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Photonic Entanglement Analysis for a Pair of Superposed Twin Beams Squeezed State

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Abstract

In this thesis, we have analyzed a pair of superposed twin squeezed states with the same or different frequency. We have found that the mean photon number for a pair of superposed twin squeezed states is the sum of that of the separate squeezed states. However, the variance of the photon number for a pair of superposed twin squeezed states does not happens to be the sum of that of the constituent light beams. And the quadrature variance of superposed light beams is 2 times that of the separate light beams. Finally, we have obtained that the maximum quadrature squeezing for both separate and superposed light beams is 50% below the vacuum state level and photons in the superposed states are entangled and highly correlated.

Keywords: Twin beams, Density operator, Entanglement, and quadrature squeezing.

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Chapter 1

Introduction

Quantum optics is one of the liveliest fields in physics at present[1]. Quantum optics deals mainly with the quantum properties of the light generated by various optical systems such as lasers and with the effect of light on the dynamics of atoms. The quantum properties of light are largely determined by the state of the light mode and the most important quantum states of light are the number state, the chaotic state, the coherent, and squeezed states[2,3].

Squeezing state of light has played a crucial role in the development of quantum optics. Squeezing is one of the non classical features of light that have been extensively studied by several authors[4,5]. Squeezed states have several potential applications such as in low-noise communications, precise measurements and detection of weak signals[6].

A general class of minimum uncertainty states is known as squeezed state. In general, squeezed state may have less noise in one quadrature than coherent state. To satisfy the requirements of a minimum uncertainty state the noise in the other quadrature is greater than that of the coherent state. The coherent state are a particular member of this more general class of minimum uncertainty states with the equal noise in both quadratures[7]. The squeezed states can be generated in non-linear optical process such as degenerate or non-degenerate amplifiers[8].

Quantum entanglement is an important role in quantum computation and communication. It allows us to teleport quantum states and reduces necessary numbers of qubits for communication. One of the most fundamentally interesting phenomena associated with a composite quantum system is called entanglement. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles are correlated in ways such that the quantum states of each particle cannot be described independently instead, a quantum state must be described for the system as a whole. Quantum entanglement is one of the central principles of quantum physics, which is the science of sub-atomic particles, Multiple particles, such as photons, are connected with each other even when they are very far apart and what happens to one particle can have an effect on the other one at the same moment, even though these effects can not be used to send information faster than light[9].

The signal and idler beams from a non-degenerate parametric amplifier have a strong quantum correlation, which are called twin beams[10]. The statistical and squeezing properties of the twin light beams with the same or different frequencies (each light beam consisting of one photon from each pair) have been investigated[11]. It is found that the twin light beams are in squeezed states, with the maximum quadrature squeezing being 50% below the vacuum state level. We recall that the twin light beams with the same frequency are represented in the conventional Hamiltonian by $\hat{a^2}$ and $\hat{a^{\dagger 2}}$, for a given pump mode and the generated twin beams have exactly the same photon statics[11].

The squeezing properties of two mode light is described by two Hermitian quadrature operators $\hat{c_+}$ and $\hat{c_-}$, satisfying the computation relation

$$[\hat{c_+}, \hat{c_-}] = 4i.$$

A two mode light is said to be squeezed if

$$(\Delta c_+)^2 < 2,$$

or

$$(\Delta c_-)^2 < 2,$$

such that

$$\Delta c_+ \Delta c_- \geq 2.$$

A non-degenerate parametric amplifier is a typical source of two mode squeezed light. In this system a pump photon of frequency ω_c is down converted into highly correlated signal and idler photons of frequency ω_a and ω_b, such that $\omega_c = \omega_a + \omega_b$. Such process can be described by the Hamiltonian

$$\hat{H} = i\lambda\eta \left(\hat{D}^\dagger \hat{a}_1 \hat{a}_2 - \hat{D} \hat{a_1^\dagger} \hat{a_2^\dagger} \right),$$

where $\hat{a}_1$ and $\hat{a_2}$ are annihilation operators for the light modes, $\hat{D}$ is the annihilation operator for the pump mode, λ is the coupling constant, and η is proportional to the amplitude of the pump mode.

In this thesis, we study the photon statistics and quadrature fluctuation of the light produced by superposed light beams. We first obtain the, Q function for twin beams squeezed state. The Q function is obtained with the aide of anti normally ordered characteristics function defined in the Heisenberg picture. Applying the Q function, we calculate photon statistics, quadrature fluctuation and entanglement analysis of twin beams squeezed state, by finding the density operator for a pair of superposed light beams.

In addition, we also determine the Q function for the superposition of light beams and employing the resulting density operator, we calculate photon statistics, quadrature fluctuation and entanglement analysis for pair of superposed light beams. We consider a quantum system with Gaussian variables with zero mean[12].

