Nondegenerate three-level laser with parametric amplifier driven by coherent light in vacuum reservoir

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(**Quantum Optics and Information)**

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented in any other university, and that all sources of material used for the thesis have been dully acknowledged.

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Abstract

In this research we have studied the squeezing and statistical properties of the cavity light produced by a coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Applying the steady state solutions of the stochastic differential equations of the atomic operators along with the quantum Langavin equations for the cavity mode operators, we have calculated the mean, variance of the photon number, photon number correlations, intensity difference fluctuations, the quadrature squeezing, and entanglement for the two-mode cavity light. We have found that the mean photon number of two mode is the sum of the mean photon number of mode a and b and the photon number variance of two mode is the square of mean photon number of two-mode. We have seen that the maximum quadrature squeezing is found to be 62.19%. Moreover, the presence of parametric amplifier enhances the quadrature squeezing.

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INTRODUCTION

Quantum optics deals mainly with the quantum properties of light generated by various optical system such laser with the effects of light on the dynamics of the atoms $[1-5]$. There has been a considerable interest in the analysis of squeezing and statistical properties of light generated by three-level lasers $[6-14]$. Squeezing is one of the interesting non classical features of light that has been attracting attention and studied by many authors $[15-22]$.

In recent years, the topic of entanglement has received a significant amount of attention as it plays an important role in all branches of quantum information processing [2]. The efficiency of quantum information schemes highly depends on degree of entanglement. Moreover, Eyob [3] has studied continuous variable entanglement in a non degenerate three-level laser with a parametric amplifier. In this model the injected atomic coherence introduced by initially preparing the atoms in a coherent superposition of the top and bottom levels. This combined system exhibits a two-mode squeezed light and produces light in an entangled state. In one model of such a laser, three-level atoms initially in the upper level are injected at a constant rate in to the cavity and removed after they have decayed due to spontaneous emission $[8, 12, 21]$. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the laser cavity. Besides, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity.

A parametric amplifier will consider as an important source of squeezed light $[19]$. It is one of the most interesting and well characterized optical devices in quantum optics. In this device a pump photon interacts with a nonlinear crystal inside an open cavity. The three-level laser may be defined as a quantum optical system in which three level atoms initially prepared in a coherent superposition of the two levels, are injected into a cavity coupled to a vacuum reservoir via a single-port mirror $[7, 14]$. The cascade system has an excited state coupled to intermediate state, and the intermediate state one coupled to the ground state.

The three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called degenerate three level atom otherwise it is called non degenerate. We consider the case in which N non degenerate three level atoms in cascade configuration are available in an open cavity. The emission of light when the atoms makes the transition from the top level to intermediate level is light mode a, and the emission of light when the atoms makes the transition from the intermediate level to the bottom level is light mode b.

In this thesis first we derive the master equation and the quantum langevin equations. Employing the master equation we obtain the stochastic differential equations. Moreover, applying the large time approximation of the stochastic differential equations,we get the steady state solutions of these equations. Finally, employing the solutions of the quantum langevin equations, we study the squeezing, the entanglement and the photon statistics of the cavity light.

2

OPERATOR DYNAMICS

2.1 Master equation

In this chapter we consider a non degenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode vacuum reservoir via a single-port mirror as shown in figure 2.1. We first obtain the master equation for a coherently driven non degenerate three-level atom with the cavity modes and the quantum langevin equations for the cavity mode operators. Moreover, employing the master equation and the large approximation scheme, we drive the stochastic differential equations of the atomic operators. Finally, we determine the steady-state solutions of the resulting equations. Here we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. We consider the the case in which N non degenerate three-level atoms in cascade configuration are available in an open cavity. We denote the top,intermediate and bottom levels of the three-level atom by $|a\rangle_k$, $|b\rangle_k$ and $|c\rangle_k$ respectively. For non degenerate three-level cascade configuration, when the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and $|b\rangle_k$ to $|c\rangle_k$ two photons with different frequencies are emitted.

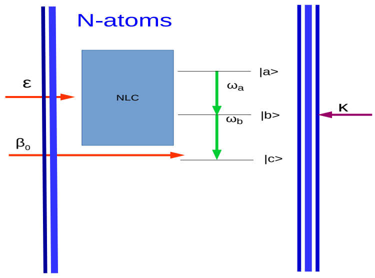

Figure 2.1: Schematic representation of a coherently driven non degenerate three-level laser coupled to a two-mode vacuum reservoir.

The emission of light, when the atoms makes a transition from top level to the intermediate level is light mode a and the emission of light,when the atoms makes a transition from intermediate level to the bottom level is light mode b. We assume the cavity mode to be at resonance with the two transitions $|a\rangle_k \rightarrow |b\rangle_k$ and $|b\rangle_k \rightarrow |c\rangle_k$ are dipole allowed, and with direct transition between levels $|a\rangle_k \rightarrow |c\rangle_k$ to be dipole forbidden. The coupling of the top and bottom levels of a non degenerate three-level atom by coherent light can be described by the Hamiltonian

$$\hat{H}_1 = \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big], \tag{2.1}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{k\,k}\langle a|, \tag{2.2}$$

is lowering atomic operator and

$$\Omega = 2\beta_0\lambda. \tag{2.3}$$

Here β_0, considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the driving coherent light and the three-level atom. The interaction of a three level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H}_2 = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big], \tag{2.4}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{k\ k}\langle a|, \tag{2.5}$$

$$\hat{\sigma}_b^k = |c\rangle_{k\ k}\langle b|. \tag{2.6}$$

This g is the coupling constant between the atom and the cavity mode a and b, and the operators $\hat{a}$ and $\hat{b}$ are the annihilation operators for light mode a and b. In addition, the interaction of Hamiltonian with parametric amplifier can be expressible as

$$\hat{H}_3 = i\varepsilon\big[\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\big], \tag{2.7}$$

where ε is the amplitude of the driving coherent light that drives NLC. Thus, combination of Eqs.(2.1), (2.4), and (2.7), we see that

$$\hat{H}_S = \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big] + ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big] + i\varepsilon\big[\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\big]. \tag{2.8}$$

The quantum analysis of the interaction of a system such as a cavity mode or a three-level atom with the external environment is a relatively complex problem. The external environment, usually referred to as a reservoir, can be thermal light, ordinary or squeezed vacuum. We are interested in the dynamics of the system and this is describable by the master equation or quantum Langevin equations. Here, we obtain the above set of dynamical equations for a cavity mode coupled to a two-mode vacuum reservoir via a single-port mirror. The resulting equations are easily adaptable to the case when the external environment is either a thermal or a vacuum reservoir. We then focus our study when the cavity mode is coupled to a two-mode vacuum reservoir. A system coupled with a two-mode vacuum reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{2.9}$$

where $\hat{H}_S$ is the Hamiltonian of the system and $\hat{H}_{SR}$ is the Hamiltonian of the system and the reservoir. Suppose $\hat{X}(t)$ is the density operator for the system and the reservoir. Then the equation of evolution of this density operator is given by

$$\frac{d}{dt}\hat{X}(t) = -i\big[\hat{H}_S + \hat{H}_{SR}, \hat{X}(t)\big]. \tag{2.10}$$

We are interested in the quantum dynamics of the system alone. Hence taking into account (2.10), we see that the density operator for the system, also known as the the reduced density operator,

$$\hat{\rho}(t) = Tr_R\hat{X}(t) \tag{2.11}$$

evolves in time according to

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}_S, \hat{\rho}(t)\big] - iTr_R\big[\hat{H}_{SR}, \hat{X}(t)\big], \tag{2.12}$$

where Tr_R indicates the trace over the reservoirs variables only. On the other hand, a formal solution of Eq. (2.10) can be written as

$$\hat{X}(t) = \hat{X}(0) - i\int_0^t \big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{X}(t')\big]dt'. \tag{2.13}$$

In order to obtain mathematically manageable that $\hat{\chi}(t')$ by some approximately valid expression. Then, in the first place, we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. This can be achieved by letting a beam of thermal light (or light in a vacuum state) of constant intensity fall continuously on the system. Moreover, we decouple the system and reservoirs density operators, so that

$$\hat{X}(t') = \hat{\rho}(t')\hat{R}. \tag{2.14}$$

Therefore, with the aid of this, one can rewrite Eq. (2.13) as

$$\hat{X}(t) = \hat{\rho}(0) - i\int_0^t \big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\big]dt'. \tag{2.15}$$

Now on substituting Eq.(2.15) into (2.12) there follows

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i\big[\hat{H}_{SR}(t), \hat{\rho}(t)\big] - i\big[\langle\hat{H}_{SR}(t)\rangle_R, \hat{\rho}(0)\big] \\ &\quad - \int_0^t \big[\langle\hat{\rho}_{SR}(t)\rangle_R, \big[\hat{H}_S(t'), \hat{\rho}(t')\big]\big]dt' \\ &\quad - \int_0^t Tr_R\big[\hat{H}_{SR}(t'), \big[\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\big]\big]dt',\end{aligned} \tag{2.16}$$

where the subscript R indicates that the expectation value is to be calculated using the reservoirs density operator $\hat{R}$. Furthermore, the master equation for a system coupled to a reservoir takes the form

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -iTr_A[\hat{H}_S, \hat{\rho}_{AR}(t,t')] - h\langle\hat{H}_{SR}^2\hat{R}\rangle_R\hat{\rho}(t) \\ &\quad +2hTr_R(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}) - h\hat{\rho}(t)\langle\hat{H}_{SR}^2\hat{R}\rangle_R.\end{aligned} \tag{2.17}$$

A light mode confined in a cavity, usually formed by two mirrors, is called a cavity mode. A commonly used cavity has a single-port mirror. One side of each cavity is a mirror through which light can enter or leave the cavity. We now proceed to obtain the equation of evolution of the reduced density operator, in short the master equation, for the cavity mode coupled to a thermal reservoir via a single port-mirror. We consider the reservoirs to be composed of large number of submodes. Thus, the interaction Hamiltonian for a two-mode cavity light coupled to thermal reservoir is written as

$$\begin{aligned}\hat{H}_{SR}(t) &= i\sum_k g_k\big(\hat{\sigma}_a^{\dagger k}\hat{a}_k e^{i(\omega_0-\omega_k)t} - \hat{a}^\dagger\hat{\sigma}_a^k e^{-i(\omega_0-\omega_k)t} \\ &+ \hat{\sigma}_b^{\dagger k}\hat{b}_k e^{i(\omega_0-\omega_k)t} - \hat{b}^\dagger\hat{\sigma}_b^k e^{-i(\omega_0-\omega_k)t}\big),\end{aligned} \tag{2.18}$$

where $\hbar = 1$, $\hat{a}_k$, and $\hat{b}_k$ are annihilation operators for the reservoir submode and

$$g_k = \big[\frac{\omega_k}{2\epsilon_0 v}\big]^{1/2} d_{ab}.u_k \tag{2.19}$$

is coupling constant. Furthermore, employing Eq. (2.18), we then see that

$$\big[\langle\hat{H}_{SR}(t)\rangle_R, \hat{\rho}(t')\big] = \big[\hat{H}_{SR}(t), \big[\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\big]\big] = 0. \tag{2.20}$$

In view of this results, expression (2.16) reduces to

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}_S, \hat{\rho}(t)\big] - i\int_0^t Tr_R\big[\hat{H}_{SR}(t), \big[\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\big]\big]dt'. \tag{2.21}$$

It then follows that

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= -i\big[\hat{H}_S, \hat{\rho}(t)\big] - \int_0^t Tr_R(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t))dt' \\ &- \int_0^t \hat{\rho}(t)Tr_R(\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t))dt' \\ &+ \int_0^t Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t)\hat{R}\hat{H}_{SR}(t'))dt' \\ &+ \int_0^t Tr_R(\hat{H}_{SR}(t')\hat{\rho}(t)\hat{R}\hat{H}_{SR}(t))dt'.\end{aligned} \tag{2.22}$$

where $\hat{\rho}(t') = \hat{\rho}(t)$. In addition, in view of Eq. (2.18), one can readily obtain

$$\begin{aligned}\hat{H}_{SR}(t') &= i\sum_{k} g_{k'}\big(\hat{\sigma}_a^{\dagger k'}\hat{a}_{k'}e^{i(\omega_0-\omega_{k'})t'} - \hat{a}_{k'}^{\dagger}\hat{\sigma}_a^{k'}e^{-i(\omega_0-\omega_{k'})t'} \\ &+ \hat{\sigma}_b^{\dagger k'}\hat{b}_{k'}e^{i(\omega_0-\omega_{k'})t'} - \hat{b}_{k'}^{\dagger}\hat{\sigma}_b^{k'}e^{-i(\omega_0-\omega_{k'})t'}\big). \qquad (2.23)\end{aligned}$$

Using Eqs.(2.18) and (2.23) and applying the cyclic property of the trace operation, one can write

$$\begin{aligned}Tr_R(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')) &= -\Gamma_1\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^{\dagger k'} + \Gamma_2\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^{k'} - \Gamma_3\hat{\sigma}_a^{\dagger k}\hat{\sigma}_b^{\dagger k'} + \Gamma_4\hat{\sigma}_a^{\dagger k}\hat{\sigma}_b^{k'} \\ &+ \Gamma_5\hat{\sigma}_a^{k}\hat{\sigma}_a^{\dagger k'} + \Gamma_6\hat{\sigma}_a^{k}\hat{\sigma}_a^{k'} - \Gamma_7\hat{\sigma}_a^{k}\hat{\sigma}^{\dagger k'} + \Gamma_8\hat{\sigma}_a^{k}\hat{\sigma}_b^{k'} \\ &+ \Gamma_9\hat{\sigma}_b^{\dagger k}\hat{\sigma}_a^{\dagger k'} + \Gamma_{10}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_a^{k'} + \Gamma_{11}\hat{\sigma}_b^{\dagger k}\hat{\sigma}^{\dagger k'} - \Gamma_{12}\hat{\sigma}_b^{\dagger k}\hat{\sigma}^{k'} \\ &- \Gamma_{13}\hat{\sigma}_b^{k}\hat{\sigma}_a^{\dagger k'} + \Gamma_{14}\hat{\sigma}_b^{k}\hat{\sigma}_a^{k'} - \Gamma_{15}\hat{\sigma}_b^{k}\hat{\sigma}_a^{k'} + \Gamma_{16}\hat{\sigma}_b^{k}\hat{\sigma}_b^{k'}, \qquad (2.24)\end{aligned}$$

where

$$\Gamma_1 = \sum_{kk'} g_k g_{k'} \langle \hat{a}_k \hat{a}_{k'} \rangle_R e^{i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.25}$$

$$\Gamma_2 = \sum_{kk'} g_k g_{k'} \langle \hat{a}_k \hat{a}^\dagger_{k'} \rangle_R e^{i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.26}$$

$$\Gamma_3 = \sum_{kk'} g_k g_{k'} \langle \hat{a}_k \hat{b}_{k'} \rangle_R e^{i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.27}$$

$$\Gamma_4 = \sum_{kk'} g_k g_{k'} \langle \hat{a}_k \hat{b}^\dagger_{k'} \rangle_R e^{i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.28}$$

$$\Gamma_5 = \sum_{kk'} g_k g_{k'} \langle \hat{a}^\dagger_k \hat{a}_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.29}$$

$$\Gamma_6 = \sum_{kk'} g_k g_{k'} \langle \hat{a}^\dagger_k \hat{a}^\dagger_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.30}$$

$$\Gamma_7 = \sum_{kk'} g_k g_{k'} \langle \hat{a}^\dagger_k \hat{b}_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.31}$$

$$\Gamma_8 = \sum_{kk'} g_k g_{k'} \langle \hat{a}^\dagger_k \hat{b}^\dagger_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.32}$$

$$\Gamma_9 = \sum_{kk'} g_k g_{k'} \langle \hat{b}_k \hat{a}_{k'} \rangle_R e^{i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.33}$$

$$\Gamma_{10} = \sum_{kk'} g_k g_{k'} \langle \hat{b}_k \hat{a}^\dagger_{k'} \rangle_R e^{i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.34}$$

$$\Gamma_{11} = \sum_{kk'} g_k g_{k'} \langle \hat{b}_k \hat{b}_{k'} \rangle_R e^{i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.35}$$

$$\Gamma_{12} = \sum_{kk'} g_k g_{k'} \langle \hat{b}_k \hat{b}^\dagger_{k'} \rangle_R e^{i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.36}$$

$$\Gamma_{13} = \sum_{kk'} g_k g_{k'} \langle \hat{b}^\dagger_k \hat{b}_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.37}$$

$$\Gamma_{14} = \sum_{kk'} g_k g_{k'} \langle \hat{b}^\dagger_k \hat{a}^\dagger_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}, \tag{2.38}$$

$$\Gamma_{15} = \sum_{kk'} g_k g_{k'} \langle \hat{b}^\dagger_k \hat{b}_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t+i(\omega_0-\omega_{k'})t'}, \tag{2.39}$$

$$\Gamma_{16} = \sum_{kk'} g_k g_{k'} \langle \hat{b}^\dagger_k \hat{a}^\dagger_{k'} \rangle_R e^{-i(\omega_0-\omega_k)t-i(\omega_0-\omega_{k'})t'}. \tag{2.40}$$

