotic objects - black holes. At the time of the discovery of Cyg X-1 by Uhuru (1970) the value of this upper limit was, however, the subject of great debate[1].

Structure of Neutron stars

The cross-section of a neutron star can roughly be divided into four distinct regions (see Fig. 1.2):

The atmosphere which is only a few cm thick.The outer crust which consists of a lattice of atomic nuclei and Fermi liquid of relativistic degenerate electrons. This is essentially white dwarf matter.The outer crust envelops the inner crust, which extends from the neutron drip density to a transition density $\rho_{tr} \simeq 1.7 \times 10^{14} g/cm^3$. Beyond the transition density one enters the core, where all atomic nuclei have been dissolved into their constituents, neutrons and protons. Due to the high Fermi pressure, the core might also contain hyperons, more massive baryon resonances, and possibly a gas of free up, down and strange quarks. Finally, π and K-meson condensates may be found there too.

The equation of state for the outer and inner crust is well-known and described by the model of BPS [1] and Negele and Vautherin [1].

Today, neutron stars come in various flavors depending on the composition of the core. In this respect, we speak now of traditional neutron stars **(or hadronic stars)**,where the core mainly consists of neutrons, protons and electrons. At high densities, however, also heavier baryons are excited, the neutron star now becomes a **hyperon star**. Since these baryons

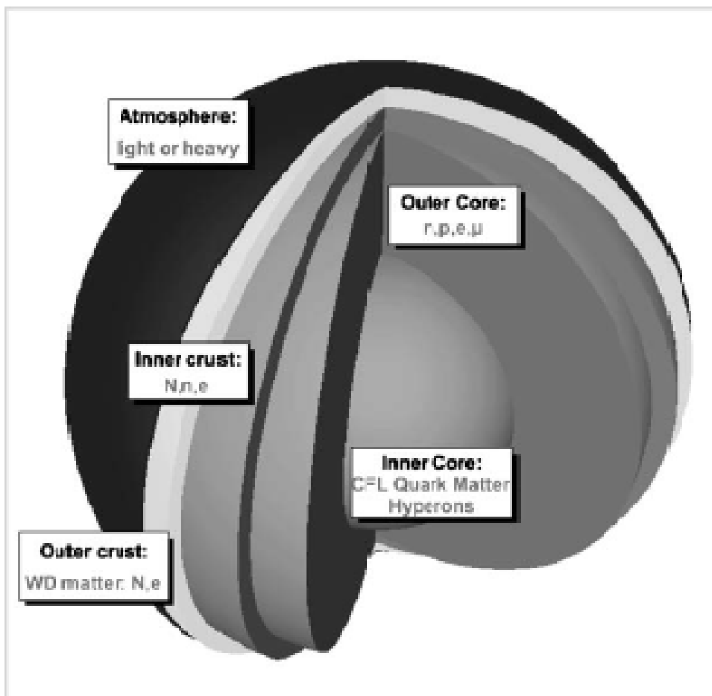

Figure 1.2: Cross-section through the interior of a neutron star. The neutron star is surrounded by a thin atmosphere and an outer crust consisting of heavy nuclei and electrons. The inner crust consists of nuclei, neutrons and electrons, which at nuclear density make a transition to a neutron fluid. The composition of the central core is still unclear, but certainly consists in the outer part only of neutrons, protons, electrons and muons
.

are so densely packed, a quark bag could be formed, and quarks are probably in a color-superconducting state. Finally, Bose condensates of pions and K mesons might occur. All these different internal structures lead to different mass-radii relations. For given mass, the traditional neutron star has the biggest radius, while neutron stars including quark cores are found to be more compact. Strange stars have the smallest radii.

1.1.3 Black Hole

Black holes were first proposed by John Michell (1784) and later by Laplace (1795) , based on the idea of an object so dense that its escape speed is greater than the speed of light. The radius at which this first occurs for an object of mass is given by:

$$C^2 = 2\frac{GM}{R} \Longrightarrow R = 2\frac{GM}{C^2}$$

This critical radius is now called the Schwarzschild radius and is the radius of the "event horizon" of non-rotating black holes.

The term "Black Hole" is often attributed to JohnA. Wheeler, who wrote (American Scientist, 56, 1, 1968) 'According to Einstein's general theory of relativity, as mass is added to a degenerate star a sudden collapse will take place and the intense gravitational field of the star will close in on itself. Such a star then forms a "black hole" in the universe.' The first plausible evidence that Black Holes actually exist came from observations of the binary X-ray source Cygnus X-1 in the early . As of , there are known stellar black holes and stellar black hole candidates.

Chapter 2

Introduction to General Theory of Relativity

2.1 Philosophical framework of the theory

General Relativity is the foundation for our understanding of compact stars. Neutron stars and black holes Can be understood correctly only in General Relativity as formulated by Einstein[5]. Dense objects like neutron stars could also exist in Newton's theory, but they would be very different objects. Chandrasekhar found (in connection with white dwarfs) that all degenerate stars have a maximum possible mass[5]. In Newton's theory such a maximum mass is attained only asymptotically when all Fermions, whose pressure supports the star, are ultra relativistic. Under such conditions, stars populated by the three heavy quarks-known as charm, truth, and beauty-would exist. However, such stars do not occur in Einstein's theory because the maximum-possible mass star is not sufficiently dense, even at its center;therefore they cannot exist in nature.[5]

In the case of relativistic stellar structure General relativity become important when considering the stability properties of white dwarfs and the equilibrium and stability properties of neutron stars and black holes. Indeed, it is largely for this reason that compact objects are of such great theoretical interest and have so many unique and fascinating dynamical features.[9]

General relativity is a relativistic theory of gravitation.One of Einsteins great insights was to make general relativity a geometric theory of gravitation. We shall not recount here all the motivations for this idea, but will simply start by examining the geometry of special relativity.

In special relativity, spacetime is the arena for physics. Spacetime consists of events, which require four numbers for their complete specification: three numbers to give the spatial location with respect to some chosen coordinate grid, and one number to give the time. Geometrically, spacetime is represented by a four-dimensional manifold (surface), each point in the manifold corresponding to an event in spacetime.

The general theory of relativity is a classical field theory of gravitation in which all variables are assumed to be continuous and are uniquely specified [10].

The basic philosophy of general relativity is to relate the geometry of space time, which determines the motion of matter, to the density of matter-energy, known as the stress energy tensor.

2.2 Expressions from General Relativity

Out of the special relativity (SR) we can deduce a generalized principle of relativity. This means that not only in every inertial reference system (IS) the physical laws are valid in the same way but also in accelerated frames of reference.

A further physical base for the GR is the equivalence principle. In conclusion this principle deals with the equivalence between gravitational and inertial forces. Therefore gravitational and inertial mass must be the same. Einstein generalized this principle in the way that in all sufficiently small free falling reference frames (local IS) everything behaves as if there is no gravitational force at all.

Based on the equivalence principle we can define the principle of covariance. With this principle we can derive physical laws containing gravitational effects out of general laws of the special relativity. The valid equations in a gravitational field must satisfy the following conditions: The equations are covariant under general transformation of coordinates and valid for a local inertial system (all laws are equal to the ones of special relativity if the metric tensor is equal to the Minkowski tensor).

General relativity is defined on a four dimensional Riemannian manifold. Coordinates in this non-Euclidian space are denoted by $x_\mu = (x^0, x^1, x^2, x^3)$. ξ^α denotes a flat tangential space where in the laws of Special Relativity hold. The indices μ,ν,λ,...describe coordinates of the Riemannian space, α, β,γ, ... coordinates of the Minkowski space. For every point in the Riemannian manifold exists a coordinate transformation $x^\mu = x^\mu(\xi)$ and it holds the connection between Lorentz vector dxi^α and Riemann vector dx^μ

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha} d\xi^\alpha \tag{2.2.1}$$

Therefore we can rewrite the invariant line element ds^2 in the following way

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{2.2.2}$$

with

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha \partial \xi^\beta}{\partial x^\mu \partial x^\nu}$$

In contrast to the Minkowski tensor $\eta^{\alpha\beta}$ in Special Relativity, the metric tensor $g_{\mu\nu}$ of General Relativity depends on the four dimensional space-time. Analogously to Eq.(2.2.2), we can define a Riemann vector $A^\mu = \partial x^\mu / \partial \xi^\alpha A^\alpha$ by the Lorentz vector A^α. The metric tensor can be used to transform any contravariant Riemann vector to its covariant counterpart and vice versa

$$A_\mu = g_{\mu\nu} A^\nu, A^\mu = g^{\mu\nu} A_\nu \tag{2.2.3}$$

where $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$.

