Non-Degenerate Three Level Laser

Coupled to Thermal Reservoir

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This is to certify that the thesis prepared by Solomon Araya, entitled: *Non-Degenerate Three Level Laser Coupled to Thermal Reservoir* and submitted in Partial fulfillment of the requirements for the Degree of Masters of Science in Physics (Quantum Optics) complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Abstract

In this research, we analyze quadrature squeezing and photon statistics of the light produced by non-degenarate three level laser coupled to thermal reservoir. It is clearly shown that the mean photon number of the system increases with $\bar{n}$ but decreases with η. Moreover, it is also found that coupling of the atoms with thermal light indeed affects the squeezing of the laser.

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Contents

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Introduction

Interaction of three-level atoms with a radiation has attracted a great deal of interest in recent years [1-12]. It is believed that an atomic coherence is found to be responsible for various important quantum features of the emitted light. A three level laser may be defined as a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels, are injected into a cavity coupled to vacuum reservoir via a single-port mirror has been studied by different authors [31]. These atoms are removed from the cavity after some time. When three level atoms make a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, the quantum optical system is said to be a degenerate three level atom; otherwise it is called a non degenerate three level atom. The two photons are highly correlated and this correlation is responsible for the squeezing of light generated by a three level laser. In a cascade three-level atom the top, intermediate, and bottom levels are conveniently denoted by $|a\rangle$, $|b\rangle$, and $|c\rangle$ in which a direct transition between levels $|a\rangle$ and $|c\rangle$ is dipole forbidden. We hence realize that a non degenerate three-level laser is a two photon device in which squeezing properties are expected to occur due to the correlation between these two photons [2, 4]. Some authors have studied the squeezing and statistical properties of the light produced by three-level laser in which the crucial role is played by the superposition of the top and bottom levels [1-7]. Ansari [7] has predicted that such a laser can generate under certain conditions squeezed light.

On the other hand, S . Tesfa has studied the squeezing and statistical properties of the light generated by a non-degenerate three-level laser coupled to squeezed vacuum reservoir [18]. Furthermore, Lu and Zhu have considered a non degenerate three-level laser with the atoms initially prepared in coherent superposition of the top and bottom levels. The coherent superposition of the top and bottom levels of injected atoms shows that the quantum optical system can generate light in a squeezed state under certain conditions [7].

A mechanism can be considered as an option for producing a squeezed light when it is difficult to prepare the atoms in an arbitrary initial superposition. Driving the atoms with an external coherent radiation affects both the degree of squeezing and intensity of the generated light. When atoms are driven externally with a strong radiation the resulting squeezing and intensity of the cavity radiation is considerably reduced. However, an intense radiation with a substantial degree of squeezing can be generated especially when the atoms are initially prepared with equal probability of being in the bottom and top levels where there is no squeezing in the absence of the driving radiation. In addition, it is possible to get a squeezed light when the atoms are initially in the bottom level by this mechanism where there is no radiation at all in the absence of driving. We hence realize that driving mechanism can be considered as an option for producing a squeezed light when it is difficult to prepare the atoms in an arbitrary initial superposition. Though it appears reasonable to expect enhancement of squeezing when we externally drive atoms with an arbitrary initial superposition, we are unable to confirm this for all possible cases from our analysis. However, driving the atoms with external radiation is found to significantly improve the squeezing, if the atoms are prepared initially with maximum atomic coherence [18]. A three level laser whose cavity contains a parametric amplifier and with the injected atoms prepared initially in coherent superposition of the top and bottom levels in a laser cavity increases significantly the mean photon number [19, 20]. The parametric amplifier, the driving coherent light, and the squeezed vacuum reservoir on the three-level laser are to enhance both the degree of squeezing and the mean photon number, a bright and highly squeezed light can be produced by the quantum optical system. Hence, injected squeezed light greatly enhances the intra-cavity squeezing in the two-mode light for suitable choice of parameters. The injected squeezed light increases the mean photon number considerably [19]. Moreover, the quantum analysis of a three-level laser is usually carried out by including the interaction of the atoms inside the cavity with a reservoir outside the cavity.

In this research, employing the pertinent Hamiltonian, we first obtain the master equation for the non degenerate three level laser coupled to thermal reservoir via a single port mirror. The resulting master equation is easily adaptable to determine the stochastic differential equations of the system under consideration. In addition, using the solutions of c-number Langevin equations, the correlation properties of the noise forces, and the Q function together with density operator will be determined.

Furthermore, with the aid of the resulting Q function along with the density operator, we calculate the mean photon number, the variance of the photon number, the photon number distribution, and the quadrature fluctuations.

2

The Q Function

The first three sections of this chapter focus on developing the master equation, the stochastic differantial equations, and the solutions of the c-number Langevin equations. In the last two sections, we determined the Q function and the density operator.

2.1 The master equation

In a non-degenarate three level laser, non-degenarate three level atoms in cascade configeration are injected at a constant rate r_a, and removed from the cavity after certain time τ. We denote the top, intermediate, and bottom levels of a three level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. In addition, we assume the cavity mode to be at resonance with two transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$, with direct transition bewteen levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden.

The interaction of a non degenerate three-level atom with the cavity modes can be described by the Hamiltonian,

$$\hat{H} = ig(|a\rangle\langle b|\hat{a} - \hat{a}^\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^\dagger|c\rangle\langle b|), \tag{2.1}$$

where g is the coupling constant and $\hat{a}(\hat{b})$ are the annihilation operators for the cavity modes. We take the initial state of a single three-level atom to be

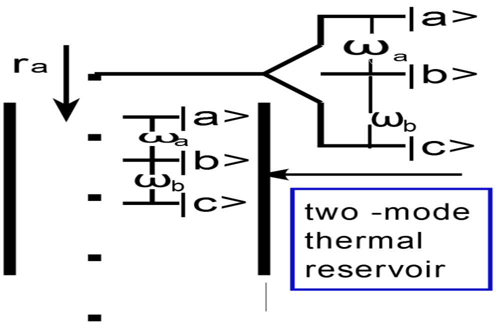

Figure 2.1: A schematic representation of non-degenarate three level laser coupled to two mode thermal reservoir.

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle \tag{2.2}$$

and hence the initial density operator of a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.3}$$

where

$$\rho_{aa}^{(0)} = C_a^* C_a, \tag{2.4}$$

$$\rho_{ac}^{(0)} = C_a C_c^*, \tag{2.5}$$

$$\rho_{ca}^{(0)} = C_c C_a^*, \tag{2.6}$$

$$\rho_{cc}^{(0)} = C_c^* C_c. \tag{2.7}$$

Suppose $\hat{\rho}_{AR}(t,t_j)$ is the dentity operator for a single atom plus the cavity mode at time t , with the atom injected at time t_j such that $(t-\tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as $\hat{\rho}_{AR}(t) = \sum_j N_j \hat{\rho}_{AR}(t,t_j)$.

Then it follows

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t,t_j)\Delta t_j, \tag{2.8}$$

where $N = r_a \Delta t_j$ represents the number of atoms injected into the cavity in a time Δt_j.

Now converting the summation into integration in the time $\Delta t_j \to 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{(t)} \hat{\rho}_{AR}(t,t'_j)dt'_j \tag{2.9}$$

and on differentiating with respect to t, there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t,t-\tau)] + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.10}$$

We observe that $\hat{\rho}_{AR}(t,t)$ is the density operator for the cavity modes plus an atom injected at time t. This operator can thus be expressed as

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(0)\hat{\rho}(t), \tag{2.11}$$

with $\hat{\rho}(t)$ being the desity operator for the cavity mode alone.

We also note that $\hat{\rho}_{AR}(t,t-\tau)$ representes the density operator for an atom plus the cavity modes at time t, with the atom being removed from the cavity at this time.

Thus Eq. (2.11) can also be put in the form

$$\hat{\rho}_{AR}(t,t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{2.12}$$

Now in view of Eqs. (2.11) and (2.12), one can rewrite Eq. (2.10) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t)=r_a[\hat{\rho}_A(0)-\hat{\rho}_A(t-\tau)]\hat{\rho}(t)+r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.13}$$

The desity operator evolves in time, in the interaction picture, as

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')=-i[\hat{H}_I,\hat{\rho}_{AR}(t,t')], \tag{2.14}$$

so that taking into account Eq. (2.14), one can put Eq. (2.13) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t)=r_a[\hat{\rho}_A(0)-\hat{\rho}_A(t-\tau)]\hat{\rho}(t)-i[\hat{H}_I,\hat{\rho}_{AR}(t)]. \tag{2.15}$$

Furthermore, tracing over the atomic variables and taking into account the damping of the cavity mode by a thermal reservoir, we have [31].

$$\frac{d}{dt}\rho(t)=-iTr_A[\hat{H}_I,\hat{\rho}_{AR}(t,t)]-h\langle\hat{H}_{SR}^2\hat{R}\rangle_R\,\rho(t)+2hTr_R(\hat{H}_{SR}\,\hat{\rho}(t)\hat{R}\hat{H}_{SR})$$
$$-h\hat{\rho}(t)\langle\hat{H}_{SR}^2\hat{R}\rangle_R, \tag{2.16}$$

in which Tr_A stands for trace over atomic variables and we have used the fact that

$$-i[H_s,\rho(t)]=-iTr_A[\hat{H}_I,\hat{\rho}_{AR}(t,t')] \tag{2.17}$$

and

$$Tr\,\hat{\rho}_A(t)=Tr\,\hat{\rho}_A(t-\tau)=1. \tag{2.18}$$

On the other hand, the interaction Hamilitonian for a two- mode cavity light coupled to a two-mode thermal reservoir is written as

$$\hat{H}_{SR}=i\lambda(\hat{a}^\dagger\hat{a}_{in}-\hat{a}_{in}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}_{in}-\hat{b}_{in}^\dagger\hat{b}), \tag{2.19}$$

where λ is the coupling constant and $\hat{a}_{in}(\hat{b}_{in})$ are the annihilation operators of cavity mode a (cavity mode b) associated with the first (second) thermal reservoirs.

Employing Eq. (2.19), we then see that

$$hTr_R(\hat{H}_{SR}^2\hat{R}) = hTr_R\bigg(i\lambda(\hat{a}^\dagger\hat{a}_{in} - \hat{a}_{in}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}_{in} - \hat{b}_{in}^\dagger\hat{b})\bigg)^2. \tag{2.20}$$

Thus Eq. (2.20) can be rewritten as

$$\begin{aligned}hTr_R(\hat{H}_{SR}^2\hat{R}) = -h\lambda^2 Tr_R\Bigg[&(\hat{a}^\dagger\hat{a}_{in}\hat{a}^\dagger\hat{a}_{in}) - (\hat{a}^\dagger\hat{a}_{in}\hat{a}_{in}^\dagger\hat{a})\\ &+(\hat{a}^\dagger\hat{a}_{in}\hat{b}^\dagger\hat{b}_{in}) - (\hat{a}^\dagger\hat{a}_{in}\hat{b}_{in}^\dagger\hat{b}) - (\hat{a}_{in}^\dagger\hat{a}\hat{a}^\dagger\hat{a}_{in})\\ &+(\hat{a}_{in}^\dagger\hat{a}\hat{a}_{in}^\dagger\hat{a}) - (\hat{a}_{in}^\dagger\hat{a}\hat{b}^\dagger\hat{b}_{in}) + (\hat{a}_{in}^\dagger\hat{a}\hat{b}_{in}^\dagger\hat{b})\\ &+(\hat{b}^\dagger\hat{b}_{in}\hat{a}^\dagger\hat{a}_{in}) - \hat{b}^\dagger\hat{b}_{in}\hat{a}_{in}^\dagger\hat{a}) + (\hat{b}^\dagger\hat{b}_{in}\hat{b}^\dagger\hat{b}_{in})\\ &-\langle\hat{b}^\dagger\hat{b}_{in}\hat{b}_{in}^\dagger\hat{b}) - (\hat{b}_{in}^\dagger\hat{b}\hat{a}^\dagger\hat{a}_{in}) + (\hat{b}_{in}^\dagger\hat{b}\hat{a}_{in}^\dagger\hat{a})\\ &-(\hat{b}_{in}^\dagger\hat{b}\hat{b}^\dagger\hat{b}_{in}) + (\hat{b}_{in}^\dagger\hat{b}\hat{b}_{in}^\dagger\hat{b})\Bigg].\end{aligned} \tag{2.21}$$

Since the cavity mode operators commute with reservior mode operators. Then we observe that

$$\begin{aligned}hTr_R(\hat{H}_{SR}^2\hat{R}) = -h\lambda^2\big[&\hat{a}^{\dagger 2}\langle\hat{a}_{in}^2\rangle_R - \hat{a}^\dagger\hat{a}\langle\hat{a}_{in}\hat{a}_{in}^\dagger\rangle_R\\ &+\hat{a}^\dagger\hat{b}^\dagger\langle\hat{a}_{in}\hat{b}_{in}\rangle_R - \hat{a}^\dagger\hat{b}\langle\hat{a}_{in}\hat{b}_{in}^\dagger\rangle_R - \hat{a}\hat{a}^\dagger\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle_R\\ &+\hat{a}^2\langle\hat{a}_{in}^{\dagger 2}\rangle_R - \hat{a}\hat{b}^\dagger\langle\hat{a}_{in}^\dagger\hat{b}_{in}\rangle_R + \hat{a}\hat{b}\langle\hat{a}_{in}^\dagger\hat{b}_{in}^\dagger\rangle_R\\ &+\hat{b}^\dagger\hat{a}^\dagger\langle\hat{b}_{in}\hat{a}_{in}\rangle_R - \hat{b}^\dagger\hat{a}\langle\hat{b}_{in}\hat{a}_{in}^\dagger\rangle_R + \hat{b}^{\dagger 2}\langle\hat{b}_{in}^2\rangle_R\\ &-\hat{b}^\dagger\hat{b}\langle\hat{b}_{in}\hat{b}_{in}^\dagger\rangle_R - \hat{b}\hat{a}^\dagger\langle\hat{b}_{in}^\dagger\hat{a}_{in}\rangle_R + \hat{b}\hat{a}\langle\hat{b}_{in}^\dagger\hat{a}_{in}^\dagger\rangle_R\\ &-\hat{b}\hat{b}^\dagger\langle\hat{b}_{in}^\dagger\hat{b}_{in}\rangle_R + \hat{b}^2\langle\hat{b}_{in}^{\dagger 2}\rangle_R\Big].\end{aligned} \tag{2.22}$$