For a vacuum reservoir

$$\langle \hat{a}_k \hat{a}_{k'} \rangle_R = \langle \hat{b}_k \hat{b}_{k'} \rangle_R = 0,$$
$$\langle \hat{a}_k^\dagger \hat{a}_{k'}^\dagger \rangle_R = \langle \hat{b}_k^\dagger \hat{b}_{k'}^\dagger \rangle_R = 0,$$
$$\langle \hat{a}_k^\dagger \hat{a}_{k'} \rangle_R = \langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle_R = 0,$$
$$\langle \hat{a}_k \hat{b}_{k'} \rangle_R = \langle \hat{a}_k \hat{b}_{k'}^\dagger \rangle_R = 0,$$
$$\langle \hat{b}_k \hat{a}_{k'} \rangle_R = \langle \hat{b}_k \hat{a}_{k'}^\dagger \rangle_R = 0, \tag{2.41}$$

and

$$\langle \hat{a}_k \hat{a}_{k'}^\dagger \rangle_R = \big(\langle \hat{a}_k^\dagger \hat{a}_{k'} \rangle_R + 1\big)\hat{\sigma}_{kk'},$$
$$\langle \hat{b}_k \hat{b}_{k'}^\dagger \rangle_R = \big(\langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle_R + 1\big)\hat{\sigma}_{kk'}. \tag{2.42}$$

In view of Eq. (2.41) for vacuum reservoir, we have

$$\Gamma_1 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 0,$$
$$\Gamma_6 = \Gamma_7 = \Gamma_8 = \Gamma_9 = \Gamma_{10} = 0,$$
$$\Gamma_{11} = \Gamma_{13} = \Gamma_{14} = \Gamma_{15} = \Gamma_{16} = 0. \tag{2.43}$$

So that on account of these results, Eq. (2.24) takes the form

$$Tr_R(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')) = \Gamma_2 \hat{\sigma}_a^{\dagger k} \hat{\sigma}_a^{k'} + \Gamma_{12} \hat{\sigma}_b^{\dagger k} \hat{\sigma}_b^{k'}. \tag{2.44}$$

For the values $k = k'$, one can easily observe that

$$\Gamma_2 = \big(\langle \hat{a}_k^\dagger \hat{a}_k \rangle_R + 1\big) \sum_k g_k^2 e^{i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t}. \tag{2.45}$$

Furthermore, in view of Eq.(2.41), we see that

$$\Gamma_2 = \sum_k g_k^2 e^{i(\omega_0 - \omega_k)t - i(\omega_0 - \omega_{k'})t}. \tag{2.46}$$

In order to evaluate the dot product involved in Eq. (2.19), we adopt spherical coordinates in k space with the electrical dipole matrix element d_{ab} taken to be along z-axis. In addition, we take the unit vector u_k to be in the plan formed by the vectors d_{ab} and k. Since u_k is normal to k^3, the angle between u_k and d_{ab} is $(\frac{\pi}{2} - \theta)$. Then, we see that

$$d_{ab}.u_k = d_{ab}\cos(\frac{\pi}{2} - \theta) = d_{ab}\sin\theta. \tag{2.47}$$

Employing the transformation, we have

$$\sum_k \to \frac{\nu}{(2\pi)^3}\int d^3k = \frac{\nu}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} k^2 dk. \tag{2.48}$$

It then follows that

$$\Gamma_2 = \frac{d_{ab}^2}{2(2\pi)^3\epsilon_0 c^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta(1-\cos^2\theta))d\theta \int_0^{\infty} \omega^3 e^{i(\omega_0-\omega_k)(t-t')}d\omega, \tag{2.49}$$

where $k = \omega/c$ and from trigonometry

$$\sin^2\theta = 1 - \cos^2\theta, \tag{2.50}$$

one can readily obtain

$$\Gamma_2 = \frac{d_{ab}^2}{2(6\pi^2\epsilon_0 c^3)}\int_0^{\infty} \omega^3 e^{i(\omega_0-\omega_k)(t-t')}d\omega, \tag{2.51}$$

in which

$$\int_0^{2\pi} d\phi = 2\pi, \tag{2.52}$$

$$\int_0^{\pi} \sin\theta(1-\cos^2\theta) = \frac{2}{3}. \tag{2.53}$$

Now, we replace ω^3 by ω_0^3 and extending the lower limit of the integration to∞.Then after setting $\omega' = \omega - \omega_0$, we obtain

$$\Gamma_2 = \gamma\sigma(t - t'), \tag{2.54}$$

where

$$\gamma = \frac{\omega_0 d_{ab}^2}{3\pi\epsilon_0 c^3} \tag{2.55}$$

is γ is the spontaneous emission decay rate. In a similar procedure, one can easily verify that

$$Tr_R(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')) = \gamma\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + \gamma\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k. \tag{2.56}$$

It then follows that

$$\int_0^t Tr_R(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t'))\hat{\rho}dt' = \gamma\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} + \gamma\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} \tag{2.57}$$

Following a similar fashion, we see that

$$\int_0^t \hat{\rho}(t)Tr_R(\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t))dt' = \gamma\hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + \gamma\hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k, \tag{2.58}$$

$$\int_0^t Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t)\hat{R}\hat{H}_{SR}(t'))dt' = \gamma\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \gamma\hat{\rho}\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k}, \tag{2.59}$$

$$\int_0^t Tr_R(\hat{H}_{SR}(t')\hat{\rho}(t)\hat{R}\hat{H}_{SR}(t))dt' = \gamma\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \gamma\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k}. \tag{2.60}$$

Taking into account Eq. (2.57), (2.58), (2.59), and (2.60), the expression in Eq. (2.22) takes the form

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}_S(t), \hat{\rho}(t)\big] - \gamma\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \gamma\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \gamma\hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k - \gamma\hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k$$

$$+ \gamma\hat{\rho}\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \gamma\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} + \gamma\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} + \gamma\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k}. \tag{2.61}$$

It then follows

$$\frac{d}{dt}\hat{\rho}(t) = -i\big[\hat{H}_S(t),\hat{\rho}(t)\big] + \frac{\gamma}{2}\big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\big] + \frac{\gamma}{2}\big[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\big]. \tag{2.62}$$

On the other hand, employing Eq. (2.8), one can readily establish that

$$-i\big[\hat{H}_S,\hat{\rho}(t)\big] = g\big(\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} + \hat{\rho}\hat{b}\hat{\sigma}_b^k\big) + \frac{\Omega}{2}\big(\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\hat{\rho} + \hat{\rho}\hat{\sigma}_c^k\big) + \varepsilon\big(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b}\big). \tag{2.63}$$

Now the combination of Eqs. (2.62) and (2.63) results in

$$\frac{d}{dt}\hat{\rho}(t) = g\big[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} + \hat{\rho}\hat{b}\hat{\sigma}_b^k\big] + \frac{\Omega}{2}\big[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\hat{\rho} + \hat{\rho}\hat{\sigma}_c^k\big) + \varepsilon\big(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b}\big] + \frac{\gamma}{2}\big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\big] + \frac{\gamma}{2}\big[2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\big]. \tag{2.64}$$

where γ is the spontaneous emission decay constant, which considered to be the same for levels $|a\rangle_k$ and $|b\rangle_k$. Therefore, Eq. (2.64) the master equation for a coherently driven nondegenerate three-level atom in an open cavity and coupled to a two-mode vacuum reservoir.

2.2 Quantum Langevin equations

We recall that the laser cavity is coupled to a two-mode vacuum reservoir via a single port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators. We can therefore drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i\big[\hat{a}, \hat{H}_S\big], \tag{2.65}$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i\big[\hat{b}, \hat{H}_S\big], \tag{2.66}$$

where κ is the cavity damping constant. In view of Eq.(2.8) the quantum Langevin equations for cavity mode $\hat{a}$ and $\hat{b}$ turns out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \varepsilon\hat{b}^\dagger, \tag{2.67}$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_b^k + \varepsilon\hat{a}^\dagger. \tag{2.68}$$

2.3 Stochastic differential equations

Here we seek to derive the stochastic differential equations or the equations of evolution of the expectation values of the atomic operators by applying the master equation and the large-time approximation scheme. Moreover, we find the steady-state solutions of the equations of evolution of the atomic operators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\big(\frac{d\hat{\rho}}{dt}\hat{A}\big), \tag{2.69}$$

along with the master equation described by Eq. (2.64), one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g\big[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle\big] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle - \gamma\langle\hat{\sigma}_a^k\rangle, \tag{2.70}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = g\big[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle\big] - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle - \frac{\gamma}{2}\langle\hat{\sigma}_b^k\rangle, \tag{2.71}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = g\big[\langle\hat{\sigma}_a^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{a}\rangle - \langle\hat{a}^k\hat{b}\rangle\big] + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_c^k\rangle\big] - \frac{\gamma}{2}\langle\hat{\sigma}_c^k\rangle, \tag{2.72}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g\big[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big] - \gamma\langle\hat{\eta}_a^k\rangle, \tag{2.73}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g\big[\langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\big] + \gamma\big[\langle\hat{\eta}_a^k\rangle, -\langle\hat{\eta}_b^k\rangle\big] \tag{2.74}$$

in which

$$\hat{\eta}_a^k = |a\rangle_{k\;k}\langle a|, \tag{2.75}$$

$$\hat{\eta}_b^k = |b\rangle_{k\;k}\langle b|, \tag{2.76}$$

$$\hat{\eta}_b^k = |c\rangle_{k\;k}\langle c|. \tag{2.77}$$

We see that Eqs. (2.70)-(2.74) are nonlinear differential equations and hence it is not possible to find the exact time dependant solutions of these equations. We intend to overcome this problem by applying the large-time approximation. Therefore, employing this approximation scheme, we get from Eqs. (2.67) and (2.68) the approximately valid relations

$$\hat{a} = \frac{-2g}{\kappa}\hat{\sigma}_a^k + \frac{2\varepsilon}{\kappa}\hat{b}^\dagger, \tag{2.78}$$

$$\hat{b} = \frac{-2g}{\kappa}\hat{\sigma}_b^k + \frac{2\varepsilon}{\kappa}\hat{a}^\dagger, \tag{2.79}$$

and the corresponding conjugates are

$$\hat{a}^{\dagger} = \frac{-2g}{\kappa}\hat{\sigma}_a^{\dagger k} + \frac{2\varepsilon}{\kappa}\hat{b}, \tag{2.80}$$

$$\hat{b}^{\dagger} = \frac{-2g}{\kappa}\hat{\sigma}_b^{\dagger k} + \frac{2\varepsilon}{\kappa}\hat{a}, \tag{2.81}$$

so that applying Eqs. (2.80) and (2.81) into (2.78) and (2.79) leads to

$$\hat{a} = \frac{-2gk\hat{\sigma}_a^k - 4g\varepsilon\hat{\sigma}_b^{\dagger k}}{\kappa^2 - 4\varepsilon^2}, \tag{2.82}$$

$$\hat{b} = \frac{-2gk\hat{\sigma}_b^k - 4g\varepsilon\hat{\sigma}_a^{\dagger k}}{\kappa^2 - 4\varepsilon^2}. \tag{2.83}$$

These are the steady-state solutions of quantum Langevin equation.

Now introducing Eqs. (2.82) and (2.83) into Eqs. (2.70)-(2.74), the stochastic differential equations of the atomic operators takes the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\langle\hat{\sigma}_a^k\rangle + (\frac{\Omega}{2} + \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2})\langle\hat{\sigma}_b^{\dagger}\rangle, \tag{2.84}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{1}{2}(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\langle\hat{\sigma}_b^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{2.85}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{1}{2}(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\langle\hat{\sigma}_c^k\rangle + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{2.86}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\langle\hat{\eta}_a^k\rangle + (\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2})\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.87}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = (\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\big[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle + \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{2.88}$$

where

$$\gamma_c = \frac{4g^2}{\kappa}, \tag{2.89}$$

is the stimulating emission constant. We next sum Eqs .(2.84)-(2.88) over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})\langle\hat{m}_a\rangle + (\frac{\Omega}{2} + \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})\langle\hat{m}_b^\dagger\rangle, \tag{2.90}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{1}{2}(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^{\dagger k}\rangle, \tag{2.91}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{1}{2}(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})\langle\hat{m}_c\rangle + \frac{\Omega}{2}\big[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\big], \tag{2.92}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})\langle\hat{N}_a\rangle + (\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^\dagger\rangle\big], \tag{2.93}$$

$$\frac{d}{dt}\langle\hat{N}_b^k\rangle = +(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})\big[\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle + \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^\dagger\rangle\big], \tag{2.94}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N}\hat{\sigma}_a^k, \tag{2.95}$$

$$\hat{m}_b = \sum_{k=1}^{N}\hat{\sigma}_b^k, \tag{2.96}$$

$$\hat{m}_c = \sum_{k=1}^{N}\hat{\sigma}_c^k, \tag{2.97}$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \tag{2.98}$$

$$\hat{N}_b = \sum_{k=1}^{N}\hat{\eta}_b^k, \tag{2.99}$$

$$\hat{N}_c = \sum_{k=1}^{N}\hat{\eta}_c^k. \tag{2.100}$$

The operators $\hat{N}_a$,$\hat{N}_b$, $\hat{N}_c$ represents the number of atoms in the top, intermediate and the bottom levels respectively. In addition, employing the completeness relation

$$\hat{\eta}_a + \hat{\eta}_a + \hat{\eta}_a = \hat{I}, \tag{2.101}$$

we easily verified that

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.102}$$

Furthermore, using the definition given by Eqs. (2.95)-(2.97) and setting for any k

$$\hat{\sigma}_a = |b\rangle\langle a|, \tag{2.103}$$

$$\hat{\sigma}_b = |c\rangle\langle b|, \tag{2.104}$$

$$\hat{\sigma}_c = |c\rangle\langle a|, \tag{2.105}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|, \tag{2.106}$$

$$\hat{m}_b = N|c\rangle\langle b|, \tag{2.107}$$

$$\hat{m}_c = N|c\rangle\langle a|. \tag{2.108}$$

Following a similar procedure, one can easily verify that

$$\hat{N}_a = N|a\rangle\langle a|, \tag{2.109}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.110}$$

$$\hat{N}_c = N|c\rangle\langle c|. \tag{2.111}$$

Using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b, \tag{2.112}$$

and taking into account Eqs. (2.106)-(2.111), it can be readily established that

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{2.113}$$

$$\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{2.114}$$

$$\hat{m}^2 = N\hat{m}_c, \tag{2.115}$$

$$\hat{m}^{\dagger 2} = N\hat{m}_c^\dagger. \tag{2.116}$$

On the other hand, combination of Eqs. (2.67) and (2.68) together with (2.78) and (2.79), the cavity mode operators $\hat{a}$ and $\hat{b}$ can be

$$\frac{d\hat{a}}{dt} = -\frac{1}{2}\Big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\Big]\hat{a}(t) - g\hat{\sigma}_a^k - \frac{2g\varepsilon}{\kappa}\hat{\sigma}_b^{\dagger k}, \tag{2.117}$$

$$\frac{d\hat{b}}{dt} = -\frac{1}{2}\Big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\Big]\hat{b}(t) - g\hat{\sigma}_b^k - \frac{2g\varepsilon}{\kappa}\hat{\sigma}_a^{\dagger k}. \tag{2.118}$$

Now applying the steady-state solutions of Eqs. (2.117) and (2.118), we get

$$\hat{a} = \frac{-2gk\hat{\sigma}_a^k - 4g\varepsilon\hat{\sigma}_b^{\dagger k}}{\kappa^2 - 4\varepsilon^2}, \tag{2.119}$$

$$\hat{b} = \frac{-2gk\hat{\sigma}_b^k - 4g\varepsilon\hat{\sigma}_a^{\dagger k}}{\kappa^2 - 4\varepsilon^2}. \tag{2.120}$$