Chapter 2

Operator Dynamics

Aim: We establish the photon statistics and quadrature squeezing of the light beams procedured by the subharmonic generation.

A non degenerate parametric amplifier is a typical source of a two-mode squeezed light. In this system a pump photon of frequency ω_c is down converted into highly correlated signal and idler photons of frequency ω_a and ω_b, such that $\omega_c = \omega_a + \omega_b$[2].

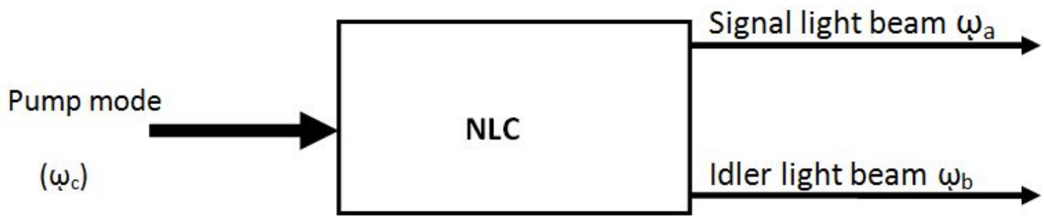

Figure 2.1: The physical scheme for subharmonic generators.

The process of subharmonic generation is described by the Hamiltonian with the pump mode treated classically [2]

$$\hat{H} = i\varepsilon\left(\hat{a}_1\hat{a}_2 - \hat{a}_1{}^{\dagger}\hat{a}_2{}^{\dagger}\right), \tag{2.0.1}$$

where $\hat{a_1}$ ($\hat{a_2}$) is the annihilation operators for the signal (idler) modes and ε=$\eta\beta$, in which η is the cavity coupling constant and β is the amplitude of the pump mode. Applying Eq. (2.0.1), the equation of evolution for the density operator can be written as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \varepsilon\left(\hat{a_1}\hat{a_2}\hat{\rho} - \hat{\rho}\hat{a_1}\hat{a_2} + \hat{\rho}\hat{a_1}^\dagger\hat{a_2}^\dagger - \hat{a_1}^\dagger\hat{a_2}^\dagger\hat{\rho}\right) \\ &+\frac{\kappa}{2}\left(2\hat{a_1}\hat{\rho}\hat{a_1}^\dagger - \hat{a_1}^\dagger\hat{a_1}^\dagger\hat{\rho} - \hat{\rho}\hat{a_1}^\dagger\hat{a_1}^\dagger\right) \\ &+\frac{\kappa}{2}\left(2\hat{a_2}\hat{\rho}\hat{a_2}^\dagger - \hat{a_2}^\dagger\hat{a_2}^\dagger\hat{\rho} - \hat{\rho}\hat{a_2}^\dagger\hat{a_2}^\dagger\right), \end{aligned} \qquad (2.0.2)$$

where κ is the cavity damping constant for light modes $\hat{a_1}$ and $\hat{a_2}$. Then by setting κ =0, Eq. (2.0.2) reduced to

$$\frac{d\hat{\rho}}{dt} = \varepsilon\left(\hat{a_1}\hat{a_2}\hat{\rho}\hat{a_1} - \hat{\rho}\hat{a_1}\hat{a_2}\hat{a_1} + \hat{\rho}\hat{a_1}^\dagger\hat{a_2}^\dagger\hat{a_1} - \hat{a_1}^\dagger\hat{a_2}^\dagger\hat{\rho}\hat{a_1}\right) \qquad (2.0.3)$$

In the Schroedinger picture, the time evolution of the expectation value in terms of the density operator is expressible as

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right). \qquad (2.0.4)$$

Introducing Eq. (2.0.3) into (2.0.4) , we see that

$$\frac{d}{dt}\langle\hat{a_1}(t)\rangle = \varepsilon Tr\left(\hat{a_1}\hat{a_2}\hat{\rho}\hat{a_1} - \hat{\rho}\hat{a_1}\hat{a_2}\hat{a_1} + \hat{\rho}\hat{a_1^\dagger}\hat{a_2^\dagger}\hat{a_1} - \hat{a_1^\dagger}\hat{a_2^\dagger}\hat{\rho}\hat{a_1}\right). \qquad (2.0.5)$$

Then applying the cyclic property of trace operation, the equation of evolution for the expectation value of mode $\hat{a}_1$ is expressible as

$$\frac{d}{dt}\langle\hat{a_1}(t)\rangle = \varepsilon Tr\left(\hat{\rho}\left[\hat{a_1}^\dagger, \hat{a_1}\right]\hat{a_2}^\dagger\right). \qquad (2.0.6)$$