The definition of covariant divergence for an arbitrary contravariant/covariant vector is given by

$$A^\mu_{;\nu} = A^\mu_{,\nu} + \Gamma^\mu_{\sigma\nu} A^\sigma \tag{2.2.4}$$

The Riemann(Christoffel curvature) tensor $R^\rho_{\sigma\mu\nu}$ is defined as

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\nu\sigma} \tag{2.2.5}$$

From the Riemann tensor we obtain the Ricci tensor

$$R_{\mu\nu} = R^\rho_{\sigma\mu\nu} \tag{2.2.6}$$

Using the definition of the Riemann tensor (2.2.5), we rewrite the Ricci tensor as

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} \tag{2.2.7}$$

We now define the scalar curvature R

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.2.8}$$

By applying the above equation, one may acquire the third covariant derivative of an arbitrary covariant vector, and use the result to obtain the Bianchi identity

$$R^\alpha_{\mu\nu\rho;\sigma} + R^\alpha_{\mu\sigma\nu;\rho} + R^\alpha_{\mu\rho\sigma;\nu} = 0 \tag{2.2.9}$$

2.3 The Einstein Field Equations

The Einstein field equations connect the space time curvature with the energy and momentum in this space time. The energy and momentum in space time is described by the energy- momentum-tensor $T_{\mu\nu}$. For an ideal fluid it has the following form

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U_\mu U_\nu + p g_{\mu\nu} \qquad (2.3.1)$$

with the density ρ, the pressure P and the fluid four-velocity $U_\mu = dx_\mu/d\tau$.

The Bianchi identity (2.2.9) can now be multiplied by $g^{\mu\nu}$ and transformed, so that we arrive at

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)_{;\nu} = 0 \qquad (2.3.2)$$

From the above equation, on may derive the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (2.3.3)$$

where $G_{\mu\nu}$ is the Einstein curvature tensor. $T_{\mu\nu}$ denotes the energy-momentum tensor.

2.4 Spherically Static Isotropic Metric solution of Einstein Field Equation

We seek solutions to Einstein's field equations in static isotropic regions of spacetime such as would be encountered in the interior and exterior regions of static stars. Under these conditions the $g_{\mu\nu}$ are independent of time ($x^o \equiv t$). We choose spatial coordinates $x^l = r$, $x^2 = \theta$, and $x^3 = \phi$.The most general form of the line element is then

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (2.4.1)$$

The metric tensor has the nonvanishing components

$$\begin{aligned} g_{rr} &= A(r) \\ g_{\theta\theta} &= r^2 \sin^2\theta \\ g_{\phi\phi} &= r^2 \sin^2\theta \end{aligned}$$

$$g_{tt} = -B(r) \tag{2.4.2}$$

with function A(r) and B(r) that are to be determined by solving the field equations. Since $g_{\mu\nu}$ is diagonal,it is easy to write down all the nonvanishing components of its inverse:

$$g^{rr} = \frac{1}{A(r)}$$
$$g^{\theta\theta} = \frac{1}{r^2}$$
$$g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta}$$

$$g^{tt} = \frac{1}{-B(r)} \tag{2.4.3}$$

Furthermore, the determinant of the metric tensor is -g, where

$$g = r^4 A(r)B(r)\sin^2\theta \tag{2.4.4}$$

so the invariant volume element is

$$\sqrt{g}drd\theta d\phi = r^2\sqrt{A(r)B(r)}\sin\theta drd\theta d\phi \tag{2.4.5}$$

The affine connection can be computed from the usual formula

$$\Gamma^{\lambda}_{\mu\nu} \quad = \quad \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right) \tag{2.4.6}$$

It is only nonvanishing components are

$$\Gamma^{r}_{rr} = \frac{1}{2A}\frac{dA(r)}{dr}$$
$$\Gamma^{r}_{tt} = \frac{1}{2A(r)}\frac{dB(r)}{dr}$$
$$\Gamma^{r}_{\theta\theta} = -\frac{r}{A(r)}$$

,

$$\Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A(r)}$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$$

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2B(r)}\frac{dB(r)}{dr} \tag{2.4.7}$$

Now we need to calculate the Ricci tensor of this metric. A silly way of doing this would be to blindly calculate all the components of the Riemann tensor and to then perform all the relevant contractions to obtain the Ricci tensor.

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \frac{\partial\Gamma^\lambda_{\mu\lambda}}{\partial\nu} - \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\eta\nu} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\eta\lambda} \tag{2.4.8}$$

Inserting in (2.4.8) the components of the affine connection given by (2.4.7),we find,

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) \tag{2.4.9}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)} \tag{2.4.10}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{2.4.11}$$

$$R_{tt} = -\frac{B''(r)}{2A(r)} + \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{-A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right) \tag{2.4.12}$$

$R_{\mu\nu} = 0$ for $\mu \neq \nu$

The prime denote differentiation with respect to r.

Chapter 3

The Tolman-Oppenheimer-Volkoff Equations and Polytropes

The hydrostatic equilibrium equation is obtained via an approximation of the known Tolmann-Oppenheimer-Volkof equation which, is derived directly from Einstein's field equations. Assuming a polytropic equation of state, these equations are the so called Lane-Emden diferential equations, which have a solution depending on the polytropic index n.

3.1 The Tolman-Oppenheimer-Volkoff Equations

In astrophysics, the Tolman-oppenheimer-Volkoff (TOV)equation constrains the structure of a spherically symmetric body of isotropic material which is in static gravitational equilibrium, as modeled by general relativity.

The equation is derived by solving the Einstein equations for a general time invariant, spherically symmetric metric. The TOV equation is the relativistic equivalent of the equation of hydrostatic equilibrium for non relativistic stellar objects. It can be derived from Einsteins field equations, (2.3.3)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{3.1.1}$$

where

- $R_{\mu\nu}$ is the Ricci tensor,
- $T_{\mu\nu}$ is the energy-momentum tensor and
- $g_{\mu\nu}$ is the metric.

The following conditions are assumed:

Static,spherically symmetric interior The interior is assumed to be static and spherically symmetric, which can be described by the diagonal metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1.2}$$

where A(r) and B(r) are functions of r to be determined.The non-vanishing components of the metric tensor are given by

$$g_{rr} = A(r)$$

$$g_{\theta\theta} = r^2$$

$$g_{\phi\phi} = r^2 \sin^2\theta$$

$$g_{tt} = -B(r) \tag{3.1.3}$$

$g_{\mu\nu}$=0 for $\mu \neq \nu$

Perfect fluid The interior is assumed to consist of a perfect fluid with energy-momentum tensor $T_{\mu\nu}$ given as

$$T_{\mu\nu} = (\rho c^2 + P)U_\mu U_\nu + Pg_{\mu\nu} \tag{3.1.4}$$

with p the proper pressure, ρ the proper total energy density, and U^μ the velocity four-vector,defined so that

$$g^{\mu\nu}U_\mu U_\nu \quad = \quad -1$$

Since the fluid is at rest, we take

$$U_r = U_\theta = U_\phi = 0,$$

$$U_t = -(-g^{tt})^{-1/2} \tag{3.1.5}$$

Hence we can rewrite the energy momentum tensor for static stars

$$T_{\mu\nu} = diag(-\rho c^2 B, pA, pr^2, pr^2 \sin^2\theta). \tag{3.1.6}$$

With the metric tensor $g_{\mu\nu}$ we can compute the trace of the energy momentum tensor:

$$T = g^{\mu\nu}T_{\mu\nu} = -(\rho c^2 - 3P) \tag{3.1.7}$$

where the normalization condition $g_{\mu\nu}U_\mu U_\nu$ = -1 and $g_{\mu\nu}g^{\mu\nu} = 4$ were used.