The commutation relation of the cavity mode operators $\hat{a}$ and $\hat{a}^\dagger$ as well as $\hat{b}$ and $\hat{b}^\dagger$ can be

$$\big[\hat{a}, \hat{a}^\dagger\big]_k = \frac{\gamma_c \kappa}{\kappa^2 - 4\varepsilon^2}\big[\kappa^2(\hat{\eta}_b^k - \hat{\eta}_a^k) + 4\varepsilon^2(\hat{\eta}_b^k - \hat{\eta}_c^k) - 2\varepsilon\kappa(\hat{\sigma}_c^k + \hat{\sigma}_c^{\dagger k})\big], \tag{2.121}$$

and on summing over all atoms, we have

$$\big[\hat{a}, \hat{a}^\dagger\big] = \frac{\gamma_c \kappa}{\kappa^2 - 4\varepsilon^2}\big[\kappa^2(\hat{N}_b - \hat{N}_a) + \varepsilon^2(\hat{N}_b - \hat{N}_c) - 2\varepsilon\kappa(\hat{m}_c + \hat{m}_c^\dagger)\big], \tag{2.122}$$

where

$$\big[\hat{a}, \hat{a}^\dagger\big] = \sum_{k=1}^{N} \big[\hat{a}, \hat{a}^\dagger\big]_k.$$

In the absence of parametric amplifier (when $\varepsilon = 0$), Eq. (2.122) takes the form

$$\big[\hat{a}, \hat{a}^{\dagger}\big] = \frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_a\big]. \tag{2.123}$$

Similarly for light mode b, we see that

$$\big[\hat{b}, \hat{b}^{\dagger}\big]_k = \frac{\gamma_c\kappa}{[\kappa^2 - 4\varepsilon^2]^2}\big[\kappa^2(\hat{\eta}_c^k - \hat{\eta}_b^k) + 4\varepsilon^2(\hat{\eta}_a^k - \hat{\eta}_b^k) + 2\varepsilon\kappa(\hat{\sigma}_c^k + \hat{\sigma}_c^{\dagger k})\big], \tag{2.124}$$

from which follows

$$\big[\hat{b}, \hat{b}^{\dagger}\big] = \frac{\gamma_c\kappa}{[\kappa^2 - 4\varepsilon^2]^2}\big[\kappa^2(\hat{N}_c - \hat{N}_b) + 4\varepsilon^2(\hat{N}_a - \hat{N}_b) + 2\varepsilon\kappa(\hat{m}_c + \hat{m}_c^{\dagger})\big]. \tag{2.125}$$

In the absence of parametric amplifier, for $\varepsilon = 0$, Eq. (2.125) reduces to

$$\big[\hat{b}, \hat{b}^{\dagger}\big] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_b\big]. \tag{2.126}$$

The cavity light modes a and b are interacting with all N three-level atoms or in the presence of N three-level atoms, we rewrite Eqs. (2.117) and (2.118) as operators as

$$\frac{d\hat{a}(t)}{dt} = -\frac{1}{2}\big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\big]\hat{a}(t) + \lambda_1\hat{m}_a + \lambda_2\hat{m}_b^{\dagger}, \tag{2.127}$$

$$\frac{d\hat{b}(t)}{dt} = -\frac{1}{2}\big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\big]\hat{b}(t) + \beta_1\hat{m}_b + \beta_2\hat{m}_a^{\dagger}, \tag{2.128}$$

in which λ and β are constants whose values remain to be fixed. We note that the steady state solutions of Eqs. (2.117) and (2.118) are,

$$\hat{a} = \frac{2\lambda_1\kappa}{\kappa^2 - 4\varepsilon^2}\hat{m}_a + \frac{2\lambda_2\kappa}{\kappa^2 - 4\varepsilon^2}\hat{m}_b^{\dagger}, \tag{2.129}$$

$$\hat{b} = \frac{2\beta_2\kappa}{\kappa^2 - 4\varepsilon^2}\hat{m}_b + \frac{2\beta_1\kappa}{\kappa^2 - 4\varepsilon^2}\hat{m}_a^{\dagger}. \tag{2.130}$$

Now taking in to account Eqs. (2.129) and (2.130), the commutation relations for the cavity mode operators are found to be

$$\big[\hat{a}, \hat{a}^{\dagger}\big] = \frac{4N\kappa^3}{[\kappa^2 - 4\varepsilon^2]^2}\big[\lambda_1^2(\hat{N}_b - \hat{N}_a) + \lambda_2^2(\hat{N}_b - \hat{N}_c) - \lambda_1\lambda_2(\hat{m}_c + \hat{m}_c^{\dagger})\big], \tag{2.131}$$

$$\big[\hat{b}, \hat{b}^{\dagger}\big] = \frac{4N\kappa^3}{[\kappa^2 - 4\varepsilon^2]^2}\big[\beta_1^2(\hat{N}_c - \hat{N}_b) + \beta_2^2(\hat{N}_b - \hat{N}_c) + \beta_1\beta_2(\hat{m}_c + \hat{m}_c^{\dagger})\big]. \tag{2.132}$$

Thus on account of Eqs. (2.122) with (2.131) and (2.125) with (2.132), we see that

$$\lambda_1 = \beta_1 = \pm\frac{g}{\sqrt{N}}, \tag{2.133}$$

$$\lambda_2 = \beta_2 = \pm\frac{2g\varepsilon}{\kappa\sqrt{N}}. \tag{2.134}$$

Hence in view of these two results, the equations of evolution of the light modes a and b operators given by Eqs. (2.127) and (2.128) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{1}{2}\Big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\Big]\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \frac{2g\varepsilon}{\kappa\sqrt{N}}\hat{m}_b^\dagger, \tag{2.135}$$

$$\frac{d\hat{b}}{dt} = -\frac{1}{2}\Big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\Big]\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \frac{2g\varepsilon}{\kappa\sqrt{N}}\hat{a}_b^\dagger. \tag{2.136}$$

Moreover, the steady-state solutions of Eqs. (2.135) and (2.136) are

$$\hat{a} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_a + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_b^\dagger, \tag{2.137}$$

$$\hat{b} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_b + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_a^\dagger. \tag{2.138}$$

Now adding Eqs. (2.135) and (2.136) results in

$$\frac{d\hat{c}(t)}{dt} = -\Big[\frac{\kappa^2 - 4\varepsilon^2}{\kappa}\Big]\hat{c}(t) + \frac{g}{\sqrt{N}}\hat{m} + \frac{2g\varepsilon}{\kappa\sqrt{N}}\hat{m}^\dagger, \tag{2.139}$$

in which

$$\hat{c} = \hat{a} + \hat{b}, \tag{2.140}$$

$$\hat{m} = \hat{m}_a + \hat{m}_b. \tag{2.141}$$

The steady state solution of Eq. (2.139) is found to be

$$\hat{c} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m} + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}^\dagger. \tag{2.142}$$

The commutation for $\hat{c}$ and $\hat{c}^\dagger$ can be expressible as

$$\big[\hat{c}, \hat{c}^\dagger\big] = \frac{\gamma_c\kappa^3(\hat{N}_c - \hat{N}_a) + 4\gamma_c\kappa\varepsilon^2(\hat{N}_a - \hat{N}_c)}{[\kappa^2 - 4\varepsilon^2]^2}, \tag{2.143}$$

Furthermore, in the absence of parametric amplifier (when $\varepsilon = 0$), we see that

$$\big[\hat{c}, \hat{c}^{\dagger}\big] = \frac{\gamma_c}{\kappa}(\hat{N}_c - \hat{N}_a).$$

Next we proceed to calculate the solution the stochastic differential equations of the cavity operators. Now , the steady-state solutions of Eqs. (2.90) - (2.94) are

$$\langle \hat{m}_a \rangle = \big(\frac{\Omega}{2} + \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}\big)\langle \hat{m}_b^{\dagger} \rangle, \tag{2.144}$$

$$\langle \hat{m}_b \rangle = -\frac{\Omega}{\gamma_c + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}\langle \hat{m}_b^{\dagger} \rangle, \tag{2.145}$$

$$\langle \hat{m}_c \rangle = -\frac{\Omega}{\gamma_c + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}\big[\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle\big], \tag{2.146}$$

$$\langle \hat{N}_a \rangle = \big(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}\big)\big[\langle \hat{m}_c \rangle + \langle \hat{m}_c^{\dagger} \rangle\big], \tag{2.147}$$

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle - \frac{\frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}{\gamma_c + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}\big[\langle \hat{m}_c \rangle + \langle \hat{m}_c^{\dagger} \rangle\big]. \tag{2.148}$$

The conjugates of Eqs. (2.144) - (2.146) are

$$\langle \hat{m}_a^{\dagger} \rangle = \big(\frac{\Omega}{2} + \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}\big)\langle \hat{m}_b \rangle, \tag{2.149}$$

$$\langle \hat{m}_b^{\dagger} \rangle = -\frac{\Omega}{\gamma_c + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}\langle \hat{m}_b \rangle, \tag{2.150}$$

$$\langle \hat{m}_c^{\dagger} \rangle = -\frac{\Omega}{\gamma_c + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}\big[\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle\big], \tag{2.151}$$

Now compering Eqs. (2.146) and (2.151) yields

$$\langle \hat{m}_c \rangle = \langle \hat{m}_c^{\dagger} \rangle. \tag{2.152}$$

It then follows that

$$\langle \hat{N}_a \rangle = 2\big(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}\big)\langle \hat{m}_c \rangle, \tag{2.153}$$

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle - 2\frac{\frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}{\gamma_c + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}}\langle \hat{m}_c \rangle, \tag{2.154}$$

Now substituting Eq. (2.150) into (2.90) and (2.149) into (2.91), we get

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\big(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2} + \frac{\Omega}{2} + \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}\big)\langle\hat{m}_a\rangle, \tag{2.155}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{1}{2}\big(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2} + \frac{\Omega}{2} + \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}\big)\langle\hat{m}_b\rangle, \tag{2.156}$$

The steady-state solution of Eqs. (2.155) and (2.156) are

$$\langle\hat{m}_a\rangle = 0, \tag{2.157}$$

$$\langle\hat{m}_b\rangle = 0. \tag{2.158}$$

Furthermore, the formal solution of Eqs.(2.135) and (2.136) are found to be

$$\begin{aligned}\langle\hat{a}(t)\rangle &= \langle\hat{a}(0)\rangle e^{-\mu_0 t'/2} + \frac{g}{\sqrt{N}}e^{-\mu_0 t/2}\int_0^t e^{\mu_0 t/2}\langle\hat{m}_a(t')\rangle dt' \\ &+ \frac{2g\varepsilon}{\kappa\sqrt{N}}e^{-\mu_0 t/2}\int_0^t e^{\mu_0 t/2}\langle\hat{m}_b^\dagger(t')\rangle dt',\end{aligned} \tag{2.159}$$

and

$$\begin{aligned}\langle\hat{b}(t)\rangle &= \langle\hat{b}(0)\rangle e^{-\mu_0 t'/2} + \frac{g}{\sqrt{N}}e^{-\mu_0 t/2}\int_0^t e^{\mu_0 t/2}\langle\hat{m}_b(t')\rangle dt' \\ &+ \frac{2g\varepsilon}{\kappa\sqrt{N}}e^{-\mu_0 t/2}\int_0^t e^{\mu_0 t/2}\langle\hat{m}_a^\dagger(t')\rangle dt',\end{aligned} \tag{2.160}$$

where $\mu_0 = \frac{k^2 - 4\varepsilon^2}{k}$.

In view of Eqs. (2.157) and (2.158), and the assumption that the cavity mode light is initially in vacuum state Eqs. (2.159) and (2.160) reduces to

$$\langle\hat{a}(t\rangle = 0, \tag{2.161}$$

$$\langle\hat{b}(t)\rangle = 0. \tag{2.162}$$

In view of these results, we see that

$$\langle\hat{c}(t)\rangle = 0. \tag{2.163}$$

We observe on the basis of equations (2.159) and (2.160) along with Eqs. (2.161), (2.162), and (2.163), $\hat{a}$, $\hat{b}$, and $\hat{c}$ are a Gaussian variable with zero mean. We next seek to calculate the steady-state solutions of the expectation values for the atomic operators $\hat{N}_a$, $\hat{N}_b$, $\hat{N}_c$, and $\hat{m}_c$, from Eqs. (2.92)-(2.94), we get

$$\langle \hat{N}_a \rangle_{ss} = [\frac{2\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2})}{(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2}]N, \tag{2.164}$$

$$\langle \hat{N}_b \rangle_{ss} = [\frac{\Omega^2}{(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2}]N, \tag{2.165}$$

$$\langle \hat{N}_c \rangle_{ss} = [\frac{(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2})}{(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2}]N, \tag{2.166}$$

$$\langle \hat{m}_c \rangle_{ss} = [\frac{\Omega(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})}{(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2}]N. \tag{2.167}$$

For the case in which when the parametric amplifier is absent (when $\varepsilon = 0$), we have

$$\langle \hat{N}_a \rangle_{ss} = [\frac{\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}]N, \tag{2.168}$$

$$\langle \hat{N}_b \rangle_{ss} = \frac{\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}N, \tag{2.169}$$

$$\langle \hat{N}_c \rangle_{ss} = [\frac{\Omega^2 + (\gamma + \gamma_c)^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}]N, \tag{2.170}$$

$$\langle \hat{m}_c \rangle_{ss} = \frac{\Omega(\gamma + \gamma_c)}{(\gamma + \gamma_c)^2 + 3\Omega^2}N. \tag{2.171}$$

These equations represent the steady-state solutions of the equations of evolution of the atomic operators. Furthermore, upon setting $\gamma = 0$, for the case in which spontaneous emission is absent, the steady-state solutions described by Eqs.

(2.168)-(2.171), has the form

$$\langle \hat{N}_a \rangle_{ss} = \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} N, \tag{2.172}$$

$$\langle \hat{N}_b \rangle_{ss} = \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} N, \tag{2.173}$$

$$\langle \hat{N}_c \rangle_{ss} = \Big[\frac{\Omega^2 + \gamma_c^2}{\gamma_c^2 + 3\Omega^2}\Big] N, \tag{2.174}$$

$$\langle \hat{m}_c \rangle_{ss} = \frac{\Omega \gamma_c}{\gamma_c^2 + 3\Omega^2} N. \tag{2.175}$$

When $\Omega >> \gamma_c$, these results take the form

$$\langle \hat{N}_a \rangle_{ss} = \frac{1}{3} N, \tag{2.176}$$

$$\langle \hat{N}_b \rangle_{ss} = \frac{1}{3} N, \tag{2.177}$$

$$\langle \hat{N}_c \rangle_{ss} = \frac{1}{3} N, \tag{2.178}$$

$$\langle \hat{m}_c \rangle_{ss} = 0. \tag{2.179}$$

Moreover, when $\Omega = 0$ Eqs. (2.172)- (2.175) turns out to be

$$\langle \hat{N}_a \rangle_{ss} = 0, \tag{2.180}$$

$$\langle \hat{N}_b \rangle_{ss} = 0, \tag{2.181}$$

$$\langle \hat{N}_c \rangle_{ss} = N, \tag{2.182}$$

$$\langle \hat{m}_c \rangle_{ss} = 0. \tag{2.183}$$

3

PHOTON STATISTICS

In this chapter, we seek to study the statistical properties of the light produced by the coherently driven non degenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. Applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the photon statistics for light modes a and b. In addition, we determine the photon statistics of the two-mode cavity light.

3.1 Single-mode photon statistics

3.1.1 The mean photon number

Here we seek to calculate the mean photon number for light mode a and b.