Now using the commutation relation

$$\left[\hat{a_1}, \hat{a_1^\dagger}\right] = \left[\hat{a_2}, \hat{a_2^\dagger}\right] = 1, \qquad (2.0.7)$$

$$\left[\hat{a_1}^\dagger, \hat{a_2}^\dagger\right] = \left[\hat{a_1}, \hat{a_2}^\dagger\right] = \left[\hat{a_2}^\dagger, \hat{a_1}\right] = 0 \qquad (2.0.8)$$

and substituting Eq. (2.0.7) into (2.0.6) , we get

$$\frac{d}{dt}\langle\hat{a_1}(t)\rangle = -\varepsilon\langle\hat{a_2}^\dagger(t)\rangle. \qquad (2.0.9)$$

Following a similar procedure, we can also show that

$$\frac{d}{dt}\langle \hat{a_2}(t)\rangle = -\varepsilon \langle \hat{a_1}^\dagger(t)\rangle. \tag{2.0.10}$$

Using the Hermitian adjoint of Eqs. (2.0.9) and (2.0.10) , one finds

$$\frac{d}{dt}\langle \hat{a_1}^\dagger(t)\rangle = -\varepsilon \langle \hat{a_2}(t)\rangle, \tag{2.0.11}$$

$$\frac{d}{dt}\langle \hat{a_2}^\dagger(t)\rangle = -\varepsilon \langle \hat{a_1}(t)\rangle. \tag{2.0.12}$$

We introduce a new operators defined by

$$\hat{A}_1(t) = \hat{a_1}(t) + \hat{a_2}^\dagger(t) \tag{2.0.13}$$

and

$$\hat{A}_2(t) = \hat{a_1}(t) - \hat{a_2}^\dagger(t). \tag{2.0.14}$$

Now from Eqs. (2.0.13) and (2.0.14) , we see that

$$\hat{a_1}(t) = \frac{\hat{A}_1(t)}{2} + \frac{\hat{A}_2(t)}{2} \tag{2.0.15}$$

and

$$\hat{a_2}^\dagger(t) = \frac{\hat{A}_1(t)}{2} - \frac{\hat{A}_2(t)}{2}. \tag{2.0.16}$$

Introducing Eqs. (2.0.15) and (2.0.16) into (2.0.9) , one can readily verify that

$$\frac{d}{dt}\langle \hat{A}_1(t)\rangle + \frac{d}{dt}\langle \hat{A}_2(t)\rangle = -\varepsilon\langle \hat{A}_1(t)\rangle + \varepsilon\langle \hat{A}_2(t)\rangle. \tag{2.0.17}$$

Similarly, one can check that

$$\frac{d}{dt}\langle \hat{A}_1(t)\rangle - \frac{d}{dt}\langle \hat{A}_2(t)\rangle = -\varepsilon\langle \hat{A}_1(t)\rangle - \varepsilon\langle \hat{A}_2(t)\rangle. \tag{2.0.18}$$

In views of Eqs. (2.0.17) and (2.0.18) , we find that

$$\frac{d}{dt}\langle \hat{A}_1(t)\rangle = -\varepsilon\langle \hat{A}_1(t)\rangle. \tag{2.0.19}$$

The solution of Eq. (2.0.19) can be written as

$$\langle \hat{A}_1(t)\rangle = \langle \hat{A}_1(0)\rangle e^{-\varepsilon t}. \tag{2.0.20}$$

It can be shown in a similar manner that

$$\langle \hat{A}_2(t) \rangle = \langle \hat{A}_2(0) \rangle e^{\varepsilon t}. \tag{2.0.21}$$

Applying Eqs. (2.0.20) and (2.0.21) into (2.0.15) , one can readily get

$$\langle \hat{a}_1(t) \rangle = \frac{\langle \hat{A}_1(0) \rangle}{2} e^{-\varepsilon t} + \frac{\langle \hat{A}_2(0) \rangle}{2} e^{\varepsilon t}. \tag{2.0.22}$$

Substituting Eqs. (2.0.13) and (2.0.14) into (2.0.22), we have

$$\langle \hat{a}_1(t) \rangle = \frac{\langle \hat{a}_1(0) \rangle + \langle \hat{a}_2{}^\dagger(0) \rangle}{2} e^{-\varepsilon t} + \frac{\langle \hat{a}_1(0) \rangle - \langle \hat{a}_2{}^\dagger(0) \rangle}{2} e^{\varepsilon t}. \tag{2.0.23}$$

Then the solution of the above differential equation can be establish that

$$\langle \hat{a}_1(t) \rangle = \langle \hat{a}_1(0) \rangle \cosh \varepsilon t - \langle \hat{a}_2{}^\dagger(0) \rangle \sinh \varepsilon t. \tag{2.0.24}$$