Inserting equation (3.1.7) into (3.1.1) gives

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} + \frac{1}{2}(\rho c^2 - 3p)g_{\mu\nu}) \tag{3.1.8}$$

By making use of Eq.(3.1.3)-(3.1.7) and the Ricci tensor components given by Eq.(2.4.9 - 2.4.12) inserting into equation (3.1.8) leads to

$$R_{tt} = -4\pi G(\rho + 3P)B \tag{3.1.9}$$

$$R_{rr} = -4\pi G(\rho - P)A \tag{3.1.10}$$

$$R_{\theta\theta} = -4\pi G(\rho - P)r^2 \tag{3.1.11}$$

$$R_{\phi\phi} = \sin^\theta R_{\theta\theta} \tag{3.1.12}$$

In addition,we may recall the equation $-\frac{\partial P}{\partial x^\lambda} = (P + \rho)\frac{\partial}{\partial x^\lambda}\ln(-g_{00})^{1/2}$ for hydrostatic equilibrium,

$$\frac{B'}{B} = -\frac{2P'}{P + \rho} \tag{3.1.13}$$

To find the TOV equation, the functions A(r) and B(r) should be found. Our first step in solving these equations is to drive an equation for A(r) alone, by forming the quantity

$$\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{tt}}{2B} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = -\frac{8\pi G}{c^2}\rho = -8\pi G\rho \tag{3.1.14}$$

This equation can be written

$$\frac{d}{dr}\frac{r}{A(r)} = 1 - 8\pi G\rho r^2 \tag{3.1.15}$$

If we integrate the above Equation from the center of the star r = 0 to the radius r with the additional condition $(r/A)\,|_{r=0} = 0$ (A must be finite at r = 0 because of a continuous mass distribution) we obtain

$$A(r) = \left[1 - \frac{2GM(r)}{r}\right]^{-1} \tag{3.1.16}$$

where M(r) is a mass function that describes the mass contained within a radius r.

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')dr' \tag{3.1.17}$$

In a similar manner to how we found equation (3.1.14) one may also find the relation

$$\frac{R_{rr}}{g_{rr}} + \frac{R_{tt}}{2g_{tt}} - \frac{R_{\theta\theta}}{r^2} \quad = \quad \frac{1}{r^2} - \frac{1}{r^2 g_{rr}} - \frac{1}{r g_{rr}}\left(\frac{g'_{tt}}{g_{tt}}\right) = -8\pi G p \tag{3.1.18}$$

An expression for B(r), inserting the expression we found for A(r) in equation (3.1.16) into equation (3.1.18) and rearranging the terms yields

$$\frac{B'}{B} = \frac{2GM(r)}{r^2}\frac{\left(1 + \frac{4\pi r^3 p}{M(r)}\right)}{1 - \frac{2GM(r)}{r}} \tag{3.1.19}$$

We can make further use of the stress-energy tensor for a perfect fluid by enforcing energy-momentum conservation as well as hydrostatic equilibrium, yielding

$$-\partial_\lambda p = (p + \rho)\partial_\lambda \ln(-g_{tt})^{1/2} \tag{3.1.20}$$

and as before only the derivative with respect to r is non-zero, so in our case equation (3.1.20) reads

$$-P' = (p+\rho)\,\frac{B'}{B} \tag{3.1.21}$$

We eliminate B'/B from equation (3.1.21) by using the expression we found in equation (3.1.19) to get the Tolman-Oppenheimer-Volkof(TOV) equation

$$-\,r^2 p'(r) \;=\; GM(r)\rho(r)\left[1+\frac{p(r)}{\rho(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{M(r)}\right]\left[1-\frac{2GM(r)}{r}\right]^{-1} \tag{3.1.22}$$

For a given equation of state $P = P(\rho)$, the TOV equations can easily be integrated from the origin with initial conditions M(0) = 0 and an arbitrary value for the central density $\rho_c = \rho(0)$, until the pressure P(r) will vanish at some radius R. To each possible equation of state, there is a unique family of stars parameterized by the central density, i.e. we obtain a sequence of stellar models $M = M(\rho_c)$.

The TOV equations (3.1.22) simplifies

$$-r^2 p'(r) = GM(r)\rho(r) \tag{3.1.23}$$

with M(r) defined by

$$\frac{dM}{dr} = 4\pi r^2 \rho \tag{3.1.24}$$

which when combined give us the Poisson equation $\nabla\phi = 4\pi G\rho$ in spherical coordinates

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho \tag{3.1.25}$$

3.2 A Variational Form of The Equilibrium Condition

A particular stellar configuration, with uniform entropy per nucleon and chemical composition, will satisfy the TOV equations for equilibrium, if and only if the quantity M defined by

$$M = \int_0^\infty 4\pi r^2 \rho(r) dr \tag{3.2.1}$$

is stationary with respect to all variations of $\rho(r)$ that leave unchanged the total number of baryons, $n(r)$ =$\rho(0)(r)$ as the baryon density,

$$N = \int_0^\infty 4\pi r^2 n(r)\left[1-\frac{2GM(r)}{r}\right]^{-1/2} dr \tag{3.2.2}$$

To derive this theorem one uses the Lagrange multiplier method[10]: M will be stationary with respect to all variations that leave N fixed if and only if there exist a constant λ for which M -λN is stationary with respect to all variations. In general, we get

$$\delta M - \lambda\delta N = \int_0^\infty 4\pi r^2 \rho(r)dr - \lambda\int_0^\infty 4\pi r^2\left[1-\frac{2GM(r)}{r}\right]^{-1/2}\delta n(r)dr \\ - \lambda G\int_0^\infty 4\pi r\left[1-\frac{2GM(r)}{r}\right]^{-3/2} n(r)\delta M(r)dr \tag{3.2.3}$$

These variations are supposed not to change the entropy per nucleon, i.e.

$$0 = \delta\left(\frac{\rho}{n}\right) + p\delta\left(\frac{1}{n}\right) \tag{3.2.4}$$

or

$$\delta n(r) = \frac{n(r)}{p(r)+\rho(r)}\delta\rho(r) \tag{3.2.5}$$

And in addition we have

$$\delta M(r) = \int_0^r 4\pi r'^2\delta\rho(r')dr' \tag{3.2.6}$$

If we interchange the r and r' integration in the last term of the total variation

$$\delta M - \lambda\delta N = \int_0^\infty 4\pi r^2\{1 - \frac{\lambda n(r)}{p(r)+\rho(r)}\left(1-\frac{2GM(r)}{r}\right)^{-1/2} \\ - \lambda G\int_0^\infty 4\pi r' n(r')\left(1-\frac{2GM(r')}{r'}\right)^{-3/2} dr'\}\delta\rho(r)dr \tag{3.2.7}$$

Thus $\delta M - \lambda\delta N$ will vanish if and only if

$$\frac{1}{\lambda} = \frac{n(r)}{p(r)+\rho(r)}\left[1-\frac{2GM(r')}{r'}\right]^{-1/2} + G\int_r^\infty 4\pi r' n(r')\left[1-\frac{2GM(r')}{r'}\right]^{-3/2} dr' \tag{3.2.8}$$

This will be the case for some multiplier λ if and only if the right-hand side is independent of r,that is, if and only if

$$0 = \{\frac{n'}{p+\rho} - \frac{n(p'+\rho')}{(p+\rho)^2}\}\left[1-\frac{2GM}{r}\right]^{-1/2} + \frac{Gn}{p+\rho}\{4\pi r\rho - \frac{M}{r^2}\}\left[1-\frac{2GM}{r}\right]^{-3/2} - 4\pi Grn\left[1-\frac{2GM}{r}\right]^{-3/2} \quad (3.2.9)$$

The condition for uniform entropy gives

$$\frac{d}{dr}\left(\frac{\rho}{n}\right) + p\frac{d}{dr}\left(\frac{1}{n}\right) = 0 \quad (3.2.10)$$

and therefore

$$n'(r) = \frac{n(r)\rho'(r)}{p(r)+\rho(r)} \quad (3.2.11)$$

3.3 Polytropes

Here we assume $P(r)= p(\rho(r))$ (with $\frac{dp}{d\rho} \neq 0$). In order that $\rho_c= \rho(0)$ be finite, it is necessary that p'(0) vanish.

$\rho'(0) = 0.$

Also note that the internal energy density is proportional to the pressure,that is

$$e = \rho - m_N n = \frac{1}{\gamma-1}p$$

Here $(\gamma-1)^{-1}$ is just a constant proportionality coefficient.

Using equation(3.2.10) we obtain

$$0 = \frac{1}{\gamma-1}\left[\gamma P\frac{d}{dr}\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)\frac{dp}{dr}\right]$$

$$\frac{dp}{p} = \gamma\frac{dn}{n}$$

$$\ln\frac{p}{n^\gamma} = constant$$

Then finally we obtain

$$P = K\rho^{1+\frac{1}{n}} \tag{3.3.1}$$

where K and n are constants, and n is called a polytropic index and γ=$1 + 1/n$.

Notice that the n = 1.5 case corresponds to an adiabatic star supported by pressure of non-relativistic gas, and the n = 3 case corresponds to an adiabatic star supported by pressure of ultra-relativistic gas.

