One-Mode Subharmonic Generator

Coupled to Thermal Reservoir

Molla Meressa

A Thesis Submitted to

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In Partial Fulfillment of the

Requirements for the Degree of

Masters of Science in Physics (Quantum Optics)

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By

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Abstract

We have analyzed the squeezing and statistical properties of the light produced by one-mode subharmonic generator coupled to thermal reservoir. We first obtain c-number Langevin equation with the aid of the master equation. The solution of the resulting c-number Langevin equation is then used to determine the anti-normally characteristic function. With the aid of the resulting characteristic function, we obtain the Q function and the density operator. In addition, employing the Q function along with the density operator, we calculate the mean photon number, the variance of the photon number, the power spectrum, the photon number distribution, and the quadrature variance. We found that the squeezing of the one-mode subharmonic generator indeed affected by the thermal light.

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Contents

1

Introduction

Light has played a special role in our attempts to understand nature both classically and quantum mechanically. Squeezing is one of the interesting nonclassical features of light that has been attracting attention and studied by many authors [3-6, 8, 11, 17, 21]. In squeezed-state of light the noise in one quadrature is below the vacuum or coherent-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in low-noise communications and precision measurements [18, 19]. Squeezed light can be generated by quantum optical processes such as parametric oscillation [3-6, 8, 11, 21], second harmonic generation [3, 11], and four-wave mixing [5, 11, 13, 27]. A subharmonic generator has been considered as an important source of squeezed light. It is one of the most interesting and well characterized optical device in quantum optics. In this device a pump photon interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons. If these photons have the same frequency the device is called a one-mode subharmonic generator, otherwise it is called a two-mode subharmonic generator. The quadrature squeezing and photon statistics of the signal mode produced by one-mode subharmonic generator coupled to a squeezed vacuum reservoir have been analyzed by a number of authors [3, 5, 8, 21]. In this thesis, we wish to study the statistical and squeezing properties of the light generated by a one-mode subharmonic generator coupled to thermal reservoir via a single port mirror. We first determine the master equation and c-number Langevin equation for the one mode subharmonic generator coupled to thermal reservoir. Employing the solution of the c -number Langevin equation, we then obtain the Q function together with the density operator, with aid of which we calculate the photon statistics and quadrature variance of the system under consideration.

2

The Q Function

In this chapter, we study the squeezing and statistical properties of the one-mode subharmonic generation coupled to thermal reservoir. In a one-mode subharmonic generator, a pump photon of frequency 2ω is down converted into a pair of signal photons each of frequency ω. We first derive the master equation of the system under consideration. Thus employing the master equation, we determine the c-number Langevin equation. The solution of the resulting c-number Langevin equation is then used to determine the antinormally ordered characteristic function. With the aid of this antinormally ordered characteristic function, we determine the Q function. Then, we obtain an expression for the density operator in terms of the resulting Q function.

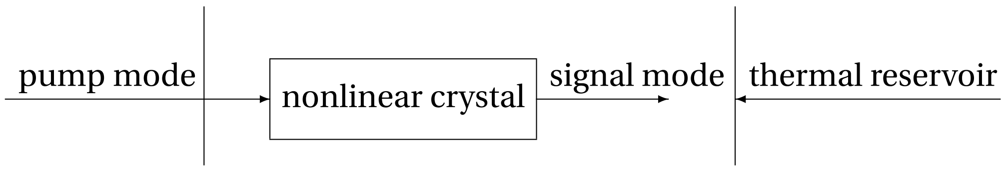

Figure 2.1 One-mode subharmonic generator coupled to thermal reservoir.

2.1 The master equation

The process of one-mode subharmonic generation is described by the Hamiltonian

$$\hat{H}_S = i\mu(\hat{b}^\dagger - \hat{b}) + \frac{i\lambda}{2}(\hat{b}\hat{a}^2 - \hat{b}^\dagger\hat{a}^{\dagger 2}), \tag{2.1}$$

where μ is proportional to the amplitude of the coherent light driving the pump mode, λ is the coupling constant, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for the signal mode and the pump mode, respectively. With a pump mode represented by a real and constant c-number β, the process of one-mode subharmonic generation can be represnted by the Hamiltonian

$$\hat{H}_S = \frac{i\varepsilon}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}), \tag{2.2}$$

where $\varepsilon = \lambda\beta$.

The master equation for a cavity mode coupled to a reservoir can be written as [3]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = &-i[\hat{H}_S, \hat{\rho}(t)] - h\langle\hat{H}_{SR}^2\rangle_R\hat{\rho}(t) - h\hat{\rho}(t)\langle\hat{H}_{SR}^2\rangle_R \\ &+2hTr_R\bigg(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}\bigg),\end{aligned} \tag{2.3}$$

where

$$\hat{H} = i\lambda'(\hat{a}^\dagger\hat{a}_{in} - \hat{a}_{in}^\dagger\hat{a}), \tag{2.4}$$

is the Hamiltonian describing the interaction of a cavity mode with a reservoir, Tr_R is the trace over reservoir variables. Taking into account Eq. (2.4) under consideration, we note that

$$\langle\hat{H}_{SR}^2\rangle = -\lambda'^2\langle\bigg[(\hat{a}^\dagger\hat{a}_{in}\hat{a}^\dagger\hat{a}_{in}) - (\hat{a}^\dagger\hat{a}_{in}\hat{a}_{in}^\dagger\hat{a}) - (\hat{a}_{in}^\dagger\hat{a}\hat{a}^\dagger\hat{a}_{in}) + (\hat{a}_{in}^\dagger\hat{a}\hat{a}_{in}^\dagger\hat{a})\bigg]\rangle. \tag{2.5}$$

Employing the commutation relation

$$[\hat{a}_{in}, \hat{a}^{\dagger}_{in}] = 1, \tag{2.6}$$

one easily obtains

$$\langle \hat{H}^2_{SR} \rangle = -\lambda'^2 \left[\hat{a}^{\dagger 2} \langle \hat{a}^2_{in} \rangle_R - \langle \hat{a}^{\dagger}_{in} \hat{a}_{in} \rangle_R \hat{a}^{\dagger} \hat{a} - \hat{a}^{\dagger} \hat{a} - \hat{a} \hat{a}^{\dagger} \langle \hat{a}^{\dagger}_{in} \hat{a}_{in} \rangle_R + \langle \hat{a}^{\dagger 2}_{in} \rangle_R \hat{a}^2 \right]. \tag{2.7}$$

Next we seek to calculate the expectation values of reservoir-mode operators. Hence the expectation value of $\hat{a}^{\dagger}_{in} \hat{a}_{in}$ can be written as

$$\langle \hat{a}^{\dagger}_{in} \hat{a}_{in} \rangle = Tr_R(\hat{\rho} \hat{a}^{\dagger}_{in} \hat{a}_{in}), \tag{2.8}$$

in which

$$\hat{\rho} = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle\langle n|, \tag{2.9}$$

is the density operator for thermal light. It then follows that

$$\langle \hat{a}^{\dagger}_{in} \hat{a}_{in} \rangle_R = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \langle n | \hat{a}^{\dagger}_{in} \hat{a}_{in} | n \rangle. \tag{2.10}$$

Eq. (2.8) leads to

$$\langle \hat{a}^{\dagger}_{in} \hat{a}_{in} \rangle = \bar{n}. \tag{2.11}$$

The expectation value of the annihilation operator $\hat{a}_{in}$ can also be written as

$$\langle \hat{a}_{in} \rangle_R = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \langle n | \hat{a}_{in} | n \rangle. \tag{2.12}$$

Then, we see that

$$\langle \hat{a}_{in} \rangle = \langle \hat{a}^{\dagger}_{in} \rangle = 0. \tag{2.13}$$

Similarly, one can easily check that

$$\langle \hat{a}^2_{in} \rangle = \langle \hat{a}^{\dagger 2}_{in} \rangle = 0. \tag{2.14}$$

Upon substituting Eqs. (2.11), (2.13), and (2.14) into (2.7), we find

$$\langle \hat{H}_{SR}^2 \rangle_R = \lambda'^2 \bigg((\bar{n}+1)\hat{a}^\dagger \hat{a} + \bar{n}\hat{a}\hat{a}^\dagger \bigg). \tag{2.15}$$

It then follows that

$$-h\langle \hat{H}_{SR}^2 \rangle_R \hat{\rho}(t) = -\frac{\kappa}{2}(\bar{n}+1)\hat{a}^\dagger \hat{a}\hat{\rho}(t) - \frac{\kappa}{2}\bar{n}\hat{a}\hat{a}^\dagger \hat{\rho}(t), \tag{2.16}$$

where

$$\kappa = 2h\lambda'^2, \tag{2.17}$$

is the cavity damping constant.