Following a similarly procedure,

$$\langle \hat{a}_2(t) \rangle = \langle \hat{a}_2(0) \rangle \cosh \varepsilon t - \langle \hat{a}_1{}^\dagger(0) \rangle \sinh \varepsilon t, \tag{2.0.25}$$

$$\langle \hat{a}_1{}^\dagger(t) \rangle = \langle \hat{a}_1{}^\dagger(0) \rangle \cosh \varepsilon t - \langle \hat{a}_2(0) \rangle \sinh \varepsilon t, \tag{2.0.26}$$

$$\langle \hat{a}_2{}^\dagger(t) \rangle = \langle \hat{a}_2{}^\dagger(0) \rangle \cosh \varepsilon t - \langle \hat{a}_1(0) \rangle \sinh \varepsilon t, \tag{2.0.27}$$

where

$$\cosh \varepsilon t = \frac{e^{\varepsilon t} + e^{-\varepsilon t}}{2} \tag{2.0.28}$$

and

$$\sinh \varepsilon t = \frac{e^{\varepsilon t} - e^{-\varepsilon t}}{2}. \tag{2.0.29}$$

2.1 The Q function

The Q function is an important tool in quantum optics. Knowing this function, all the non classical effect can be predicted and the different moments of the operators can be evaluated[13]. We now proceed to determine the Q function for the signal-idler modes. The Q function for two-mode light beams is expressible as

$$Q(\alpha_1, \alpha_2, t) = \frac{1}{\pi^4} \int d^2 z d^2 \eta \phi_a(z, \eta, t) exp\left[z^* \alpha_1 - z\alpha_1^* + \eta^* \alpha_2 - \eta\alpha_2^*\right], \quad (2.1.1)$$

in which the anti-normally-ordered characteristics function $\phi_a(z, \eta, t)$ is defined in the Heisenberg picture by

$$\phi_a(z, \eta, t) = Tr(\hat{\rho}(0) e^{-z^* \hat{a_1}(t)} e^{z\hat{a_1}^\dagger(t)} e^{-\eta^* \hat{a_2}(t)} e^{\eta \hat{a_2}^\dagger(t)}). \quad (2.1.2)$$

Employing Baker-Hausdorff identity

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \quad (2.1.3)$$

we see that

$$\begin{aligned} \phi_a(z, \eta, t) &= exp[-\frac{1}{2}(z^* z + \eta^* \eta)] \\ & \quad Tr(\hat{\rho}(0) exp[z\hat{a_1}^\dagger(t) - z^* \hat{a_1}(t) + \eta \hat{a_2}^\dagger(t) - \eta^* \hat{a_2}(t)]), \end{aligned} \quad (2.1.4)$$

so that on account of (2.0.24) , (2.0.25) , (2.0.26) , and (2.0.27) , we obtain

$$\begin{aligned} \phi_a(z, \eta, t) &= exp[-\frac{1}{2}(z^* z + \eta^* \eta)] Tr(\hat{\rho}(0) exp[(z \cosh \varepsilon t + \eta^* \sinh \varepsilon t)\hat{a_1}^\dagger(0) \\ & \quad -(z^* \cosh \varepsilon t + \eta \sinh \varepsilon t)\hat{a_1}(0)] exp[(\eta \cosh \varepsilon t + z^* \sinh \varepsilon t)\hat{a_2}^\dagger(0) \\ & \quad -(\eta^* \cosh \varepsilon t + z \sinh \varepsilon t)\hat{a_2}(0)]). \end{aligned} \quad (2.1.5)$$

Applying once more (2.1.3) , we easily find

$$\begin{aligned} \phi_a(z, \eta, t) &= exp[-(z^* z + \eta^* \eta)\cosh^2 \varepsilon t - (z\eta + z^* \eta^*) \cosh \varepsilon t \sinh \varepsilon t] \\ & \quad \times Tr\big(\hat{\rho}(0) exp(z \cosh \varepsilon t + \eta^* \sinh \varepsilon t)\hat{a_1}^\dagger(0) \\ & \quad \times exp(\eta \cosh \varepsilon t + z^* \sinh \varepsilon t)\hat{a_2}^\dagger(0) \\ & \quad \times exp[-(\eta \sinh \varepsilon t + z^* \cosh \varepsilon t)\hat{a_1}(0)] \\ & \quad \times exp[-(z \sinh \varepsilon t + \eta^* \cosh \varepsilon t)\hat{a_2}(0)]\big) \end{aligned} \quad (2.1.6)$$

and assuming the signal-idler modes to be initially in a two modes coherent state, we have

$$\hat{\rho}(0) = |\gamma_1, \gamma_2\rangle\langle\gamma_1, \gamma_2|, \quad (2.1.7)$$