Any star for which the equation of state takes the form of the above equation is called a polytrope.

Differentiation of Eq.(3.3.1) gives

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{n+1}{n}\frac{r^2}{\rho}K\rho^{1/n}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

We can now rewrite this differential equation in its dimensionless form, by selecting the transformations

$$\rho \;=\; \rho(0)\theta^{1/\gamma-1}, \tag{3.3.2}$$

$$P \;=\; K\rho(0)^{\gamma}\theta^{\gamma/\gamma-1} \tag{3.3.3}$$

The differential equation is now given by

$$(n+1)\frac{K\rho_0^{1+1/n}}{4\pi G\rho_0^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

By transforming variables from r,ρ to ξ, θ , we get a universal equation for polytropes parameterized by the polytropic index γ.

The transformation is,

$$r = \alpha\xi \tag{3.3.4}$$

where

$$\alpha = \left(\frac{k\gamma}{4\pi G(\gamma-1)}\right)^{1/2} \rho(0)^{(\gamma-2)/2} \tag{3.3.5}$$

then the appropriate transformation takes the form

$$r = \left(\frac{k\gamma}{4\pi G(\gamma-1)}\right)^{1/2} \rho(0)^{(\gamma-2)/2}\xi \tag{3.3.6}$$

and the result is the Lane-Emden equation of index$(\gamma-1)^{-1} = n$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^{1/\gamma-1} \tag{3.3.7}$$

The boundary conditions are

$$\theta(0) = 1 \tag{3.3.8}$$

$$\theta'(0) = 0 \tag{3.3.9}$$

The function $\theta(\xi)$ defined by the above is known as the Lane-Emden function of index $(\gamma-1)^{-1}$ or n.

we can use the Lane-Emden Equation for any polytropic index n. Only three analytic solutions exist, i.e for n = 0, 1 and 5, for which the solutions are:

$$\begin{aligned}
\theta_0(\xi) &= 1 - \frac{\xi^2}{6}, n = 0, \xi_1 = \sqrt{6} \approx 2.45 \\
\theta_1(\xi) &= \frac{\sin\xi}{\xi}, n = 1, \xi_1 = \pi \approx 3.14 \\
\theta_5(\xi) &= \left(1 + \frac{\xi^2}{3}\right)^{-1/2}, n = 5, \xi_1 = \infty
\end{aligned} \tag{3.3.10}$$

The solution is constructed monotonically decreases from the center,and for $n < 5$ has a zero for some finite $\xi = \xi_1$, θ has its first zero, and thus the configuration has a definite boundary.For n=5, the configuration extends to infinity.

For $\xi(0)$ with in series solution the Lane-Emden equation is integrated to yield

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120(\gamma-1)}... \tag{3.3.11}$$

For $\gamma > 6/5$,$\theta(\xi)$ vanishes at some finite ξ_1:

$$\theta(\xi_1) = 0 \tag{3.3.12}$$

The Radius relation

The radius R of the star is given by R $= r_1 = \alpha\xi_1$ which translates to

$$R = \alpha\xi_1 = \left(\frac{K\gamma}{4\pi G(\gamma - 1)}\right)^{1/2} \rho(0)^{(\gamma-2)/2}\xi_1 \tag{3.3.13}$$

where ξ_1 is defines the first zero of θ_n.

The Mass relation

We can also use the Lane-Emden solutions to calculate the stellar mass:

$$M = \int_0^R 4\pi r^2 \rho(r) dr$$

$$M = 4\pi \left(\frac{K\gamma}{4\pi G(\gamma - 1)}\right)^{3/2} \rho_0^{(3\gamma-4)/2} \int_0^{\xi_1} \xi^2 \theta^{1/(\gamma-1)}(\xi) d\xi \tag{3.3.14}$$

Using the Lane-Emden equation, the integral is easily evaluated:

$$\int_0^{\xi_1} \xi^2 \theta^{1/(\gamma-1)} d\xi = -\int_0^1 \frac{d}{d\xi}\xi^2 \frac{d\theta}{d\xi} d\xi = -\xi^2\theta'(\xi_1) \tag{3.3.15}$$

Hence, the mass is found as

$$M = 4\pi\rho_0^{(3\gamma-4)/2} \left(\frac{K\gamma}{4\pi G(\gamma - 1)}\right)^{3/2} \xi^2 |\theta'(\xi_1)| \tag{3.3.16}$$

Mass-Radius Relation for Polytropes

By eliminating the central density between (3.3.13) and (3.3.16), we obtain a relation between mass M and radius R:

$$M = 4\pi R^{(3\gamma-4)/(\gamma-2)} \left(\frac{K\gamma}{4\pi G(\gamma - 1)}\right)^{-1/(\gamma-2)} \xi^{-(3\gamma-4)/(\gamma-2)} \xi_1^2 |\theta'(\xi_1)|. \tag{3.3.17}$$

or

$$GM^{-\frac{(\gamma-2)}{(\gamma-1)^2}}R^{(3\gamma-4)} = \frac{\gamma K}{(\gamma-1)4\pi^{(\gamma-1)}}\left[-\xi^{-\frac{\gamma}{\gamma-2}}\frac{d\theta}{d\xi}|_{\xi=\xi_1}\right]^{-(\gamma-2)} \qquad (3.3.18)$$

We shall denote by w_{0n} the quantity

$$w_{on} = -\xi_1^{-(\frac{\gamma}{\gamma-2})}\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1} \qquad (3.3.19)$$

We can rewrite eq.(3.3.18)as

$$K = N_n GM^{-\frac{\gamma-2}{(\gamma-1)^2}} \qquad (3.3.20)$$

Where N_n stands for the numerical coefficient

$$N_n = \frac{1}{n+1}\left[\frac{4\pi}{w_{0n}^{n-1}}\right]^{1/n} \qquad (3.3.21)$$

or

$$N_n = \frac{\gamma-1}{\gamma}\left(\frac{4\pi}{w_{0n}^{-\frac{(\gamma-2)}{(\gamma-1)}}}\right)^{\gamma-1} \qquad (3.3.22)$$

The Relation of the mean density to the central density Let $\overline{\rho}$ denotes the mean density of matter interior to r=$\alpha\xi$ .Then

$$\overline{\rho} = \frac{M}{\frac{4}{3}\pi\alpha^3\xi^3} \qquad (3.3.23)$$

or

$$\overline{\rho} = -\frac{3}{\xi}\left(\frac{d\theta}{d\xi}\right)\rho(0) \qquad (3.3.24)$$

From this the central density is

$$\rho(0) = -\left|\frac{\xi}{3}\frac{1}{d\theta_n/d\xi}\right|_{\xi=\xi_1}\overline{\rho} \qquad (3.3.25)$$

This relation shows that for a polytrope of a given index ,n the central density is a definite multiple of the mean density.

$$\frac{\rho(0)}{\overline{\rho}} = -\left[\frac{3}{\xi_1}(\frac{d\theta}{d\xi})\xi_1\right]^{-1} \qquad (3.3.26)$$

A second useful relationship is between mass and radius. We start by expressing the central density ρ_0 in terms of the other constants in the problem and our length scale α:

$$\rho_0 = \left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{n/(n-1)}$$

The central pressure can be expressed trivially as

$$p_c = K\rho_0^{(n+1)/n} = W_n \frac{GM}{R^4} \tag{3.3.27}$$

where W_n stands for the quantity

$$W_n = \frac{1}{4\pi(n+1)\left[\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}\right]^2} \tag{3.3.28}$$

The value of W_n are given in table.