Following the same procedure, we observe that

$$-h\hat{\rho}(t)\langle \hat{H}_{SR}^2 \rangle_R = -\frac{\kappa}{2}(\bar{n}+1)\hat{\rho}(t)\hat{a}^\dagger \hat{a} - \frac{\kappa}{2}\bar{n}\hat{\rho}(t)\hat{a}\hat{a}^\dagger. \tag{2.18}$$

Moreover, employing the Hamiltonian described by Eq. (2.4), we obtain

$$\begin{aligned} 2hTr_R\bigg(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR} \bigg) = -2h\lambda^2 Tr_R\bigg(& \hat{a}^\dagger \hat{a}_{in}\hat{\rho}(t)\hat{R}\hat{a}^\dagger \hat{a}_{in} - \hat{a}^\dagger \hat{a}_{in}\hat{\rho}(t)\hat{R}\hat{a}_{in}^\dagger \hat{a} \\ & -\hat{a}_{in}^\dagger \hat{a}\hat{\rho}(t)\hat{R}\hat{a}^\dagger \hat{a}_{in} + \hat{a}_{in}^\dagger \hat{a}\hat{\rho}(t)\hat{R}\hat{a}_{in}^\dagger \hat{a} \bigg). \end{aligned} \tag{2.19}$$

It then follows that

$$\begin{aligned} 2hTr_R\bigg(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR} \bigg) = -2h\lambda^2 Tr_R\bigg(& \hat{a}^\dagger \hat{\rho}(t)\hat{a}^\dagger \hat{R}\hat{a}_{in}^2 - \hat{a}^\dagger \hat{\rho}(t)\hat{a}\hat{R}\hat{a}_{in}^\dagger \hat{a}_{in} \\ & -\hat{a}\hat{\rho}(t)\hat{a}^\dagger \hat{R}\hat{a}_{in}\hat{a}_{in}^\dagger + \hat{a}\hat{\rho}(t)\hat{a}\hat{R}\hat{a}_{in}^{\dagger 2} \bigg). \end{aligned} \tag{2.20}$$

One then readily obtains

$$2hTr_R\bigg(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR} \bigg) = \frac{\kappa}{2}(\bar{n}+1)2\hat{a}\hat{\rho}(t)\hat{a}^\dagger + \frac{\kappa}{2}\bar{n}(2\hat{a}^\dagger \hat{\rho}(t)\hat{a}). \tag{2.21}$$

Upon combining Eqs. (2.16), (2.18), (2.21), and (2.3), we arrive at

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & -i[\hat{H}_S, \hat{\rho}(t)] + \frac{\kappa}{2}(\bar{n}+1)\bigg[2\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger \hat{a} \bigg] \\ & + \frac{\kappa}{2}\bar{n}\bigg[2\hat{a}^\dagger \hat{\rho}(t)\hat{a} - \hat{\rho}(t)\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger \hat{\rho}(t) \bigg]. \end{aligned} \tag{2.22}$$

This represents the master equation for a cavity mode coupled to thermal reservoir in which the effect of the reservoir is incorporated via the parameter $\bar{n}$. Finally, in view of Eq. (2.2), the master equation for one-mode subharmonic generator coupled to thermal reservoir takes the form

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} = &\frac{\varepsilon}{2}\Big[\hat{a}^2\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^2 + \hat{\rho}(t)\hat{a}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{\rho}(t)\Big]\\ &+\frac{\kappa}{2}(\bar{n}+1)\Big[2\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}^\dagger\hat{a}\Big]\\ &+\frac{\kappa}{2}\bar{n}\Big[2\hat{a}^\dagger\hat{\rho}(t)\hat{a} - \hat{\rho}(t)\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}(t)\Big].\end{aligned} \tag{2.23}$$

2.2 C-number Langevin equation

Now employing the relation

$$\frac{d}{dt}\langle\hat{A}(t)\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}(t)\right) \tag{2.24}$$

and applying the cyclic property of the trace operation along with the master equation, we readily obtain

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\varepsilon\langle\hat{a}^\dagger(t)\rangle - \frac{\kappa}{2}\langle\hat{a}(t)\rangle, \tag{2.25}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\varepsilon\langle\hat{a}^2(t)\rangle - \varepsilon\langle\hat{a}^{\dagger 2}(t)\rangle - \kappa\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \kappa\bar{n}, \tag{2.26}$$

and

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -\kappa\langle\hat{a}^2(t)\rangle - 2\varepsilon\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle - \varepsilon. \tag{2.27}$$

The c- number equation corresponding to Eqs. (2.25), (2.26), and (2.27) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\varepsilon\langle\alpha^*(t)\rangle - \frac{\kappa}{2}\langle\alpha(t)\rangle, \tag{2.28}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\varepsilon\langle\alpha^2(t)\rangle - \varepsilon\langle\alpha^{*2}(t)\rangle - \kappa\langle\alpha^*(t)\alpha(t)\rangle + \kappa\bar{n}, \tag{2.29}$$

and

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon. \tag{2.30}$$

On the basis of Eq. (2.28), one can write

$$\frac{d}{dt}\alpha(t) = -\varepsilon\alpha^*(t) - \frac{\kappa}{2}\alpha(t) + f_\alpha(t), \tag{2.31}$$

where $f_\alpha(t)$ is a noise force whose correlation properties remain to be determined. We note that Eq. (2.28) and the expectation value of Eq. (2.31) will have the same form if

$$\langle f_\alpha(t)\rangle = 0. \tag{2.32}$$

Using the mathematical relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \left\langle\frac{d\alpha^*(t)}{dt}\alpha(t)\right\rangle + \left\langle\alpha^*(t)\frac{d\alpha(t)}{dt}\right\rangle, \tag{2.33}$$

along with Eq. (2.31), and its complex conjugate, one can easily establish that

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = &-\varepsilon\langle\alpha^{*2}(t)\rangle - \varepsilon\langle\alpha^2(t)\rangle - \kappa\langle\alpha^*(t)\alpha(t)\rangle \\ &+\langle\alpha(t)f^*(t)\rangle + \langle\alpha^*(t)f(t)\rangle.\end{aligned} \tag{2.34}$$

On comparing this with Eq. (2.29), we have

$$\langle\alpha(t)f^*(t)\rangle + \langle\alpha^*(t)f(t)\rangle = \kappa\bar{n}. \tag{2.35}$$

A formal solution of Eq. (2.31) and its complex conjugate can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\kappa(t-t')/2}(f_\alpha(t') - \varepsilon\alpha^*(t'))dt' \tag{2.36}$$

and

$$\alpha^*(t) = \alpha^*(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\kappa(t-t')/2}(f_\alpha^*(t') - \varepsilon\alpha(t'))dt'. \tag{2.37}$$