A, Mean photon number for light mode a

Now we seek to calculate the mean photon number of light mode a. Employing the steady-state solution of Eq. (2.135)

$$\hat{a} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_a + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_b^\dagger, \tag{3.1}$$

and it's conjugate is

$$\hat{a}^{\dagger} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_a^{\dagger} + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_b. \tag{3.2}$$

The mean photon number of light mode a is defined by

$$\bar{n}_a = \langle \hat{a}^{\dagger}\hat{a}\rangle. \tag{3.3}$$

In view Eqs. (3.1) and (3.2), we get

$$\bar{n}_a = \frac{\gamma_c\kappa^3\langle \hat{N}_a\rangle + 4\gamma_c\kappa\varepsilon^2\langle \hat{N}_c\rangle + 4\gamma_c\kappa^2\varepsilon\langle \hat{m}_c\rangle}{(\kappa^2 - 4\varepsilon^2)^2}. \tag{3.4}$$

On account of Eqs. (2.164), (2.166), and (2.67) we found

$$\begin{aligned}\bar{n}_a &= q\big[2\kappa^2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) \\ &+4\varepsilon^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2})] + 4\kappa\varepsilon\Omega(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\big],\end{aligned} \tag{3.5}$$

where

$$q = \frac{\gamma_c\kappa N}{[\kappa^2 - 4\varepsilon^2]^2((\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2)}.$$

For the case $\varepsilon = 0$, we see that

$$\bar{n}_a = \frac{\gamma_c}{\kappa}N\big[\frac{\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}\big], \tag{3.6}$$

in the absence of spontaneous emission ($\gamma = 0$), we have

$$\bar{n}_a = \frac{\gamma_c}{\kappa}N\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}, \tag{3.7}$$

in addition, for $\Omega >> \gamma_c$, we get

$$\bar{n}_a = \frac{\gamma_c}{3\kappa}N. \tag{3.8}$$

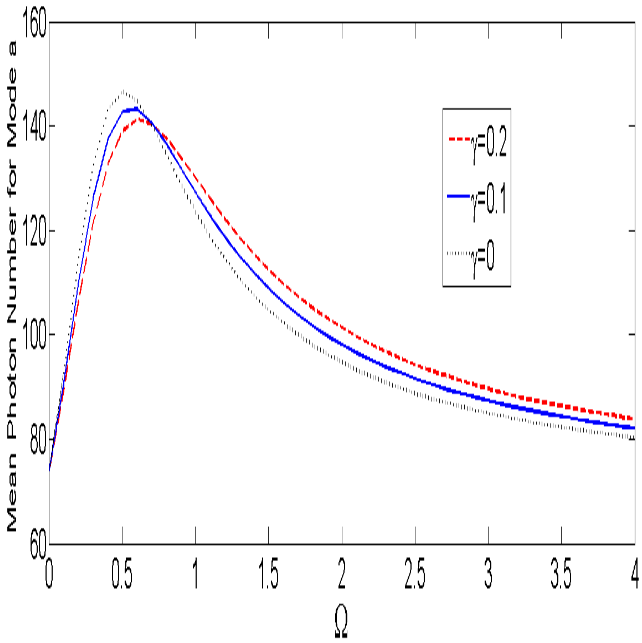

Figure 3.1: plots of the mean photon number of light mode a [Eq. (3.5)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

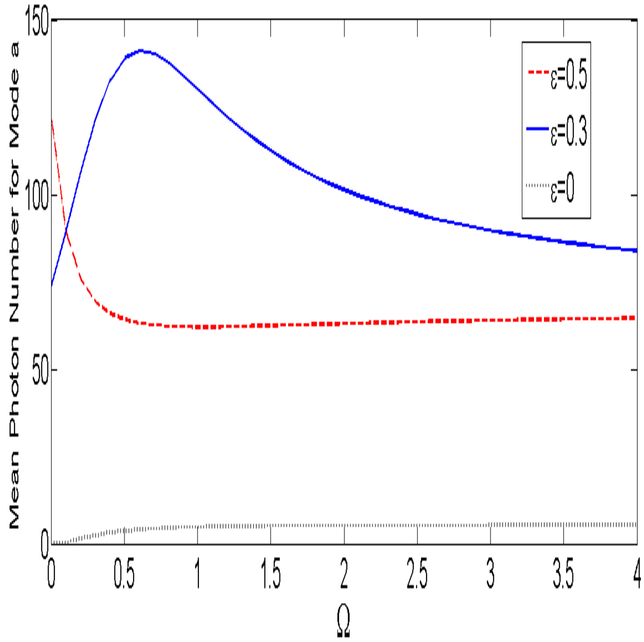

Figure 3.2: plots of the mean photon number of light mode a [Eq.(3.5)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$, for different values of ε.

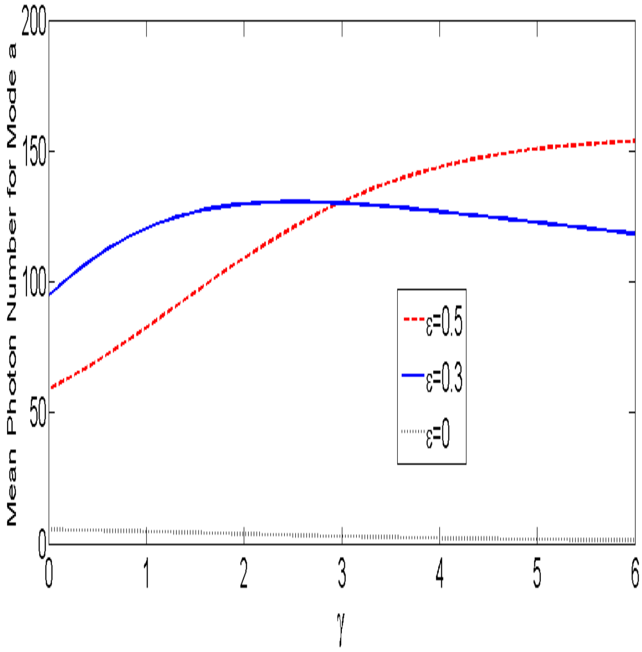

Figure 3.3: plots of the mean photon number of light mode a [eq.(3.5)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\Omega = 2$, $N = 50$ and for different values of ε.

B. Mean photon number for light mode b

Here we seek to calculate the mean photon number of light mode b. Employing the steady-state solution of Eq. (2.136)

$$\hat{b} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_b + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_a^\dagger \tag{3.9}$$

and it's conjugate is

$$\hat{b}^\dagger = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_b^\dagger + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}_a. \tag{3.10}$$

The mean photon number of light mode b can be expressed as

$$\bar{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle. \tag{3.11}$$

On account of Eqs. (3.9) and (3.10), we get

$$\bar{n}_b = \frac{\gamma_c\kappa^3\langle \hat{N}_b\rangle + 4\gamma_c\kappa\varepsilon^2\langle \hat{N}_b\rangle}{(\kappa^2 - 4\varepsilon^2)^2}. \tag{3.12}$$

In view of Eq. (2.165) we found

$$\bar{n}_b = q\big[\kappa^2 + 4\varepsilon^2\big], \tag{3.13}$$

where

$$q = \frac{\gamma_c\kappa N}{[\kappa^2 - 4\varepsilon^2]^2((\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) + \Omega^2)}.$$

For the case $\varepsilon = 0$, we have

$$\bar{n}_b = \frac{\gamma_c}{\kappa}N\big[\frac{\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}\big], \tag{3.14}$$

in the absence of spontaneous emission $(\gamma = 0)$, we get

$$\bar{n}_b = \frac{\gamma_c}{\kappa}N\big[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\big], \tag{3.15}$$

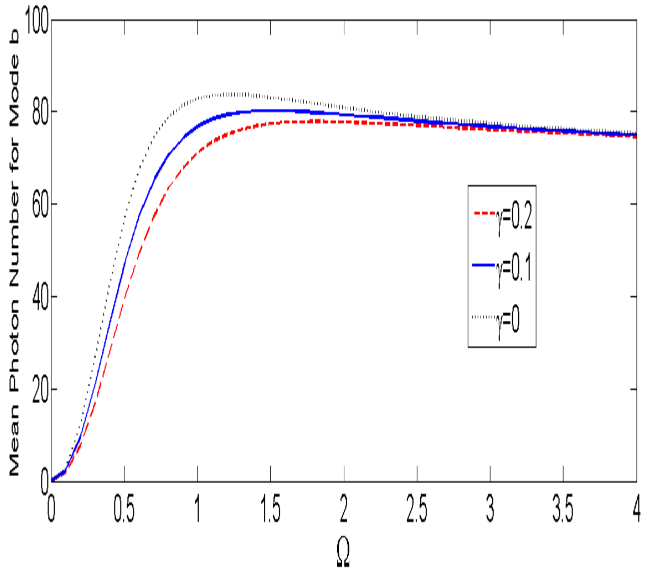

Figure 3.4: plots of the mean photon number of light mode b [Eq.(3.13)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

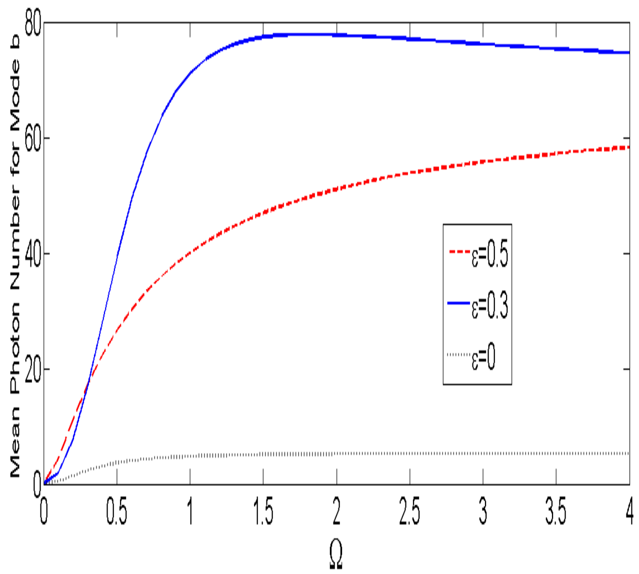

Figure 3.5: plots of the mean photon number of light mode b [Eq.(3.13)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$ and for different values of ε.

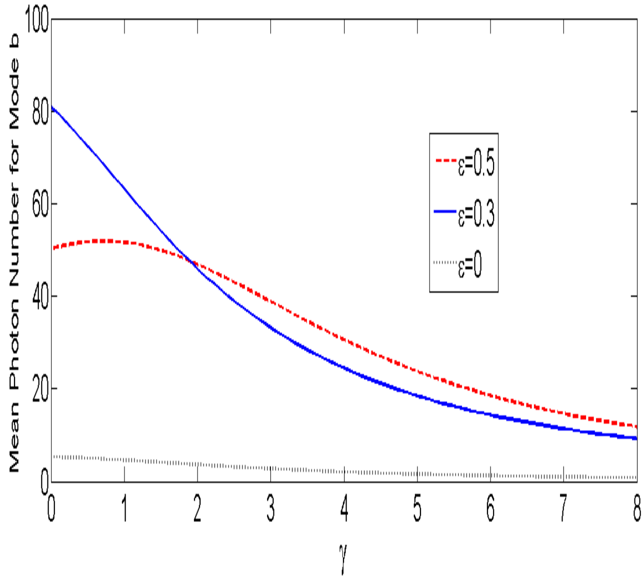

Figure 3.6: plots of the mean photon number of light mode b [Eq.(3.13)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\Omega = 2$, $N = 50$ and for different values of ε.

moreover, for $\Omega >> \gamma_c$, we see that

$$\bar{n}_b = \frac{\gamma_c}{3\kappa}N. \tag{3.16}$$

3.1.2 The photon number variance

Now, we proceed to obtain the photon number variance of light mode a and b.

A, The photon number variance of light mode a

The photon number variance of light mode b is defined by

$$(\Delta n_a)^2 = \langle n_a^2 \rangle - \langle n_a \rangle^2. \tag{3.17}$$

We have

$$\hat{n}_a = \hat{a}^\dagger \hat{a}.$$

Since the operator $\hat{a}$ is a Gaussian variable with zero mean, we verified that

$$(\Delta n_a)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}\hat{a}^\dagger \rangle. \tag{3.18}$$

We easily establish that

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c \kappa}{[\kappa^2 - 4\varepsilon^2]^2} \big[\kappa^2 \langle \hat{N}_b \rangle + 4\varepsilon^2 \langle \hat{N}_b \rangle \big]. \tag{3.19}$$

Now, combining Eqs. (3.4) and (3.19), we obtain

$$(\Delta n_a)^2 = \Big(\frac{\gamma_c \kappa^3 \langle \hat{N}_a \rangle + \gamma_c \kappa \varepsilon^2 \langle \hat{N}_c \rangle + 4\gamma_c \kappa^2 \varepsilon \langle \hat{m}_c \rangle}{[\kappa^2 - 4\varepsilon^2]^2}\Big)\Big(\frac{\gamma_c \kappa^3 \langle \hat{N}_b \rangle + \gamma_c \kappa \varepsilon^2 \langle \hat{N}_b \rangle}{[\kappa^2 - 4\varepsilon^2]^2}\Big). \tag{3.20}$$

In view of Eqs. (2.165)-(2.167), we get

$$\begin{aligned} (\Delta n_a)^2 \ &= \ q^2 \big[(2\kappa^2 \Omega (\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + 4\kappa\varepsilon\Omega(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2}) \\ &\quad + 4\varepsilon^2 [(\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2})])(\kappa^2 \Omega^2 + 4\varepsilon^2 \Omega^2) \big], \end{aligned} \tag{3.21}$$

where

$$q = \frac{\gamma_c \kappa N}{[\kappa^2 - 4\varepsilon^2]^2 ((\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2)}.$$

When $\varepsilon = 0$, Eq. (3.21) reduces to

$$(\Delta n_a)^2 = \big[\frac{\gamma_c}{\kappa} N\big]^2 \big[\frac{\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}\big]^2, \tag{3.22}$$

in the absence of spontaneous emission $(\gamma = 0)$, we get

$$(\Delta n_a)^2 = \big[\frac{\gamma_c}{\kappa} N\big]^2 \big[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\big]^2, \tag{3.23}$$

in addition, we note that, for $\Omega >> \gamma_c$ we obtain

$$(\Delta n_a)^2 = \big[\frac{\gamma_c}{3\kappa} N\big]^2. \tag{3.24}$$

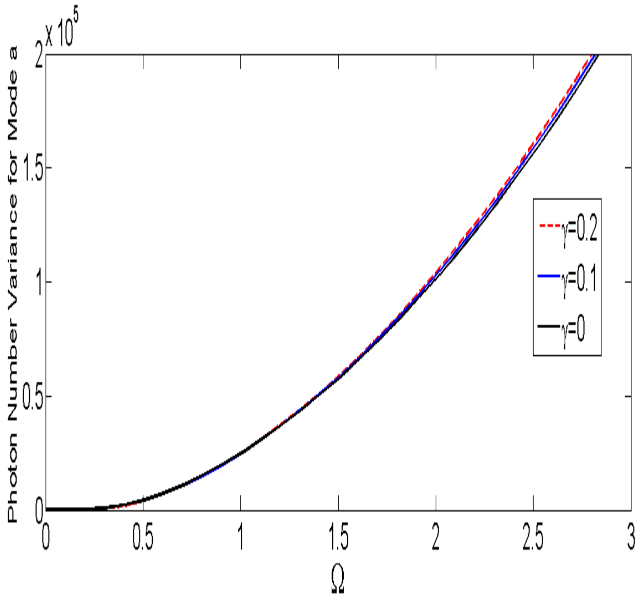

Figure 3.7: plots of the photon number variance for mode a [Eq.(3.21)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

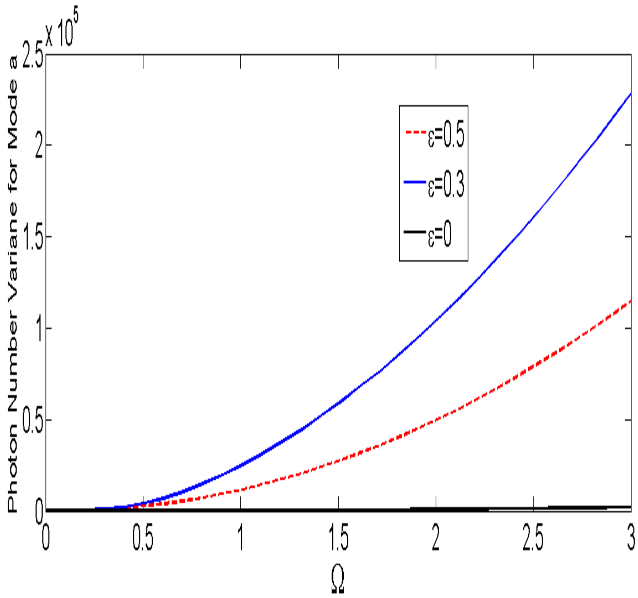

Figure 3.8: plots of the photon number variance for mode a [Eq.(3.21)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$ and for different values of ε.