New using the density operator of the thermal reservior

$$\hat{R} = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(1+\bar{n})^{n+1})}|n\rangle\langle n|, \tag{2.23}$$

one finds that

$$\langle\hat{a}_{in}^2\rangle_R = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(1+\bar{n})^{n+1}}|n\rangle\langle n|\hat{a}_{in}^2|n\rangle. \tag{2.24}$$

It then follows

$$\langle \hat{a}_{in}^2 \rangle_R = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \sqrt{n(n-1)} \langle n|n-2\rangle = 0, \tag{2.25}$$

where

$$\langle n|n-2\rangle = 0.$$

Follwing the same procedure, we obtain

$$\langle \hat{a}_{in}^2 \rangle = \langle \hat{b}_{in}^2 \rangle = \langle \hat{a}_{in}^{\dagger 2} \rangle = \langle \hat{b}_{in}^{\dagger 2} \rangle = 0 \tag{2.26}$$

and

$$\begin{aligned} \langle \hat{a}_{in}^\dagger \hat{b}_{in} \rangle_R = & \langle \hat{a}_{in} \hat{b}_{in}^\dagger \rangle_R = \langle \hat{b}_{in}^\dagger \hat{a}_{in} \rangle_R = \langle \hat{b}_{in} \hat{a}_{in}^\dagger \rangle_R \\ & = \langle \hat{b}_{in} \hat{a}_{in}^\dagger \rangle_R = \langle \hat{b}_{in} \hat{a}_{in} \rangle_R = \langle \hat{b}_{in}^\dagger \hat{a}_{in}^\dagger \rangle_R = \langle \hat{a}_{in}^\dagger \hat{b}_{in}^\dagger \rangle_R = 0. \end{aligned} \tag{2.27}$$

In addition, applying the commutation relation

$$[\hat{a}_{in}, \hat{a}_{in}^\dagger] = 1, \tag{2.28}$$

we then note that

$$\langle \hat{a}_{in} \hat{a}_{in}^\dagger \rangle = \bar{n}_a + 1, \tag{2.29}$$

$$\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle = \bar{n}_a, \tag{2.30}$$

$$\langle \hat{b}_{in} \hat{b}_{in}^\dagger \rangle = \bar{n}_b + 1, \tag{2.31}$$

$$\langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle = \bar{n}_b, \tag{2.32}$$

$\bar{n}_a$ and $\bar{n}_b$ are the mean photon numbers of mode a and mode b of a two-mode thermal reservoir, respectively and we assumme $\bar{n}_a = \bar{n}_b = \bar{n}$.

Hence on substituting Eqs. (2.26), (2.27), (2.29) - (2.32) into Eq. (2.22), there follows

$$hTr_R(\hat{H}_{SR}^2\hat{R})\hat{\rho}(t) = h\lambda^2[(\bar{n}+1)(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{b}^\dagger\hat{b}\hat{\rho} + \bar{n}(\hat{a}\hat{a}^\dagger\rho + \hat{b}\hat{b}^\dagger\hat{\rho})]. \quad (2.33)$$

In the same manner, we observe that

$$h\hat{\rho}(t)Tr_R(\hat{H}_{SR}^2\hat{R}) = h\lambda^2[(\bar{n}+1)(\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{b}^\dagger\hat{b}) + \bar{n}(\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{b}\hat{b}^\dagger)]. \quad (2.34)$$

In addition, one readly find

$$\begin{aligned} 2hTr_R[\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}] = -2h\lambda^2\Big[& \hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\langle\hat{a}_{in}^2\rangle_R - \hat{a}^\dagger\hat{\rho}\hat{a}\langle\hat{a}_{in}^\dagger\hat{a}_{in}\rangle_R + \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\langle\hat{b}_{in}\hat{a}_{in}\rangle_R \\ & -\hat{a}^\dagger\hat{\rho}\hat{b}\langle\hat{b}_{in}^\dagger\hat{a}_{in}\rangle_R - \hat{a}\hat{\rho}\hat{a}^\dagger\langle\hat{a}_{in}\hat{a}_{in}^\dagger\rangle_R + \hat{a}\hat{\rho}\hat{a}\langle\hat{a}_{in}^{\dagger 2}\rangle_R \\ & -\hat{a}\hat{\rho}\hat{b}^\dagger\langle\hat{b}_{in}\hat{a}_{in}^\dagger\rangle_R + \hat{a}\hat{\rho}\hat{b}\langle\hat{b}_{in}^\dagger\hat{a}_{in}^\dagger\rangle_R + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\langle\hat{a}_{in}\hat{b}_{in}\rangle_R \\ & -\hat{b}^\dagger\hat{\rho}\hat{a}\langle\hat{a}_{in}^\dagger\hat{b}_{in}\rangle_R + \hat{b}^\dagger\hat{\rho}\hat{b}^\dagger\langle\hat{b}_{in}\hat{b}_{in}\rangle_R - \hat{b}^\dagger\hat{\rho}\hat{b}\langle\hat{b}_{in}^\dagger\hat{b}_{in}\rangle_R \\ & -\hat{b}\hat{\rho}\hat{a}^\dagger\langle\hat{a}_{in}\hat{b}_{in}^\dagger\rangle_R + \hat{b}\hat{\rho}\hat{a}\langle\hat{a}_{in}^\dagger\hat{b}_{in}^\dagger\rangle_R - \hat{b}\hat{\rho}\hat{b}^\dagger\langle\hat{b}_{in}\hat{b}_{in}^\dagger\rangle_R \\ & +\hat{b}\hat{\rho}\hat{b}\langle\hat{b}_{in}^\dagger\hat{b}_{in}^\dagger\rangle_R\Big], \end{aligned} \quad (2.35)$$

so that application of Eqs. (2.27), (2.29) - (2.32) in Eq. (2.35) leads to

$$\begin{aligned} 2hTr_R[\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}] = 2\lambda^2 h[\bar{n}(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{b}) \\ +(\bar{n}+1)(\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho}\hat{b}^\dagger)]. \end{aligned} \quad (2.36)$$

Taking into account Eq. (2.33), (2.34), and (2.36), the expression in Eq. (2.16) takes the form

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) = -iTr_A[\hat{H}_I, \rho_{AR}(t,t)] + \frac{\kappa}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ +\frac{\kappa}{2}\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \frac{\kappa}{2}(\bar{n}+1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ +\frac{\kappa}{2}\bar{n}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\rho - \hat{\rho}\hat{b}\hat{b}^\dagger), \end{aligned} \quad (2.37)$$

where $\kappa_a = \kappa_b = \kappa = 2h\lambda^2$ is cavity damping constant.

Moreover, employing Eq. (2.1), the master equation for the cavity modes coupled to thermal reservior, can be put in the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = & g(\rho_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \rho_{bc}\hat{b}^\dagger - \hat{b}^\dagger\rho_{bc} + \hat{a}\rho_{ba} - \rho_{ba}\hat{a} + \hat{b}\rho_{cb} - \rho_{cb}\hat{b}) \\ & + \frac{\kappa}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) \\ & + \frac{\kappa}{2}\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{\kappa}{2}(\bar{n}+1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ & + \frac{\kappa}{2}\bar{n}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\rho - \hat{\rho}\hat{b}\hat{b}^\dagger), \end{aligned} \tag{2.38}$$

in which the matrix element $\rho_{\alpha\beta}$ is defined by

$$\rho_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.39}$$

with $\alpha, \beta = a,\ b,\ and\ c.$

On the other hand, from Eq. (2.15), we see that

$$\begin{aligned}\frac{d}{dt}\rho_{\alpha\beta}(t) = & [r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - r_a\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle]\hat{\rho}(t) \\ & -i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}(t,t')|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}(t,t')\hat{H}_I|\beta\rangle] - \gamma\rho_{\alpha\beta}, \end{aligned} \tag{2.40}$$

where the last term is included to account for the decay of atoms due to spontaneous emission. Here γ, considered to be the same for all the three levels, is the atomic decay constant. We assume that the atoms are removed from the cavity after they have decayed to a level other than the intermedate or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \tag{2.41}$$

and hence Eq. (2.40) reduces to

$$\frac{d}{dt}\rho_{\alpha\beta}(t) = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t)$$
$$-i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}(t,t')|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}(t,t')\hat{H}_I|\beta\rangle] - \gamma\rho_{\alpha\beta}. \tag{2.42}$$

Applying this equation and taking into account Eqs. (2.1) and (2.3), one readily obtains

$$\frac{d}{dt}\rho_{ab} = g(\rho_{ab}^{(o)}\hat{b}^\dagger + \hat{a}\rho_{bb}^{(o)} - \rho_{aa}^{(o)}\hat{a}) - \gamma\rho_{ab}, \tag{2.43}$$

$$\frac{d}{dt}\rho_{bc} = g(\hat{b}\rho_{cc}^{(o)} - \rho_{bb}^{(o)}\hat{b} - \hat{a}^\dagger\rho_{cc}^{(o)}) - \gamma\rho_{bc}, \tag{2.44}$$

$$\frac{d}{dt}\rho_{aa} = r_a\rho_{aa}^{(o)}\hat{\rho}(t) + g(\hat{a}\rho_{ba}^{(o)} + \rho_{ab}^{(o)}\hat{a}^\dagger) - \gamma\rho_{aa}, \tag{2.45}$$

$$\frac{d}{dt}\rho_{bb} = g(\hat{b}\rho_{cb}^{(o)} - \rho_{ba}^{(o)}\hat{a} - \hat{a}^\dagger\rho_{ab}^{(o)} + \rho_{bc}^{(o)}\hat{b}^\dagger) - \gamma\rho_{bb}, \tag{2.46}$$

$$\frac{d}{dt}\rho_{ac} = r_a\rho_{ac}^{(o)}\hat{\rho}(t) + g(\hat{a}\rho_{bc}^{(o)} - \rho_{ab}^{(o)}\hat{b}) - \gamma\rho_{ac}, \tag{2.47}$$

$$\frac{d}{dt}\rho_{cc} = r_a\rho_{cc}^{(o)}\hat{\rho}(t) - g(\hat{b}^\dagger\rho_{bc}^{(o)} + \rho_{cb}^{(o)}\hat{b}) - \gamma\rho_{cc}. \tag{2.48}$$

We confine ourselves in a linear analysis and this can be achived by dropping the g terms in Eqs. (2.45)-(2.48). Applying the large-time aproximation scheme Eqs. (2.45) - (2.48), we get

$$\rho_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}(t), \tag{2.49}$$

$$\rho_{bb} = 0, \tag{2.50}$$

$$\rho_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}(t), \tag{2.51}$$

$$\rho_{cc} = \frac{r_a \rho_{cc}^{(0)}}{\gamma}\hat{\rho}(t). \tag{2.52}$$

Now combination of Eqs. (2.43), (2.44), and (2.49)-(2.52) leads to

$$\frac{d}{dt}\rho_{ab} = \frac{g r_a}{\gamma}(\rho_{ac}^{(o)}\hat{\rho}\hat{b}^\dagger - \rho_{aa}^{(o)}\hat{\rho}\hat{a}) - \gamma\rho_{ab}, \tag{2.53}$$

$$\frac{d}{dt}\rho_{bc} = \frac{g r_a}{\gamma}(\rho_{cc}^{(o)}\hat{b}\hat{\rho} - \rho_{ac}^{(o)}\hat{a}^\dagger\hat{\rho}) - \gamma\rho_{bc}. \tag{2.54}$$

Using once more the large-time approximation scheme to Eqs. (2.53) - (2.54), we easly find

$$\rho_{ab} = \frac{g r_a}{\gamma^2}(\rho_{ac}^{(o)}\hat{\rho}\hat{b}^\dagger - \rho_{aa}^{(o)}\hat{\rho}\hat{a}), \tag{2.55}$$

$$\rho_{bc} = \frac{g r_a}{\gamma^2}(\rho_{cc}^{(o)}\hat{b}\hat{\rho} - \rho_{ac}^{(o)}\hat{a}^\dagger\hat{\rho}). \tag{2.56}$$

Finally, on acount of Eqs. (2.55) and (2.56), the equation of evolution for the reduced density operator for the cavity modes coupled to two-mode thermal reservior turns out to be

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = {} & \frac{\kappa}{2}(\bar{n}+1)\Big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big) + \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa\bar{n})\Big(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\Big) \\ & + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa(\bar{n}+1)\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\Big) + \frac{1}{2}\kappa\bar{n}\Big(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\rho - \hat{\rho}\hat{b}\hat{b}^\dagger\Big) \\ & + \frac{1}{2}A\rho_{ac}^{(0)}\Big(\hat{a}\hat{b}\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{b}\hat{\rho}\hat{a}\Big) + \frac{1}{2}A\rho_{ca}^{(0)}\Big(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a}\hat{b} - \hat{b}\hat{\rho}\hat{a}\Big),\end{aligned} \tag{2.57}$$

where $A = \frac{2r_a g^2}{\gamma^2}$ is a linear gain coefficient.

This is the master equation for the non-degenarate three level laser coupled to thermal reservoir.