Multiplying Eq. (2.36) by $f_\alpha^*(t)$ and Eq. (2.37) by $f_\alpha(t)$ from the right and taking the expectation value of the resulting expressions

$$\langle\alpha(t)f_\alpha^*(t)\rangle = \langle\alpha(0)f_\alpha^*(t)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\kappa(t-t')/2}\Big(\langle f_\alpha(t')f_\alpha^*(t)\rangle - \langle\varepsilon\alpha^*(t')f_\alpha^*(t)\rangle\Big)dt \tag{2.38}$$

and

$$\langle\alpha^*(t)f_\alpha(t)\rangle = \langle\alpha^*(0)f_\alpha(t)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\kappa(t-t')/2}\Big(\langle f_\alpha^*(t')f_\alpha(t)\rangle - \langle\varepsilon\alpha(t')f_\alpha(t)\rangle\Big)dt'. \tag{2.39}$$

On account of the assertion that a noise force at a later time does not affect system variables at earlier times, we have

$$\langle\alpha(0)f_\alpha^*(t)\rangle = \langle\alpha(0)\rangle\langle f_\alpha^*(t)\rangle = 0, \tag{2.40}$$

$$\langle\alpha^*(t')f_\alpha^*(t)\rangle = \langle\alpha^*(t')\rangle\langle f_\alpha^*(t)\rangle = 0, \tag{2.41}$$

$$\langle\alpha^*(0)f_\alpha(t)\rangle = \langle\alpha^*(0)\rangle\langle f_\alpha(t)\rangle = 0, \tag{2.42}$$

and

$$\langle\alpha(t')f_\alpha(t)\rangle = \langle\alpha(t')\rangle\langle f_\alpha(t)\rangle = 0. \tag{2.43}$$

With the aid of these relations Eqs. (2.38) and (2.39) reduce to

$$\langle\alpha(t)f_\alpha^*(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t')f_\alpha^*(t)\rangle dt' \tag{2.44}$$

and

$$\langle\alpha^*(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha^*(t')f_\alpha(t)\rangle dt'. \tag{2.45}$$

Now taking into account Eqs. (2.35), (2.44), (2.45) and assuming that

$$\langle f_\alpha(t')f^*_\alpha(t)\rangle = \langle f^*_\alpha(t')f_\alpha(t)\rangle, \tag{2.46}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t)f_\alpha(t')\rangle dt' = \frac{\kappa\bar{n}}{2}. \tag{2.47}$$

Now on the basis of the relation

$$\int_0^t e^{-\kappa(t-t')/2}\langle f(t)g(t')\rangle dt' = D, \tag{2.48}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'). \tag{2.49}$$

Then on the basis of Eq. (2.48), we find

$$\langle f_\alpha(t)f^*_\alpha(t')\rangle = \langle f^*_\alpha(t)f_\alpha(t')\rangle \ = \ \kappa\bar{n}\delta(t-t'). \tag{2.50}$$

Furthermore, employing the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \langle\frac{d\alpha(t)}{dt}\alpha(t) + \langle\alpha(t)\frac{d}{dt}\alpha(t)\rangle, \tag{2.51}$$

along with Eq. (2.31), one can readily verify that

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{2.52}$$

Now comparison of Eq. (2.30) and (2.52), we see that

$$\langle\alpha(t)f_\alpha(t)\rangle = -\frac{\varepsilon}{2}. \tag{2.53}$$

Multiplying Eq. (2.36) by $f_\alpha(t)$ from the right and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned}\langle\alpha(t)f_\alpha(t)\rangle = \langle\alpha(0)f_\alpha(t)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\kappa(t-t')/2}\Big(\langle(f_\alpha(t')f_\alpha(t)\rangle \\ -\varepsilon\langle\alpha^*(t')f_\alpha(t)\rangle\Big)dt'.\end{aligned} \tag{2.54}$$

Since a noise force at a later time does not affect system variables at earlier times, one can write

$$\langle\alpha(t)f_\alpha(t)\rangle=\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t')f_\alpha(t)\rangle dt', \tag{2.55}$$

so that in view of Eq. (2.53), it follows that

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\alpha(t')\rangle dt'=-\frac{\varepsilon}{2}. \tag{2.56}$$

Now with the aid of Eq. (2.48) and (2.49), we find

$$\langle f_\alpha(t)f_\alpha(t')\rangle=-\varepsilon\delta(t-t'). \tag{2.57}$$

It is worth mentioning that Eqs. (2.32), (2.50), and (2.57) describe the correlation properties of the noise force $f_\alpha(t)$ associated with the normal ordering.

In order to obtain the solution of Eq. (2.31), we introduce a new variable defined by

$$\alpha_\pm(t)=\alpha^*(t)\pm\alpha(t). \tag{2.58}$$

Applying Eq. (2.31) and its complex conjugate, we see that

$$\frac{d}{dt}\alpha_\pm(t)=-\frac{1}{2}\xi_\pm\alpha_\pm(t)+f^*(t)\pm f(t), \tag{2.59}$$

where

$$\xi_\pm=\kappa\pm 2\varepsilon. \tag{2.60}$$

According to Eq. (2.59) and Eq. (2.60), the equation of evolution of $\alpha_-(t)$ does not have a well behaved solution for $\kappa<2\varepsilon$. We then identify $\kappa=2\varepsilon$ as a threshold condition. For $2\varepsilon<\kappa$, the solution of Eq. (2.59) can be written as

$$\alpha_\pm(t)=\alpha_\pm(0)e^{-\xi_\pm t/2}+\int_0^t e^{-\xi_\pm(t-t')/2}\Big(f^*(t')\pm f(t')\Big)dt'. \tag{2.61}$$

Finally, applying Eqs. (2.59) and (2.60), it can be readily established that

$$\alpha(t) = F_+(t)\alpha(0) + F_-(t)\alpha^*(0) + G_+(t) - G_-(t), \tag{2.62}$$

in which

$$F_\pm = \frac{1}{2}\left(e^{-\xi_+ t/2} \pm e^{-\xi_- t/2}\right) \tag{2.63}$$

and

$$G_\pm(t) = \frac{1}{2}\int_0^t e^{-\xi_\pm (t-t')/2}\left(f^*(t') \pm f(t')\right)dt'. \tag{2.64}$$

2.3 The Q function

We now proceed to calculate the Q function applying the anti-normally ordered characteristic function for the signal mode assumed to be initially in the vacuum state. The Q function is expressible in terms of the anti-normally ordered characteristic function as

$$Q(\alpha, \alpha^*, t) = \frac{1}{\pi^2}\int d^2z\phi_a(z^*, z, t)e^{z^*\alpha - z\alpha^*}. \tag{2.65}$$

Employing the identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{2.66}$$

the characteristic function $\phi_a(z^*, z, t)$, defined in the Heisenberg picture by

$$\phi_a(z^*, z, t) = Tr\left(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}\right), \tag{2.67}$$

can be written in terms of c-number variable associated with the normall ordering as

$$\phi_a(z^*, z, t) = e^{-z^*z}\langle e^{z\alpha^*(t) - z^*\alpha(t)}\rangle. \tag{2.68}$$

It is easy to see that $\alpha(t)$ is Gaussian variable with zero mean. Thus according to Ref. [3], one can put Eq. (2.68) in the form

$$\phi_a(z^*, z, t) = e^{-z^* z} exp\left(\frac{1}{2}\langle[z\alpha^*(t) - z^*\alpha(t)]^2\rangle\right). \tag{2.69}$$

It then follows that

$$\phi_a(z^*, z, t) = e^{-z^* z} exp\left[\frac{1}{2}\langle(z^2\alpha^{*2}(t) + z^{*2}\alpha^2(t) - 2z^* z\alpha^*(t)\alpha(t))\rangle\right]. \tag{2.70}$$