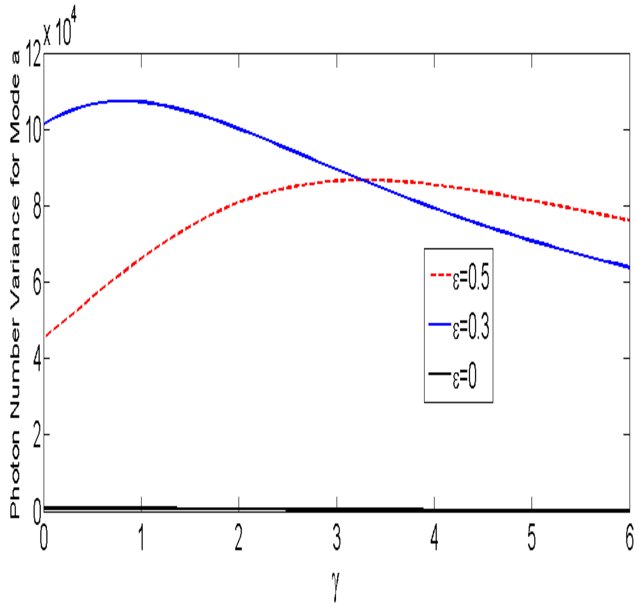

Figure 3.9: plots of the photon number variance for mode a [Eq.(3.21)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\Omega = 2$, $N = 50$ and for different values of ε.

In view of eq.(3.8) we have

$$(\Delta n_a)^2 = \bar{n}_a^2. \tag{3.25}$$

This represents the normally ordered variance of the photon number for chaotic light.

B, The photon number variance of light mode b

The photon number variance of light mode b is defined by

$$(\Delta n_b)^2 = \langle n_b^2 \rangle - \langle n_b \rangle^2. \tag{3.26}$$

We have

$$\hat{n}_b = \hat{b}^\dagger \hat{b}.$$

Since the operator $\hat{b}$ is a Gaussian variable with zero mean, we verified that

$$(\Delta n_b)^2 = \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{b}\hat{b}^\dagger\rangle. \tag{3.27}$$

We easily establish that

$$\langle\hat{b}\hat{b}^\dagger\rangle = \frac{\gamma_c\kappa^3\langle\hat{N}_c\rangle + 4\gamma_c\kappa\varepsilon^2\langle\hat{N}_a\rangle + 4\gamma_c\kappa^2\varepsilon\langle\hat{m}_c\rangle}{[\kappa^2-4\varepsilon^2]^2}. \tag{3.28}$$

Now, combining Eqs. (3.12) and (3.28), we have

$$(\Delta n_b)^2 = (\frac{\gamma_c\kappa^3\langle\hat{N}_b\rangle + 4\gamma_c\kappa^3\varepsilon^2\langle\hat{N}_b\rangle}{[\kappa^2-4\varepsilon^2]^2})(\frac{\gamma_c\kappa^3\langle\hat{N}_c\rangle + 4\gamma_c\kappa\varepsilon^2\langle\hat{N}_a\rangle + 4\gamma_c\kappa^2\varepsilon\langle\hat{m}_c\rangle}{[\kappa^2-4\varepsilon^2]^2}). \tag{3.29}$$

In view of Eqs. (2.164) - (2.167) we get

$$\begin{aligned}(\Delta n_b)^2 &= q^2\Big[\big(\kappa^2\Omega^2 + 4\kappa^2\varepsilon^2\Omega^2\big)\big(\kappa^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})] \\ &\quad + 4\kappa\varepsilon\Omega(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2}) + 8\varepsilon^2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})\big)\Big],\end{aligned} \tag{3.30}$$

where

$$q = \frac{\gamma_c\kappa N}{[\kappa^2-4\varepsilon^2]^2((\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) + \Omega^2)}.$$

For the case $\varepsilon = 0$, Eq. (3.30) reduces to

$$(\Delta n_b)^2 = \Big[\frac{\gamma_c}{\kappa}N\Big]^2\Big[\frac{\Omega^2 + (\gamma+\gamma_c)^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\Big]^2, \tag{3.31}$$

in the absence of spontaneous emission $(\gamma = 0)$, we get

$$(\Delta n_b)^2 = \Big[\frac{\gamma_c}{\kappa}N\Big]^2\Big[\frac{\Omega^2 + \gamma_c^2}{\gamma_c^2 + 3\Omega^2}\Big]^2, \tag{3.32}$$

in addition, we note that, for $\Omega >> \gamma_c$ we see that

$$(\Delta n_b)^2 = \Big[\frac{\gamma_c}{3\kappa}N\Big]^2. \tag{3.33}$$

In view of Eq. (3.16) we have

$$(\Delta n_b)^2 = \bar{n}_b^2. \tag{3.34}$$

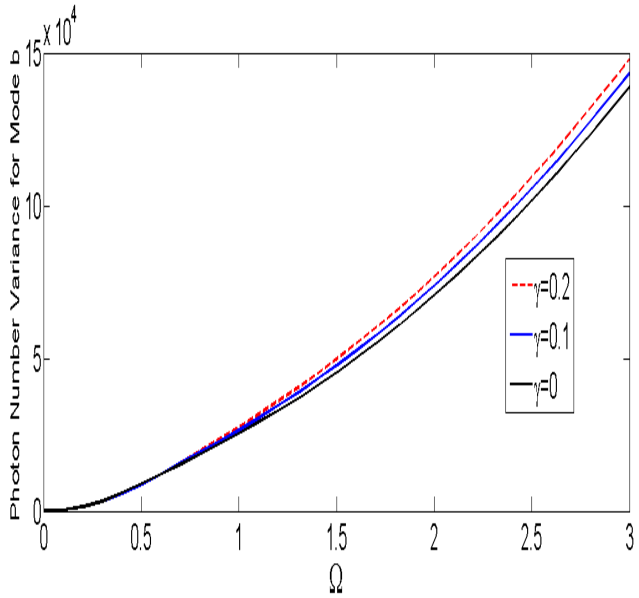

Figure 3.10: plots of the photon number variance for mode b [Eq.(3.30)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

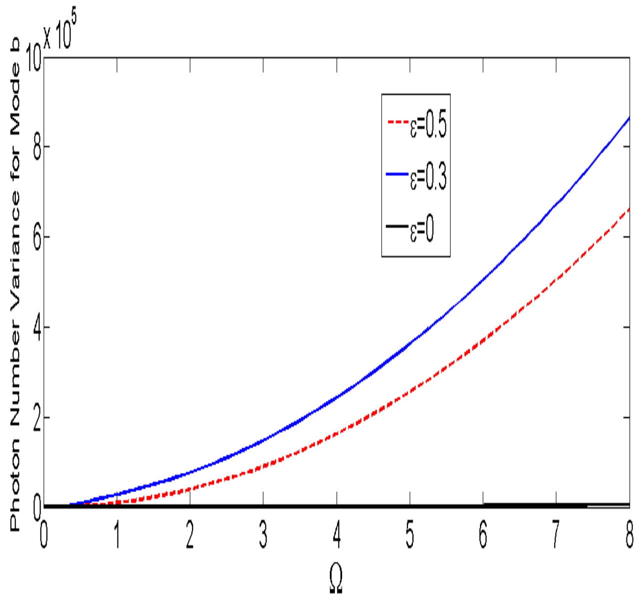

Figure 3.11: plots of the photon number variance for mode b [Eq.(3.30)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$ and for different values of ε.

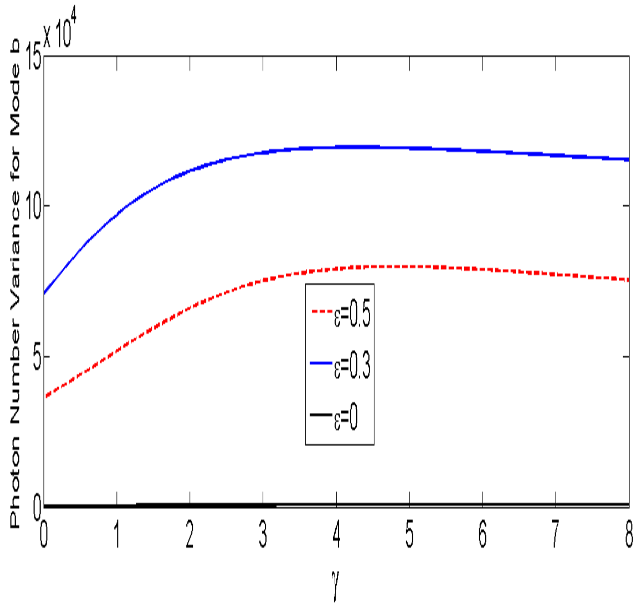

Figure 3.12: plots of the photon number variance for mode b [Eq.(3.30)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\Omega = 2$, $N = 50$ and for different values of ε.

3.2 Two-mode photon statistics

Applying the steady state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators. We seek to obtain the mean and variance of the photon number for the two mode light.

3.2.1 Two mode mean photon number

Here we seek to calculate the steady-state solution of the mean photon number of the two mode cavity light beam.

The mean photon number of two mode light represented by the operators $\hat{c}$ and $\hat{c}^\dagger$

is defined by

$$\bar{n} = \langle \hat{c}^{\dagger}\hat{c}\rangle. \tag{3.35}$$

From Eq. (2.142), we have

$$\hat{c} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m} + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}^{\dagger} \tag{3.36}$$

and it's conjugates

$$\hat{c}^{\dagger} = \frac{2g\kappa}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}^{\dagger} + \frac{4g\varepsilon}{\sqrt{N}(\kappa^2 - 4\varepsilon^2)}\hat{m}. \tag{3.37}$$

Eq. (3.35) gives as

$$\bar{n} = \frac{\gamma_c\kappa^3[\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle] + 4\gamma_c\kappa^2\varepsilon\langle \hat{m}_c\rangle + 4\gamma_c\kappa\varepsilon^2[\langle \hat{N}_b\rangle + \langle \hat{N}_c\rangle]}{(\kappa^2 - 4\varepsilon^2)^2}. \tag{3.38}$$

In view of Eq.(2.164) - Eq.(2.167) we obtain

$$\begin{aligned} \bar{n} &= q\big[2\kappa^2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + \kappa^2\Omega^2 + 4\kappa\varepsilon\Omega(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2}) \\ &\quad + 4\varepsilon^2[\Omega^2 + (\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2})]\big], \end{aligned} \tag{3.39}$$

where

$$q = \frac{\gamma_c\kappa N}{[\kappa^2 - 4\varepsilon^2]^2((\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2)}$$

For the case $\varepsilon = 0$, we easily shows

$$\bar{n} = \frac{\gamma_c}{\kappa}N\big[\frac{2\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}\big], \tag{3.40}$$

in the absence of spontaneous emission $(\gamma = 0)$, we get

$$\bar{n} = \frac{\gamma_c}{\kappa}N\big[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2}\big]. \tag{3.41}$$

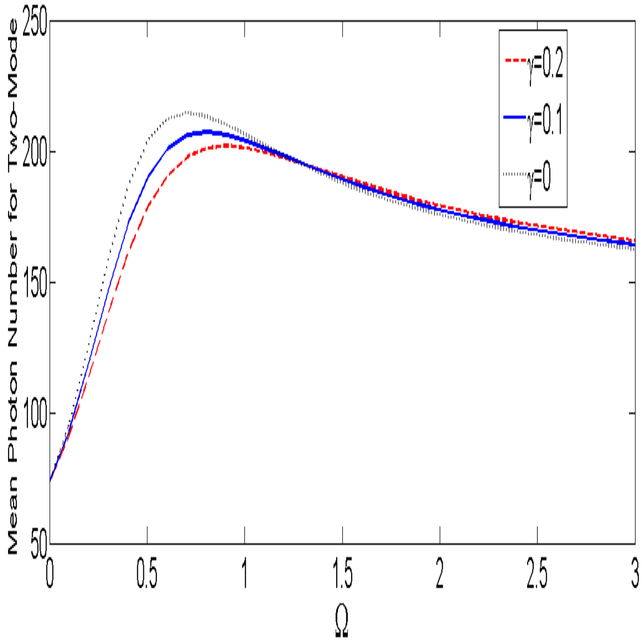

Figure 3.13: plots of the mean photon number for two mode [Eq.(3.39)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

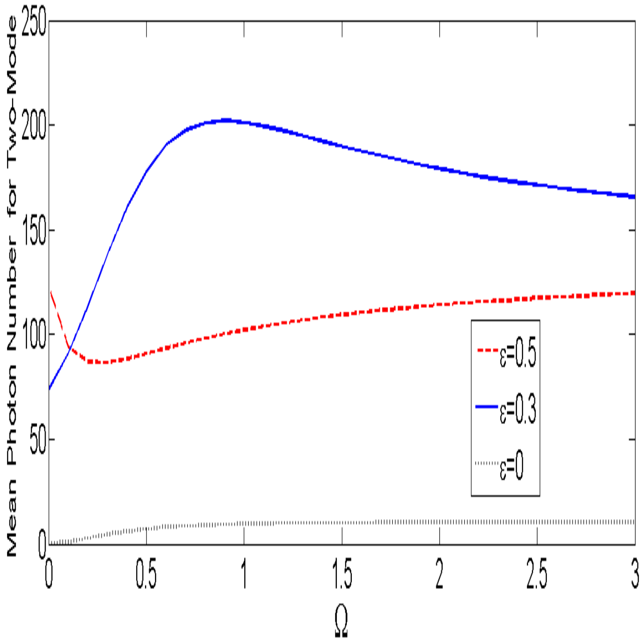

Figure 3.14: plots of the mean photon number for two mode [Eq.(3.39)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\gamma = 0.2$, $N = 50$ and for different values of ε.

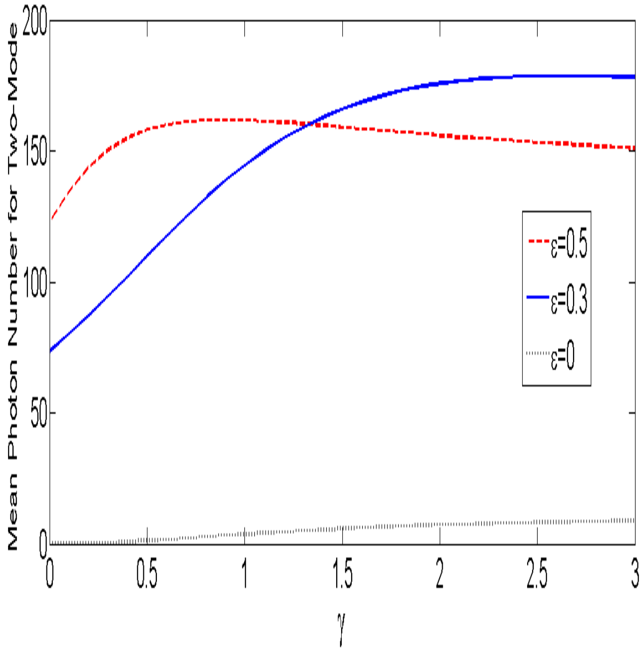

Figure 3.15: plots of the mean photon number for two mode [Eq.(3.39)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\Omega = 2$, $N = 50$ and for different values of ε.

Moreover, for $\Omega >> \gamma_c$, we obtain

$$\bar{n} = 2\frac{\gamma_c}{3\kappa}N. \tag{3.42}$$

Therefore

$$\bar{n} = \bar{n}_a + \bar{n}_b. \tag{3.43}$$

This is, the mean photon number of two-mode light is the sum of the mean photon number of the separate single-mode light.

Fig. 3.13, shows that the mean photon number for two-mode cavity light in the presence of spontaneous emission (when $\gamma \neq 0$) and in the absence of spontaneous emission (when $\gamma = 0$). It is found that the mean photon number of the two-mode cavity light increases with Ω in both cases. Moreover, the mean photon number of the two-mode cavity light is greater when $\gamma = 0$, than when $\gamma \neq 0$ at $\Omega \leq 1.25$, and the mean photon number of the two-mode cavity light is smaller when $\gamma = 0$, than when $\gamma \neq 0$ for $\varepsilon = 0.3$ at $1.25 \leq \Omega \leq 3$.

fig. 3.14, shows that the mean photon number for two-mode cavity light is greater when in the presence of parametric amplifier (when $\varepsilon \neq 0$), than when in the absence of parametric amplifier (when $\varepsilon = 0$). Therefore the parametric amplifier increase the mean photon number of two-mode cavity light.