2.2 c-number Langevin equations

Next we seek to determine the solutions of the c-number Langevin equations. Thus employing

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}(t)}{dt}\hat{A}\right), \tag{2.58}$$

along with Eq. (2.57) and applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1 \tag{2.59}$$

and

$$[\hat{a}, \hat{a}] = [\hat{b}, \hat{b}] = [\hat{a}, \hat{b}] = 0, \tag{2.60}$$

we readily obtain

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^{\dagger}(t)\rangle, \tag{2.61}$$

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^2(t)\rangle - A\rho_{ac}^{(0)}\langle\hat{b}^{\dagger}(t)\hat{a}(t)\rangle, \tag{2.62}$$

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}(t)\rangle, \tag{2.63}$$

$$\frac{d}{dt}\langle\hat{b}^2(t)\rangle = -(A\rho_{cc}^{(0)} + \kappa)\langle\hat{b}^2(t)\rangle + A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle, \tag{2.64}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = &-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}(t)\hat{b}(t)\rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\hat{a}(t)b(t)\rangle \\ &+\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)},\end{aligned} \tag{2.65}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = &-(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}(t)\hat{b}(t)\rangle \\ &-\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\rangle + A\rho_{aa}^{(0)} + \kappa\bar{n},\end{aligned} \tag{2.66}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle$$
$$-\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^{2}(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger 2}(t)\rangle, \quad (2.67)$$

$$\frac{d}{dt}\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle = -(A\rho_{cc}^{(0)} + \kappa)\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}(t)b^{\dagger}(t)\rangle$$
$$+\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}(t)\hat{b}(t)\rangle + \kappa\bar{n}, \quad (2.68)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^{\dagger}(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}(t)\rangle, \quad (2.69)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger 2}(t)\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^{\dagger 2}(t)\rangle - A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}(t)\hat{b}(t)\rangle, \quad (2.70)$$

$$\frac{d}{dt}\langle\hat{b}^{\dagger}(t)\rangle = -\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\hat{b}^{\dagger}(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^(t)\rangle, \quad (2.71)$$

$$\frac{d}{dt}\langle\hat{b}^{\dagger 2}(t)\rangle = -(A\rho_{cc}^{(0)} + \kappa)\langle\hat{b}^{\dagger 2}(t)\rangle + A\rho_{ac}^{(0)}\langle\hat{b}^{\dagger}(t)\hat{a}(t)\rangle. \quad (2.72)$$

The c-number stochastic differential equations corresponding to Eqs. (2.61)-(2.72) in normal order are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\alpha(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^{*}(t)\rangle, \quad (2.73)$$

$$\frac{d}{dt}\langle\alpha^{2}(t)\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\alpha^{2}(t)\rangle - A\rho_{ac}^{(0)}\langle\hat{\beta}^{*}(t)\alpha(t)\rangle, \quad (2.74)$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\beta(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^{*}(t)\rangle, \quad (2.75)$$

$$\frac{d}{dt}\langle\beta^{2}(t)\rangle = -(A\rho_{cc}^{(0)} + \kappa)\langle\beta^{2}(t)\rangle + A\rho_{ac}^{(0)}\langle\alpha^{*}(t)\beta(t)\rangle, \quad (2.76)$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\alpha(t)\beta(t)\rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\alpha(t)\beta(t)\rangle$$
$$+\frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^{*}(t)\alpha(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^{*}(t)\beta(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}, \quad (2.77)$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -(\kappa - A\rho^{(0)}_{aa})\langle\alpha^*(t)\alpha(t)\rangle - \frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha(t)\beta(t)\rangle$$
$$-\frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha^*(t)\beta^*(t)\rangle + (A\rho^{(0)}_{aa} + \kappa\bar{n}), \tag{2.78}$$

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = -\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})\langle\alpha^*(t)\beta(t)\rangle - \frac{1}{2}(A\rho^{(0)}_{cc} + \kappa)\langle\alpha^*(t)\beta(t)\rangle$$
$$-\frac{1}{2}A\rho^{(0)}_{ac}\langle\beta^2(t)\rangle + \frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha^{*2}(t)\rangle, \tag{2.79}$$

$$\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle = -(A\rho^{(0)}_{cc} + \kappa)\langle\beta^*(t)\beta(t)\rangle + \frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha^*(t)\beta^*(t)\rangle$$
$$+\frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha(t)\beta(t)\rangle + \kappa\bar{n}, \tag{2.80}$$

$$\frac{d}{dt}\langle\alpha^*(t)\rangle = -\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})\langle\alpha^*(t)\rangle - \frac{1}{2}A\rho^{(0)}_{ac}\langle\beta(t)\rangle, \tag{2.81}$$

$$\frac{d}{dt}\langle\alpha^{*2}(t)\rangle = -(\kappa - A\rho^{(0)}_{aa})\langle\alpha^{*2}(t)\rangle - A\rho^{(0)}_{ac}\langle\alpha^*(t)\beta(t)\rangle, \tag{2.82}$$

$$\frac{d}{dt}\langle\beta^*(t)\rangle = -\frac{1}{2}(A\rho^{(0)}_{cc} + \kappa)\langle\beta^*(t)\rangle + \frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha(t)\rangle, \tag{2.83}$$

$$\frac{d}{dt}\langle\beta^{*2}(t)\rangle = -(A\rho^{(0)}_{cc} + \kappa)\langle\beta^{*2}(t)\rangle + A\rho^{(0)}_{ac}\langle\beta^*(t)\alpha(t)\rangle. \tag{2.84}$$

With the aid of Eqs. (2.73) and (2.75), one can write the c-number Langevin equations in the form

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})\alpha(t) - \frac{1}{2}A\rho^{(0)}_{ac}\beta^*(t) + f_\alpha(t), \tag{2.85}$$

$$\frac{d}{dt}\beta(t) = -\frac{1}{2}(A\rho^{(0)}_{cc} + \kappa)\beta(t) + \frac{1}{2}A\rho^{(0)}_{ac}\alpha^*(t) + f_\beta(t), \tag{2.86}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces where their correlation properties remain to be determined. Taking the expection value of Eqs. (2.85) and (2.86), we see that

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})\langle\alpha(t)\rangle - \frac{1}{2}A\rho^{(0)}_{ac}\langle\beta^*(t)\rangle + \langle f_\alpha(t)\rangle, \tag{2.87}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\beta(t)\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*(t)\rangle + \langle f_\beta(t)\rangle. \tag{2.88}$$

Eqs. (2.73) and (2.87) as well as Eqs. (2.75) and (2.88) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.89}$$

Moreover, using the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \langle\alpha(t)\frac{d}{dt}\alpha(t)\rangle + \langle(\frac{d}{dt}\alpha(t))\alpha(t)\rangle, \tag{2.90}$$

along with Eq. (2.85), one can readily verify that

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\alpha^2(t)\rangle - A\rho_{ac}^{(0)}\langle\beta^*(t)\alpha(t)\rangle + \langle 2\alpha(t)f_\alpha(t)\rangle. \tag{2.91}$$

Now comparing Eqs. (2.74) and (2.91), we observe that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{2.92}$$

Employing Eq. (2.86) in relation

$$\frac{d}{dt}\langle\beta(t)\rangle = \langle\beta(t)\frac{d}{dt}\beta(t)\rangle + \langle(\frac{d}{dt}\beta(t))\beta(t)\rangle,$$

one can readily verify that

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -(A\rho_{cc}^{(0)} + \kappa)\langle\beta^2(t)\rangle + A\rho_{ca}^{(0)}\langle\alpha^*(t)\beta(t)\rangle + \langle 2\beta(t)f_\beta(t)\rangle. \tag{2.93}$$

Upon comparing of Eqs. (2.76) and (2.93), we see that

$$\langle\beta(t)f_\beta(t)\rangle = 0. \tag{2.94}$$

Furthermore, a formal solution of Eq. (2.85) can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})t} + \int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}\left(f_\alpha(t') - \frac{1}{2}A\rho_{ac}^{(0)}\beta^*(t')\right)dt'. \tag{2.95}$$

Multiplying both sides of Eq. (2.95) by $f_\alpha(t)$ on the right and taking the expectation value of the resulting expression, we have

$$\langle\alpha(t)f_\alpha(t)\rangle = \langle\alpha(0)f_\alpha(t)\rangle e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})t} + \int_0^t e^{-\frac{1}{2}(\kappa - A\rho(0)_{aa})(t-t')} \left(\langle f_\alpha(t')f_\alpha(t)\rangle - \frac{1}{2}A\rho^{(0)}_{ac}(\langle\beta^*(t')f_\alpha(t)\rangle\right)dt'. \tag{2.96}$$

Since a noise force at latter time does not affect c-number variabile in earlier time, one can write

$$\langle\alpha(0)f_\alpha(t)\rangle = \langle\alpha(0)\rangle\langle f_\alpha(t)\rangle = 0 \tag{2.97}$$

and

$$\langle\beta^*(t)f_\alpha(t) = \langle\beta^*(t)\rangle\langle f_\alpha(t)\rangle = 0. \tag{2.98}$$

Upon substituting Eqs. (2.92), (2.97), and (2.98) into Eq. (2.96), we get

$$\int_0^t e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})(t-t')}\langle f_\alpha(t)f_\alpha(t)\rangle dt' = 0. \tag{2.99}$$

Applying the relation

$$\int_0^t e^{-\frac{1}{2}a(t-t')}\langle f(t)g(t')\rangle dt' = D, \tag{2.100}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'), \tag{2.101}$$

where a and D are a constants or some function of time t. We then see that

$$\langle f_\alpha(t)f_\alpha(t')\rangle = 0. \tag{2.102}$$

It can also be established in similar manner that

$$\langle f_\beta(t)f_\beta(t')\rangle = 0. \tag{2.103}$$

Using Eqs. (2.85) and (2.86) togther with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\alpha(t)\frac{d}{dt}\beta(t)\rangle + \langle(\frac{d}{dt}\alpha(t))\beta(t)\rangle, \tag{2.104}$$

we find

$$\begin{aligned}\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = &-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\alpha(t)\beta(t)\rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\alpha(t)\beta(t)\rangle \\ &+\frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*(t)\alpha(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*(t)\beta(t)\rangle \\ &+\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle.\end{aligned} \tag{2.105}$$

Compering Eqs. (2.77) and (2.105) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = \frac{1}{2}A\rho_{ac}^{(0)}. \tag{2.106}$$

The formal solution of Eq. (2.86) can also be written as

$$\beta(t) = \beta(0)e^{-\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa)t} + \int_0^t e^{-\frac{1}{2}(A\rho_{aa}^{(0)}+\kappa)(t-t')}\left(f_\beta(t') - \frac{1}{2}A\rho_{ac}^{(0)}\alpha^*(t')\right)dt'. \tag{2.107}$$

We then note that

$$\begin{aligned}\langle\beta(t)f_\alpha(t) = \langle\beta(0)f_\alpha(t)\rangle e^{-\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa)t} + \int_0^t e^{-\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa)(t-t')} \\ \left(\langle f_\beta(t')f_\alpha(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*(t')f_\alpha(t)\rangle\right)dt'.\end{aligned} \tag{2.108}$$

Assumming the noise force $f_\alpha(t)$ at time t does not affect a c-number variable at the earlier times, we see that

$$\langle\beta(0)f_\alpha(t)\rangle = \langle\alpha(0)\rangle\langle f_\alpha(t)\rangle = 0 \tag{2.109}$$

and

$$\langle\alpha^*(t')f_\alpha(t)\rangle = \langle\alpha^*(t')\rangle\langle f_\alpha(t)\rangle = 0. \tag{2.110}$$

Thus on account of Eqs. (2.89), (2.108), (2.109), and (2.110), we obtain

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa)(t-t')}\langle f_\beta(t')f_\alpha(t)\rangle dt'. \tag{2.111}$$

It can also be verified in a similar fashion that

$$\langle\alpha(t)f_\beta(t)\rangle=\int_0^t e^{-\frac{1}{2}(\kappa-A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha(t')f_\beta(t)\rangle dt'. \tag{2.112}$$

Therefore, in view of Eqs. (2.106), (2.111), and (2.112) and the assumption that

$$\langle f_\beta(t')f_\alpha(t)\rangle=\langle f_\alpha(t')f_\beta(t)\rangle, \tag{2.113}$$

we arrive at

$$\int_0^t e^{-\frac{1}{2}a(t-t')}\Big(\langle f_\alpha(t)f_\beta(t')\rangle+\langle f_\beta(t)f_\alpha(t')\rangle\Big)dt'=\tfrac{1}{2}A\rho_{ac}^{(0)}, \tag{2.114}$$

from which follows

$$\langle f_\alpha(t)f_\beta(t')\rangle=\langle f_\beta(t)f_\alpha(t')\rangle=\frac{1}{2}A\rho_{ac}^{(0)}\delta(t-t'). \tag{2.115}$$

Moreover, employing Eq. (2.85) and its complex conjugate along with the relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle=\langle\alpha^*(t)\frac{d}{dt}\alpha(t)\rangle+\langle(\frac{d}{dt}\alpha^*(t))\alpha(t)\rangle, \tag{2.116}$$

we have

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle=-(\kappa-A\rho_{aa}^{(0)})\langle\alpha^*(t)\alpha(t)\rangle-\frac{1}{2}A\rho_{ac}^{(0)}\Big(\langle\alpha^*(t)\beta^*(t)\rangle+\langle\alpha(t)\beta(t)\rangle\Big)$$
$$+\langle\alpha^*(t)f_\alpha(t)\rangle+\langle\alpha(t)f_\alpha^*(t)\rangle. \tag{2.117}$$

Thus upon comparing Eqs. (2.78) and (2.117) shows that

$$\langle\alpha^*(t)f_\alpha(t)\rangle+\langle\alpha(t)f_\alpha^*(t)\rangle=A\rho_{aa}^{(0)}+\kappa\bar{n}. \tag{2.118}$$

With aid of Eqs. (2.95) and its complex conjugate, we note that

$$\langle\alpha(t)f_\alpha^*(t)=\langle\alpha(0)f_\alpha^*(t)\rangle e^{-\frac{1}{2}(\kappa-A\rho_{aa}^{(0)})t}+\int_0^t e^{-\frac{1}{2}(\kappa-A\rho_{aa}^{(0)})(t-t')}$$
$$\Big(\langle f_\alpha(t')f_\alpha^*(t)\rangle-\frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*(t')f_\alpha^*(t)\rangle\Big)dt' \tag{2.119}$$

and

$$\langle\alpha^*(t)f_\alpha(t) = \langle\alpha^*(0)f_\alpha(t)\rangle e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})t} + \int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}$$
$$\left(\langle f_\alpha^*(t')f_\alpha(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta(t')f_\alpha(t)\rangle\right)dt'. \tag{2.120}$$

Assumming a noise force at some time does not affect the c-number variable at earlier times, we notice that

$$\langle\alpha(0)f_\alpha^*(t)\rangle = \langle\alpha(0)\rangle\langle f_\alpha^*(t)\rangle = 0 \tag{2.121}$$

and

$$\langle\beta^*(t')f_\alpha^*(t)\rangle = \langle\beta^*(t')\rangle\langle f_\alpha^*(t)\rangle = 0. \tag{2.122}$$

Using the complex conjugete of of Eqs. (2.121) and (2.120), we write Eqs. (2.119) and (2.120) as

$$\langle\alpha(t)f_\alpha^*(t)\rangle = \int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha(t')f_\alpha^*(t)\rangle dt' \tag{2.123}$$

and

$$\langle\alpha^*(t')f_\alpha(t)\rangle = \int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha^*(t')f_\alpha(t)\rangle dt'. \tag{2.124}$$

Now in the view of Eqs. (2.118), (2.123), and (2.124) and the assumption that

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\alpha^*(t')\rangle, \tag{2.125}$$

we get

$$2\int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha^*(t')f_\alpha(t)\rangle dt' = A\rho_{aa}^{(0)} + \kappa\bar{n}. \tag{2.126}$$