Using Eqs. (2.62) and (2.64), one obtains

$$\begin{aligned}\langle\alpha^*(t)\alpha(t)\rangle = &\langle G_+^*(t)G_+(t)\rangle - \langle G_+^*(t)G_-(t)\rangle \\ &-\langle G_-^*(t)G_+(t)\rangle + \langle G_-^*(t)G_-(t)\rangle,\end{aligned} \tag{2.71}$$

$$\langle\alpha^2(t)\rangle = \langle G_+^2(t)\rangle + \langle G_-^2(t)\rangle - \langle G_+(t)G_-(t)\rangle - \langle G_-(t)G_+(t)\rangle, \tag{2.72}$$

and

$$\langle\alpha^{*2}(t)\rangle = \langle G_+^{*2}(t)\rangle + \langle G_-^{*2}(t)\rangle - \langle G_+^*(t)G_-^*(t)\rangle - \langle G_-^*(t)G_+^*(t)\rangle. \tag{2.73}$$

Applying Eq. (2.64), it can be readily established that

$$\begin{aligned}\langle G_+^2(t)\rangle = \frac{1}{4}\int_0^t e^{-\xi_+(2t-t'-t'')/2}\Big[&\langle f_\alpha^*(t')f_\alpha^*(t'')\rangle + \langle f_\alpha^*(t')f_\alpha(t'')\rangle \\ &+\langle f_\alpha(t')f_\alpha^*(t'')\rangle + \langle f_\alpha(t')f_\alpha(t'')\rangle\Big]dt'dt''.\end{aligned} \tag{2.74}$$

Taking into account Eqs. (2.50) and (2.57), we get

$$\langle G_+^2(t)\rangle = \frac{1}{4}\int_0^t e^{-\xi_+(2t-t'-t'')/}\Big[2\kappa\bar{n}\delta(t''-t') - 2\varepsilon\delta(t''-t')\Big]dt'dt'', \tag{2.75}$$

so that upon carrying out the integration, we find

$$\langle G_+^2(t)\rangle = \frac{(\kappa\bar{n} - \varepsilon)}{2\xi_+}(1 - e^{-\xi_+ t}). \tag{2.76}$$

Similarly, one can check that

$$\langle G_-^2(t)\rangle = \frac{-(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}), \tag{2.77}$$

$$\langle G_+^*(t)G_+(t)\rangle = \frac{(\kappa\bar{n}-\varepsilon)}{2\xi_+}(1-e^{-\xi_+ t}), \tag{2.78}$$

$$\langle G_-^*(t)G_-(t)\rangle = \frac{-(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}), \tag{2.79}$$

and

$$\langle G_+(t)G_-(t)\rangle = \langle G_-(t)G_+(t)\rangle = 0. \tag{2.80}$$

Now on account of these results, we have

$$\langle \alpha^2(t)\rangle = \langle \alpha^{*2}(t)\rangle = \frac{(\kappa\bar{n}-\varepsilon)}{2\xi_+}(1-e^{-\xi_+ t}) - \frac{(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}) \tag{2.81}$$

and

$$\langle \alpha^*(t)\alpha(t)\rangle = \frac{(\kappa\bar{n}-\varepsilon)}{2\xi_+}(1-e^{-\xi_+ t}) + \frac{(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}). \tag{2.82}$$

Thus in view of Eqs. (2.81) and (2.82), Eq. (2.68) takes the form

$$\phi_a(z^*, z, t) = exp\bigg(-az^*z + \frac{1}{2}b(z^{*2}+z^2)\bigg), \tag{2.83}$$

in which

$$a = 1 + \frac{(\kappa\bar{n}-\varepsilon)}{2\xi_+}(1-e^{-\xi_+ t}) + \frac{(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}) \tag{2.84}$$

and

$$b = \frac{(\kappa\bar{n}-\varepsilon)}{2\xi_+}(1-e^{-\xi_+ t}) - \frac{(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}). \tag{2.85}$$

We now proceed to determine the Q function for the one-mode subharmonic generator coupled to thermal reservoir. Then upon substituting Eq. (2.83) into Eq. (2.65), we see that

$$Q(\alpha, \alpha^*, t) = \frac{1}{\pi^2}\int d^2z e^{(-az^*z+z^*\alpha-z\alpha^*+b(z^{*2}+z^2)/2)}. \tag{2.86}$$

Thus upon performing the integration by employing the relation

$$\int \frac{d^2z}{\pi} exp\bigg(-azz^* + bz + cz^* + Az^2 + Bz^{*2}\bigg) = \bigg[\frac{1}{a^2 - 4AB}\bigg]^{\frac{1}{2}} exp\bigg[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\bigg], a > 0 \tag{2.87}$$

we find

$$Q(\alpha, \alpha^*, t) = \frac{1}{\pi}(u^2 - v^2)^{\frac{1}{2}} exp\bigg[-u\alpha\alpha^* - v(\alpha^2 + \alpha^{*2})/2\bigg], \tag{2.88}$$

where

$$u = \frac{a}{(a^2 - b^2)} \tag{2.89}$$

and

$$v = \frac{b}{(a^2 - b^2)}. \tag{2.90}$$

This represents the Q function for the one-mode subharmonic generator coupled to thermal reservoir.

2.4 The density operator

Here we seek to determine the density operator for a signal mode. Suppose $\hat{\rho}(\hat{a}^\dagger, \hat{a})$ is density operator for a certain light beam. The normally-ordered density operator has the form

$$\hat{\rho}(t) = \sum_{kl} C_{kl} \hat{a}^{\dagger k} \hat{a}^l. \tag{2.91}$$

Now, we introduce the completeness relation for coherent state as [3]

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| = \hat{I}. \tag{2.92}$$

The expectation value of an operator function $\hat{A}(\hat{a}^\dagger, \hat{a}, t)$ can be expressed in the form

$$\langle \hat{A}(\hat{a}^\dagger, \hat{a}, t)\rangle = Tr(\hat{\rho}(\hat{a}^\dagger, \hat{a}, t)\hat{A}(0)). \tag{2.93}$$

Then, using the completeness relation for coherent state twice, we have [3]

$$\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t) = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}|\alpha\rangle\langle\alpha|\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)|\beta\rangle\langle\beta|. \tag{2.94}$$

Thus we obtain

$$\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t) = \frac{1}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta, t)\langle\alpha|\beta\rangle|\alpha\rangle\langle\beta|, \tag{2.95}$$

in which

$$Q(\alpha^*, \beta, t) = \frac{1}{\pi}\langle\alpha|\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)|\beta\rangle. \tag{2.96}$$

Therefore, in view of Eq. (2.93) and (2.95), the expectation value of a given operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a})$ is expressible as [3]

$$\langle\hat{A}(\hat{a}^{\dagger}, \hat{a})\rangle = \frac{1}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta, t) \times exp\Big[-\alpha^*\alpha - \beta^*\beta + \beta^*\alpha + \alpha^*\beta\Big]A_n(\alpha, \beta^*), \tag{2.97}$$

where $\hat{A}_n(\alpha, \beta^*)$ is a c-number function corresponding to the operator $\hat{A}(\hat{a}^{\dagger}, \hat{a})$ in the normal order. Then the expectation value of a given operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a}, t)$ for the signal mode coupled to thermal reservoir in terms of the Q function can be rewritten as

$$\langle\hat{A}(\hat{a}^{\dagger}, \hat{a}, t)\rangle = (u^2 - v^2)^{\frac{1}{2}}\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} exp\Big[-u\alpha^*\beta - \frac{v}{2}(\alpha^{*2} + \beta^2) - \alpha^*\alpha - \beta^*\beta + \beta^*\alpha + \alpha^*\beta\Big]A_n(\alpha, \beta^*). \tag{2.98}$$

3

Photon Statistics

It would be helpful to classify the Photon statistics of light modes based on the relation between the variance of the photon number and the mean photon number. Hence the photon statistics of a light mode for which $(\Delta n)^2 = \bar{n}$ referred to as Poissonian and the photon statistics of a light mode for which $(\Delta n)^2 > \bar{n}$ is called supper Poissonian. Otherwise the photon statistics is said to be sub-Poissonian [3]. Then in this chapter we seek to calculate the mean photon number, variance of the photon number, power spectrum, and photon number distribution for the light produced by one mode subharmonic generation.