3.3.1 Thermal and Gravitational Energy for Polytropes

For general Newtonian Stars total internal energy as

$$E = T + V \tag{3.3.29}$$

with the thermal energy T and gravitational energy V given by

$$T = \int_0^R 4\pi r^2 e(r) dr \tag{3.3.30}$$

$$V = -\int_0^R 4\pi G M(r)\rho(r) dr \tag{3.3.31}$$

Derive these energies for polytropes. From fundamental hydrostatic equilibrium (TOV)condition

$$\frac{dP(r)}{dr} = \frac{GM(r)\rho(r)}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi r^3 P}{M}\right)\left(1-\frac{2GM(r)}{r}\right)^{-1}$$

For Newtonian Star

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{3.3.32}$$

Then

$$\begin{aligned} V &= -\int_0^R 4\pi r G M \rho dr \\ &= -4\pi \int_0^R (\frac{dP}{dr}) r^3 dr \end{aligned}$$

Integrating this by part

$$V = -12\pi \int_0^R r^2 P(r) dr \tag{3.3.33}$$

Multiplying and dividing in the integral by $\rho(r)$

$$V = 3\int_0^R M(r) d\left(\frac{P(r)}{\rho(r)}\right) dr$$

We assume here that $\gamma > 1$, so that P/ρ vanishes at R. This can be evaluated by using the equation of state to calculate

$$\begin{aligned} \frac{d}{dr}\left(\frac{P(r)}{\rho(r)}\right) &= \frac{1}{\rho}\frac{dP}{dr} - \frac{P}{\rho}\frac{dp}{dr} \\ &= \left(\frac{1}{\rho} - \frac{P}{\rho}\right)\frac{dP}{dr} \\ &= \frac{1}{\rho}\left(1 - \frac{P}{\rho}\right)\frac{dP}{dr} \\ &= -\left(\frac{\gamma - 1}{\gamma}\right)\frac{GM(r)}{r^2} \end{aligned}$$

So

$$V = -3\left(\frac{\gamma - 1}{\gamma}\right)\int_0^R \frac{GM^2(r)}{r^2} dr \tag{3.3.34}$$

Since $dr/r^2 =$ -d(1/r),we can integrate by parts once again, and find

$$V = 3\left(\frac{\gamma - 1}{\gamma}\right)\frac{GM^2}{R} - 2\int_0^R \frac{GMdM}{r}$$

$$V = 3\left(\frac{\gamma - 1}{\gamma}\right)\frac{GM^2}{R} + 2V$$

$$V = -\frac{3(\gamma-1)}{(5\gamma-6)}\frac{GM^2}{R} \tag{3.3.35}$$

To calculate T, we use the approximation, $e(r) \simeq \rho(r)$

$$\begin{aligned} T &= \int_0^R 4\pi r^2 \rho(r) dr \\ &= \int_0^R 4\pi r^2 \frac{P(r)}{\gamma-1} dr \\ &= \frac{4\pi}{\gamma-1}\int_0^R r^2 P(r) dr \end{aligned}$$

Multiplying both sides by (-3/3)

$$T = \frac{1}{3(\gamma-1)}\left[12\int_0^R r^2 P(r) dr\right]$$

$$T = \frac{-V}{3(\gamma-1)} \tag{3.3.36}$$

By using eq.(3.3.35) finally we obtain

$$T = \frac{1}{(5\gamma-6)}\frac{GM^2}{R} \tag{3.3.37}$$

From eqn.(3.3.36) gravitational energy rewrite as

$$V = -3(\gamma-1)T \tag{3.3.38}$$

So the total internal energy is

$$\begin{aligned} E &= T + V \\ &= -(3\gamma-4)T \end{aligned}$$

$$E = -\frac{(3\gamma-4)}{(5\gamma-6)}\frac{GM^2}{R} \tag{3.3.39}$$

To see the stability configuration

we try the assumption of

$$\rho \equiv constant$$

$$T = \int 4\pi r^2 e(r) dr$$

but we have $e(r) \simeq \rho(r)$ and $\rho(r) \simeq \frac{\rho}{\gamma-1}$

$$\begin{aligned} T &= \frac{1}{\gamma-1} \int 4\pi r^2 \rho(r) dr \\ &= \frac{4\pi\rho}{(\gamma-1)} \int_0^R r^2 dr \\ &= \frac{4\pi}{3} \frac{\rho R^3}{(\gamma-1)} \end{aligned}$$

$$T = \frac{4\pi}{3} (\gamma-1)^{-1} K\rho^\gamma R^3 \tag{3.3.40}$$

From this and the above finally we obtain

$$T = \frac{KM}{\gamma-1} \rho^{\gamma-1} \tag{3.3.41}$$

$$V = -\frac{16\pi^2}{15} G\rho^2 R^5 \tag{3.3.42}$$

So,eliminating R,

$$E = T + V \tag{3.3.43}$$

$$= a\rho^{\gamma-1} - b\rho^{1/3} \tag{3.3.44}$$

Where

$$a = \frac{KM}{(\gamma-1)} \tag{3.3.45}$$

$$b = \frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3} GM^{5/3} \tag{3.3.46}$$

For $\gamma > 4/3$, E has a minimum at

$$\rho \quad = \quad \left(\frac{b}{3a(\gamma-1)}\right)^{1/(\gamma-4/3)} = \left(\frac{M^{2/3}G(4\pi/3)^{1/3}}{5K}\right)^{1/(\gamma-4/3)} \tag{3.3.47}$$

corresponding to a configuration of stable equilibrium.

For $\gamma = 4/3$, E is stationary with respect to ρ only if it vanishes every where, which requires that a=b, or

$$M \quad = \quad \left(\frac{5K}{G}\right)^{3/2}\left(\frac{4\pi}{3}\right)^{-1/2} \tag{3.3.48}$$

For $\gamma < 4/3$, E has a maximum at the point(3.3.47), corresponding to a state of unstable equilibrium.

Incidentally,Eq.(3.3.47) gives an estimate for the mass

$$M \simeq \frac{4\pi}{3}\rho^{(3\gamma-4)/2}\left(\frac{15K}{4\pi G}\right)^{3/2}$$

which may be compared with the exact result (3.3.16).The ratio of these two expressions is

$$\frac{M(variational)}{M(exact)} = \frac{(15(\gamma-1)/\gamma)^3/2}{3\xi_1^2|\theta'(\xi_1)|}$$

For γ=5/3 this ratio is 1.8; for $\gamma = 4/3$ it is 1.2.Not only does the variational method give the correct dependence of M on ρ (including the fact that for $\gamma = 4/3$, M is independent of ρ, and E vanishes),but it even provides a fair approximation to the exact numerical results. We can accept with confidence its prediction that a polytropic is stable or unstable according to whether $\gamma > 4/3$ or $\gamma < 4/3$.

Chapter 4

Mass- Radius Relation of Compact Objects

4.1 White Dwarfs

The equation of state of a degenerate gas

When the temperature is sufficiently law ($T \rightarrow 0$) the electrons will be frozen into the lowest available energy levels. The Pauli Principle tell us: that there will be two electrons in each levels(because of the two spin states available) and there are $\frac{4\pi k^2}{(2\pi\hbar)^3}dk$ levels per unit volume with momenta between k and k+dk

The number of electrons per unit volume related to the maximum momentum K_F is

$$n = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F} k^2 dk = \frac{k_F^3}{3\pi^2\hbar^3} \tag{4.1.1}$$

The mass density of the star is given by

$$\rho = n m_N \mu \tag{4.1.2}$$

where m_N =1.67 $\times 10^{-24} gm$ is the mass of the nucleons and μ is the number of nucleons per electron $\mu \simeq 2$ for stars that have used up their hydrogen.This gives

$$kT \ll [k_F^2 + m_e^2]^{1/2} - m_e \tag{4.1.3}$$

$$k_F = \hbar \left(\frac{3\pi^3 \rho}{m_N \mu} \right)^{1/3} \tag{4.1.4}$$

Then the kinetic energy density and pressure of these electrons are

$$e = \frac{8\pi}{3(2\pi\hbar)^3} \int_0^{k_F} [(k^2 + m_e^2 - m_e]^{1/2} k^2 dk \tag{4.1.5}$$

$P = \frac{\rho}{3}$, $\rho = T^{00} = \int_0^{k_F} \frac{k^2}{(k^2+m_e^2)^{1/2}} dk$

$$P = \frac{1}{3} \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} \frac{k^2}{(k^2 + m_e^2)^{1/2}} k^2 dk \tag{4.1.6}$$

However, the equation of state reduces to a polytrope in the two extreme cases, distinguished by the criteria $\rho \ll \rho_c$ or $\rho \gg \rho_{crit}$,where ρ_{crit} is the critical density at which k_F becomes equal to m_e ($k_F = m_e$) (in c.g.s. units):

$$\rho_{crit} = \frac{m_N \mu m_e^3 c^3}{3\pi^2 \hbar^3} = 0.97 \times 10^6 \mu gm/cm^3 \tag{4.1.7}$$