Hence on account of Eqs. (2.100) and (2.101), we obtain

$$\langle f_\alpha^*(t')f_\alpha(t)\rangle = \langle f_\alpha(t')f_\alpha^*(t)\rangle = (A\rho_{aa}^{(0)} + \kappa\bar{n})\delta(t - t'). \tag{2.127}$$

It can also be established in a similar procedure that

$$\langle f_\beta(t) f^*_\beta(t')\rangle = \langle f^*_\beta(t) f_\beta(t')\rangle = \kappa\bar{n}\delta(t-t'). \tag{2.128}$$

We next proceede to obtain the correlation properties of the noise forces $f^*_\alpha(t)$ and $f^*_\beta(t)$. Taking the complex conjugete of Eqs. (2.85) and (2.86), one can write

$$\frac{d}{dt}\alpha^*(t) = -\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})\alpha^*(t) - \frac{1}{2}A\rho^{(0)}_{ca}\beta(t) + f^*_\alpha(t) \tag{2.129}$$

and

$$\frac{d}{dt}(\beta^*(t)) = -\frac{1}{2}(A\rho^{(0)}_{cc} + \kappa)\beta^*(t) + \frac{1}{2}A\rho^{(0)}_{ca}\alpha(t) + f^*_\beta(t). \tag{2.130}$$

We note that Eq. (2.81) and the expectation value of Eq. (2.129) as well as Eq. (2.83) and the expectation value of Eq. (2.130) will have the same form if

$$\langle f^*_\alpha(t)\rangle = \langle f^*_\beta(t)\rangle = 0. \tag{2.131}$$

Applying the relation

$$\frac{d}{dt}\langle\alpha^{*2}(t)\rangle = \langle\alpha^*(t)\frac{d}{dt}\alpha^*(t) + \langle\frac{d\alpha^*(t)}{dt}\alpha^*(t), \tag{2.132}$$

along with the conjugate complex of Eq. (2.85), we obtain

$$\frac{d}{dt}\langle\alpha^{*2}(t)\rangle = -(\kappa - A\rho^{(0)}_{aa})\langle\alpha^{*2}(t)\rangle - A\rho^{(0)}_{ac}\langle\alpha^*(t)\beta(t)\rangle$$
$$+\langle\alpha^*(t) f^*_\alpha(t)\rangle + \langle f^*_\alpha(t)\alpha^*(t)\rangle. \tag{2.133}$$

Comparson of Eqs. (2.82) and (2.133) leads to

$$\langle f^*_\alpha(t)\alpha^*(t)\rangle + \alpha^*(t)\rangle + \langle\alpha^*(t) f^*_\alpha(t)\rangle = 0. \tag{2.134}$$

Using the complex conjugate of Eq. (2.95) along with Eq. (2.134), we get

$$\langle f^*_\alpha(t)\alpha^*(t)\rangle + \langle\alpha^*(t) f^*_\alpha(t)\rangle = \langle\alpha^*(0) f^*_\alpha(t)\alpha(t)\rangle e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})t}$$
$$+\int_0^t e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})(t-t')}\bigg(\langle f^*_\alpha(t') f^*_\alpha(t)\rangle$$
$$-\frac{1}{2}A\rho^{(0)}_{ac}\langle\beta^*(t') f^*_\alpha(t)\rangle\bigg)dt'. \tag{2.135}$$

Then it follows

$$\langle\alpha^*(0)f_\alpha^*(t)\rangle = \langle\alpha^*(0)\rangle\langle f_\alpha^*(t)\rangle = 0 \quad (2.136)$$

and

$$\langle\beta^*(t')f_\alpha^*(t)\rangle = \langle\beta^*(t')\rangle\langle f_\alpha^*(t)\rangle = 0. \quad (2.137)$$

Now taking into account (2.134)-(2.137), we find

$$\int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha^*(t)f_\alpha^*(t')\rangle dt' = 0, \quad (2.138)$$

then it follows that

$$\langle f_\alpha^*(t)f_\alpha^*(t')\rangle = 0. \quad (2.139)$$

It can also be established in a similar manner that

$$\langle f_\beta^*(t)f_\beta^*(t')\rangle = 0. \quad (2.140)$$

Moreover, employing Eqs. (2.86) and (2.129) along with the relation

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = \langle\alpha^*(t)\frac{d}{dt}\beta(t)\rangle + \langle\frac{d}{dt}\alpha^*(t))\beta(t)\rangle, \quad (2.141)$$

we find

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = -\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\alpha^*(t)\beta(t) + \frac{1}{2}A\rho_{ca}^{(0)}\langle\alpha^{*2}(t)\rangle + \langle\alpha^*(t)f_\beta(t)\rangle$$
$$-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\alpha^*(t)\beta(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^2(t)\rangle + \langle\beta(t)f_\alpha^*(t)\rangle. \quad (2.142)$$

Upon comparing Eqs. (2.79) and (2.142), we note that

$$\langle\alpha^*(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha^*(t)\rangle = 0. \quad (2.143)$$

In addition, multiplying the complex cojugate of Eq. (2.95) and Eq. (2.107) from the right by $f_\beta(t)\ and\ f_\alpha^*(t)$ respectively, we note that

$$\langle\alpha^*(t)f_\beta(t)\rangle = \langle\alpha^*(0)f_\beta(t)\rangle e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})t} + \int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}$$
$$\left(\langle f_\alpha^*(t')f_\beta(t)\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta(t')f_\beta(t)\rangle\right)dt' \quad (2.144)$$

and

$$\langle\beta(t)f_\alpha^*(t)\rangle = \langle\beta^*(0)f_\alpha^*(t)\rangle e^{-\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa)t} + \int_0^t e^{-\frac{1}{2}(A\rho_{cc}^{(0)}}$$
$$\left(f_\beta(t')\langle f_\alpha^*(t)\rangle - \frac{1}{2}A\rho_{ca}^{(0)}\langle\alpha^*(t')f_\alpha^*(t)\rangle\right)dt'. \qquad (2.145)$$

Now on account of the fact that

$$\langle\alpha^*(0)f_\beta(t)\rangle = \langle\alpha^*(0)\rangle\langle f_\beta(t)\rangle = 0 \qquad (2.146)$$

and

$$\langle\beta^*(t')f_\beta(t)\rangle = \langle\beta^*(t')\rangle\langle f_\beta(t)\rangle = 0. \qquad (2.147)$$

In addition using the complex conjugete Eqs. (2.46) and (2.147), Eqs. (2.144) and (2.145) become

$$\langle\alpha^*(t)f_\beta(t)\rangle = \int_0^t e^{-\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha^*(t')f_\beta(t)f\rangle dt' \qquad (2.148)$$

and

$$\langle\beta(t')f_\alpha^*(t)\rangle = \int_0^t e^{-\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa)(t-t')}\langle f_\alpha^*(t)f_\beta(t')\rangle dt'. \qquad (2.149)$$

With the aid of Eqs. (2.143), (2.148), and (2.149), and the assumption that

$$\langle f_\beta(t)f_\alpha^*(t')\rangle = \langle f_\alpha^*(t)f_\beta(t')\rangle, \qquad (2.150)$$

we arrive at

$$\langle f_\beta(t)f_\alpha^*(t')\rangle = \langle f_\beta(t')f_\alpha^*(t)\rangle = 0. \qquad (2.151)$$

It is worth mentioning that Eqs. (2.89), (2.102), (2.103), (2.115), (2.127), (2.128), (2.139), (2.140), (2.250), and (2.251) are the correlation properties of the noise forces.

2.3 Solutions of the c- number Langevin equations

Next we seek to establish the solutions of the c-number Langevin equations what we have obtain is see in 2.2.

We have found two differetial equations for $\alpha(t)$ and $\beta^*(t)$.

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\alpha(t) - \frac{1}{2}A\rho_{ac}^{(0)}\beta^*(t) + f_\alpha(t) \tag{2.152}$$

and

$$\frac{d}{dt}(\beta^*(t)) = -\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\beta^*(t) + \frac{1}{2}A\rho_{ca}^{(0)}\alpha(t) + f_\beta^*(t). \tag{2.153}$$

Next we seek to obtain the solutions of these coupled c-number Langevin equations using matrix method. On account of Eqs. (2.152) and (2.153), we can write the matrix equation

$$\frac{d}{dt}J(t) = MJ(t) + E(t), \tag{2.154}$$

where

$$J(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}, \tag{2.155}$$

$$M = \begin{pmatrix} Q - U & S \\ T & -(U + R) \end{pmatrix}, \tag{2.156}$$

and

$$E(t) = \begin{pmatrix} f_\alpha \\ f_\beta^* \end{pmatrix}, \tag{2.157}$$

with

$$Q = \frac{1}{2}A\rho_{aa}^{(0)}, \tag{2.158}$$

$$U = \frac{1}{2}\kappa, \tag{2.159}$$

$$S = \frac{1}{2}A\rho_{ac}^{(0)}, \tag{2.160}$$

$$T = \frac{1}{2}A\rho_{ca}^{(0)}, \tag{2.161}$$

$$R = \frac{1}{2}A\rho_{cc}^{(0)}. \tag{2.162}$$

To solve Eq. (2.154), we need to find the eigenvectors and eigenvalues of M such that

$$MV_i = \lambda_i V_i, \tag{2.163}$$

where

$$V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \tag{2.164}$$

with normalazation condition

$$x_i^2 + y_i^2 = 1. \tag{2.165}$$

Eq. (2.163) can be written as

$$(M - \lambda_i I)V_i = 0. \tag{2.166}$$

Eq. (2.166) has a non-trvial solution provided that the determinant of the matrix $(M - \lambda I)V_i = 0$, that is,

$$|M - \lambda_i I| = 0. \tag{2.167}$$

This implies that

$$\lambda^2 + (2U + R - Q)\lambda + \Big((U - Q)(U + R) + TS\Big) = 0. \tag{2.168}$$

Solving the quadratic equation for λ, the eigenvalues of the matrix M to be

$$\lambda_1 = \frac{-(2U+R-Q)+\sqrt{\Big((2U+R-Q)^2-4((U-Q)(U+R)+TS)\Big)}}{2} \tag{2.169}$$

and

$$\lambda_2 = \frac{-(2U+R-Q)-\sqrt{\Big((2U+R-Q)^2-4((U-Q)(U+R)+TS)\Big)}}{2}. \tag{2.170}$$

Using Eqs. (2.163) and (2.164) along with the normalization condition given by Eq. (2.165), we easily find eigenvectors to be

$$V_1 = \frac{1}{\sqrt{S^2+(Q-U-\lambda_1)^2}} \begin{pmatrix} S \\ Q-U-\lambda_1 \end{pmatrix} \tag{2.171}$$

and

$$V_2 = \frac{1}{\sqrt{S^2+(Q-U-\lambda_2)^2}} \begin{pmatrix} S \\ Q-U-\lambda_2 \end{pmatrix}. \tag{2.172}$$

We construct a matrix V consisting of the eigenvectors of M as column matrix

$$V = (V_1 V_2). \tag{2.173}$$

We then find

$$V = \begin{pmatrix} \frac{S}{\sqrt{S^2+(Q-U-\lambda_1)^2}} & \frac{S}{\sqrt{S^2+(Q-U-\lambda_2)^2}} \\ \frac{Q-U-\lambda_1}{\sqrt{S^2+(Q-U-\lambda_1)^2}} & \frac{Q-U-\lambda_2}{\sqrt{S^2+(Q-U-\lambda_2)^2}} \end{pmatrix}. \tag{2.174}$$

The inverse of this matrix has the form

$$V^{-1} = \begin{pmatrix} \frac{-(Q-U-\lambda_2)\sqrt{S^2+(Q-U-\lambda_1)^2}}{S\lambda} & \frac{\sqrt{S^2+(Q-U-\lambda_1)^2}}{\lambda} \\ \frac{(Q-U-\lambda_1)\sqrt{S^2+(Q-U-\lambda_2)^2}}{S\lambda} & \frac{-\sqrt{S^2+(Q-U-\lambda_2)^2}}{\lambda} \end{pmatrix}, \tag{2.175}$$

where

$$\lambda = \lambda_2 - \lambda_1. \tag{2.176}$$

Applying the identity operator $\hat{I} = VV^{-1}$ in Eq. (2.154), we have

$$\frac{d}{dt} J(t) = VV^{-1}MVV^{-1}J(t) + E(t) \tag{2.177}$$

and multipying Eq. (2.177) on the left by V^{-1}, we get

$$\frac{d}{dt}\Big(V^{-1}J(t)\Big) = DV^{-1}J(t) + V^{-1}E(t), \tag{2.178}$$

in which

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.179}$$

For physically realizable solution λ_1 and λ_2 must have negative values.