3.1 The mean photon number

The mean photon number of a light mode is expressible as

$$\langle \hat{a}^\dagger \hat{a} \rangle = Tr(\hat{\rho}(t)\hat{a}^\dagger \hat{a}). \tag{3.1}$$

With the aid of Eq. (2.97), we can write

$$\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle = \frac{1}{\pi}\int d^2\alpha d^2\beta Q(\alpha^*, \beta, t) \times exp\biggl(- \alpha^*\alpha - \beta^*\beta + \beta^*\alpha + \alpha^*\beta\biggr)\beta^*\alpha, \tag{3.2}$$

so that taking into account Eq. (2.98), we have

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2-v^2)^{\frac{1}{2}}\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} exp\Big[-u\beta\alpha^* - \frac{v}{2}(\beta^2+\alpha^{*2}) \\ -\alpha^*\alpha - \beta^*\beta + \beta^*\alpha + \alpha^*\beta\Big]\beta^*\alpha. \tag{3.3}$$

This can be rewritten as

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2-v^2)^{\frac{1}{2}}\frac{d}{dp}\int \frac{d^2\alpha}{\pi} exp\Big(-\alpha^*\alpha - \frac{v}{2}\alpha^{*2} + p\alpha\Big) \\ \times\frac{d}{dq}\int\frac{d^2\beta}{\pi} exp\Big(-\beta^*\beta + \alpha^*\beta + \beta^*\alpha - u\beta\alpha^* - \frac{v}{2}\beta^2 + q\beta^*\Big)_{p=q=0}. \tag{3.4}$$

Then leads to

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2-v^2)^{\frac{1}{2}}\frac{d}{dp}\int \frac{d^2\alpha}{\pi} exp\Big(-\alpha^*\alpha + p\alpha - \frac{v}{2}\alpha^{*2}\Big) \\ \times\frac{d}{dq}\int\frac{d^2\beta}{\pi} exp\Big(-\beta^*\beta + ((1-u)\alpha^*)\beta + (\alpha+q)\beta^* - \frac{v}{2}\beta^2\Big)_{p=q=0}. \tag{3.5}$$

Upon performing the integration over β and carrying out the differentiation with respect to q, one finds

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2-v^2)^{\frac{1}{2}}\frac{d}{dp}\int \frac{d^2\alpha}{\pi} e^{-u\alpha^*\alpha + p\alpha - \frac{v}{2}(\alpha^2+\alpha^{*2})}\Big((1-u)\alpha^* - v\alpha\Big)_{p=0}. \tag{3.6}$$

This can be written as

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = (u^2-v^2)^{\frac{1}{2}}\Big[(1-u)\frac{d^2}{dpd\gamma}\int\frac{d^2\alpha}{\pi}e^{-u\alpha^*\alpha + p\alpha + \gamma\alpha^* - \frac{v}{2}(\alpha^2+\alpha^{*2})} \\ -v\frac{d^2}{dqd\gamma}\int\frac{d^2\alpha}{\pi}e^{-u\alpha^* + (p+\gamma)\alpha - \frac{v}{2}(\alpha^2+\alpha^{*2})}\Big]_{p=\gamma=0}. \tag{3.7}$$

One can then readily verify that

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{u}{u^2-v^2} - 1, \tag{3.8}$$

so that using Eqs. (2.89) and (2.90), we obtain

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = a - 1. \tag{3.9}$$

And in view of Eq. (2.84), the mean photon number takes the form

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{(\kappa\bar{n}-\varepsilon)}{2\xi_+}(1-e^{-\xi_+ t}) + \frac{(\kappa\bar{n}+\varepsilon)}{2\xi_-}(1-e^{-\xi_- t}). \tag{3.10}$$

Thus at steady state, we see that

$$\langle \hat{a}^\dagger\hat{a}\rangle = \frac{\kappa^2\bar{n}}{\kappa^2-4\varepsilon^2} + \frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2}. \tag{3.11}$$

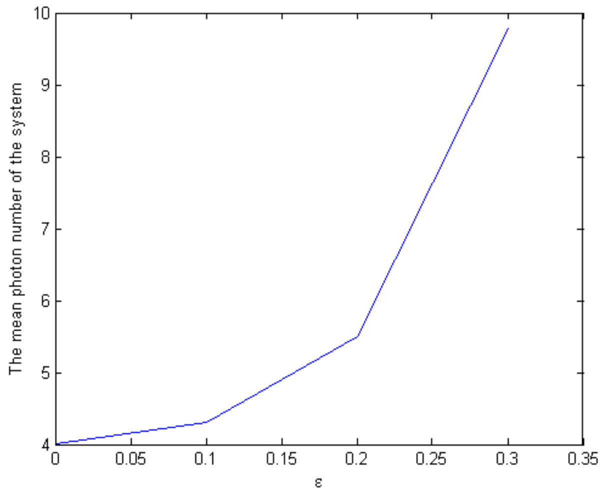

Figure 3.1: A plot of the mean photon number versus ε [Eq. 3.11] for κ=0.8 and $\bar{n}$=10.

This shows that the mean photon number of the system under consideration does not happens to be the sum of the mean photon number of the thermal light and the signal mode. If we consider the case in which $\bar{n}=0$, we see that

$$\langle \hat{a}^\dagger\hat{a}\rangle = \frac{2\varepsilon^2}{\kappa^2-4\varepsilon^2}. \tag{3.12}$$

This is the mean photon number of the signal mode coupled to vacuum reservoir.

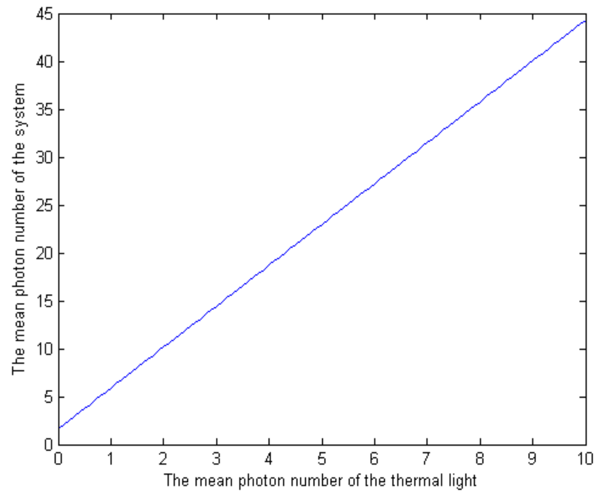

Figure 3.2: A plot of the mean photon number versus $\bar{n}$ [Eq. 3.11] for κ=0.8 and ε=0.35.

The plots in fig. 3.1 and fig. 3.2 shows the mean photon number of the system increases rapidly with increasing ε and $\bar{n}$. Moreover, upon comparing Eqs. (3.11) and (3.12), we see that the intensity of the light produced by the system under consideration increases by $\frac{\kappa^2\bar{n}}{\kappa^2-4\varepsilon^2}$.