3.2.2 Two-mode photon number variance

Here we seek to obtain the photon number variance of the two mode light beam. The photon number variance of two mode cavity light can be expressed as

$$(\Delta n)^2 = \langle(\hat{c}^\dagger\hat{c})^2\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2. \tag{3.44}$$

Since $\hat{c}$ is a Gaussian variable with zero mean, the variance of the photon number can be put in the form

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle \langle \hat{c}^2 \rangle. \tag{3.45}$$

Using Eq. (3.36) and Eq. (3.37), we get

$$\langle \hat{c}\hat{c}^\dagger \rangle = \frac{\gamma_c \kappa^3 [\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle] + 4\gamma_c \kappa^2 \varepsilon \langle \hat{m}_c \rangle + 4\gamma_c \kappa \varepsilon^2 [\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle]}{(\kappa^2 - 4\varepsilon^2)^2}, \tag{3.46}$$

$$\langle \hat{c}^{\dagger 2} \rangle = \frac{\gamma_c \kappa}{\kappa^2 - 4\varepsilon^2} \big[2\kappa\varepsilon (N + \langle \hat{N}_b \rangle) + (\kappa^2 + 4\varepsilon^2) \langle \hat{m}_c \rangle \big], \tag{3.47}$$

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c \kappa}{\kappa^2 - 4\varepsilon^2} \big[2\kappa\varepsilon (N + \langle \hat{N}_b \rangle) + (\kappa^2 + 4\varepsilon^2) \langle \hat{m}_c \rangle \big]. \tag{3.48}$$

Now, combining Eq. (3.38), (3.46), (3.47), (3.48), and taking into account $\varepsilon = 0$, we obtain

$$(\Delta n)^2 = [\frac{\gamma_c N}{\kappa}]^2 \big[\frac{4\Omega^4 + 3\Omega^2 (\gamma + \gamma_c)^2}{[(\gamma + \gamma_c)^2 + 3\Omega^2]^2} \big]. \tag{3.49}$$

In the absence of spontaneous emission ($\gamma = 0$) we have

$$(\Delta n)^2 = [\frac{\gamma_c N}{\kappa}]^2 \big[\frac{4\Omega^4 + 3\Omega^2 \gamma_c^2}{[(\gamma_c^2 + 3\Omega^2]^2} \big], \tag{3.50}$$

moreover, for $\Omega >> \gamma_c$ we get

$$(\Delta n)^2 = [\frac{2\gamma_c N}{3\kappa}]^2. \tag{3.51}$$

On account of Eq. (3.42) we see that

$$(\Delta n)^2 = \bar{n}^2. \tag{3.52}$$

fig. 3.16, shows that the photon number variance for two-mode is greater when $\gamma = 0$, than when $\gamma \neq 0$ for $\varepsilon = 0.3$ at $0 \leq \Omega \leq 0.75$. Therefore the presence of

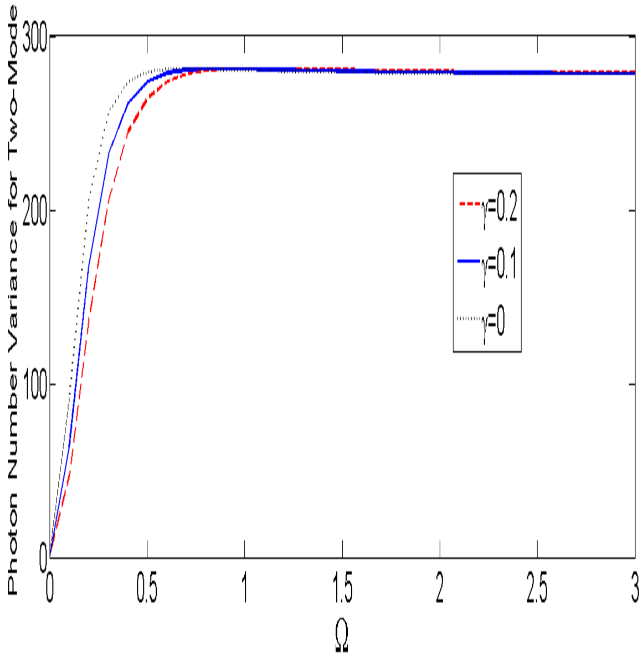

Figure 3.16: plots of the photon number Variance for two mode [Eq.(3.49)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0$, $N = 50$ and for different values of γ.

spontaneous decrease the photon number variance of two-mode cavity light.

Photon number correlation

The photon number correlation is defined by

$$g^{(2)}(\bar{n}_a,\bar{n}_b)(0)=\frac{\langle\bar{n}_a\bar{n}_b\rangle}{\langle\bar{n}_a\rangle\langle\bar{n}_b\rangle}, \tag{3.53}$$

where

$$\langle\bar{n}_a\rangle=\langle\hat{a}^\dagger\hat{a}\rangle,$$

$$\langle\bar{n}_b\rangle=\langle\hat{b}^\dagger\hat{b}\rangle$$

Since $\hat{a}$ and $\hat{b}$ are a Gaussian variable

$$\langle\bar{n}_a\bar{n}_b\rangle=\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{a}\hat{b}\rangle+\langle\hat{a}^\dagger\hat{b}\rangle\langle\hat{a}\hat{b}^\dagger\rangle, \tag{3.54}$$

$$g^{(2)}(\bar{n}_a,\bar{n}_b)(0)=1+\frac{\langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{a}\hat{b}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle}, \tag{3.55}$$

where

$$\langle\hat{a}^\dagger\hat{b}\rangle\langle\hat{a}\hat{b}^\dagger\rangle=0. \tag{3.56}$$

Therefore, it is correlated.

Fluctuations of intensity difference

Intensity difference is defined as

$$\hat{I}_D=\hat{n}_a-\hat{n}_b, \tag{3.57}$$

then

$$\hat{I}_D=\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}. \tag{3.58}$$

The fluctuations of intensity difference can be expressed as

$$(\Delta\hat{I}_D)^2 = \langle\hat{I}_D^2\rangle - \langle\hat{I}_D\rangle^2. \tag{3.59}$$

$$\langle\hat{I}_D^2\rangle = \langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\rangle - \langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle - \langle\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{a}\rangle \tag{3.60}$$

and

$$\langle\hat{I}_D\rangle^2 = \langle\hat{a}^\dagger\hat{a}\rangle^2 - \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle - \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle^2. \tag{3.61}$$

Since the cavity mode operators $\hat{a}$ and $\hat{b}$ are a Gaussian variables

$$\langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle^2 + \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\hat{a}^\dagger\rangle, \tag{3.62}$$

$$\langle\hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b}\rangle = \langle\hat{b}^\dagger\hat{b}\rangle^2 + \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{b}\hat{b}^\dagger\rangle, \tag{3.63}$$

$$\langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{a}\hat{b}\rangle, \tag{3.64}$$

$$\langle\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{a}\rangle = \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}^\dagger\rangle\langle\hat{b}\hat{a}\rangle. \tag{3.65}$$

In view of Eqs. (3.62) - (3.65), Eq. (3.59) reduces to

$$(\Delta\hat{I}_D)^2 = \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle\langle\hat{b}\hat{b}^\dagger\rangle - \langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{a}\hat{b}\rangle - \langle\hat{b}^\dagger\hat{a}^\dagger\rangle\langle\hat{b}\hat{a}\rangle. \tag{3.66}$$

We vitrified that

$$\langle\hat{a}\hat{b}\rangle = \frac{4\gamma_c\kappa^2\varepsilon\langle\hat{N}_b\rangle}{[\kappa^2 - 4\varepsilon^2]^2}, \tag{3.67}$$

$$\langle\hat{b}^\dagger\hat{a}^\dagger\rangle = \frac{4\gamma_c\kappa^2\varepsilon\langle\hat{N}_b\rangle}{[\kappa^2 - 4\varepsilon^2]^2}, \tag{3.68}$$

$$\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle = \frac{2\gamma_c\kappa^2\varepsilon(\langle\hat{N}_a\rangle + \langle\hat{N}_c\rangle) + (\gamma_c\kappa^3 + 4\gamma_c\kappa\varepsilon^2)\langle\hat{m}_c\rangle}{[\kappa^2 - 4\varepsilon^2]^2}, \tag{3.69}$$

$$\langle\hat{b}\hat{a}\rangle = \frac{2\gamma_c\kappa^2\varepsilon(\langle\hat{N}_a\rangle + \langle\hat{N}_c\rangle) + (\gamma_c\kappa^3 + 4\gamma_c\kappa\varepsilon^2)\langle\hat{m}_c\rangle}{[\kappa^2 - 4\varepsilon^2]^2}. \tag{3.70}$$

Now combining Eqs. (3.4), (3.12), (3.19), and (3.28), together with Eqs. (3.67)-(3.70) and for the case $\varepsilon = 0$, we obtain

$$(\Delta\hat{I}_D)^2 = [\frac{\gamma_c}{\kappa}N]^2\big[\frac{2\Omega^4 + \Omega^2(\gamma+\gamma_c)^2}{[(\gamma+\gamma_c)^2 + 3\Omega^2]^2}\big], \tag{3.71}$$

for $\gamma = 0$

$$(\Delta\hat{I}_D)^2 = [\frac{\gamma_c}{\kappa}N]^2\big[\frac{2\Omega^4 + \Omega^2\gamma_c^2}{[\gamma_c^2 + 3\Omega^2]^2}\big], \tag{3.72}$$

in addition, for $\Omega >> \gamma_c$ we get

$$(\Delta\hat{I}_D)^2 = 2[\frac{\gamma_c}{3\kappa}N]^2. \tag{3.73}$$

In view of Eq. (3.8) we see that

$$(\Delta\hat{I}_D)^2 = 2\bar{n}_a^2. \tag{3.74}$$

Fig. 3.17, shows that the fluctuations of intensity difference is greater when $\gamma = 0$, than when $\gamma \neq 0$ at $0 \leq \Omega \leq 3$. Therefore the presence of spontaneous emission decrease the fluctuations of intensity difference.

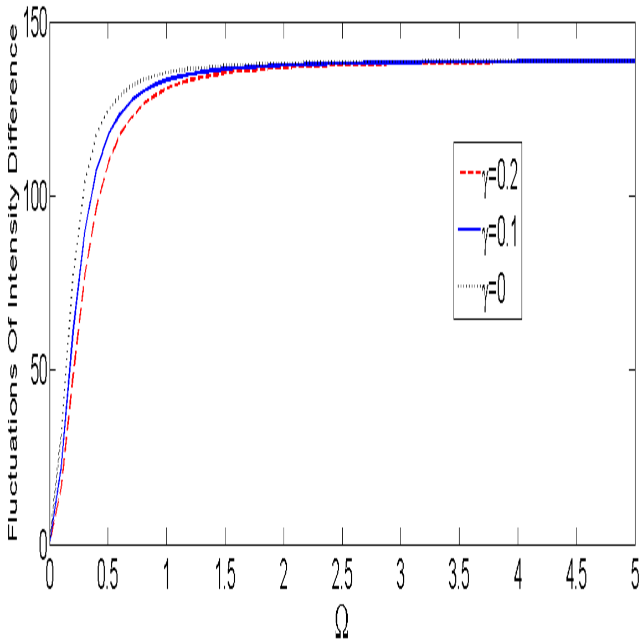

Figure 3.17: plots of Intensity Fluctuation Difference [Eq.(3.71)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0$, $N = 50$ and for different values of γ.

4

QUADRATURE SQUEEZING

In this chapter we seek to study the quadrature variance and the quadrature squeezing of the light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the quadrature variances for light modes a and b. In addition, we determine the quadrature squeezing of the two-mode cavity light.

4.1 Single-mode quadrature variance

In this section we seek to study the quadrature variance of the light mode a and b.

4.1.1 The quadrature variance of light mode a

Now we proceed to calculate the quadrature variance of light mode a in the entire frequency interval.

The squeezing properties of light mode a are described by two quadrature operators

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a} \tag{4.1}$$

and

$$\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a}). \tag{4.2}$$

where $\hat{a}_{+}$ and $\hat{a}_{-}$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators for light mode a. With the help of Eqs. (4.1) and (4.2), we can show that the two quadrature operators satisfy the commutation relation

$$\big[\hat{a}_{-}, \hat{a}_{+}\big] = -2i\big[\hat{a}, \hat{a}^{\dagger}\big], \tag{4.3}$$

On account of Eq. (2.122), we have

$$\big[\hat{a}, \hat{a}^{\dagger}\big] = \frac{\gamma_c\kappa^3(\hat{N}_b - \hat{N}_a) + 4\gamma_c\kappa\varepsilon^2(\hat{N}_b - \hat{N}_c) - 2\gamma_c\kappa^2\varepsilon(\hat{m}_c + \hat{m}_c^{\dagger})}{(\kappa^2 - 4\varepsilon^2)^2}, \tag{4.4}$$

then

$$\big[\hat{a}_{-}, \hat{a}_{+}\big] = -2i\frac{\gamma_c\kappa^3(\hat{N}_b - \hat{N}_a) + 4\gamma_c\kappa\varepsilon^2(\hat{N}_b - \hat{N}_c) - 2\gamma_c\kappa^2\varepsilon(\hat{m}_c + \hat{m}_c^{\dagger})}{(\kappa^2 - 4\varepsilon^2)^2}. \tag{4.5}$$

When $\varepsilon = 0$ one can write

$$\big[\hat{a}_{-}, \hat{a}_{+}\big] = 2i\frac{\gamma_c}{\kappa}\big[\hat{N}_a - \hat{N}_b\big]. \tag{4.6}$$

In view of this result, the uncertainty relation for the plus and minus quadrature operators of mode a is expressed as

$$\Delta\hat{a}_{+}\Delta\hat{a}_{-} \geq \frac{1}{2}\big|\langle[\hat{a}_{+}, \hat{a}_{-}]\rangle\big|. \tag{4.7}$$

It then follows

$$\Delta\hat{a}_+\Delta\hat{a}_- \geq \left|\langle\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\hat{a}\rangle\right|, \tag{4.8}$$

On account of Eqs. (3.19) and (3.4) we get

$$\Delta\hat{a}_+\Delta\hat{a}_- \geq \frac{[\gamma_c\kappa^3\langle\hat{N}_b\rangle + 4\gamma_c\kappa\varepsilon^2\langle\hat{N}_b\rangle]}{[\kappa^2-4\varepsilon^2]^2} - \frac{\gamma_c\kappa^3\langle\hat{N}_a\rangle - \gamma_c\kappa\varepsilon^2\langle\hat{N}_c\rangle - 4\gamma_c\kappa^2\langle\hat{m}_c\rangle}{(\kappa^2-4\varepsilon^2)^2}. \tag{4.9}$$

In view of Eqs. (2.164) - (2.167) it shows

$$\begin{aligned}\Delta\hat{a}_+\Delta\hat{a}_- \;\; \geq \;\; & q\big[\kappa^2\Omega^2 + 4\varepsilon^2\Omega^2 - 2\kappa^2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) \\ & -\varepsilon^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})] \\ & -4\kappa\varepsilon(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2\big],\end{aligned} \tag{4.10}$$

where

$$q = \frac{\gamma_c\kappa N}{[\kappa^2-4\varepsilon^2]^2((\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) + \Omega^2)},$$

When $\varepsilon = 0$ we see that

$$\Delta\hat{a}_+\hat{a}_- \geq \frac{\gamma_c}{\kappa}N\big[\frac{\Omega^2}{(\gamma+\gamma_c)^2 + 3\Omega^2} - \frac{\Omega^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\big]. \tag{4.11}$$

Therefore

$$\Delta\hat{a}_+\hat{a}_- \geq 0. \tag{4.12}$$

Then, we proceed to calculate the quadrature variance of light mode a.