To this end, the formal solution of Eq. (2.178), can be written as

$$(V^{-1}J(t)) = e^{Dt}V^{-1}J(0) + \int_0^t e^{D(t-t')}V^{-1}E(t')dt', \tag{2.180}$$

from which follows

$$J(t) = Ve^{Dt}V^{-1}J(0) + \int_0^t Ve^{D(t-t')}V^{-1}E(t')dt'. \tag{2.181}$$

Since D is digonal, we have

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}. \tag{2.182}$$

It is easy to see that

$$Ve^{Dt}V^{-1}J(0) = \begin{pmatrix} \frac{1}{\lambda}\Big(Be^{\lambda_2 t} - Ce^{\lambda_1 t}\Big)\alpha(0) & \frac{S}{\lambda}\Big(e^{\lambda_1 t} - e^{\lambda_2 t}\Big)\beta^*(0) \\ \frac{BC}{S\lambda}\Big(e^{\lambda_2 t} - e^{\lambda_1 t}\Big)\alpha(0) & \frac{1}{\lambda}\Big(Be^{\lambda_1 t} - Ce^{\lambda_2 t}\Big)\beta^*(0) \end{pmatrix} \tag{2.183}$$

and

$$Ve^{Dt}V^{-1}E(t) = \begin{pmatrix} \frac{1}{\lambda}J_1 & \frac{S}{\lambda}J_2 \\ \frac{BC}{S\lambda}J_3 & \frac{1}{\lambda}J_4 \end{pmatrix}, \tag{2.184}$$

where

$$B = Q - U - \lambda_1, \tag{2.185}$$

$$C = Q - U - \lambda_2, \tag{2.186}$$

$$J_1 = \left(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')}\right) f_\alpha(t'), \tag{2.187}$$

$$J_2 = \left(e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}\right) f_\beta(t'), \tag{2.188}$$

$$J_3 = \left(e^{\lambda_2(t-t')} - e^{\lambda_1(t-t')}\right) f_\alpha(t'), \tag{2.189}$$

$$J_4 = \left(Be^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')}\right) f^*_\beta(t'). \tag{2.190}$$

Integrating Eq. (2.184), we have

$$\int_0^t Ve^{D(t-t')}V^{-1}E(t')dt' = \begin{pmatrix} \frac{1}{\lambda}\int_0^t J_1 dt' & \frac{S}{\lambda}\int_0^t J_2 dt' \\ \frac{BC}{S\lambda}\int_0^t J_3 dt' & \frac{1}{\lambda}\int_0^t J_4 dt' \end{pmatrix}. \tag{2.191}$$

Introducing Eqs. (2.183) and (2.191) into Eq. (2.181), we get

$$J(t) = \begin{pmatrix} \frac{1}{\lambda}\left(Be^{\lambda_2 t} - Ce^{\lambda_1 t}\right)\alpha(0) & \frac{S}{\lambda}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)\beta^*(0) \\ \frac{BC}{S\lambda}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)\alpha(0) & \frac{1}{\lambda}\left(Be^{\lambda_1 t} - Ce^{\lambda_2 t}\right)\beta^*(0) \end{pmatrix} + \begin{pmatrix} \frac{1}{\lambda}\int_0^t J_1 dt' & \frac{S}{\lambda}\int_0^t J_2 dt' \\ \frac{BC}{S\lambda}\int_0^t J_3 dt' & \frac{1}{\lambda}\int_0^t J_4 dt' \end{pmatrix}. \tag{2.192}$$

With the aid of Eqs. (2.155), (2.187)-(2.190), and (2.192), one can readily see that

$$\begin{aligned}\alpha(t) = &\frac{1}{\lambda}\left(Be^{\lambda_2 t} - Ce^{\lambda_1 t}\right)\alpha(0)\\ &+\frac{S}{\lambda}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)\beta^*(0)\\ &+\frac{1}{\lambda}\int_0^t \left(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')}\right) f_\alpha(t')dt'\\ &+\frac{S}{\lambda}\int_0^t \left(e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}\right) f_\beta^*(t')dt' \end{aligned} \tag{2.193}$$

and

$$\begin{aligned}\beta^*(t) = &\frac{BC}{S\lambda}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)\alpha(0) + \frac{1}{\lambda}\left(Be^{\lambda_1 t} - Ce^{\lambda_2 t}\right)\beta^*(0)\\ &+\frac{BC}{S\lambda}\int_0^t \left(e^{\lambda_2(t-t')} - e^{\lambda_1(t-t')}\right) f_\alpha(t')dt'\\ &+\frac{1}{\lambda}\int_0^t \left(Be^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')}\right) f_\beta^*(t')dt'. \end{aligned} \tag{2.194}$$

The above two equatons can be rewritten as

$$\begin{aligned}\alpha(t) = &\frac{1}{\lambda}\left(Be^{\lambda_2 t} - Ce^{\lambda_1 t}\right)\alpha(0)\\ &+\frac{S}{\lambda}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)\beta^*(0) + F(t) \end{aligned} \tag{2.195}$$

and

$$\begin{aligned}\beta^*(t) = &\frac{BC}{S\lambda}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)\alpha(0)\\ &+\frac{1}{\lambda}\left(Be^{\lambda_1 t} - Ce^{\lambda_2 t}\right)\beta^*(0) + G(t). \end{aligned} \tag{2.196}$$

Eqs. (2.195) and (2.196) can be rewritten as

$$\alpha(t) = p_1\alpha(0) + q_1\beta^*(0) + F(t) \tag{2.197}$$

and

$$\beta^*(t) = p_2\alpha(0) + q_2\beta^*(0) + G(t), \tag{2.198}$$

where

$$F(t) = \frac{1}{\lambda}\int_0^t \left(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')}\right) f_\alpha(t')dt' \\ + \frac{S}{\lambda}\int_0^t \left(e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}\right) f_\beta^*(t')dt' \quad (2.199)$$

and

$$G(t) = \frac{BC}{S\lambda}\int_0^t \left(e^{\lambda_2(t-t')} - e^{\lambda_1(t-t')}\right) f_\alpha(t')dt' \\ + \frac{1}{\lambda}\int_0^t \left(Be^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')}\right) f_\beta^*(t')dt', \quad (2.200)$$

with

$$p_1 = \frac{1}{\lambda}\left(Be^{\lambda_2 t} - Ce^{\lambda_1 t}\right), \quad (2.201)$$

$$q_1 = \frac{S}{\lambda}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right), \quad (2.202)$$

$$p_2 = \frac{BC}{S\lambda}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right), \quad (2.203)$$

$$q_2 = \frac{1}{\lambda}\left(Be^{\lambda_1 t} - Ce^{\lambda_2 t}\right). \quad (2.204)$$

2.4 The Q function

Next we wish to calculate the Q function for non-degenerate three level laser coupled to two-mode thermal reservior. The Q function for a two-mode cavity light can be defined as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z d^2w \phi_a(z,w,t) exp[z^*\alpha \\ + w^*\beta - z\alpha^* - w\beta^*], \quad (2.205)$$

where the antinormally ordered characterstic function $\phi_a(z,w,t)$ for the two-mode cavity light is

$$\phi_a(z,w,t) = Tr\left(\rho(0)e^{-z^*\hat{a}(t)}e^{-w^*\hat{b}(t)}e^{z\hat{a}^\dagger(t)}e^{w\hat{b}^\dagger(t)}\right). \tag{2.206}$$

Now making use of the operator identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{2.207}$$

it is possible to express Eq. (2.206) in terms of operator variables associated with the normal ordering as

$$\begin{aligned}\phi_a(z,w,t) = & exp[-z^*z - w^*w] \\ & \times Tr\left(\rho(0)exp[z\hat{a}^\dagger(t) + w\hat{b}^\dagger(t)) - z^*\hat{a}(t) - w^*\hat{b}(t)]\right).\end{aligned} \tag{2.208}$$

The c-number function corresponding to the antinormally ordered characteristic function described by Eq. (2.208) is expressed

$$\begin{aligned}\phi_a(z,w,t) = & exp[-z^*z - w^*w] \\ & \times\langle exp[z\alpha^*(t) + w\beta^*(t) - z^*\alpha(t) - w^*\beta(t)]\rangle.\end{aligned} \tag{2.209}$$

Since $\alpha(t)$ and $\beta(t)$ are Gaussian variables with zero mean, one can verify that [31]

$$\begin{aligned}\langle exp[z\alpha^*(t) + w\beta^*(t) - z^*\alpha(t) - w^*\beta(t)]\rangle \equiv exp\Bigg[\frac{1}{2}\Bigg\langle\Bigg(& z\alpha^*(t) + w\beta^*(t) \\ & -z^*\alpha(t) - w^*\beta(t)\Bigg)^2\Bigg\rangle\Bigg].\end{aligned} \tag{2.210}$$

In view of Eq. (210), Eq. (2.209) becomes

$$\begin{aligned}\phi_a(z,w,t) \;=&\; exp[-z^*z - w^*w]\\ &\times exp\bigg[\frac{1}{2}(z^2\langle\alpha^{*2}(t)\rangle + z^{*2}\langle\alpha^2(t)\rangle + w^2\langle\beta^{*2}(t)\rangle + w^{*2}\langle\beta^2(t)\rangle)\\ &-2z^*z\langle\alpha^*(t)\alpha(t)\rangle - 2w^*w\langle\beta^*(t)\beta(t)\rangle + 2zw\langle\alpha^*(t)\beta^*(t)\rangle\\ &+2z^*w^*\langle\alpha(t)\beta(t)\rangle\rangle - 2zw^*\langle\alpha^*(t)\beta(t)\rangle - 2z^*w\langle\alpha(t)\beta^*(t)\rangle\bigg].\end{aligned} \tag{2.211}$$

Now considering that the cavity modes are initially in a vacuum state, we observe that

$$\langle\alpha(0)\rangle = \langle\alpha^*(0)\alpha(0)\rangle = 0, \tag{2.212}$$

$$\langle\beta(0)\rangle = \langle\beta^*(0)\beta(0)\rangle = 0, \tag{2.213}$$

$$\langle\beta(0)\alpha(0)\rangle = \langle\beta^*(0)\alpha(0)\rangle = 0. \tag{2.214}$$

Applying Eqs. (2.197) and (2.198) along with Eqs. (2.212)-(2.214) and the fact that a noise force at a latter time does not affect the cavity mode variables at earlier time, one can easily check that

$$\begin{aligned}\langle\alpha^2(t)\rangle =& p_1^2\langle\alpha^2(0)\rangle + p_1q_1\langle\alpha^*(0)\beta^*(0)\rangle + p_1\langle\alpha(0)F(t)\rangle\\ &+q_1p_1\langle\beta^*(0)\alpha(0) + q_1^2\beta^{*2}(0) + q_1\langle\beta^*(0)F(t)\rangle\\ &+p_1\langle\alpha(0)F(t)\rangle + q_1\langle\beta^*(0)F(t)\rangle + \langle F^2(t)\rangle.\end{aligned} \tag{2.215}$$

Introducing Eqs. (2.212)-(2.214) into Eq. (2.115), we get

$$\langle\alpha^2(t)\rangle = \langle F^2(t)\rangle. \tag{2.216}$$

Similarly, we easily obtain

$$\langle \alpha^{*^2}(t)\rangle = \langle F^{*2}(t)\rangle, \tag{2.217}$$

$$\langle \beta^2(t)\rangle = \langle G^{*^2}(t)\rangle, \tag{2.218}$$

$$\langle \beta^{*2}(t)\rangle = \langle G^2(t)\rangle, \tag{2.219}$$

$$\langle \alpha^*(t)\alpha(t)\rangle = \langle F^*(t)F(t)\rangle, \tag{2.220}$$

$$\langle \beta^*(t)\beta(t)\rangle = \langle G(t)G^*(t)\rangle, \tag{2.221}$$

$$\langle \alpha(t)\beta(t)\rangle = \langle F(t)G^*(t)\rangle, \tag{2.222}$$

$$\langle \alpha^*(t)\beta^*(t)\rangle = \langle F^*(t)G(t)\rangle, \tag{2.223}$$

$$\langle \alpha^*(t)\beta(t)\rangle = \langle F^*(t)G^*(t)\rangle, \tag{2.224}$$

$$\langle \alpha(t)\beta^*(t)\rangle = \langle F(t)G(t)\rangle. \tag{2.225}$$

Using Eq. (2.199), we easily fond

$$\begin{aligned}\langle F^2(t)\rangle = {} & \frac{1}{\lambda^2}\int_0^t \left(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(Be^{\lambda_2(t-t'')} - Ce^{\lambda_1(t-t'')}\right) \\ & \langle f_\alpha^*(t')f_\alpha^*(t'')\rangle dt'dt'' \\ & + \frac{T}{\lambda^2}\int_0^t \left(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')}\right) \\ & \langle f_\alpha^*(t')f_\beta(t'')\rangle dt'dt'' \\ & + \frac{T}{\lambda^2}\int_0^t \left(Be^{\lambda_2(t-t'')} - Ce^{\lambda_1(t-t'')})(e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}\right) \\ & \langle f_\beta(t')f_\alpha^*(t'')\rangle dt'dt'' \\ & + \frac{T^2}{\lambda^2}\int_0^t \left(e^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')})(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')}\right) \\ & \langle f_\beta(t')f_\beta(t'')\rangle dt'dt''.\end{aligned} \tag{2.226}$$

Substituting Eqs. (2.103), (2.139), and (2.151) into Eq. (2.226) and taking Eq. (2.216) into account, we obtain

$$\langle\alpha^2(t)\rangle = \langle F^2(t)\rangle = 0. \tag{2.227}$$

It can also be established in the same way that

$$\langle\alpha^{*^2}(t)\rangle = \langle F^{*2}(t)\rangle = 0, \tag{2.228}$$

$$\langle\beta^2(t)\rangle = \langle G^{*^2}(t)\rangle = 0, \tag{2.229}$$

$$\langle\beta^{*2}(t)\rangle = \langle G^2(t)\rangle = 0, \tag{2.230}$$

$$\langle\alpha^*(t)\beta(t)\rangle = \langle F^*(t)G^*(t)\rangle = 0, \tag{2.231}$$

$$\langle\alpha(t)\beta^*(t)\rangle = \langle F(t)G(t)\rangle = 0. \tag{2.232}$$

Hence in veiw of Eqs. (2.227)-(2.232), the antinormally ordered characteristic function reduced to

$$\begin{aligned}\phi_a(z,w,t) = exp\Big[&- z^*z - w^*w - (1 + \langle\alpha^*(t)\alpha(t)\rangle)z^*z - (1 + \langle\beta^*(t)\beta(t)\rangle)w^*w \\ &+ zw\langle\alpha^*(t)\beta^*(t)\rangle + z^*w^*\langle\alpha(t)\beta(t)\rangle\Big].\end{aligned} \tag{2.233}$$

Now using Eq. (2.220) along with Eq. (2.199), we have

$$\langle\alpha^*(t)\alpha(t)\rangle = \langle F^*(t)F(t)\rangle, \tag{2.234}$$

then it follows

$$\langle F^*(t)F(t)\rangle = \frac{1}{\lambda^2}\int_0^t \Big(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(Be^{\lambda_2(t-t'')} - Ce^{\lambda_1(t-t'')}\Big)$$
$$\langle f_\alpha^*(t'')f_\alpha(t')\rangle dt'dt''$$
$$+\frac{s}{\lambda^2}\int_0^t \Big(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')})(Be^{\lambda_2(t-t'')} - Ce^{\lambda_1(t-t'')}\Big)$$
$$f_\alpha^*(t'')f_\beta^*(t')\rangle dt'dt''$$
$$+\frac{T}{\lambda^2}\int_0^t \Big(Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')}\Big)$$
$$\langle f_\beta(t'')f_\alpha(t')\rangle dt'dt''$$
$$+\frac{TS}{\lambda^2}\int_0^t \Big(e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')}\Big)$$
$$\langle f_\beta(t'')f_\beta^*(t')\rangle dt'dt''. \tag{2.235}$$