3.2 The variance of the photon number

We next proceed to obtain the variance of the photon number of the signal mode. The photon number variance is defined by

$$(\Delta n)^2 = \langle(\hat{a}^\dagger(t)\hat{a}(t))^2\rangle - \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle^2. \tag{3.13}$$

Since $\hat{a}(t)$ is Gaussian variable with zero mean, one can easily check that

$$\langle(\hat{a}^\dagger(t)\hat{a}(t))^2\rangle = 2\langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle^2 + \langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle\hat{a}^{\dagger 2}(t)\rangle\langle\hat{a}^2(t)\rangle. \tag{3.14}$$

The expectation value of $\hat{a}^2(t)$ can be written as

$$\langle\hat{a}^2(t)\rangle = (u^2 - v^2)^{\frac{1}{2}} \int \frac{d^2\alpha}{\pi} exp(-\alpha^*\alpha - \frac{v}{2}\alpha^{*2})$$
$$\times \frac{d}{dp} \int \frac{d^2\beta}{\pi} exp(-\beta^*\beta + ((1-u)\alpha^*)\beta + \alpha\beta^* - (\frac{v}{2} + p)\beta^2)_{p=0}. \quad (3.15)$$

Upon carrying out the the integration and performing the differentiation, we find

$$\langle\hat{a}^2(t)\rangle = (u^2 - v^2)^{\frac{1}{2}} \int \frac{d^2\alpha}{\pi} exp\bigg(-u\alpha^*\alpha + \frac{v}{2}\alpha^2 - \frac{v}{2}\alpha^{*2}\bigg)\alpha^2. \quad (3.16)$$

Then Eq. (3.16) can be rewritten as

$$\langle\hat{a}^2(t)\rangle = (u^2 - v^2)^{\frac{1}{2}} \frac{d}{dp} \int \frac{d^2\alpha}{\pi} exp\bigg(-u\alpha^*\alpha + (\frac{v}{2} + p)\alpha^2 - \frac{v}{2}\alpha^{*2}\bigg)_{p=0}. \quad (3.17)$$

Upon carrying out the integration and performing the differentiation, one can get

$$\langle\hat{a}^2(t)\rangle = b. \quad (3.18)$$

In a similar way, we find

$$\langle\hat{a}^{\dagger 2}(t)\rangle = b. \quad (3.19)$$

On account of (3.18) and (3.19), Eq. (3.14) reduced to

$$\langle(\hat{a}^\dagger(t)\hat{a}(t))^2\rangle = 2\langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle^2 + \langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle + b^2. \quad (3.20)$$

Upon substituting Eq. (3.20) into (3.13), we get

$$(\Delta n)^2 = \langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle^2 + \langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle + b^2. \quad (3.21)$$

In view of Eqs. (3.10) and (2.85), we observe that

$$(\Delta n)^2 = \bigg[\frac{(\kappa\bar{n} - \varepsilon)}{2\xi_+}(1 - e^{-\xi_+ t}) - \frac{(\kappa\bar{n} + \varepsilon)}{2\xi_-}(1 - e^{-\xi_- t})\bigg]^2$$
$$+ \bigg[\frac{(\kappa\bar{n} - \varepsilon)}{2\xi_+}(1 - e^{-\xi_+ t}) + \frac{(\kappa\bar{n} + \varepsilon)}{2\xi_-}(1 - e^{-\xi_- t})\bigg]^2$$
$$+ \frac{(\kappa\bar{n} - \varepsilon)}{2\xi_+}(1 - e^{-\xi_+ t}) + \frac{(\kappa\bar{n} + \varepsilon)}{2\xi_-}(1 - e^{-\xi_- t}). \quad (3.22)$$

Finally, the variance of the photon number, at steady state, takes the form

$$(\Delta n)^2 = \left[\frac{2\kappa\bar{n}\varepsilon + \kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}\right]^2 + \left[\frac{\kappa^2\bar{n}}{\kappa^2 - 4\varepsilon^2} + \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2}\right]\left[\frac{\kappa^2\bar{n}}{\kappa^2 - 4\varepsilon^2} + \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + 1\right]. \quad (3.23)$$

We observe from Eqs. (3.11) and (3.23) that the photon statistics of the light produced by one-mode subharmonic generator coupled to thermal reservoir is super-Poissonian. One can easily check that the variance of the photon number of the signal mode coupled to vacuum reservoir is found to be

$$(\Delta n)^2 = \left[\frac{\kappa\varepsilon}{\kappa^2 - 4\varepsilon^2}\right]^2 + \left[\frac{2\varepsilon}{\kappa^2 - 4\varepsilon^2}\right]^2 + \left[\frac{2\varepsilon}{\kappa^2 - 4\varepsilon^2}\right]. \quad (3.24)$$

3.3 Power spectrum

In nearly all cases the frequency of a single-mode light is not sharply defined. In general, there is some variation about the central frequency. We wish here to obtain the spectrum of the mean photon number, usually known as the power spectrum, of a light mode represented by the operators $\hat{a}$ and $\hat{a}^\dagger$. We define the power spectrum of a single-mode light with central frequency ω_0 by [3]

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle(\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau. \quad (3.25)$$

We now proceed to determine the two time correlation function that appears in Eq. (3.25) for the cavity light. To this end, we realize that Eq. (2.62) can be expressible as

$$\alpha(t+\tau) = F_+(\tau)\alpha(t) + F_-(\tau)\alpha^*(t) + G_+(t+\tau) - G_-(t+\tau). \quad (3.26)$$

Then multiplying both sides of this on the left by $\alpha^*(t)$ and taking the expectation value of the resulting equation, we have

$$\langle\alpha^*(t)\alpha(t+\tau)\rangle_{ss} = F_+(\tau)\langle\alpha^*(t)\alpha(t)\rangle_{ss} + F_-(\tau)\langle\alpha^{*2}(t)\rangle_{ss}$$
$$+\langle\alpha^*(t)G_+(t+\tau)\rangle - \langle\alpha^*(t)G_-(t+\tau)\rangle, \quad (3.27)$$

so that in view of Eq. (2.63) and the fact that

$$\langle \alpha^*(t)G_+(t+\tau)\rangle = \langle \alpha^*(t)G_-(t+\tau)\rangle = 0, \tag{3.28}$$

there follows

$$\begin{aligned}\langle \alpha^*(t)\alpha(t+\tau)\rangle_{ss} = &\langle \alpha^*(t)\alpha(t)\rangle_{ss}\frac{1}{2}\left[e^{-\frac{1}{2}(\kappa+2\varepsilon)\tau} + e^{-\frac{1}{2}(\kappa-2\varepsilon)\tau}\right]\\ &+\langle \alpha^{*2}(t)\rangle_{ss}\frac{1}{2}\left[e^{-\frac{1}{2}(\kappa+2\varepsilon)\tau} - e^{-\frac{1}{2}(\kappa-2\varepsilon)\tau}\right],\end{aligned} \tag{3.29}$$

in which $\langle \alpha^*(t)\alpha(t+\tau)\rangle_{ss}$ is the c-number function corresponding to $\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}$ in the normal order. On account of Eqs. (2.81) and (2.82), we can write Eq. (3.29) at steady state as

$$\langle \alpha^*(t)\alpha(t+\tau)\rangle_{ss} = \frac{(\kappa\bar{n}-\varepsilon)}{2(\kappa+2\varepsilon)}e^{-\frac{1}{2}(\kappa+2\varepsilon)\tau} + \frac{(\kappa\bar{n}+\varepsilon)}{2(\kappa-2\varepsilon)}e^{-\frac{1}{2}(\kappa-2\varepsilon)\tau}. \tag{3.30}$$