The variance of the plus and minus quadrature operators are

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}_+^2\rangle - \langle\hat{a}_+\rangle^2. \tag{4.13}$$

On account of Eq. (4.1) we get

$$(\Delta\hat{a}_+)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\hat{a}^\dagger\rangle. \tag{4.14}$$

And

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}_-^2\rangle - \langle\hat{a}_-\rangle^2. \tag{4.15}$$

On account of Eq. (4.2) we obtain

$$(\Delta\hat{a}_-)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\hat{a}^\dagger\rangle. \tag{4.16}$$

In view of Eq. (4.12) and (4.14) one can write

$$(\Delta\hat{a}_\pm)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle \pm \langle\hat{a}^2\rangle \pm \langle\hat{a}^{\dagger 2}\rangle \mp \langle\hat{a}\rangle^2 \mp \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\hat{a}^\dagger\rangle, \tag{4.17}$$

but

$$\langle\hat{a}\rangle = \langle\hat{a}^\dagger\rangle = \langle\hat{a}^2\rangle = \langle\hat{a}^{\dagger 2}\rangle = 0.$$

Therefore

$$(\Delta\hat{a}_\pm)^2 = \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle. \tag{4.18}$$

In view of Eq. (3.19) and Eq. (3.4) it shows

$$(\Delta\hat{a}_\pm)^2 = \frac{\gamma_c\kappa^3\langle\hat{N}_b\rangle + 4\gamma_c\kappa\varepsilon^2\langle\hat{N}_b\rangle}{[\kappa^2 - 4\varepsilon^2]^2} + \frac{\gamma_c\kappa^3\langle\hat{N}_a\rangle + \gamma_c\kappa\varepsilon^2\langle\hat{N}_c\rangle + 4\gamma_c\kappa^2\varepsilon\langle\hat{m}_c\rangle}{(\kappa^2 - 4\varepsilon^2)^2}. \tag{4.19}$$

In view of Eq. (2.164), Eq. (2.165), and Eq. (2.167) we get

$$\begin{aligned}(\Delta\hat{a}_\pm)^2 &= q\big[2\kappa^2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) \\ &\quad + \varepsilon^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2})] \\ &\quad + 4\kappa\varepsilon\Omega(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2}) + \kappa^2\Omega^2 + 4\varepsilon^2\Omega^2\big],\end{aligned} \tag{4.20}$$

where

$$q = \frac{\gamma_c \kappa N}{[\kappa^2 - 4\varepsilon^2]^2((\gamma + \frac{\gamma_c \kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c \kappa \varepsilon}{\kappa^2 - 4\varepsilon^2}) + \Omega^2)}.$$

When $\varepsilon = 0$ we obtain

$$(\Delta \hat{a}_\pm)^2 = \frac{2\gamma_c}{\kappa} N \Big[\frac{\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}\Big], \tag{4.21}$$

for $\gamma = 0$ and $\Omega >> \gamma_c$ we can write

$$(\Delta \hat{a}_\pm)^2 = \frac{2\gamma_c}{3\kappa} N. \tag{4.22}$$

In view of Eq. (3.8) we see that

$$(\Delta \hat{a}_\pm)^2 = 2\bar{n}_a. \tag{4.23}$$

4.1.2 The quadrature variance of light mode b

Here we wish to obtain the quadrature variance of light mode b in entire frequency interval.

The squeezing properties of light mode b are described by two quadrature operators

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b} \tag{4.24}$$

and

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}), \tag{4.25}$$

where $\hat{b}_+$ and $\hat{b}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{b}^\dagger$ and $\hat{b}$ are the creation and annihilation operators for light mode b. With the help of Eqs. (4.24) and (4.25), we can show that the two quadrature operators satisfy the commutation relation

$$\big[\hat{b}_{-},\hat{b}_{+}\big] = -2i\big[\hat{b},\hat{b}^{\dagger}\big]. \tag{4.26}$$

On account of Eq. (2.125) we have

$$\big[\hat{b},\hat{b}^{\dagger}\big] = \frac{\gamma_c\kappa^3(\hat{N}_c-\hat{N}_a)+4\gamma_c\kappa\varepsilon^2(\hat{N}_a-\hat{N}_b)+2\gamma_c\kappa^2\varepsilon(\hat{m}_c+\hat{m}_c^{\dagger})}{(\kappa^2-4\varepsilon^2)^2}, \tag{4.27}$$

then

$$\big[\hat{b}_{-},\hat{b}_{+}\big] = -2i\frac{\gamma_c\kappa^3(\hat{n}_c-\hat{N}_a)+4\gamma_c\kappa\varepsilon^2(\hat{N}_a-\hat{N}_b)-2\gamma_c\kappa^2\varepsilon(\hat{m}_c+\hat{m}_c^{\dagger})}{(\kappa^2-4\varepsilon^2)^2}. \tag{4.28}$$

when $\varepsilon = 0$

$$\big[\hat{b}_{-},\hat{b}_{+}\big] = 2i\frac{\gamma_c}{\kappa}\big[\hat{N}_b-\hat{N}_c\big]. \tag{4.29}$$

In view of this result, the uncertainty relation for the plus and minus quadrature operators of mode b is expressed as

$$\Delta\hat{b}_{+}\Delta\hat{b}_{-} \geq \frac{1}{2}\big|\langle[\hat{b}_{+},\hat{b}_{-}]\rangle\big|, \tag{4.30}$$

it then follows

$$\Delta\hat{b}_{+}\Delta\hat{b}_{-} \geq \big|\langle\hat{b}\hat{b}^{\dagger}\rangle-\langle\hat{b}^{\dagger}\hat{b}\rangle\big|. \tag{4.31}$$

In view of Eq. (3.12) and Eq. (3.28) we found

$$\Delta\hat{b}_{+}\Delta\hat{b}_{-} \geq \frac{\gamma_c\kappa^3\langle\hat{N}_c\rangle+4\gamma_c\kappa\varepsilon^2\langle\hat{N}_a\rangle+4\gamma_c\kappa^2\varepsilon\langle\hat{m}_c\rangle}{[\kappa^2-4\varepsilon^2]^2}-\frac{\gamma_c\kappa^3\varepsilon\langle\hat{N}_b\rangle-4\gamma_c\kappa\varepsilon^2\langle\hat{N}_b\rangle}{[\kappa^2-4\varepsilon^2]^2}. \tag{4.32}$$

On account of Eqs. (2.164) - (2.167) we get

$$\begin{aligned}\Delta\hat{b}_{+}\Delta\hat{b}_{-} \;\;\geq\;\; & q\big[\kappa^2[(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2+2\Omega(\frac{\Omega}{2}-\frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})]+4\kappa\varepsilon\Omega(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2}) \\ & +8\varepsilon^2\Omega(\frac{\Omega}{2}-\frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})-\kappa^2\Omega^2-4\varepsilon^2\Omega^2\big].\end{aligned} \tag{4.33}$$

When $\varepsilon = 0$ it shows

$$\Delta\hat{b}_+\Delta\hat{b}_- \geq \frac{\gamma_c}{\kappa}N|\frac{(\gamma+\gamma_c)^2}{(\gamma+\gamma_c)^2+3\Omega^2}|, \tag{4.34}$$

for $\gamma = 0$ we have

$$\Delta\hat{b}_+\Delta\hat{b}_- \geq \frac{\gamma_c}{\kappa}N|\frac{\gamma_c^2}{\gamma_c^2+3\Omega^2}|, \tag{4.35}$$

for $\Omega >> \gamma_c$ we get

$$\Delta\hat{b}_+\Delta\hat{b}_- \geq 0. \tag{4.36}$$

Therefore the product of the uncertainties in the two quadratures satisfies the minimum uncertainty relation.

Then, we proceed to calculate the quadrature variance of light mode b.

The variance of the plus and minus quadrature operators of light mode b are

$$(\Delta\hat{b}_+)^2 = \langle\hat{b}_+^2\rangle - \langle\hat{b}_+\rangle^2 \tag{4.37}$$

and

$$(\Delta\hat{b}_-)^2 = \langle\hat{b}_-^2\rangle - \langle\hat{b}_-\rangle^2. \tag{4.38}$$

In view of eqs. (4.24) and (4.25) it shows

$$(\Delta\hat{b}_+)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}^2\rangle + \langle\hat{b}^{\dagger 2}\rangle - \langle\hat{b}\rangle^2 - \langle\hat{b}^\dagger\rangle^2 - 2\langle\hat{b}\hat{b}^\dagger\rangle, \tag{4.39}$$

$$(\Delta\hat{b}_-)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - \langle\hat{b}^2\rangle - \langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b}\rangle^2 + \langle\hat{b}^\dagger\rangle^2 + 2\langle\hat{b}\hat{b}^\dagger\rangle. \tag{4.40}$$

On account of Eqs. (4.39) and (4.40) we can write

$$(\Delta\hat{b}_\pm)^2 = \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle \pm \langle\hat{b}^2\rangle \pm \langle\hat{b}^{\dagger 2}\rangle \mp \langle\hat{b}\rangle^2 \mp \langle\hat{b}^\dagger\rangle^2 \mp 2\langle\hat{b}\hat{b}^\dagger\rangle, \tag{4.41}$$

we have

$$\langle\hat{b}\rangle = \langle\hat{b}^{\dagger}\rangle = \langle\hat{b}^{2}\rangle = \langle\hat{b}^{\dagger 2}\rangle = 0.$$

Therefore

$$(\Delta\hat{b}_{\pm})^{2} = \langle\hat{b}\hat{b}^{\dagger}\rangle + \langle\hat{b}^{\dagger}\hat{b}\rangle. \tag{4.42}$$

In view of Eq. (3.12) and Eq. (3.28) we get

$$(\Delta\hat{b}_{\pm})^{2} = \frac{\gamma_c\kappa}{[\kappa^2 - 4\varepsilon^2]^2}\big[\kappa^2\langle\hat{N}_b\rangle + 4\kappa\varepsilon^2\langle\hat{N}_b\rangle] + \frac{\gamma_c\kappa^3\langle\hat{N}_c\rangle + \gamma_c\kappa\varepsilon^2\langle\hat{N}_a\rangle + 4\gamma_c\kappa^2\varepsilon\langle\hat{m}_c\rangle}{(\kappa^2 - 4\varepsilon^2)^2} \tag{4.43}$$

In view of Eq. (2.164) - Eq. (2.167) we obtain

$$\begin{aligned}(\Delta\hat{b}_{\pm})^{2} &= q\big[\kappa^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2})] \\ &\quad + 4\kappa\varepsilon\Omega(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2}) \\ &\quad + 8\varepsilon^2\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + \kappa^2\Omega^2 + 4\varepsilon^2\Omega^2\big],\end{aligned} \tag{4.44}$$

where

$$q = \frac{\gamma_c\kappa N}{[\kappa^2 - 4\varepsilon^2]^2((\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) + \Omega^2)}.$$

when $\varepsilon = 0$ we get

$$(\Delta\hat{b}_{\pm})^{2} = \frac{\gamma_c}{\kappa}N\big[\frac{(\gamma + \gamma_c)^2 + 2\Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}\big], \tag{4.45}$$

for $\gamma = 0$ one can write

$$(\Delta\hat{b}_{\pm})^{2} = \frac{\gamma_c}{\kappa}N\big[\frac{\gamma_c^2 + 2\Omega^2}{\gamma_c^2 + 3\Omega^2}\big], \tag{4.46}$$

for $\Omega >> \gamma_c$ we see that

$$(\Delta\hat{b}_{\pm})^{2} = 2\frac{\gamma_c}{3\kappa}N. \tag{4.47}$$

In view of Eq. (3.16) we have

$$(\Delta\hat{b}_{\pm})^2 = 2\bar{n}_b. \tag{4.48}$$

4.2 Two-mode quadrature squeezing

Now, we proceed to study the quadrature variance and the quadrature squeezing of the two mode light beam produced by the coherently driven non degenerate three level laser with an open cavity and coupled to a two mode vacuum reservoir.

Here we seek to determine the quadrature variance of the two mode light beam.

The squeezing properties of the two mode cavity light described by two operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{4.49}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.50}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{c}^\dagger$ and $\hat{c}$ are the creation and annihilation operators for light two-mode. With the help of Eqs. (4.49) and (4.50), we can show that the two quadrature operators satisfy the commutation relation

$$\big[\hat{c}_-, \hat{c}_+\big] = -2i\big[\hat{c}, \hat{c}^\dagger\big], \tag{4.51}$$

$$\big[\hat{c}_-, \hat{c}_+\big] = (-2i)\frac{\gamma_c\kappa^3(\hat{N}_c - \hat{N}_a) + 4\gamma_c\kappa\varepsilon^2(\hat{N}_a - \hat{N}_c)}{[\kappa^2 - 4\varepsilon^2]^2}. \tag{4.52}$$

For $\varepsilon = 0$

$$\big[\hat{c}_-, \hat{c}_+\big] = 2i\frac{\gamma_c}{\kappa}\big[\hat{N}_a - \hat{N}_c\big]. \tag{4.53}$$

Then, the uncertainty relation for the plus and minus quadrature operators of two mode is expressed as

$$\Delta\hat{c}_+\Delta\hat{c}_- \geq \frac{1}{2}\big|\langle[\hat{c}_+,\hat{c}_-]\rangle\big|, \tag{4.54}$$

it then follows

$$\Delta\hat{c}_+\Delta\hat{c}_- \geq \big|\langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^\dagger\hat{c}\rangle\big|. \tag{4.55}$$

In view of Eqs. (3.38) and (3.46) we obtain

$$\Delta\hat{c}_+\Delta\hat{c}_- \geq \frac{\gamma_c\kappa^3\langle\hat{N}_c\rangle + 4\gamma_c\kappa\varepsilon^2\langle\hat{N}_a\rangle}{[\kappa^2-4\varepsilon^2]^2} - \frac{\gamma_c\kappa^3\langle\hat{N}_c\rangle - 4\gamma_c\kappa\varepsilon^2\langle\hat{N}_a\rangle}{[\kappa^2-4\varepsilon^2]^2}, \tag{4.56}$$

therefore

$$\Delta\hat{c}_+\Delta\hat{c}_- \geq 0. \tag{4.57}$$

Next, we proceed to calculate the quadrature variance of two mode cavity light. The variance of the plus and minus quadrature operators of two mode cavity light is expressed as

$$(\Delta\hat{c}_+)^2 = \langle\hat{c}_+^2\rangle - \langle\hat{c}_+\rangle^2 \tag{4.58}$$

and

$$(\Delta\hat{c}_-)^2 = \langle\hat{c}_-^2\rangle - \langle\hat{c}_-\rangle^2. \tag{4.59}$$

In view of Eqs.(4.49) and (4.50), and since $\hat{c}$ is a Gaussian variable

$$(\Delta\hat{c}_+)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}^2\rangle + \langle\hat{c}^{\dagger 2}\rangle - \langle\hat{c}\rangle^2 - \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle \tag{4.60}$$

and

$$(\Delta\hat{c}_-)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle - \langle\hat{c}^2\rangle - \langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}\rangle^2 + \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle. \tag{4.61}$$

On account of Eqs.(4.60) and (4.61) we get

$$(\Delta\hat{c}_{\pm})^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \mp \langle\hat{c}\rangle^2 \mp \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle, \tag{4.62}$$

we have

$$\langle\hat{c}\rangle = \langle\hat{c}^\dagger\rangle = 0.$$

Therefore

$$(\Delta\hat{c}_{\pm})^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle. \tag{4.63}$$

On account of Eqs. (3.39), (3.46), (3.47), and (3.48) we get

$$\begin{aligned}(\Delta\hat{c}_{\pm})^2 &= \frac{\gamma_c\kappa}{[\kappa^2-4\varepsilon^2]^2}\big[\kappa^2(N+\langle\hat{N}_b\rangle)+4\varepsilon^2(N+\langle\hat{N}_b\rangle)\\ &\quad \pm 4\kappa\varepsilon(N+\langle\hat{N}_b\rangle) \pm 2(\kappa^2+4\varepsilon^2+4\kappa\varepsilon)\langle\hat{m}_c\rangle\big].\end{aligned} \tag{4.64}$$

In view of Eqs. (2.165) - (2.167) the quadrature variance two mode cavity light takes the form

$$\begin{aligned}(\Delta\hat{c}_{-})^2 &= \frac{\gamma_c\kappa N}{[\kappa^2-4\varepsilon^2]^2[(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2+4\Omega(\frac{\Omega}{2}-\frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})+\Omega^2]}\\ &\quad \big[\kappa^2[(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2+4\Omega(\frac{\Omega}{2}-\frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})+2\Omega^2]\\ &\quad +4\varepsilon^2[(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2+4\Omega(\frac{\Omega}{2}-\frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})+2\Omega^2]\\ &\quad -4\kappa\varepsilon[(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2+4\Omega(\frac{\Omega}{2}-\frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2})+2\Omega^2]\\ &\quad -2\Omega(\kappa^2+4\varepsilon^2+4\kappa\varepsilon)(\gamma+\frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})\big].\end{aligned} \tag{4.65}$$

When $\varepsilon = 0$, we obtain

$$(\Delta\hat{c}_{-})^2 = \frac{\gamma_c}{\kappa}N\big[\frac{(\gamma+\gamma_c)^2+4\Omega^2-2\Omega(\gamma+\gamma_c)}{(\gamma+\gamma_c)^2+3\Omega^2}\big], \tag{4.66}$$

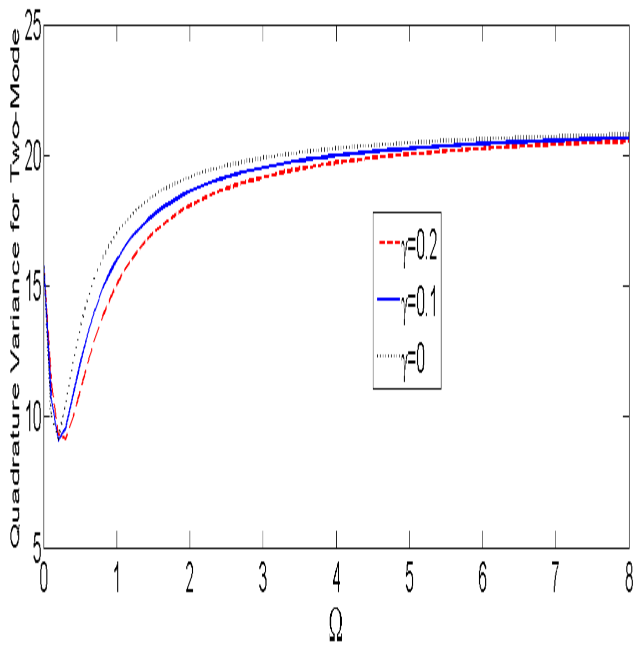

Figure 4.1: plots of the Quadrature variance of two mode [Eq.(4.65)] versus Ω for $\gamma_c = 0.4$,$\kappa = 0.8$, $\varepsilon = 0.5$, $N = 50$ and for different values of γ.