Using the correlation properties of the noise forces given by Eqs. (2.115), (2.127), and (2.128) and then carrying out the integration, at steady state, we obtain

$$\langle \alpha^*(t)\alpha(t)\rangle_{ss} = \frac{(2BC(2Q+\kappa\bar{n}) - 2ST(B+C+\kappa\bar{n}))}{\lambda^2(\lambda_1+\lambda_2)} - \frac{B^2(2Q+\kappa\bar{n}) - ST(2B-\kappa\bar{n})}{\lambda^2\lambda_2} - \frac{C^2(2Q+\kappa\bar{n}) - ST(B+C+\kappa\bar{n})}{\lambda^2\lambda_1}. \tag{2.236}$$

Similarly, we can readily find

$$\langle \beta^*(t)\beta(t)\rangle_{ss} = \frac{2B^2C^2(2Q+\kappa\bar{n})}{\lambda^2 ST(\lambda_1+\lambda_2)} - \frac{BC(2(B+C-\kappa\bar{n})}{\lambda^2(\lambda_1+\lambda_2)} - \frac{B^2C^2(2Q+\kappa\bar{n})}{\lambda^2 ST\lambda_1} - \frac{B(B\kappa\bar{n}-2BC)}{\lambda^2\lambda_1} - \frac{B^2C^2(2Q+\kappa\bar{n})}{\lambda^2 ST\lambda_1} - \frac{C(C\kappa\bar{n}-2BC)}{\lambda^2\lambda_2}, \tag{2.237}$$

$$\langle\alpha(t)\beta(t)\rangle_{ss} = \frac{2BS(C-\kappa\bar{n})}{\lambda^2\lambda_1} - \frac{BC^2(2Q+\kappa\bar{n})}{\lambda^2 T\lambda_1} + \frac{2CS(B-\kappa\bar{n})}{\lambda^2\lambda_2} - \frac{B^2C(2Q+\kappa\bar{n})}{\lambda^2 T\lambda_2} + \frac{BC(2Q+\kappa\bar{n})(B+C)}{\lambda^2 T(\lambda_1+\lambda_2)} - \frac{S\Big((B+C)^2-\kappa\bar{n}(B+C)\Big)}{\lambda^2(\lambda_1+\lambda_2)}. \tag{2.238}$$

Now on account of Eqs. (2.236)-(2.238), the characteristic function described by Eq. (2.233) can be put in the form

$$\phi_a(z,w,t) = exp[-az^*z - bw^*w + c(z^*w^* + zw)], \tag{2.239}$$

where

$$a = 1 + \langle\alpha^*(t)\alpha(t)\rangle, \tag{2.240}$$

$$b = 1 + \langle\beta^*(t)\beta(t)\rangle, \tag{2.241}$$

$$c = \langle\alpha(t)\beta(t)\rangle = \langle\alpha^*(t)\beta^*(t)\rangle. \tag{2.242}$$

Finally, applying Eq. (2.239) in Eq. (2.205), we have

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z d^2w\, exp[-az^*z - bw^*w + c(z^*w^* + zw) + z^*\alpha - z\alpha^* + w^*\beta - w\beta^*]. \tag{2.243}$$

Then it leads to

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z\, exp\Big[-bz^*z + z^*\beta - z\beta^*\Big] \times \int d^2w\, exp\Big[-bw^*w + w(cz-\beta^*) + w^*(cz^*+\beta)\Big]. \tag{2.244}$$

Performing the integration employing the relation

$$\int \frac{d^2z}{\pi} exp[-az^*z + bz + cz^* + Az^2 + Bz^*2] = \left[\frac{1}{a^2 - 4AB}\right]^{\frac{1}{2}} exp(\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}), a > 0, \tag{2.245}$$

we readily find

$$Q(\alpha, \beta, t) = \frac{u_1u_2 - v^2}{\pi^2}\, exp\left[- u_1\, \beta^*\beta - u_2\, \alpha^*\alpha + v(\alpha^*\beta^* + \alpha\beta)\right], \tag{2.246}$$

where

$$u_1 = \frac{a}{ab - c^2}\ , \tag{2.247}$$

$$u_2 = \frac{b}{ab - c^2}\ , \tag{2.248}$$

$$v = \frac{c}{ab - c^2}\ . \tag{2.249}$$

This is the Q function for the non-degenarate three level laser coupled to thermal reservior.

2.5 The density operator

Here we seek to determine the density operator for a two-mode light beams. Suppose $\hat{\rho}'(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)$ is the density operator for a certain two mode light beam. Employing the completeness relation

$$I = \frac{1}{\pi^2}\int d^2\alpha d^2\beta|\alpha, \beta\rangle\langle\beta, \alpha|, \tag{2.250}$$

for two-mode coherent states twice, we can write [32]

$$I = \frac{1}{\pi^4}\int d^2\alpha d^2\beta d^2\eta d^2\lambda|\alpha, \beta\rangle\langle\beta, \alpha|\rho'|\eta, \lambda\rangle\langle\lambda, \eta|. \tag{2.251}$$

Then expanding the density operator in the normal order as

$$\rho'(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t) = \sum_{p,q,r,s} C_{pqrs}\hat{a}^{\dagger p}(t)\hat{b}^{\dagger q}(t)\hat{a}^r(t)\hat{b}^s(t). \tag{2.252}$$

Now applying Eq. (2.251) in (2.252), one can easily obtain, the density operator for a two mode light beam as in the form [32]

$$\rho'(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t) = \frac{1}{\pi^2}\int d^2\alpha d^2\beta d^2\lambda d^2\eta\, Q(\alpha^*, \beta^*, \eta, \lambda, t)$$
$$\langle\beta, \alpha|\eta, \lambda\rangle|\alpha, \beta\rangle\langle\lambda, \eta|, \tag{2.253}$$

where

$$Q(\alpha^*, \beta^*, \eta, \lambda, t) = \frac{1}{\pi^2}\sum_{p,q,r,s} C_{pqrs}\alpha^{*p}\beta^{*q}\eta^r\lambda^s. \tag{2.254}$$

On the other hand, the expectation value of an operator $\hat{A}(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)$ can be expressed in terms of the density operator in the form

$$\langle\hat{A}(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)\rangle = Tr(\hat{\rho}'(t)\hat{A}(0)). \tag{2.255}$$

Now substituting Eq. (2.253) into Eq. (2.256) yields

$$\langle\hat{A}(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)\rangle = \frac{1}{\pi^2}\int d^2\alpha d^2\beta d^2\eta d^2\lambda\, Q(\alpha^*, \beta^*, \eta, \lambda, t)$$
$$\langle\alpha, \beta|\lambda, \eta\rangle Tr(\langle\beta, \alpha|\eta, \lambda\rangle\hat{A}) \tag{2.256}$$

Eq. (2.256) can be rewritten as

$$\langle\hat{A}(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b}, t)\rangle = \frac{1}{\pi^2}\int d^2\alpha d^2\beta d^2\eta d^2\lambda\, Q(\alpha^*, \beta^*, \eta, \lambda, t)$$
$$\langle\beta, \alpha|\eta, \lambda\rangle\langle\lambda, \eta|\hat{A}|\alpha, \beta\rangle. \tag{2.257}$$

Upon expanding $\hat{A}(a^\dagger, b^\dagger, a, b, t)$ in the normal order once, we obtain [32]

$$\langle\hat{A}(a^\dagger, b^\dagger, a, b, t)\rangle = \frac{1}{\pi^2}\int d^2\alpha d^2\beta d^2\eta d^2\lambda\, Q(\alpha^*, \beta^*, \eta, \lambda, t)$$
$$\times exp\Big[-\alpha^*\alpha - \beta^*\beta - \eta^*\eta - \lambda^*\lambda + \eta^*\alpha$$
$$+\alpha^*\eta + \beta^*\lambda + \lambda^*\beta\Big]A_n(\eta^*, \lambda^*, \alpha, \beta), \tag{2.258}$$

where

$$|\langle\beta,\alpha|\eta,\lambda\rangle|^2=exp[-|\alpha-\eta|^2]exp[-|\beta-\lambda|^2], \tag{2.259}$$

in which $A_n(\eta^*,\lambda^*,\alpha,\beta)$ is the c-number function corresponding to $A(\hat{a}^\dagger,\hat{b}^\dagger,\hat{a},\hat{b})$ in the normal order.

Thus employing Eq. (2.246), Eq. (2.259) leads to

$$\begin{aligned}\langle\hat{A}(\hat{a}^\dagger,\hat{b}^\dagger,\hat{a},\hat{b},t)\rangle=(\frac{u_1u_2-v^2}{\pi^4})\int d^2\alpha d^2\beta d^2\eta d^2\lambda \\ exp\Big[-u_1\beta^*\lambda-u_2\alpha^*\eta+v(\alpha^*\beta^*+\eta\lambda) \\ -\alpha^*\alpha-\beta^*\beta+\eta^*\eta-\lambda^*\lambda+\eta^*\alpha+\alpha^*\eta \\ +\beta^*\lambda+\lambda^*\beta\Big]A_n(\eta^*,\lambda^*,\alpha,\beta),\end{aligned} \tag{2.260}$$

is the normally ordered expectation value of any operator $\hat{A}(\hat{a}^\dagger,\hat{b}^\dagger,\hat{a},\hat{b},t)$ for the non-degenerate three level laser coupled to thermal reservoir.

3

Photon Statistics

In this chapter, we wish to calculate the mean photon number, the variance of the photon number, and the photon number distribution. Here it proves to be more convenient to introduce a new parameter η. Then we defined $\rho_{aa}^{(0)}$ in terms of η as

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{3.1}$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{3.2}$$

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{3.3}$$

one easly finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{3.4}$$

$$\rho_{ac}^{(0)} = \frac{1}{2}\sqrt{(1-\eta^2)}e^{i\theta}, \tag{3.5}$$

$$|\rho_{ca}^{(0)}| = \frac{1}{2}\sqrt{(1-\eta^2)}, \tag{3.6}$$

where θ is the phase angle. Since the value of $\rho_{aa}^{(0)}$ and $\rho_{cc}^{(0)}$ are in between 0 and 1, it is not diffcult to see the value of η lies in the interval -1$\leq\eta\leq 1$.

Applying Eqs. (3.1)- (3.6) in Eqs. (2.158) and (2.160)-(2.162), we have

$$Q=\frac{1}{4}A(1-\eta), \tag{3.7}$$

$$R=\frac{1}{4}A(1+\eta), \tag{3.8}$$

$$S=\frac{1}{4}A\sqrt{(1-\eta^2)}e^{i\theta}, \tag{3.9}$$

$$T=\frac{1}{4}A\sqrt{(1-\eta^2)}e^{-i\theta}. \tag{3.10}$$

Employing Eqs. (2.159) and (3.7)-(3.10) along with Eqs. (2.169) and (2.170), we obtain

$$\lambda_1=\frac{-1}{2}\kappa, \tag{3.11}$$

$$\lambda_2=\frac{-1}{2}(\kappa\ +A\eta), \tag{3.12}$$

$$\lambda=\lambda_2-\lambda_1=\frac{-1}{2}A\eta. \tag{3.13}$$

Furthermore, substituting Eqs. (2.159) and (3.7)-(3.13) into Eqs. (2.185) and (2.186), we get

$$B=\frac{1}{4}A(1-\eta), \tag{3.14}$$

$$C=\frac{1}{4}A(1+\eta). \tag{3.15}$$

Now introducing Eqs. (3.7)-(3.15) in Eqs. (2.236)-(2.238), we readily obtain

$$\langle\alpha^*(t)\alpha(t)\rangle_{ss}=\frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)}+\frac{\bar{n}(1+\eta)(3\eta-1)}{2\eta^2}+\frac{(1-\eta)(A\eta(\eta-1)+4\kappa\bar{n})}{2\eta^2(\kappa+A\eta)}, \tag{3.16}$$

$$\langle\beta^*(t)\beta(t)\rangle_{ss} = \frac{\bar{n}(1-\eta)}{\eta^2} + \frac{(1+\eta)(A\eta(\eta-1)+4\kappa\bar{n})}{2\eta^2(\kappa+A\eta)} + \frac{(1-\eta^2)(A\eta-4\kappa\bar{n})}{2\eta(2\kappa+A\eta)}, \tag{3.17}$$

$$\langle\alpha(t)\beta(t)\rangle_{ss} = \sqrt{1-\eta^2}e^{i\theta}\left[\frac{\bar{n}}{\eta^2} + \frac{A\eta(\eta-1)+4\kappa\bar{n}}{2\eta^2(\kappa+A\eta)} + \frac{(2(\eta-1)-4\kappa\bar{n}+A)}{2\eta^2(2\kappa+A\eta)}\right]. \tag{3.18}$$

Then for $\bar{n}=0$, Eqs. (3.16)-(3.18), can be written as

$$\langle\alpha^*(t)\alpha(t)\rangle_{ss} = \frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)} + \frac{(1-\eta)[A\eta(\eta-1)]}{2\eta^2(\kappa+A\eta)}, \tag{3.19}$$

$$\langle\beta^*(t)\beta(t)\rangle_{ss} = \frac{(1+\eta)[A\eta(\eta-1)]}{2\eta^2(\kappa+A\eta)} + \frac{A(1-\eta^2)}{2(2\kappa+A\eta)}, \tag{3.20}$$

$$\langle\alpha(t)\beta(t)\rangle_{ss} = \sqrt{1-\eta^2}e^{i\theta}\left[\frac{A\eta(\eta-1)}{2\eta^2(\kappa+A\eta)} + \frac{2(\eta-1)+A}{2\eta^2(2\kappa+A\eta)}\right]. \tag{3.21}$$

3.1 The mean photon number

The mean photon number for the two-mode cavity light can be defined as [32]

$$\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle = Tr\bigg(\hat{\rho}(t)\hat{c}^\dagger(o)\hat{c}(o)\bigg), \tag{3.22}$$

in which

$$\hat{c}(t) = \hat{a}(t) + \hat{b}(t). \tag{3.23}$$

Then it follows

$$\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle a^\dagger(t)\hat{b}(t)\rangle + \langle\hat{b}^\dagger(t)\hat{a}(t) + \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle. \tag{3.24}$$

In view of Eqs. (2.231) and (2.232), we observe that

$$\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle. \tag{3.25}$$

Therefore, the mean of the photon number of the first light beam in terms of Q function can be put in the form

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle = \int d^2\alpha d^2\beta Q(\alpha^*, \beta, t)(\alpha^*\alpha - 1). \tag{3.26}$$