Substituting Eq. (3.30) into Eq. (3.25), we get

$$\begin{aligned}P(\omega) = &\frac{1}{2\pi}\left(\frac{\kappa\bar{n}-\varepsilon}{\kappa+2\varepsilon}\right)Re\int_0^\infty d\tau\, e^{-\left(\frac{1}{2}(\kappa+2\varepsilon)-i(\omega-\omega_0)\right)\tau}\\ &+\frac{1}{2\pi}\left(\frac{\kappa\bar{n}+\varepsilon}{\kappa-2\varepsilon}\right)Re\int_0^\infty d\tau\, e^{-\left(\frac{1}{2}(\kappa-2\varepsilon)-i(\omega-\omega_0)\right)\tau}.\end{aligned} \tag{3.31}$$

Upon carrying out the integration by employing the identity

$$P(\eta) = \frac{1}{\pi}Re\int_0^\infty dz\, e^{-\left(\frac{\Gamma}{2}-i(\eta-\eta_0)\right)z} \equiv \frac{\frac{\Gamma}{2\pi}}{(\eta-\eta_0)^2+(\frac{\Gamma}{2})^2}, \tag{3.32}$$

we find

$$\begin{aligned}P(\omega) = &\frac{1}{2\pi}\left(\frac{\kappa\bar{n}-\varepsilon}{\kappa+2\varepsilon}\right)\left[\frac{\frac{(\kappa+2\varepsilon)}{2}}{(\omega-\omega_0)^2+(\frac{\kappa+2\varepsilon}{2})^2}\right]\\ &+\frac{1}{2\pi}\left(\frac{\kappa\bar{n}+\varepsilon}{\kappa-2\varepsilon}\right)\left[\frac{\frac{(\kappa-2\varepsilon)}{2}}{(\omega-\omega_0)^2+(\frac{\kappa-2\varepsilon}{2})^2}\right].\end{aligned} \tag{3.33}$$

Upon integrating both sides of Eq. (3.33) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega)d\omega = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} \tag{3.34}$$

in which

$$\langle \hat{a}^\dagger \hat{a} \rangle_{ss} = \frac{\kappa^2 \bar{n}}{\kappa^2 - 4\varepsilon^2} + \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2}, \tag{3.35}$$

is the steady state mean photon number of the one-mode subharmonic generator coupled to thermal reservoir. From this result, we observe that $P(\omega)d\omega$ is the steady state mean photon number in the interval between ω and $\omega + d\omega$.

We next seek to calculate the mean photon number in a given frequency interval. We thus realize that the steady state mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ can be written as [3]

$$\langle \hat{a}^\dagger \hat{a} \rangle_{\pm\lambda} = \int_{-\lambda}^{+\lambda} P(\omega') d\omega', \tag{3.36}$$

where $\omega' = \omega - \omega_0$. Therefore, using Eq. (3.33) and the fact that [3]

$$\int_{-\lambda}^{+\lambda} \frac{d\omega'}{\omega^2 + a^2} = \frac{2}{a} tan^{-1}(\frac{\lambda}{a}), \tag{3.37}$$

we readily obtain

$$\langle \hat{a}^\dagger \hat{a} \rangle_{\pm\lambda} = \langle \hat{a}^\dagger \hat{a} \rangle_{ss} z(\lambda), \tag{3.38}$$

where

$$z(\lambda) = \frac{1}{\pi(\kappa^2 \bar{n} + 2\varepsilon^2)} \Bigg[(\kappa\bar{n} - \varepsilon)(\kappa - 2\varepsilon) tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon}\right) + (\kappa\bar{n} + \varepsilon)(\kappa + 2\varepsilon) tan^{-1}\left(\frac{2\lambda}{\kappa - 2\varepsilon}\right) \Bigg]. \tag{3.39}$$

On the other hand, when $\bar{n} = 0$, $z(\lambda)$ tends to

$$z(\lambda) = \frac{1}{2\pi\varepsilon} \left[(\kappa + 2\varepsilon) tan^{-1}\left(\frac{2\lambda}{\kappa - 2\varepsilon}\right) - (\kappa - 2\varepsilon) tan^{-1}\left(\frac{2\lambda}{\kappa + 2\varepsilon}\right) \right]. \tag{3.40}$$

One can easily get from fig. 3.3 that

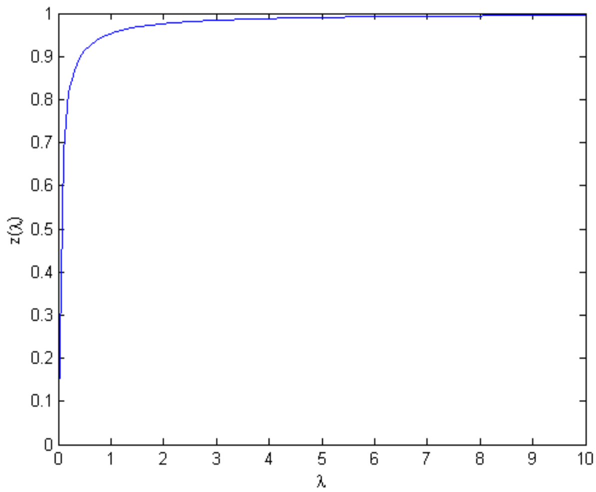

Figure 3.3: A plot of $z(\lambda)$ versus λ [Eq. 3.39] for κ=0.8 and $\bar{n}$=2.

$z(0.2) = 0.8165$, $z(0.8) = 0.9425$, $z(2.2) = 0.9779$, $z(3.6) = 0.9865$, $z(4.8) = 0.9899$. Then combination of these results with Eq. (3.38) yields

$\langle\hat{a}^{\dagger}\hat{a}\rangle_{\pm 0.2} = 0.8165\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}$, $\langle\hat{a}^{\dagger}\hat{a}\rangle_{\pm 0.8} = 0.9425\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}$,

$\langle\hat{a}^{\dagger}\hat{a}\rangle_{\pm 2.2} = 0.9779\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}$, $\langle\hat{a}^{\dagger}\hat{a}\rangle_{\pm 3.6} = 0.9865\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}$,

$\langle\hat{a}^{\dagger}\hat{a}\rangle_{\pm 4.8} = 0.9899\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}$.

We observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

3.4 The photon number distribution

We wish to obtain the explicit expression for the photon number distribution employing the Q function along with the density operator for the signal mode coupled to thermal reservoir. The photon number distribution for the single mode light can be defined as

$$P(n) = \langle n|\hat{\rho}(\hat{a}^{\dagger}, \hat{a}, t)|n\rangle. \tag{3.41}$$

Introducing Eq. (2.95) into (3.41), we see that

$$P(n) = \frac{1}{\pi}\int d^2z d^2\eta Q(z^*,\eta,t)\langle n|z\rangle\langle\eta|n\rangle\langle z|\eta\rangle. \tag{3.42}$$

Now using the Q function described by Eq. (2.88), Eq. (3.42) can be rewritten as

$$\begin{aligned} P(n) = &\frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\int\frac{d^2z}{\pi}exp\left[-z^*z-\frac{v}{2}z^{*2}+\alpha^*z\right] \\ &\times\int\frac{d^2\eta}{\pi}exp\left[-\eta^*\eta-\frac{v}{2}\eta^2-uz^*\eta+z^*\eta+\alpha\eta^*\right]_{\alpha^*=\alpha=0}, \end{aligned} \tag{3.43}$$

where

$$\langle z|\eta\rangle = e^{-\frac{z^*z}{2}-\frac{\eta^*\eta}{2}+z^*\eta} \tag{3.44}$$

$$\langle n|z\rangle = e^{\frac{-z^*z}{2}}\frac{z^n}{\sqrt{n!}}, \tag{3.45}$$

and

$$\langle\eta|\ n\rangle = e^{\frac{-\eta^*\eta}{2}}\frac{\eta^{*n}}{\sqrt{n!}}. \tag{3.46}$$

Upon carrying out the integration, we readily obtain

$$P(n) = \frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}exp\left[(1-u)\alpha^*\alpha-\frac{v}{2}(\alpha^{*2}+\alpha^2)\right]_{\alpha^*=\alpha=0}. \tag{3.47}$$