A.If $\rho \ll \rho_c$. In non In the relativistic,$k_F \ll m_e$,and electrons are non relativistic.Then equation (4.1.5)and (4.1.6) respectively yield

$$e = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} m_e[1 + \frac{1}{2}\frac{k^2}{m_e^2} + ...] - m_e k^2 dk$$

$$e \simeq \frac{8\pi}{(2\pi\hbar)^3} \times \frac{1}{m_e^2} \times \frac{k_F^5}{5} \tag{4.1.8}$$

$$P \simeq \frac{1}{3} \frac{8\pi}{(2\pi\hbar)^3} \times \frac{1}{m_e} \times \frac{k_F^5}{5} \tag{4.1.9}$$

$$e = \frac{3}{2} p \tag{4.1.10}$$

$$P = \frac{8\pi k_F^5}{15 m_e (2\pi\hbar)^3}$$

$$P = \frac{\hbar^2}{15 m_e \pi^2} \left(\frac{3\pi^2 \rho}{m_N \mu} \right)^{5/3} \tag{4.1.11}$$

This is a polytrope, with $\gamma = \frac{5}{3}$,

$$K = \frac{\hbar^2}{15 m_e \pi^2} \left(\frac{3\pi^2}{m_N \mu} \right)^{5/3} \tag{4.1.12}$$

B. If $\rho \gg \rho_c$.In the relativistic limit $k_F \gg m_e$,then

$$e \simeq \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^3 dk = \frac{8\pi}{(2\pi\hbar)^3} \times \frac{k_F^4}{4}$$

$$P \simeq \frac{1}{3} \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^3 dk = \frac{8\pi}{3(2\pi\hbar)^3} \times \frac{k_F^4}{4} \tag{4.1.13}$$

$$e = 3P \tag{4.1.14}$$

$$P = \frac{8\pi k_F^4}{12(2\pi\hbar)^3}$$

$$P = \frac{\hbar}{12\pi^2} \left(\frac{3\pi^2 \rho}{m_N \mu} \right)^{4/3} \tag{4.1.15}$$

This is a polytrope,with $\gamma = \frac{4}{3}$

$$K = \frac{\hbar}{12\pi^2} \left(\frac{3\pi^2}{m_N \mu} \right)^{4/3} \tag{4.1.16}$$

4.2 Neutron Stars

Equation Of State(EOS)

In order to formulate the quantitative theory of neutron stars,we begin by writing down expressions for the total energy density and pressure of an ideal Fermi gas of neutrons with maximum momentum K_F:

$$\rho = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}(k^2+m_n^2)^{1/2}k^2dk \quad (4.2.1)$$

$$= 3\rho_c\int_0^{k_F/m_n}(u^2+1)^{1/2}u^2du \quad (4.2.2)$$

$$P = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F}\frac{k^2}{(k^2+m_n^2)^{1/2}}k^2dk \quad (4.2.3)$$

$$= \rho_c\int_0^{k_F/m_n}(u^2+1)^{-1/2}u^4du \quad (4.2.4)$$

where

$$\rho_c = \frac{8\pi m_n^4c^3}{3(2\pi\hbar)^3} = 6.11\times10^{15}gm/cm^3 \quad (4.2.5)$$

Now by eliminating k_F/m_n in the above equation,we obtain the equation of state in the form

$$\frac{p}{\rho_c} = F\left(\frac{\rho}{\rho_c}\right) \quad (4.2.6)$$

where F is a definite transcendental function.The structure of neutron star with given central density $\rho(0)$ is to be calculated by the TOV equation with p given as a function of ρ by the above equation. Since the only dimensional quantities in these equations are $\rho(0)$,ρ_c,and G, the solution must give the mass and radius as functions of $\rho(0)$ of the form,

$$M = M_0 f\left(\frac{\rho(0)}{\rho_c}\right) \quad (4.2.7)$$

$$R = R_0 g\left(\frac{\rho(0)}{\rho_c}\right) \quad (4.2.8)$$

where(in c.g.s.units)

$$R_0 = c(8\pi G\rho_c)^{-1/2} = 3.0km \quad (4.2.9)$$

$$M_0 = \frac{c^2R_0}{G} = 2M_\odot \quad (4.2.10)$$

where f and g are unknown dimensionless functions.

For $\rho(0) \ll \rho_c$,

$$M = \frac{1}{2}\left(\frac{3\pi}{8}\right)^{1/2}(2.71406)\left(\frac{\hbar^{3/2}}{m_n^2 G^{3/2}}\right)\left(\frac{\rho(0)}{\rho_c}\right)^{1/2}$$

$$M = \frac{1}{2}(2.71406)M_0\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} \tag{4.2.11}$$

and the radius of the star is

$$R = \left(\frac{3\pi}{8}\right)^{1/2}(3.65375)\left(\frac{\hbar^{3/2}}{m_n^2 G^{3/2}}\right)\left(\frac{\rho(0)}{\rho_c}\right)^{1/6}$$

$$R = (3.65375)R_0\left(\frac{\rho(0)}{\rho_c}\right)^{1/6} \tag{4.2.12}$$

with ρ_c given by eq.(4.2.5)

For $\rho(0) \gg \rho_c$, the neutrons near the center of the star have $k_F \gg m_n$, so the above two equations give

$$\rho = \frac{3\rho_c}{4}\left(\frac{k_F}{m_n}\right)^5, \tag{4.2.13}$$

$$p = \frac{\rho_c}{4}\left(\frac{k_F}{m_n}\right)^5 \tag{4.2.14}$$

Therefore,from equation(4.2.13) and (4.2.14) the relation between P and ρ is

$$p = \frac{\rho}{3} \tag{4.2.15}$$

as would be expected for a gas of highly relativistic particles.Using this equation of state in the fundamental differential equation(3.1.22) gives

$$-r^2\rho'(r) = 4\pi M(r)\rho(r)\left[1 + \frac{4\pi r^{3\rho(r)}}{3M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1} \tag{4.2.16}$$

$$\rho'(r) = \frac{-1}{r^2}\left[\frac{4GM(r)\rho(r)\left[1+\frac{4\pi^3\rho(r)}{3M(r)}\right]}{[1-\frac{2GM(r)}{r}]}\right]$$
$$= \frac{-1}{r^2}\left[\frac{4GM(r)\rho(r)+\frac{16\pi^3\rho^2(r)G}{3}}{[1-\frac{2GM(r)}{r}]}\right]$$
$$\frac{d\rho(r)}{dr} = \frac{-1}{r^2}\left[\frac{4GM(r)\rho(r)+\frac{16\pi^3\rho^2(r)G}{3}}{[r-2GM(r)]}\right]$$

$$\int d\rho(r) = \int_0^r \frac{-1}{r^2}\left[\frac{4GM(r)\rho(r)+\frac{16\pi^3\rho^2(r)G}{3}}{[r-2GM(r)]}\right] dr \qquad (4.2.17)$$

Then the direct integration of equation(4.2.17) and rearranging, we get exact solution.

$$\rho(r) = \frac{3}{56\pi G r^2} \qquad (4.2.18)$$

Chapter 5

Results and Discussion

Using General Theory of Relativity we have developed the fundamental TOV equations. Then, with simplifying boundary conditions such as high density and low temperature characteristics were used to develop the compact objects equation of state as polytropic kinds. Here, we summarize the important equations so far developed to analyze and describe the Mass-Radius relationships of the compact objects to exist. Furthermore, the differential equations are numerically integrated to discuss the limiting masses of the compact objects with their boundary conditions. Finally, the results be compared with the work of the others.

5.1 Summary of the important equations

5.1.1 Tolman-Oppenheimer-Volkof(TOV) equations from EFEs

The Tolman-Oppenheimer-Volkof(TOV) equation are

$$-r^2p'(r) = GM(r)\rho(r)\left[1+\frac{p(r)}{\rho(r)}\right]\left[1+\frac{4\pi r^3p(r)}{M(r)}\right]\left[1-\frac{2GM(r)}{r}\right]^{-1} \tag{5.1.1}$$

$$\frac{dM}{dr} = 4\pi r^2\rho \tag{5.1.2}$$

5.1.2 Application of Variational technique and polytropes

$$P = K\rho^{1+\frac{1}{n}} \tag{5.1.3}$$

where K and n are constants, and n is called a polytropic index and γ=$1 + 1/n$.

By transforming variables from r,ρ to ξ, θ , we get a universal equation for polytropes parameterized by the polytropic index γ.