On account of Eq. (2.246), we see that

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle = \left(u_1u_2 - v^2\right) \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} exp[-u_1\beta^*\beta - u_2\alpha^*\alpha$$
$$+v\alpha\beta + v\alpha^*\beta^*](\alpha^*\alpha - 1). \tag{3.27}$$

Eq. (3.27) can be rewritten as

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle = u_1u_2 - v^2\left(\frac{\partial^2}{\partial p^*\partial p}\right) \int \frac{d^2\alpha}{\pi} exp[-u_2\alpha^*\alpha + p\alpha^* + p^*\alpha]$$
$$\times \int \frac{d^2\beta}{\pi} exp[-u_1\beta^*\beta + v\alpha\beta + v\alpha^*\beta^*]\bigg|_{p=q=0} - 1. \tag{3.28}$$

Upon carrying out the integration over α and β employing the relation described by Eq. (2.245), we obtain

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle = \left(\frac{\partial^2}{\partial p^*\partial p}\right) exp[\frac{(u_1u_2 - v^2)\alpha^*\alpha + u_1\alpha^* p + u_1\alpha p^*}{u_1}]\bigg|_{p=q=0} - 1, \tag{3.29}$$

we also noice that

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle = a - 1, \tag{3.30}$$

where

$$a = \langle \alpha^*(t)\alpha(t)\rangle + 1, \tag{3.31}$$

we have

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle = \langle \alpha^*(t)\alpha(t)\rangle. \tag{3.32}$$

Moreover, it can also be established in a similar manner that

$$\langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle = b - 1, \tag{3.33}$$

with

$$b = \langle\beta^*(t)\beta(t)\rangle + 1, \tag{3.34}$$

then we get

$$\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = \langle\beta^*(t)\beta(t)\rangle. \tag{3.35}$$

Finally, on account of Eq. (3.25), we arrive

$$\Big\langle\hat{c}^\dagger(t)\hat{c}(t)\Big\rangle = \Big\langle\alpha^*(t)\alpha(t)\Big\rangle + \Big\langle\beta^*(t)\beta(t)\Big\rangle. \tag{3.36}$$

In veiw of Eqs. (3.16) and (3.17), we can put the cavity mean photon number in the form

$$\begin{aligned}\Big\langle\hat{c}^\dagger(t)\hat{c}(t)\Big\rangle = &\frac{(1-\eta)[A\eta(\eta-1)+4\kappa\bar{n}]}{2\eta^2(\kappa+A\eta)} + \frac{(1+\eta)[\bar{n}(3\eta-1)]}{2\eta^2} + \frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)}\\ &+\frac{(1-\eta)\bar{n}}{\eta^2} + \frac{(1+\eta)[A\eta(\eta-1)+4\kappa\bar{n}]}{2\eta^2(\kappa+A\eta)} + \frac{(1-\eta^2)[A\eta-4\kappa\bar{n}]}{2\eta(2\kappa+A\eta)}.\end{aligned} \tag{3.37}$$

We immediately observe from Fig. (3.1) that the mean photon number of the system decreases with η. But Fig. (3.2) clearly indicates that the mean photon number of the system increases with $\bar{n}$.

With $\bar{n} = 0$, Eq. (3.37) can be put in the form

$$\begin{aligned}\bar{n}'_{ss} = &\frac{(1-\eta)(A\eta(\eta-1))}{2\eta^2(\kappa+A\eta)} + \frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)}\\ &+\frac{(1+\eta)(A\eta(\eta-1))}{2\eta^2(\kappa+A\eta)} + \frac{A(1-\eta^2)}{2(2\kappa+A\eta)},\end{aligned} \tag{3.38}$$

is the mean photon number of a non-degenerate three-level laser coupled to vaccum reservoir.

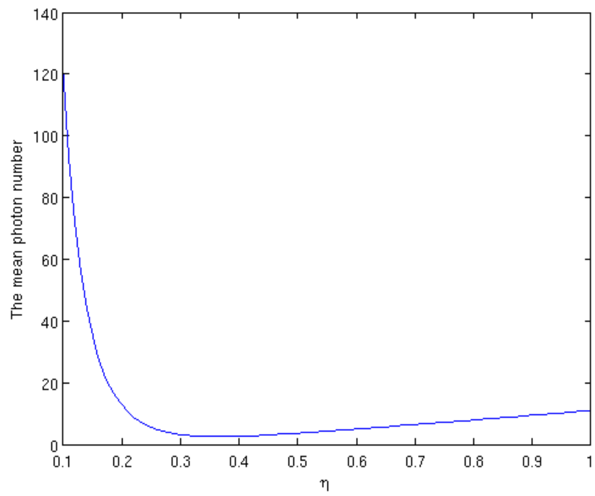

Figure 3.1: A plot of the mean photon number of the thermal light versus η [Eq. 3.37] for κ=0.8, A=5, and $\bar{n}$=2.

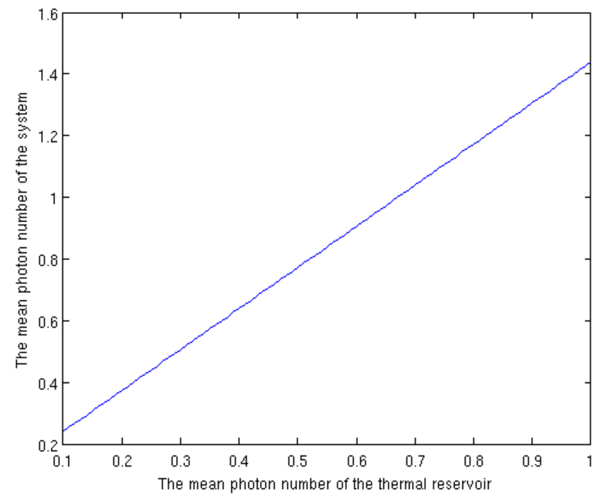

Figure 3.2: A plot of the mean photon number of the system versus $\bar{n}$ [Eq. 3.37] for κ=0.8, A=5, and η=0.5.

3.2 The variance of the photon number

The variance of the photon number for the two-mode caviy light can be defined as [32]

$$\Big(\Delta n\Big)^2 = \Big\langle (\hat{c}^\dagger(t)\hat{c}(t))^2 \Big\rangle - \Big\langle \hat{c}^\dagger(t)\hat{c}(t) \Big\rangle^2. \tag{3.39}$$

Since $\hat{a}(t)$ and $\hat{b}(t)$ are Gaussian variabiles with zero mean, then $\hat{c}(t)$ also be Gaussian variabile with zero mean. Then it follows

$$\Big\langle \Big(\hat{c}^\dagger(t)\hat{c}(t)\Big)^2 \Big\rangle = 2\Big\langle \hat{c}^\dagger(t)\hat{c}(t) \Big\rangle^2 + 2\Big\langle \hat{c}^\dagger(t)\hat{c}(t) \Big\rangle + \Big\langle \hat{c}^{\dagger^2}(t) \Big\rangle \Big\langle \hat{c}^2(t) \Big\rangle. \tag{3.40}$$

Upon substituting Eq. (3.40) into Eq. (3.39), we also see that

$$\Big(\Delta n\Big)^2 = \Big\langle \hat{c}^\dagger(t)\hat{c}(t) \Big\rangle^2 + 2\Big\langle \hat{c}^\dagger(t)\hat{c}(t) \Big\rangle + \Big\langle \hat{c}^{\dagger^2}(t) \Big\rangle \Big\langle \hat{c}^2(t) \Big\rangle. \tag{3.41}$$

On account of Eq. (3.23), we can write

$$\begin{aligned}\Big(\Delta n\Big)^2 = & \Big\langle \hat{a}^\dagger(t)\hat{a}(t) \Big\rangle^2 + 2\Big\langle \hat{a}^\dagger(t)\hat{a}(t) \Big\rangle \Big\langle \hat{b}^\dagger(t)\hat{b}(t) \Big\rangle \\ & +2\Big\langle \hat{a}^\dagger(t)\hat{a}(t) \Big\rangle + 2\Big\langle \hat{b}^\dagger(t)\hat{b}(t) \Big\rangle + \Big\langle \hat{b}^\dagger(t)\hat{b}(t) \Big\rangle^2 \\ & +4\Big\langle \hat{a}^\dagger(t)\hat{b}^\dagger(t) \Big\rangle \Big\langle \hat{a}(t)\hat{b}(t) \Big\rangle.\end{aligned} \tag{3.42}$$

Now the expectation value of $\hat{a}(t)\hat{b}(t)$ can be written as

$$\langle \hat{a}(t)\hat{b}(t) \rangle = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} Q(\alpha,\beta,t)\alpha\beta. \tag{3.43}$$

Introducing Eq. (2.246) into Eq. (3.43), we obtain

$$\begin{aligned}\langle \hat{a}(t)\hat{b}(t) \rangle = & (u_1u_2 - v^2)\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} exp\Big[- u_1\beta^*\beta - u_2\alpha^*\alpha + v\alpha\beta \\ & +v\alpha^*\beta^*\Big]\alpha\beta.\end{aligned} \tag{3.44}$$

This integral can be rewritten as

$$\langle\hat{a}(t)\hat{b}(t)\rangle = (u_1u_2 - v^2)\frac{\partial}{\partial p}\frac{\partial}{\partial q}\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}exp\bigg[-u_1\beta^*\beta - u_2\alpha^*\alpha + v\alpha\beta + v\alpha^*\beta^* + p\alpha + q\beta\bigg]. \tag{3.45}$$

Then it follows

$$\langle\hat{a}(t)\hat{b}(t)\rangle = \left(\frac{u_1u_2 - v^2}{u_1}\right)\frac{\partial}{\partial p}\frac{\partial}{\partial q} \times\int\frac{d^2\alpha}{\pi}exp\left[\frac{-(u_1u_2 - v^2)\alpha^*\alpha + u_1p\alpha^* + vq\alpha}{u_1}\right]\bigg|_{p=q=0}. \tag{3.46}$$

Hence upon carrying out the integration over α employing the relation given by Eq. (2.245), we get

$$\langle\hat{a}(t)\hat{b}(t)\rangle = \frac{v}{u_1u_2 - v^2}exp\left[\frac{pvq}{u_1u_2 - v^2}\right]\bigg|_{p=q=0}. \tag{3.47}$$

Applying the condition $p = q = 0$, we easily obtain

$$\langle\hat{a}(t)\hat{b}(t)\rangle = \frac{v}{u_1u_2 - v^2}. \tag{3.48}$$

Finally, we find

$$\langle\hat{a}(t)\hat{b}(t)\rangle = c, \tag{3.49}$$

in which

$$c = \langle\alpha(t)\beta(t)\rangle = \langle\alpha^*(t)\beta^*(t)\rangle. \tag{3.50}$$

Now with the aid of Eqs. (3.30), (3.33), (3.49), the variance of the photon number can be put in the form

$$(\Delta n)^2 = a^2 + b^2 + 2ab - 2(a+b) + 4c^2. \tag{3.51}$$

On account of Eqs. (2.240)-(2.242), Eq. (3.51) can be rewritten as

$$\left(\Delta n\right)^2 = \left\langle \alpha^*(t)\alpha(t)\right\rangle^2 + \left\langle \beta^*(t)\beta(t)\right\rangle^2 + 2\left\langle \alpha^*(t)\alpha(t)\right\rangle\left\langle \beta^*(t)\beta(t)\right\rangle + 4\left\langle \alpha(t)\beta(t)\right\rangle^2. \tag{3.52}$$

Finally, in veiw of Eqs. (3.16)-(3.18), the variance of the photon number, at steady state, can be put in the form

$$\begin{aligned}\left(\Delta n\right)^2 =& \left[\frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)} + \frac{\bar{n}(1+\eta)(3\eta-1)}{2\eta^2} + \frac{(1-\eta)(A\eta(\eta-1)+4\kappa\bar{n})}{2\eta^2(\kappa+A\eta)}\right]^2 \\ &+\left[\frac{\bar{n}(1-\eta)}{\eta^2} + \frac{(1+\eta)(A\eta(\eta-1)+4\kappa\bar{n})}{2\eta^2(\kappa+A\eta)} + \frac{(1-\eta^2)(A\eta-4\kappa\bar{n})}{2\eta(2\kappa+A\eta)}\right]^2 \\ &+2\left[\frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)} + \frac{\bar{n}(1+\eta)(3\eta-1)}{2\eta^2} + \frac{(1-\eta)(A\eta(\eta-1)+4\kappa\bar{n})}{2\eta^2(\kappa+A\eta)}\right] \\ &\left[\frac{\bar{n}(1-\eta)}{\eta^2} + \frac{(1+\eta)(A\eta(\eta-1)+4\kappa\bar{n})}{2\eta^2(\kappa+A\eta)} + \frac{(1-\eta^2)(A\eta-4\kappa\bar{n})}{2\eta(2\kappa+A\eta)}\right] \\ &+4\left[\sqrt{1-\eta^2}e^{i\theta}\left[\frac{\bar{n}}{\eta^2} + \frac{A\eta(\eta-1)+4\kappa\bar{n}}{2\eta^2(\kappa+A\eta)} + \frac{(2(\eta-1)-4\kappa\bar{n}+A)}{2\eta^2(2\kappa+A\eta)}\right]\right]^2.\end{aligned} \tag{3.53}$$

Comparison of Eqs. (3.37) and (3.53) shows that the photon statistics is supper possonian. For the case in which $\bar{n} = 0$ one can easily arrive at,

$$\begin{aligned}\left(\Delta n\right)^2 =& \left[\frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)} + \frac{(1-\eta)[A\eta(\eta-1)]}{2\eta^2(\kappa+A\eta)}\right]^2 + \left[\frac{(1+\eta)[A\eta(\eta-1)]}{2\eta^2(\kappa+A\eta)} + \frac{A(1-\eta^2)}{2(2\kappa+A\eta)}\right]^2 \\ &+2\left[\frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)} + \frac{(1-\eta)[A\eta(\eta-1)]}{2\eta^2(\kappa+A\eta)}\right]\left[\frac{(1+\eta)[A\eta(\eta-1)]}{2\eta^2(\kappa+A\eta)} + \frac{A(1-\eta^2)}{2(2\kappa+A\eta)}\right] \\ &+4\left[\sqrt{1-\eta^2}e^{i\theta}\left[\frac{A\eta(\eta-1)}{2\eta^2(\kappa+A\eta)} + \frac{2(\eta-1)+A}{2\eta^2(2\kappa+A\eta)}\right]\right]^2.\end{aligned} \tag{3.54}$$

is the variance of the photon number for a two-mode laser light coupled to vaccum reservoir.