Upon expanding the exponential functions in power series, we have

$$\begin{aligned} P(n,t) = &\frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\sum_{l,k,p}\frac{(-1)^{(k+p)}(1-u)^l v^{k+p}}{2^{k+p}l!k!p!} \\ &\times\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}(\alpha^{*l+2k}\alpha^{l+2p})_{\alpha^*=\alpha=0}. \end{aligned} \tag{3.48}$$

Upon performing the differentiation by employing the relation

$$\frac{\partial^m}{\partial\alpha^m}x^n = \frac{n!}{(n-m)!}x^{n-m}, \tag{3.49}$$

we notice that

$$\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^n}\alpha^{*l+2k}\alpha^{l+2p} = \frac{(l+2k)!\alpha^{*l+2k-n}}{(l+2k-n)!}\frac{(l+2p)!\alpha^{l+2p-n}}{(l+2p-n)!}. \tag{3.50}$$

Thus the combination of Eq. (3.41) and (3.43) leads to

$$P(n,t) = \frac{(u^2-v^2)^{\frac{1}{2}}}{n!} \sum_{l,k,p} \frac{(-1)^{(k+p)}(1-u)^l v^{(k+p)}(l+2k)!(l+2p)!}{2^{k+p} l! k! p! (l+2k-n)!(l+2p-n)!} \times (\alpha^{*(l+2k-n)} \alpha^{(l+2p-n)})_{\alpha^*=\alpha=0}. \tag{3.51}$$

Imposing the condition $\alpha^* = \alpha = 0$, we find

$$P(n,t) = \frac{(u^2-v^2)^{\frac{1}{2}}}{n!} \sum_{l,k,p} \frac{(-1)^{(k+p)}(1-u)^l v^{k+p}(l+2k)!(l+2p)!}{2^{k+p} l! k! p! (l+2k-n)!(l+2p-n)!} \times \delta_{l+2k,n} \delta_{l+2p,n}. \tag{3.52}$$

Finally, in view of the fact that $p = k$ and $l = n - 2k$, the photon number distribution can be written as

$$P(n) = (u^2-v^2)^{\frac{1}{2}} \sum_{k=0}^{[n]} \frac{n!(1-u)^{n-2k} v^{2k}}{2^{2k}(k!)^2(n-2k)!}, \tag{3.53}$$

where $[n] = n/2$ for even n and $[n] = (n-1)/2$ for odd n. From this result, we note that there is a finite probability to find odd number of signal photons. Although the signal photons are generated in pairs, it is possible for an odd number of signal photons to leave the cavity via the port mirror. This must be then the reason for the possibility to observe an odd number of signal photons inside the cavity [3].

4

Quadrature Fluctuations

4.1 Quadrature variance

We now proceed to calculate the variance of the plus and minus quadratures of the signal mode produced by the one-mode subharmonic generator coupled to thermal reservoir. The variance of the plus and minus quadratures is expressible as

$$(\Delta a_{\pm})^2 = 1 + \langle : (\hat{a}_{\pm}(t), \hat{a}_{\pm}(t)) : \rangle, \tag{4.1}$$

where :: stands for normal ordering, $\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a}$ and $\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a})$ are respectively the plus and minus quadrature operators.

Since $\hat{a}(t)$ is Gaussian variable with zero mean, Eq. (4.1) can be rewritten as

$$(\Delta a_{\pm})^2 = 1 + 2\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle \pm 2\langle \hat{a}^2(t)\rangle. \tag{4.2}$$

In view of Eqs. (2.81) and (3.10), we readily obtain

$$(\Delta a_{\pm})^2 = 1 + \frac{2\kappa\bar{n} \mp 2\varepsilon}{\kappa \pm 2\varepsilon}, \tag{4.3}$$

is the steady-state quadrature variance of the signal-mode coupled to thermal reservoir.

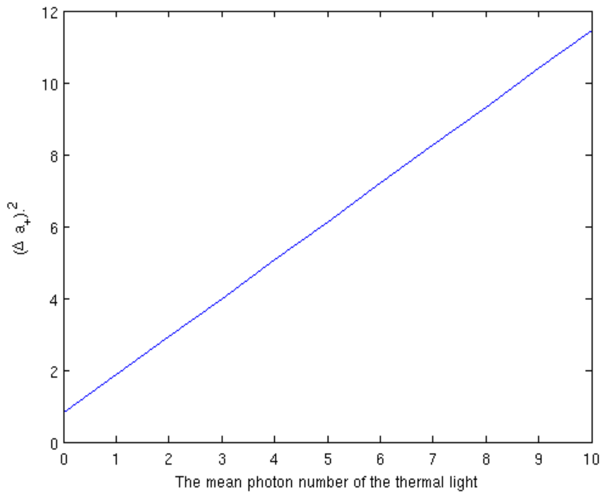

Figure 4.1: A plot of $(\Delta a_+)^2$ versus $\bar{n}$ [Eq. 4.3] for κ=0.8 and $\varepsilon = 0.35$.

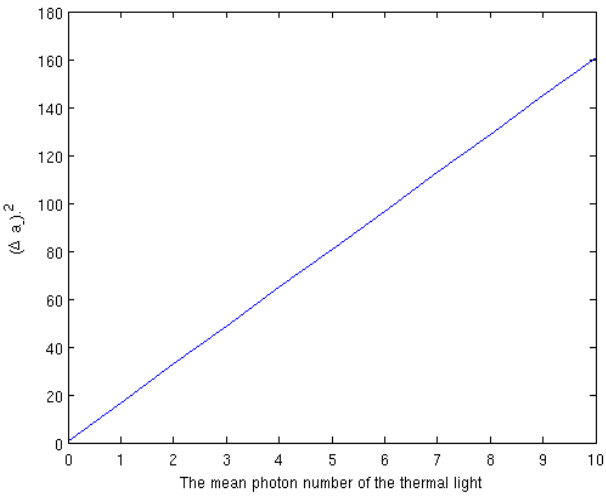

Figure 4.2: A plot of $(\Delta a_-)^2$ versus $\bar{n}$ [Eq. 4.3] for κ=0.8 and $\varepsilon = 0.35$.

We immediately observe from fig. 4.1 and fig. 4.2 that the signal-mode coupled to thermal reservoir is not in a squeezed state. In light of this, the squeezing of the signal mode indeed affected by the presence of the thermal light. It is practically seen that squeezing in optical fibers is limited by phase noise associated with thermal fluctuations of the refractive index [28].

Upon setting $\bar{n} = 0$, we observe that

$$(\Delta a_{\pm})^2 = 1 \mp \frac{2\varepsilon}{\kappa \pm 2\varepsilon}. \tag{4.4}$$

We see that the signal mode is in a squeezed state and the squeezing occurs in the plus quadrature.

5

Conclusion

We have obtained the master equation and c-number Langevin equation for a one-mode subharmonic generator coupled to thermal reservoir. Applying the solution of the resulting c-number Langevin equation, we have obtained the anti-normally ordered characteristic function. By using this characteristic function, we have determined the Q function as well as the density operator for the light produced by the one-mode subharmonic generator coupled to thermal reservoir.

Furthermore, employing the Q function together with the density operator, we have calculated the mean photon number, the variance of the photon number, the power spectrum, the photon number distribution, and the quadrature variance. We have found that the intensity of the light produced by the system under consideration increases due to the thermal light. On the other hand, the squeezing of the subharmonic light indeed affected by the presence of the thermal light. Moreover, we have clearly shown that the mean photon number is confined in a relatively small frequency interval.

6

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DECLARATION

I hereby declare that this M.Sc. thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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This M.Sc thesis has been submitted for examination with our approval as University advisor and co-advisor.

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