so that one can write

$$\begin{aligned}\phi_a(z,\eta,t) &= exp[-(z^*z+\eta^*\eta)\cosh^2\varepsilon t-(z\eta+z^*\eta^*)\cosh\varepsilon t\sinh\varepsilon t]\\ &\times\langle\gamma_1,\gamma_2|exp(z\cosh\varepsilon t+\eta^*\sinh\varepsilon t)\hat{a_1}^\dagger(0)\\ &\times exp(\eta\cosh\varepsilon t+z^*\sinh\varepsilon t)\hat{a_2}^\dagger(0)\\ &\times exp(\eta\sinh\varepsilon t+z^*\cosh\varepsilon t)\hat{a_1}(0)\\ &\times exp[-(z\sinh\varepsilon t+\eta^*\cosh\varepsilon t)\hat{a_2}(0)]|\gamma_1,\gamma_2\rangle.\end{aligned}\tag{2.1.8}$$

It then follows that

$$\begin{aligned}\phi_a(z,\eta,t) &= exp[-(z^*z+\eta^*\eta)\cosh^2\varepsilon t-(z\eta+z^*\eta^*)\cosh\varepsilon t\sinh\varepsilon t\\ &+(\gamma_{a_1}^*\cosh\varepsilon t-\gamma_{a_2}\sinh\varepsilon t)z-(\gamma_{a_1}\cosh\varepsilon t-\gamma_{a_2}^*\sinh\varepsilon t)z^*\\ &+(\gamma_{a_2}^*\cosh\varepsilon t-\gamma_{a_1}\sinh\varepsilon t)\eta-(\gamma_{a_2}\cosh\varepsilon t-\gamma_{a_1}^*\sinh\varepsilon t)\eta^*].\end{aligned}\tag{2.1.9}$$

Furthermore, substituting (2.1.9) into (2.1.1) leads to

$$\begin{aligned}Q(\alpha_1,\alpha_2,t) &= \frac{1}{\pi^4}\int d^2zd^2\eta exp[-(zz^*+\eta\eta^*)\cosh^2\varepsilon t\\ &-(z\eta+z^*\eta^*)\cosh\varepsilon t\sinh\varepsilon t\\ &+(\gamma_{a_1}^*\cosh\varepsilon t-\gamma_{a_2}\sinh\varepsilon t-\alpha_1^*)z\\ &-(\gamma_{a_1}\cosh\varepsilon t-\gamma_{a_2}^*\sinh\varepsilon t-\alpha_1)z^*\\ &+(\gamma_{a_2}^*\cosh\varepsilon t-\gamma_{a_1}\sinh\varepsilon t-\alpha_2^*)\eta\\ &-(\gamma_{a_2}\cosh\varepsilon t-\gamma_{a_1}^*\sinh\varepsilon t-\alpha_2)\eta^*].\end{aligned}\tag{2.1.10}$$

Moreover, using the relation

$$\int d^2\alpha e^{(-a\alpha^*\alpha+b\alpha+c\alpha^*+A\alpha^2+B\alpha^{*2})} = \frac{\pi}{(a^2-4AB)^{\frac{1}{2}}}e^{\frac{(abc+Ac^2+Bb^2}{a^2-4AB})}, a>o \tag{2.1.11}$$

and carrying out the integration over η, we get

$$\begin{aligned}Q(\alpha_1,\alpha_2,t) &= \frac{sech^2\varepsilon t}{\pi^3}exp[-\gamma_2\gamma_2^*-\gamma_1\gamma_1^*\tanh^2\varepsilon t+\tanh\varepsilon t(\gamma_1^*\gamma_2+\gamma_1^*\gamma_2^*)\\ &+sech\varepsilon t(\gamma_2^*-\gamma_1\tanh\varepsilon t)\beta\\ &+sech\varepsilon t(\gamma_2-\gamma_2^*\tanh\varepsilon t)\alpha_2^*-\alpha_2\alpha_2^*sech^2\varepsilon t]\\ &\int d^2zexp[-z^*z\cosh^2\varepsilon t+(\gamma_1^*sech\varepsilon t-\gamma_2\tanh\varepsilon t-\alpha_1^*)z\\ &-(\gamma_1 sech\varepsilon t-\alpha_2^*\tanh\varepsilon t-\alpha_1)z^*]\end{aligned}\tag{2.1.12}$$

and upon performing the integration over z, there follows

$$Q(\alpha_1,\alpha_2,t) = \frac{sech^2\varepsilon t}{\pi^2} exp[-\gamma_1\gamma_1^* - \gamma_2\gamma_2^* + \tanh\varepsilon t(\gamma_1\gamma_1 + \gamma_1^*\gamma_2^*)$$
$$-\alpha_1\alpha_1^* - \alpha_2\alpha_2^* - \tanh\varepsilon t(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)$$
$$+sech\varepsilon t(\gamma_1^*\alpha_1 + \gamma_1\alpha_1^* + \gamma_2^*\alpha_2 + \gamma_2\alpha_2^*)]. \quad (2.1.13)$$

By setting $\gamma_1 = \gamma_2 = 0$, we see that

$$Q(\alpha_1,\alpha_2,t) = \frac{sech^2\varepsilon t}{\pi^2} \exp[-\alpha_1\alpha_1^* - \alpha_2\alpha_2^* - \tanh\varepsilon t(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)]. \quad (2.1.14)$$

This is the Q function for signal-idler modes initially in vacuum state.