The transformation is,

$$r = \alpha\xi \tag{5.1.4}$$

where

$$\alpha = \left(\frac{k\gamma}{4\pi G(\gamma - 1)}\right)^{1/2} \rho(0)^{(\gamma-2)/2} \tag{5.1.5}$$

then the appropriate transformation takes the form

$$r = \left(\frac{k\gamma}{4\pi G(\gamma - 1)}\right)^{1/2} \rho(0)^{(\gamma-2)/2}\xi \tag{5.1.6}$$

and a new dependent variables θ,by

$$\rho = \rho(0)\theta^{1/\gamma-1}, \tag{5.1.7}$$

$$P = K\rho(0)^{\gamma}\theta^{\gamma/\gamma-1} \tag{5.1.8}$$

5.1.3 Lane-Emden equation

The Lane-Emden equation given as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^{1/\gamma-1} \tag{5.1.9}$$

The boundary conditions are

$$\theta(0) = 1 \tag{5.1.10}$$

$$\theta'(0) = 0 \tag{5.1.11}$$

we can use the Lane-Emden Equation for any polytropic index n. Only three analytic solutions exist, i.e for n = 0, 1 and 5, for which the solutions are:

$$\begin{aligned}\theta_0(\xi) &= 1-\frac{\xi^2}{6}, n=0, \xi_1=\sqrt{6}\approx 2.45\\ \theta_1(\xi) &= \frac{\sin\xi}{\xi}, n=1, \xi_1=\pi\approx 3.14\\ \theta_5(\xi) &= (1+\frac{\xi^2}{3})^{-1/2}, n=5, \xi_1=\infty\end{aligned} \tag{5.1.12}$$

For $\xi(0)$ with in series solution the Lane-Emden equation is integrated to yield

$$\theta(\xi)=1-\frac{\xi^2}{6}+\frac{\xi^4}{120(\gamma-1)}... \tag{5.1.13}$$

For $\gamma > 6/5$,$\theta(\xi)$ vanishes at some finite ξ_1:

$$\theta(\xi_1)=0 \tag{5.1.14}$$

5.2 Obtaining the Mass and Radius of the compacts in terms of the homology transformation parameters ξ and θ)

The radius R of the star is given by R = $r_1 = \alpha\xi_1$ which translates to

$$R=\left(\frac{K\gamma}{4\pi G(\gamma-1)}\right)^{1/2}\rho(0)^{(\gamma-2)/2}\xi_1 \tag{5.2.1}$$

The Mass relation

We can also use the Lane-Emden solutions to calculate the stellar mass:

$$M = 4\pi\rho_0^{(3\gamma-4)/2}\left(\frac{K\gamma}{4\pi G(\gamma-1)}\right)^{3/2}\xi^2|\theta'(\xi_1)| \tag{5.2.2}$$

Mass-Radius Relation for Polytropes

By eliminating the central density between (5.1.15) and (5.1.16), we obtain a relation between mass M and radius R:

$$M = 4\pi R^{(3\gamma-4)/(\gamma-2)}\left(\frac{K\gamma}{4\pi G(\gamma-1)}\right)^{-1/(\gamma-2)}\xi^{-(3\gamma-4)/(\gamma-2)}\xi_1^2|\theta'(\xi_1)|. \tag{5.2.3}$$

The Relation of the mean density to the central density Let $\overline{\rho}$ denotes the mean density of matter interior to r=$\alpha\xi$ .Then

$$\overline{\rho} = -\frac{3}{\xi}\left(\frac{d\theta}{d\xi}\right)\rho(0) \tag{5.2.4}$$

From this the central density is

$$\rho(0) = -\left|\frac{\xi}{3}\frac{1}{d\theta_n/d\xi}\right|_{\xi=\xi_1}\overline{\rho} \tag{5.2.5}$$

This relation shows that for a polytrope of a given index ,n the central density is a definite multiple of the mean density.

$$\frac{\rho(0)}{\overline{\rho}} = -\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)\xi_1\right]^{-1} \tag{5.2.6}$$

5.3 Energy and Stability conditions

The thermal energy T and gravitational energy V for polytropes are given by

$$V = -\frac{3(\gamma-1)}{(5\gamma-6)}\frac{GM^2}{R} \tag{5.3.1}$$

$$T = \frac{-V}{3(\gamma-1)}$$

$$T = \frac{1}{(5\gamma-6)}\frac{GM^2}{R} \tag{5.3.2}$$

The total internal energy is

$$E = T + V \tag{5.3.3}$$

Then Combining eqn.(5..21) and eqn.(5.1.22) we obtain,

$$E = -(3\gamma-4)T = -\frac{(3\gamma-4)}{(5\gamma-6)}\frac{GM^2}{R} \tag{5.3.4}$$

For Stability Configuration

$$E = T + V \quad (5.3.5)$$

$$= a\rho^{\gamma-1} - b\rho^{1/3} \quad (5.3.6)$$

Where

$$a = \frac{KM}{(\gamma - 1)} \quad (5.3.7)$$

$$b = \frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3} GM^{5/3} \quad (5.3.8)$$

For $\gamma > 4/3$, E has a minimum corresponding to a configuration of stable equilibrium.

For $\gamma = 4/3$, E is stationary with respect to ρ only if it vanishes every where.

For $\gamma < 4/3$, E has a maximum corresponding to a state of unstable equilibrium.

Some numerical solutions of Lane-Emden equation is generated computationally by Mathematica and the roots of the equation for a range of polytropic indices(n=0 , 1 , ... , 6) are as the seen below.

This graph shows numerical solutions to the Lane-Emden equation for n = 0,1,2,3,4,5,6 . Using this graph by numerical integration of the Lane-Emden equation we find a family of solutions parameterized with different values of the central density , the radii and masses of compact objects.

The Lane-Emden equation solutions exist only for n=0, 1 and 5.

The rest can be integrated.

The radius of the star is determined from the numerical integration by the fast convergence to $\theta(\xi_1)$=0.

Then ξ_1 is converted back to give the radius of the compact. and M is determined by equation $\xi^2|\theta'(\xi_1)|$.

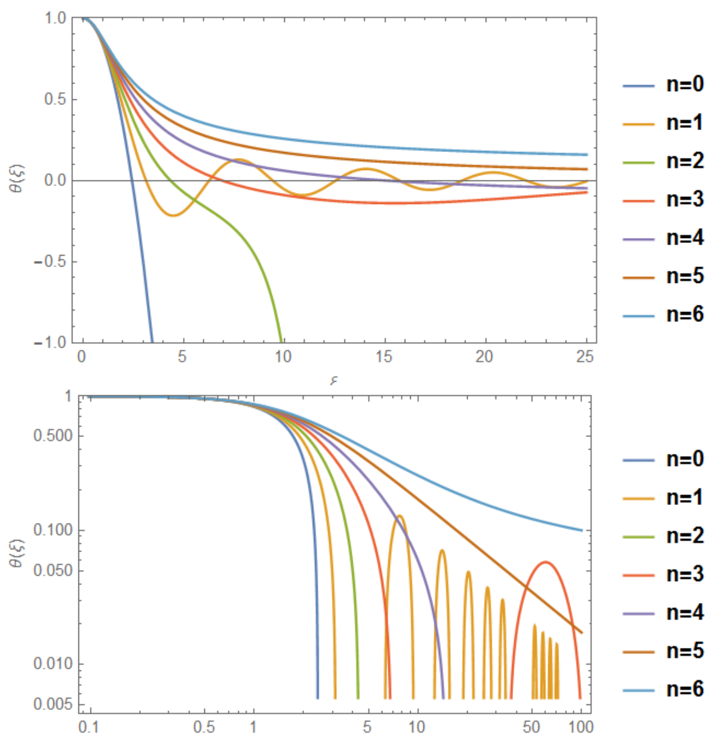

Figure 5.1: The plot of polytropic solutions for Mass-Radius relationship for compact objects with integral polytropic index, n = 0-6. As we see from the plot stable compact object solutions exit for the case of integral polytropic indices of n = 0 - 4, but higher indices n = 5 - 6 do not have. The lower plot is the LogLogPlot of the upper one, where it is used just for clarity indication of the behavior of the polytropes

n	ξ	$-\xi^2 \frac{d\theta(\xi)}{d\xi_1}$
0	2.449489	4.89898
1	3.141561	3.14159
2	4.35287	2.41104
3	6.89685	2.01824
4	14.97153	1.79723
5	∞	1.73204

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$
0............	2.4494	4.8988
0.5..........	2.7528	3.7871
1.0..........	3.14159	3.14159
1.5..........	3.65375	2.71406
2.0..........	4.35287	2.41105
2.5..........	5.35528	2.18720
3.0..........	6.89685	2.01824
3.25.........	8.01894	1.94980
3.5..........	9.53581	1.89056
4.0..........	14.97155	1.79723
4.5..........	31.83646	1.73780
4.9..........	169.47	1.7355
5.0..........	∞	1.73205

Figure 5.2: The upper table is the numerical solutions of the Lane-Emden equation that determine the radius and mass of the polytropes for n = 0, 1,2,3,4 & 5 produced by numerical integration using Mathematica 11 in this work. The lower table is taken from the work of Schuster-Emeden, Chandrasekhar presented in the book by Chandrasekhar 1967

5.4 Limiting Radius and Mass of White Dwarfs and Neutron stars

5.4.1 White Dwarfs

For the nonrelativistic case ,$\gamma = 5/3$.