3.3 The photon number distribution

we now wish to obtain an explicit expression of the photon number distribution employing the Q function along with the density operator for the two-mode cavity light. The photon number distribution for two-mode cavity light can be defined as [29]

$$P(n,m,t) = \langle m,n|\hat{\rho}|n,m\rangle. \tag{3.55}$$

Introducing Eq. (2.253) in Eq. (3.55), we see that

$$P(n,m,t) = \frac{1}{\pi^2}\int d^2\sigma d^2\varepsilon d^2\eta d^2\lambda\, Q(\sigma^*,\varepsilon^*,\eta,\lambda)$$
$$\langle m,n|\sigma,\varepsilon\rangle\langle\lambda,\eta|n,m\rangle\langle\varepsilon,\sigma|\eta,\lambda\rangle. \tag{3.56}$$

Now using Q function described by Eq. (2.246), Eq. (3.56) can be rewritten as

$$P(n,m,t) = \left(\frac{u_1u_2 - v^2}{n!m!}\right)\int \frac{d^2\sigma}{\pi}\frac{d^2\varepsilon}{\pi}\frac{d^2\eta}{\pi}\frac{d^2\lambda}{\pi} exp\bigg[-u_1\varepsilon^*\lambda - u_2\sigma^*\eta$$
$$+v\sigma^*\varepsilon^* + v\eta\lambda - \sigma^*\sigma - \varepsilon^*\varepsilon - \lambda^*\lambda - \eta^*\eta$$
$$+\sigma^*\eta + \varepsilon^*\lambda\bigg]\sigma^n\varepsilon^m\eta^{*n}\lambda^{*m}, \tag{3.57}$$

where

$$\langle m,n|\sigma,\varepsilon\rangle = e^{-\frac{\sigma^*\sigma}{2}}e^{-\frac{\varepsilon^*\varepsilon}{2}}\frac{\sigma^n}{\sqrt{n!}}\frac{\varepsilon^m}{\sqrt{m!}}\ , \tag{3.58}$$

$$\langle\varepsilon,\sigma|\eta,\lambda\rangle = exp\bigg[-\frac{\sigma^*\sigma}{2} - \frac{\varepsilon^*\varepsilon}{2} - \frac{\eta^*\eta}{2} - \frac{\lambda^*\lambda}{2} + \sigma^*\eta + \varepsilon^*\lambda\bigg], \tag{3.59}$$

and

$$\langle\lambda,\eta|n,m\rangle = e^{\frac{-\lambda^*\lambda}{2}}e^{\frac{-\eta^*\eta}{2}}\frac{\eta^{*n}}{\sqrt{n!}}\frac{\lambda^{*m}}{\sqrt{m!}}\ . \tag{3.60}$$

Eq. (3.57) can be rewritten as

$$P(n,m,t) = \left(\frac{u_1u_2 - v^2}{n!m!}\right)\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\frac{\partial^m}{\partial\beta^m}\frac{\partial^m}{\partial\beta^{*m}}\int\frac{d^2\sigma}{\pi}\frac{d^2\varepsilon}{\pi}\frac{d^2\eta}{\pi}\frac{d^2\lambda}{\pi}$$
$$\times exp\bigg[-u_1\varepsilon^*\lambda - u_2\sigma^*\eta + v\sigma^*\varepsilon^* + v\eta\lambda$$
$$-\sigma^*\sigma - \varepsilon^*\varepsilon - \lambda^*\lambda - \eta^*\eta + \sigma^*\eta + \varepsilon^*\lambda$$
$$+\alpha\eta^* + \alpha^*\sigma + \beta\lambda^* + \beta^*\varepsilon\bigg]\bigg|_{\alpha=\alpha^*=\beta=\beta^*=0}. \tag{3.61}$$

Upon carrying out the integration, we readily obtain

$$P(n,m,t) = \left(\frac{(u_1u_2 - v^2)}{n!m!}\right)\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\frac{\partial^m}{\partial\beta^m}\frac{\partial^m}{\partial\beta^{*m}} \times exp\left[(1-u_2)\alpha^*\alpha + (1-u_1)\beta^*\beta + v\alpha\beta + v\alpha^*\beta^*\right]\Bigg|_{\alpha=\beta=\alpha^*=\beta^*=0}. \tag{3.62}$$

Expanding in power series, we have

$$P(n,m,t) = \left(\frac{(u_1u_2 - v^2)}{n!m!}\right)\sum_{i,j,k,l}^{\infty}\frac{(1-u_1)^i(1-u_2)^j v^{k+l}}{i!j!k!l!} \times\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\frac{\partial^m}{\partial\beta^m}\frac{\partial^m}{\partial\beta^{*m}}\left[\alpha^{j+k}\alpha^{*j+l}\beta^{i+k}\beta^{*i+l}\right|_{\alpha=\beta=\alpha^*=\beta^*=0}. \tag{3.63}$$

Then performing the differentiation, employing the relation

$$\frac{\partial^m}{\partial\alpha^n}X^n = \frac{n!}{(n-m)!}X^{n-m} \tag{3.64}$$

and applying the condition $\alpha= \beta = \alpha = \beta =0$, the photon number distribution found to be

$$P(n,m,t) = (u_1u_2 - v^2)\sum_{i,j,k,l}^{\infty} n!m!\frac{(1-u_1)^i(1-u_2)^j v^{k+l}}{i!j!k!l!} \times \left[\frac{(j+k)!}{(j+k-n)!} \times\frac{(j+l)!}{(j+l-n)!}\times\frac{(i+k)!}{(i+k-m)!}\times\frac{(i+l)!}{(i+l-m)!}\right] \times\delta_{j+k,n}\delta_{j+l,n}\delta_{i+k,m}\delta_{i+l,m}. \tag{3.65}$$

Now we note that k = l = m-i = n-j. Hence for $n = m$, we see that

$$P(n,n,t) = (u_1u_2 - v^2)\sum_{i=0}^{n} n!^2\frac{\left((1-u_1)(1-u_2)\right)^i v^{2(n-i)}}{i!^2[(n-i)!]^2}. \tag{3.66}$$

It is possible to infer that there is a finite probability for getting equal number of photons in the two-modes.

4

Quadrature Fluctuations

In the previous chapter we have found the explicit forms of the Q function together with density operator. Here applying these results , we calculate quadrature flactuations of the two mode cavity light generated by optical system under consideration.

4.1 Quadrature variance

In order to study the squeezing properties of two -mode cavity light , we introduce the plus and minus quadrature operators defined by [31]

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{4.1}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.2}$$

Using Eq. (3.23) along with Eqs. (2.59) and (2.560), it can be readily verified that

$$\left[\hat{c}, \hat{c}^\dagger\right] = 2 \tag{4.3}$$

and

$$\left[\hat{c}_+, \hat{c}_-\right] = 4i. \tag{4.4}$$

Thus the quadrature variance of the two mode cavity light can be defined as

$$\left(\Delta c_{\pm}\right)^2 = 2 + \langle : \hat{c}_{\pm}, \hat{c}_{\pm} : \rangle \tag{4.5}$$

Using the above definition, the quadrature variance of the two quadrature operators in the normal ordering have the form

$$\left(\Delta c_{\pm}\right)^2 = 2 + 2\langle \hat{c}^{\dagger}\hat{c}\rangle \pm \langle \hat{c}^{\dagger 2}\rangle \pm \langle \hat{c}^{2}\rangle \mp \langle \hat{c}^{\dagger}\rangle^2 \mp \langle \hat{c}\rangle^2 - 2\langle \hat{c}^{\dagger}\rangle\langle \hat{c}\rangle. \tag{4.6}$$

Since $\hat{a}(t)$ and $\hat{b}(t)$ are Gaussian variables with zero mean, $\hat{c}(t)$ also be a Gaussian variable with zero mean. Hence Eq. (4.6) can be rewritten as

$$\left(\Delta c_{\pm}\right)^2 = 2 + 2\langle \hat{c}^{\dagger}\hat{c}\rangle \pm \langle \hat{c}^{\dagger 2}\rangle \pm \langle \hat{c}^{2}\rangle. \tag{4.7}$$

On account of Eq. (3.23), Eq. (4.7) can be put in the form

$$\left(\Delta c_{\pm}\right)^2 = 2 + 2\left[\langle \hat{a}^{\dagger}\hat{a}\rangle + \langle \hat{b}^{\dagger}\hat{b}\rangle \pm 2\langle \hat{a}\hat{b}\rangle\right]. \tag{4.8}$$

The c-number function corresponding to Eq. (4.9) associated with the normal ordering can be written as

$$\left(\Delta c_{\pm}\right)^2 = 2 + 2\left[\langle \alpha^*(t)\alpha(t)\rangle + \langle \beta^*(t)\beta(t)\rangle \pm 2\langle \alpha(t)\beta(t)\rangle\right]. \tag{4.9}$$

Now in veiw of Eqs. (3.17)-(3.19), Eq. (4.9) can be written as

$$\begin{aligned}\left(\Delta c_{\pm}\right)^2 = 2 + 2\Bigg[&\frac{(1-\eta)[A\eta(\eta-1)+4\kappa\bar{n}]}{2\eta^2(\kappa+A\eta)} + \frac{(1+\eta)[\bar{n}(3\eta-1)]}{2\eta^2} \\ &+\frac{A(1-\eta^2)}{2\eta(2\kappa+A\eta)} + \frac{(1-\eta)\bar{n}}{\eta^2} + \frac{(1+\eta)[A\eta(\eta-1)+4\kappa\bar{n}]}{2\eta^2(\kappa+A\eta)} \\ &+\frac{(1-\eta^2)[A\eta-4\kappa\bar{n}]}{2\eta(2\kappa+A\eta)}\Bigg] \pm 4\sqrt{1-\eta^2}e^{i\theta}\Bigg[\frac{\bar{n}}{\eta^2} + \frac{A\eta(\eta-1)+4\kappa\bar{n}}{2\eta^2(\kappa+A\eta)} \\ &+\frac{(2(\eta-1)-4\kappa\bar{n}+A)}{2\eta^2(2\kappa+A\eta)}\Bigg].\end{aligned} \tag{4.10}$$

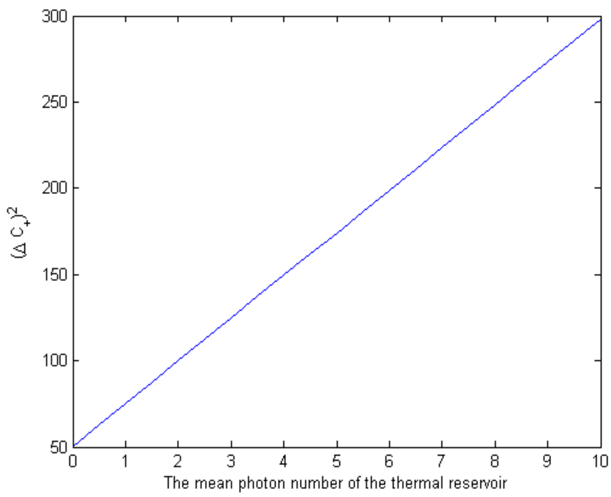

Figure 4.1: A plot of $(\Delta c_+)^2$ versus $\bar{n}$ [Eq. 4.10] for $\eta = 0.5$, $\theta = 0^0$, κ=0.8, and A=5.

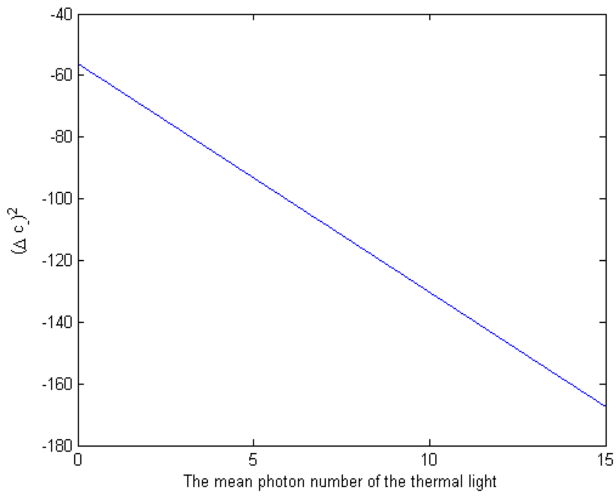

Figure 4.2: A plot of $(\Delta c_-)^2$ versus $\bar{n}$ [Eq. 4.10] for $\eta = 0.5$, $\theta = 0^0$, κ=0.8, and A=5.

We clearly observe from Fig. (4.1) and (4.2) that the quadrature squeezing of the non-degenerate three level laser light is indeed affected by the thermal light. For the case in which $\bar{n}$=0, we see that

$$\left(\Delta c_{\pm}\right)^{2}=2+2\left[\frac{(1-\eta)[A\eta(\eta-1)]}{2\eta^{2}(\kappa+A\eta)}+\frac{A(1-\eta^{2})}{2\eta(2\kappa+A\eta)}+\frac{(1+\eta)[A\eta(\eta-1)}{2\eta^{2}(\kappa+A\eta)}+\frac{(1-\eta^{2})A\eta}{2\eta(2\kappa+A\eta)}\right]\pm 4\sqrt{1-\eta^{2}}e^{i\theta}\left[\frac{A\eta(\eta-1)}{2\eta^{2}(\kappa+A\eta)}+\frac{(2(\eta-1)+A)}{2\eta^{2}(2\kappa+A\eta)}\right]. \tag{4.11}$$

5

Conclussion

In this thesis, we have studied the squeezing and the statistical properties of the light produced by a nonodegenerate three level laser coupled to a two-mode thermal reservoir. We obtained stochastic differential equations with aid of the master eqution. Employing stochastic differential equations, we obtained the correlation properties of the noise forces and then we determined the solutions of Langevin equations. Using the resulting solutions, we have established the Q function together with density operator of the system under consideration. Furthermore, with the aid of the resulting Q function along with the density operator, we have calculated the mean photon number, the variance of the photon number, the photon number distribution, and the quadrature fluctuations. We found that the two-mode thermal reservoir vanishs the squeezing and we have also shown that the noise in both quadrature is alarmly enhanced due to the thermal reservoir. Moreover, we have clearly observed that the mean photon number increases with the mean photon number of a two-mode thermal reservoir and decreases with a parametr η.

6

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DECLARATION

I hereby declare that this MSC thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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