One can easily check the normalization of the Q-function as follows

$$\int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1,\alpha_2,t) = \int d^2\alpha_1 d^2\alpha_2 \frac{sech^2\varepsilon t}{\pi^2}$$
$$\exp[-\alpha_1\alpha_1^* - \alpha_2\alpha_2^* - \tanh\varepsilon t(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)]. \quad (2.1.15)$$

$$= \frac{sech^2\varepsilon t}{\pi^2}\int d^2\alpha_1 exp(-\alpha_1\alpha_1^*)\int d^2\alpha_2$$
$$exp(-\alpha_2\alpha_2^* - \tanh\varepsilon t(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)). \quad (2.1.16)$$

Thus upon performing the integration employing the relation described by Eq. (2.1.11), we get

$$\int d^2\alpha_2 \exp(-\alpha_2\alpha_2^* - \tanh\varepsilon t(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)) = \pi \exp\alpha_1\alpha_1^* \tanh^2\varepsilon t. \quad (2.1.17)$$

Upon substituting Eq. (2.1.16) into Eq (2.1.17), we find

$$\pi\int d^2\alpha_1 \exp(-\alpha_1\alpha_1^* + \alpha_1\alpha_1^*\tanh^2\varepsilon t =$$
$$\pi\int d^2\alpha_1 \exp(-\alpha_1\alpha_1^*(1 - \tanh^2\varepsilon t) \quad (2.1.18)$$

$$= \pi\int d^2\alpha_1 \exp(-\alpha_1\alpha_1^* sech^2\varepsilon t) \quad (2.1.19)$$

and applying Eq. (2.1.11), one can readily show the normalization condition is written in the form

$$\int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1,\alpha_2,t) = \frac{sech^2\varepsilon t}{\pi^2}\frac{\pi^2}{sech^2\varepsilon t} = 1. \quad (2.1.20)$$

This shows that the Q-function is normalized.

2.2 Photon statistics

In this section we wish to calculate the mean photon number and the variance of the photon number for the signal-idler modes employing the Q function.

2.2.1 The mean photon number

Here we wish to calculate the mean photon number for the signal-idler modes. The mean photon number for the signal-idler modes can be written as

$$\bar{n} = \langle \hat{a}^{\dagger}\hat{a} \rangle, \tag{2.2.1}$$

in which

$$\hat{a} = \hat{a_1} + \hat{a_2} \tag{2.2.2}$$

and

$$\hat{a}^{\dagger} = \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}. \tag{2.2.3}$$

Substituting Eqs. (2.2.2) and (2.2.3) into (2.2.1), we have

$$\bar{n} = \left\langle \hat{a_1}^{\dagger}\hat{a_1} \right\rangle + \left\langle \hat{a_1}^{\dagger}\hat{a_2} \right\rangle + \left\langle \hat{a_2}^{\dagger}\hat{a_1} \right\rangle + \left\langle \hat{a_2}^{\dagger}\hat{a_2} \right\rangle. \tag{2.2.4}$$

We can calculate the expectation values of the operators employing the c-number variable corresponding to operators is anti-normally ordered, hence we see that

$$\left\langle \hat{a}_1^{\dagger}\hat{a}_1 \right\rangle = \int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1, \alpha_2, t)(\alpha_1^*\alpha_1 - 1). \tag{2.2.5}$$

Inserting the value of Q function from Eq. (2.1.14) into (2.2.5) , one can easily obtain

$$\begin{aligned} \left\langle \hat{a}_1^{\dagger}\hat{a}_1 \right\rangle &= \frac{sech^2\varepsilon t}{\pi^2} \int d^2\alpha_1 d^2\alpha_2 exp[-\alpha_1^*\alpha_1 - \alpha_2^*\alpha_2 - (\tanh\varepsilon t)(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)] \\ & (\alpha_1^*\alpha_1 - 1). \end{aligned} \tag{2.2.6}$$

Upon carrying out the integration, we readily obtain the mean photon number of the signal light beam to be

$$\left\langle \hat{a}_1^{\dagger}\hat{a}_1 \right\rangle = \sinh^2 \varepsilon t. \tag{2.2.7}$$