$\xi^2|\theta'(\xi_1)|$=2.71406 and $\xi_1 = 3.65375$ Mass and radius of white dwarfs are

By using eq.(3.3.16)

$$M = \frac{1}{2}\left(\frac{3\pi}{8}\right)^{1/2}(2.71406)\left(\frac{\hbar^{3/2}c^{3/2}}{m_N\mu^2G^{3/2}}\right)\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} \tag{5.4.1}$$

$$M = 2.79\mu^{-2}\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot$$

By using equation(3.3.13) the radius of the star is

$$R = \left(\frac{3\pi}{8}\right)^{1/2}(3.65375)\left(\frac{\hbar^{3/2}}{c^{1/2}G^{1/2}m_em_N\mu}\right)\left(\frac{\rho(0)}{\rho_c}\right)^{-1/6} \tag{5.4.2}$$

$$R = 2.0 \times 10^4\mu^{-1}\left(\frac{\rho(0)}{\rho_c}\right)^{-1/6} km$$

In the $\gamma = 5/3$ range of white dwarfs, the mass is an increasing function of central density, and the radius is a decreasing function (for any $\gamma < 2$). A decrease of radius with an increase of mass is quite a general property of degenerate stars, and the physical reason is simple. The gravitational attraction grows as the mass increases, causing a greater compaction of the star.

For the relativistic case From the graph with the numerical value for $\gamma = 4/3$.

$\xi^2|\theta'(\xi_1)|$= 2.01824 and $\xi_1 = 6.89685$ gives a unique mass

$$M = \frac{1}{2}(3\pi)^{1/2}(2.01824)\left(\frac{\hbar^{3/2}c^{3/2}}{G^{3/2}m_N^2\mu^2}\right) \tag{5.4.3}$$

$$M = 5.87\mu^{-2}M_\odot = 1.4675M_\odot$$

Whereas equation of the radius of the star is

$$R = \frac{1}{2}(3\pi)^{1/2}(6.89685)\left(\frac{\hbar^{3/2}}{c^{1/2}G^{1/2}m_e m_N \mu}\right)\left(\frac{\rho_c}{\rho(0)}\right)^{1/3} \tag{5.4.4}$$

$$R = 5.3 \times 10^4 \mu^{-1}\left(\frac{\rho_c}{\rho 0}\right)^{1/3} km$$

We note that $\gamma > 4/3$ for $\rho(0) \ll \rho_c$,so the least massive white dwarfs are definitely stable.We also see that M appears to grow monotonically with increasing central density,reaching a maximum when $\rho(0) \to \infty$,so there is no point where the star can become unstable.So the stable white dwarfs can exist for any mass less than the above unique mass equation.This maximum mass is known as the chandrasekhar limit

For a typical white dwarf composition $\simeq 2$ and the limiting mass works out to be $1.46M_{\odot}$.

5.4.2 Neutron Star

For $\rho(0) \ll \rho_c$,

$$M = \frac{1}{2}\left(\frac{3\pi}{8}\right)^{1/2}(2.71406)\left(\frac{\hbar^{3/2}}{m_n^2 G^{3/2}}\right)\left(\frac{\rho(0)}{\rho_c}\right)^{1/2}$$

$$M = \frac{1}{2}(2.71406)M_0\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} \tag{5.4.5}$$

and the radius of the star is

$$R = \left(\frac{3\pi}{8}\right)^{1/2}(3.65375)\left(\frac{\hbar^{3/2}}{m_n^2 G^{3/2}}\right)\left(\frac{\rho(0)}{\rho_c}\right)^{1/6}$$

$$R = (3.65375)R_0\left(\frac{\rho(0)}{\rho_c}\right)^{1/6} \tag{5.4.6}$$

with ρ_c given by eq.(4.2.5)

From equation (4.2.18)

$$\rho(r) = \frac{3}{56\pi G r^2}$$

corresponding to the limit $\rho(0) \to \infty$. However,even in the limit of infinite central density, this $\rho(r)$ will drop ρ_0 at a radius r of order R_0,so that the equation of state (4.2.13) is not valid for the outer layers of any neutron star.To deal with the crust of nonrelativistic neutrons,it is necessary to solve the full equation of (3.1.22) using the equation of state (4.2.6);the condition of infinite central density is imposed by (4.2.15) for r $\ll R_0$.The important points are that the solution has a finite radius R where ρ vanishes, and that the mass M within this radius is finite, because the singularity is integrable at r=0. Thus the mass and radius of a neutron star approach finite limits as $\rho(0) \to \infty$.Numerical solution of the fundamental equation (3.1.22) gives the limits as

$$M_\infty = 0.171 M_0 \tag{5.4.7}$$

$$R_\infty = 1.06 R_0 \tag{5.4.8}$$

There remains the question of stability. For $\rho(0) \ll \rho_c$, a pure neutron star is simply a Newtonian polytrope with $\gamma = 5/3$(like a small white dwarf) and is therefore stable.

Equation eq.(4.2.11) shows that M is a monotonically increasing function of $\rho(0)$ for these small central densities. If M continues to increase monotonically to the value M_∞, then no transition to instability can occur.But eq.4.2.11) shows that when $\rho(0)$=0.016 ρ_c (which is small enough for eq.(4.2.11) to be a good approximation),the mass M is already greater than M_∞.Thus we expect that M rises to a maximum value $M > M_\infty$ at some central density ρ_m of order ρ_c,and then dropes to the value M_∞ at finite central density.

The mass M of a pure ideal-gas neutron star reaches a maximum

$$M_m = 0.7 M_\odot \tag{5.4.9}$$

at a radius

$$R_m = 9.6 km \tag{5.4.10}$$

Since this is a point where $\partial M/\partial\rho(0)$ vanishes, we expect the transition here from stability to instability with respect to radial oscillations.Thus characterize a neutron star with the greatest mass and central density allowed by the requirement that the star be stable.Thus mass is known as the Oppenheimer-Volkoff limit.

Chapter 6

Summary and Conclusion

General Theory of Relativity is used to derive the appropriate TOV-equations. With simplifying boundary conditions, the resulting TOV-equations be used to develop equation of state. Since, compact objects are characterized with high densities and low temperatures.By applying boundary conditions of high density,Polytropic equation of states are developed.Using the polytrope we develops the Lane-Emden equations and together with quantum mechanical theory(Fermi gas) to find to an expression for the EOS. This provides results for mass and radii of stars with different central pressure(i.e central density) the more accurate results from General Relativity of the TOV equations.

We have modeled white dwarfs and neutron stars as a non-interacting Fermi gas at zero temperature, supported against gravitational collapse by electron and neutron degeneracy pressure, respectively. With a non-relativistic polytropic equation of state, P $= K\rho^{5/3}$, this model produced white dwarf radii and masses which fit observational data for low mass white dwarfs. Using a relativistic polytropic equation of state, P $= K\rho 4/3$, reproduced the Chandrasekhar limit, but gave no predictions for low mass white dwarfs. This showed that a simple polytropic equation of state worked well in the regime it was made for, but that the difference in conditions within a white dwarf are too large to be wholly described by such a simple equation of state.

We calculated mass-radius relations for White Dwarfs using for relativistic and non relativistic polytropic approximations.For the relativistic case we find the computed masses to be constant the result is $1.4675 M_{\odot}$ and in a very good agreement with the chandrasekhar mass limit.

The properties of neutron star since the correct description of the equilibrium structure of these kinds of compact objects imposes the consideration of general relativity effects.The correct description of the equilibrium properties of neutron stars are provided by the TOV equations, which are derived from Einstein field equations applied to a time-invariant and spherically symmetric mass distribution. As a purely general relativity effect,neutron stars must have a maximum mass, for finite values of the central density and radius.For cold(T=0) neutron star entirely composed of free neutrons,the maximum mass is around $0.7 M_{\odot}$ with the radius $9.6 M_{\odot}$.

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