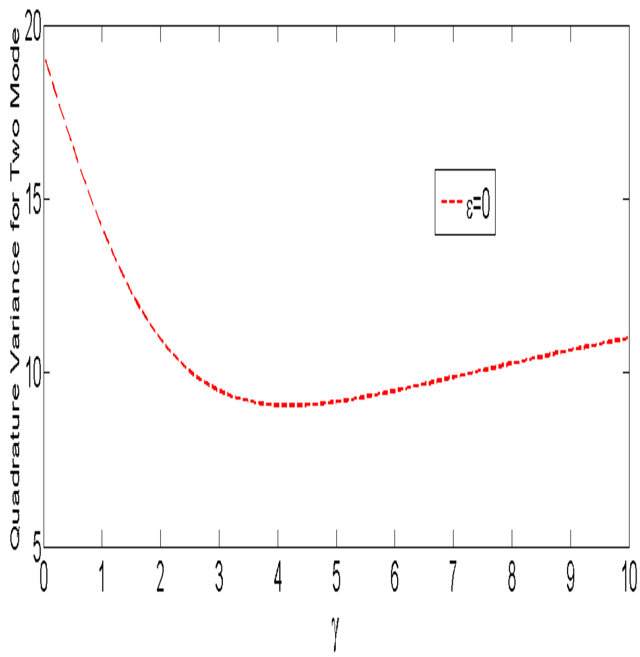

Figure 4.2: plots of the Quadrature variance of two mode [Eq.(4.65)] versus γ for $\gamma_c = 0.4$, $\kappa = 0.8$, $\Omega = 2$, $N = 50$ and for $\varepsilon = 0$.

for $\gamma = 0$

$$(\Delta \hat{c}_{-})^2 = \frac{\gamma_c}{\kappa} N \Big[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\Big]. \tag{4.67}$$

In addition, we consider the case in which the driving coherent light is absent, thus $\Omega = 0$, Eq. (4.67) reduces to

$$(\Delta \hat{c}_{-})^2 = \frac{\gamma_c}{\kappa} N. \tag{4.68}$$

Therefore

$$(\Delta \hat{c}_{+})_v^2 = \frac{\gamma_c}{\kappa} N, \tag{4.69}$$

and

$$(\Delta \hat{c}_{-})_v^2 = \frac{\gamma_c}{\kappa} N. \tag{4.70}$$

Which is the normal ordered quadrature variance of the two-mode cavity vacuum state.

The uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation.

Fig. 4.1, shows that quadrature variance for two mode cavity light is greater when $\gamma = 0$, than when $\gamma \neq 0$ for $\varepsilon = 0.5$ at $\Omega > 0.25$. Therefore the presence of spontaneous emission decrease the quadrature variance of two-mode cavity light.

Next, we proceed to calculate the quadrature squeezing of the two mode cavity light in the entire frequency interval relative to the quadrature variance of the two mode vacuum state.

The quadrature squeezing of the two mode cavity light is expressed as

$$S = \frac{(\Delta \hat{c}_{-})_v^2 - (\Delta \hat{c}_{-})^2}{(\Delta \hat{c}_{-})_v^2}, \tag{4.71}$$

it then follows

$$S = 1 - \frac{(\Delta \hat{c}_-)^2}{(\Delta \hat{c}_-)_v^2}. \tag{4.72}$$

In view of Eqs.(4.65) and (4.70), we obtain

$$\begin{aligned} S &= 1 - \frac{1}{[\kappa^2 - 4\varepsilon^2]^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) + \Omega^2]} \\ &\quad \big[\kappa^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + 2\Omega^2] \\ &\quad +4\varepsilon^2[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + 2\Omega^2] \\ &\quad -4\kappa\varepsilon[(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}) + 2\Omega^2] \\ &\quad -2\Omega(\kappa^2 + 4\varepsilon^2 + 4\kappa\varepsilon)(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2 - 4\varepsilon^2})\big]. \end{aligned} \tag{4.73}$$

For $\varepsilon = 0$, this reduces to

$$S = \frac{2\Omega(\gamma + \gamma_c) - \Omega^2}{(\gamma + \gamma_c)^2 + 3\Omega^2}. \tag{4.74}$$

Moreover, for $\gamma = 0$, we get

$$S = \frac{2\Omega\gamma_c - \Omega^2}{\gamma_c^2 + 3\Omega^2}. \tag{4.75}$$

Fig. 4.3, shows that the quadrature squeezing for two-mode cavity light is greater when $\gamma \neq 0$, than when $\gamma = 0$ for $\varepsilon = 0.3$ at $\Omega > 0.2$. Therefore the presence of spontaneous emission increase the quadrature squeezing for two-mode cavity light. Moreover, the maximum quadrature squeezing when $\gamma = 0.3$ and $\gamma = 0.2$ is 62.19%.

Entanglement

A quantum system is said to be entangled, if it can not be separable. That is if the

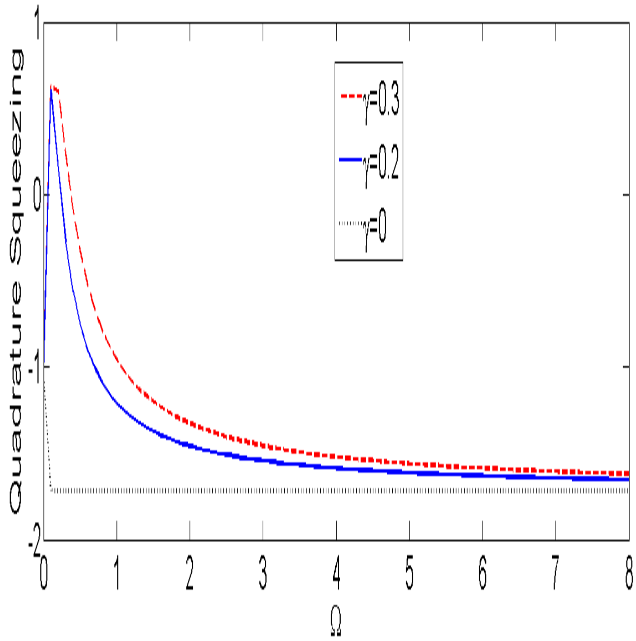

Figure 4.3: plots of the Quadrature Squeezing [Eq.(4.73)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.1$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

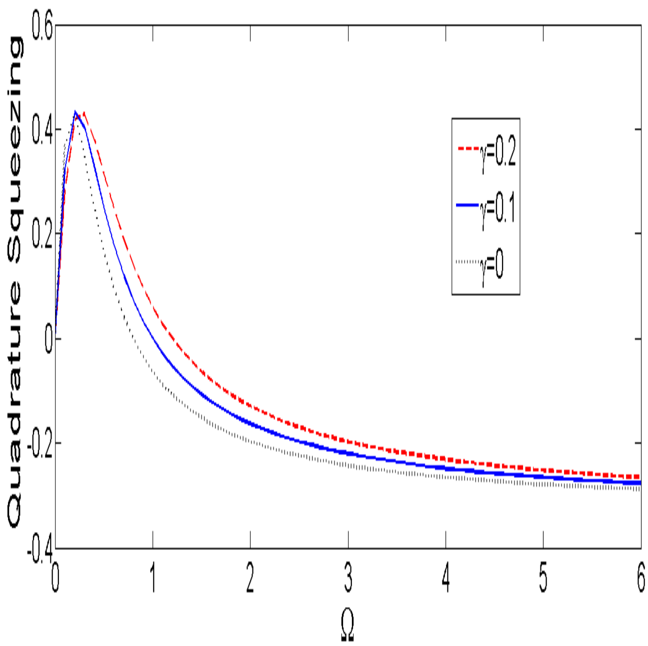

Figure 4.4: plots of the Quadrature Squeezing [Eq.(4.74)] versus Ω for $\gamma_c = 0.4$, $\varepsilon = 0$, $N = 50$ and for different values of γ.

density operator for the combined state can not be described as a combination of the product density operators of the constituents.

$$\hat{\rho} \neq \sum_k p_k \hat{p}_k^1 \otimes \hat{p}_k^2, \tag{4.76}$$

in which $p_k >> 0$ and $\sum_k p_k = 1$ to verify the normalization of the combined density states, on the other hand , a maximally entangled continuous variable (cv) state can be expressed as the quadrature operator,such as $\hat{x}_2 - \hat{x}_1$ and $\hat{p}_2 - \hat{p}_1$.

The total variance of these two operators reduces to zero for maximally entangled cv states.According to the criteria given by Duan [12], the cavity photons of a system are entangled, if the sum of the variance of the quadrature operators

$$\hat{s} = \hat{x}_2 - \hat{x}_1, \tag{4.77}$$

$$\hat{t} = \hat{p}_2 - \hat{p}_1, \tag{4.78}$$

where

$$\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \tag{4.79}$$

$$\hat{x}_2 = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger), \tag{4.80}$$

$$\hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger), \tag{4.81}$$

$$\hat{p}_2 = \frac{i}{\sqrt{2}}(\hat{b} - \hat{b}^\dagger), \tag{4.82}$$

are quadrature operators $\hat{a}$ and $\hat{b}$ satisfy,

$$(\Delta s)^2 + (\Delta t)^2 < 2N. \tag{4.83}$$

Since the cavity operators $\hat{a}$ and $\hat{b}$ are a Gaussian variable with zero mean, we get

$$(\Delta s)^2 + (\Delta t)^2 = \left[\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{b}^\dagger \hat{b}\rangle + \langle \hat{b}\hat{b}^\dagger\rangle\right] - \left[\langle \hat{a}\hat{b}\rangle + \langle \hat{a}^\dagger \hat{b}^\dagger\rangle + \langle \hat{b}\hat{a}\rangle + \langle \hat{b}^\dagger \hat{a}^\dagger\rangle\right], \tag{4.84}$$

this gives

$$(\Delta s)^2 + (\Delta t)^2 < 2(\Delta a_-)^2. \tag{4.85}$$

On the other hand, cavity atomic states of a system are entangled, if sum of the variance of the quadrature operators

$$\hat{u} = \hat{x}_2' - \hat{x}_1', \tag{4.86}$$

$$\hat{\nu} = \hat{p}_2' + \hat{p}_1', \tag{4.87}$$

where

$$\hat{x}_1' = \frac{1}{\sqrt{2}}(\hat{m}_a + \hat{m}_a^\dagger), \tag{4.88}$$

$$\hat{x}_2' = \frac{1}{\sqrt{2}}(\hat{m}_b + \hat{m}_b^\dagger), \tag{4.89}$$

$$\hat{p}_1' = \frac{i}{\sqrt{2}}(\hat{m}_a^\dagger - \hat{m}_a), \tag{4.90}$$

$$\hat{p}_2' = \frac{i}{\sqrt{2}}(\hat{m}_b^\dagger - \hat{m}_b), \tag{4.91}$$

are the quadrature operators for the cavity atoms, satisfy,

$$(\Delta u)^2 + (\Delta \nu)^2 < 2N. \tag{4.92}$$

Since $\hat{m}_a$ and $\hat{m}_b$ are a Gaussian atomic operators with zero means, we can find

$$(\Delta u)^2 + (\Delta \nu)^2 = \left[\langle \hat{m}_a^\dagger \hat{m}_a\rangle + \langle \hat{m}_a \hat{m}_a^\dagger\rangle + \langle \hat{m}_b^\dagger \hat{m}_b\rangle + \langle \hat{m}_b \hat{m}_b^\dagger\rangle\right] - \left[\langle \hat{m}_a \hat{m}_b\rangle + \langle \hat{m}_b^\dagger \hat{m}_a^\dagger\rangle\right], \tag{4.93}$$

this gives

$$(\Delta u)^2 + (\Delta\nu)^2 = \langle\hat{N}_a\rangle + 2\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle, \tag{4.94}$$

$$(\Delta u)^2 + (\Delta\nu)^2 = N + \langle\hat{N}_b\rangle, \tag{4.95}$$

it then follows

$$(\Delta u)^2 + (\Delta\nu)^2 = 2N - \langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle. \tag{4.96}$$

From Eq. (4.96) we have

$$(\Delta u)^2 + (\Delta\nu)^2 = N\big[1 + \frac{\Omega^2}{(\gamma + \frac{\gamma_c\kappa^2}{\kappa^2-4\varepsilon^2})^2 + 4\Omega(\frac{\Omega}{2} - \frac{\gamma_c\kappa\varepsilon}{\kappa^2-4\varepsilon^2}) + \Omega^2}\big]. \tag{4.97}$$

The plots 4.5 shows that $(\Delta u)^2 + (\Delta\nu)^2$ decrease when the spontaneous emission increase.

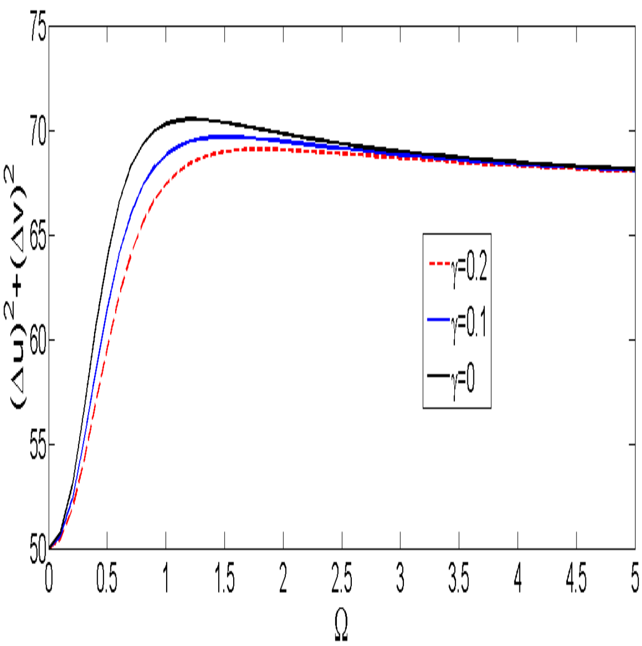

Figure 4.5: plots of the Sum of Variance of Operators [Eq.(4.97)] versus Ω for $\gamma_c = 0.4$, $\kappa = 0.8$, $\varepsilon = 0.3$, $N = 50$ and for different values of γ.

5

CONCLUSION

In this thesis we have studied the squeezing and statistical properties of the light produced by the coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. We have carried out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Applying the steady state solutions of the atomic operators and the quantum Langavin equations for the cavity mode operators, we have determined the mean , the variance of the photon number,the quadrature squeezing, photon state entanglement as well as atom and photon number correlation. It is found that the photons and the atoms in the system are strongly entangled at steady state. Results show that the presence of parametric amplifier is to increase the squeezing and the mean photon number of the two- mode cavity light significantly.

Moreover, we have shown that the mean photon number of the two-mode light is the sum of the mean photon numbers of the separate single-mode light. However, we have observed that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beams. We have found that the light generated by the three-level laser is in a squeezed state and the squeezing occurs in the minus quadrature. The plots in Fig 4.3, show that the maximum quadrature squeezing when $\gamma = 0.2$ and $\gamma = 0.3$ is 62.19% at $\Omega = 0.101$ for $\varepsilon = 0.3$. The quadrature squeezing when $\gamma = 0.3$ is greater than when $\gamma = 0.2$ at $\Omega > 0.102$.

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