It can be shown in a similar manner the mean photon number of the idler light beam to be

$$\left\langle \hat{a}_2^{\dagger}\hat{a}_2 \right\rangle = \sinh^2 \varepsilon t. \tag{2.2.8}$$

The other expectation values of the given operators takes the form

$$\left\langle \hat{a}_1{}^\dagger \hat{a}_2 \right\rangle = 0, \tag{2.2.9}$$

$$\left\langle \hat{a}_2{}^\dagger \hat{a}_1 \right\rangle = 0, \tag{2.2.10}$$

$$\langle \hat{a}_1 \hat{a}_1 \rangle = 0, \tag{2.2.11}$$

$$\langle \hat{a}_2 \hat{a}_2 \rangle = 0, \tag{2.2.12}$$

$$\left\langle \hat{a}_1{}^\dagger \hat{a}_1{}^\dagger \right\rangle = 0, \tag{2.2.13}$$

$$\left\langle \hat{a}_2{}^\dagger \hat{a}_2{}^\dagger \right\rangle = 0, \tag{2.2.14}$$

$$\left\langle \hat{a}_2 \hat{a}_1{}^\dagger \right\rangle = 0, \tag{2.2.15}$$

$$\left\langle \hat{a}_1 \hat{a}_2{}^\dagger \right\rangle = o, \tag{2.2.16}$$

$$\left\langle \hat{a}_1{}^\dagger \hat{a}_2{}^\dagger \right\rangle = -\cosh \varepsilon t \sinh \varepsilon t, \tag{2.2.17}$$

$$\left\langle \hat{a}_2{}^\dagger \hat{a}_1{}^\dagger \right\rangle = -\cosh \varepsilon t \sinh \varepsilon t, \tag{2.2.18}$$

$$\langle \hat{a}_1 \hat{a}_2 \rangle = -\cosh \varepsilon t \sinh \varepsilon t, \tag{2.2.19}$$

$$\langle \hat{a}_2 \hat{a}_1 \rangle = -\cosh \varepsilon t \sinh \varepsilon t, \tag{2.2.20}$$

$$\left\langle \hat{a}_1 \hat{a^\dagger}_1 \right\rangle = \cosh^2 \varepsilon t, \tag{2.2.21}$$

$$\left\langle \hat{a}_2 \hat{a^\dagger}_2 \right\rangle = \cosh^2 \varepsilon t. \tag{2.2.22}$$

Combinations of Eqs. (2.2.7) , (2.2.8) , (2.2.9), (2.2.10) and (2.2.4) yields

$$\bar{n} = 2 \sinh^2 \varepsilon t. \tag{2.2.23}$$

This represents the mean photon number for the signal-idler modes initially in a vacuum state, which is the sum of the mean photon number of the signal light beams and idler light beams.

2.2.2 The variance of the photon number

We next proceed to determine the variance of the photon number for the signal-idler modes. Then we define the photon number variance for the signal-idler modes by

$$(\Delta n)^2 = \langle \hat{n^2} \rangle - \langle \hat{n} \rangle^2. \qquad (2.2.24)$$

This can be rewritten as

$$(\Delta n)^2 = \langle (\hat{a}^\dagger(t)\hat{a}(t))^2 \rangle - \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^2 \qquad (2.2.25)$$

Now using the commutation relation, (2.0.7) we find

$$\left[\hat{a}, \hat{a}^\dagger\right] = 2. \qquad (2.2.26)$$

Hence employing Eqs. (2.2.26) the variance of the photon number becomes

$$(\Delta n)^2 = \langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t) \rangle + 2\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle - \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^2. \qquad (2.2.27)$$

We note that $\hat{a}(t)$ is a Gaussian variables with zero mean, we see that

$$\langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t) \rangle = 2\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^2 + \langle \hat{a}^{\dagger 2}(t) \rangle \langle \hat{a}^2(t) \rangle. \qquad (2.2.28)$$

Substituting Eq. (2.2.28) into (2.2.27) , we have

$$(\Delta n)^2 = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^2 + 2\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle + \langle \hat{a}^{\dagger 2}(t) \rangle \langle \hat{a}^2(t) \rangle. \qquad (2.2.29)$$

Upon introducing Eqs. (2.0.7) , (2.1.11) , (2.1.14) , (2.2.2) , and (2.2.3) into (2.2.29) , there follows

$$\begin{aligned}
(\Delta n)^2 = \; & \langle \hat{a_1^\dagger}(t)\hat{a}_1(t) \rangle^2 + 2\langle \hat{a_1^\dagger}(t)\hat{a}_2(t) \rangle \langle \hat{a_1^\dagger}(t)\hat{a}_2(t) \rangle + 2\langle \hat{a_1^\dagger}(t)\hat{a}_1(t) \rangle \langle \hat{a_2^\dagger}(t)\hat{a}_1(t) \rangle \\
& + 2\langle \hat{a_1^\dagger}(t)\hat{a}_1(t) \rangle \langle \hat{a_2^\dagger}(t)\hat{a}_1(t) \rangle + \langle \hat{a_1^\dagger}(t)\hat{a}_2(t) \rangle^2 + 2\langle \hat{a_1^\dagger}(t)\hat{a}_2(t) \rangle \langle \hat{a_2^\dagger}(t)\hat{a}_1(t) \rangle \\
& + 2\langle \hat{a_1^\dagger}(t)\hat{a}_2(t) \rangle \langle \hat{a_2^\dagger}(t)\hat{a}_2(t) \rangle + \langle \hat{a_2^\dagger}(t)\hat{a}_1(t) \rangle^2 + 2\langle \hat{a_1^\dagger}(t)\hat{a}_1(t) \rangle \langle \hat{a_2^\dagger}(t)\hat{a}_2(t) \rangle \\
& + \langle \hat{a_2^\dagger}(t)\hat{a}_2(t) \rangle^2 + 4\langle \hat{a_1^\dagger}(t)\hat{a}_1(t) \rangle + 4\langle \hat{a_1^\dagger}(t)\hat{a}_2(t) \rangle + 4\langle \hat{a_2^\dagger}(t)\hat{a}_1(t) \rangle + 4\langle \hat{a_2^\dagger}(t)\hat{a}_2(t) \rangle \\
& + \langle \hat{a_1^{\dagger 2}}(t) \rangle \langle \hat{a_1^2}(t) \rangle + 2\langle \hat{a_1^{\dagger 2}}(t) \rangle \langle \hat{a}_1(t)\hat{a}_2(t) \rangle + \langle \hat{a_1^{\dagger 2}}(t) \rangle \langle \hat{a_2^2}(t) \rangle \\
& + 2\langle \hat{a}_1(t)\hat{a}_2(t) \rangle \langle \hat{a_1^2}(t) \rangle + 4\langle \hat{a_1^\dagger}(t)\hat{a_2^\dagger}(t) \rangle \langle \hat{a}_1(t)\hat{a}_2(t) \rangle + 2\langle \hat{a_1^\dagger}(t)\hat{a_2^\dagger}(t) \rangle \langle \hat{a_2^2}(t) \rangle \\
& + \langle \hat{a_2^{\dagger 2}}(t) \rangle \langle \hat{a_2^2}(t) \rangle + 2\langle \hat{a_2^{\dagger 2}}(t) \rangle \langle \hat{a}_1(t)\hat{a}_2(t) \rangle + \langle \hat{a_2^{\dagger 2}}(t) \rangle \langle \hat{a_1^2}(t) \rangle. \qquad (2.2.30)
\end{aligned}$$

In view of fact that operator $\hat{a_1}(t)$ and $\hat{a_2}(t)$ are Gaussian variables with zero mean, we see that

$$\langle \hat{a_1}(t) \rangle = \langle \hat{a_2}(t) \rangle = \langle \hat{a_1^\dagger}(t) \rangle = \langle \hat{a_2^\dagger}(t) \rangle = 0. \qquad (2.2.31)$$

Thus applying Eqs. (2.2.7) - (2.2.16) and (2.2.31) into (2.2.30) , we get

$$\begin{aligned}(\Delta n)^2 &= \langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle^2 + \langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle^2 + 2\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + 2\langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle \\ &\quad +2\langle \hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle \hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle + 4\langle \hat{a}_1^\dagger(t)\hat{a}_2^\dagger(t)\rangle\langle \hat{a}_1(t)\hat{a}_2(t)\rangle. \end{aligned} \tag{2.2.32}$$

Substituting Eqs. (2.2.7) , (2.2.8) , (2.2.17) , (2.2.19) , and (2.2.20) into (2.2.32) , there follows

$$(\Delta n)^2 = 4\sinh^4 \varepsilon t + 4\sinh^2 \varepsilon t + 4\cosh^2 \varepsilon t \sinh^2 \varepsilon t. \tag{2.2.33}$$

This is the variance of the photon number for signal-idler modes initially in vacuum state. Eq. (2.2.33) leads to

$$(\Delta n)^2 = 2\bar{n} + \bar{n}^2 + 2\bar{n}\cosh^2 \varepsilon t. \tag{2.2.34}$$

This result shows that the photon statistics is super-Poissonian

2.3 Quadrature fluctuation

In this section, we determine the quadrature variance and quadrature squeezing for the signal-idler modes.

2.3.1 Quadrature variance

We wish here to determine the quadrature variance for signal-idler modes. The plus and minus quadrature operators for signal-idler modes are defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{2.3.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{2.3.2}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively. The quadrature variance can be expressed in terms of the quadratures as

$$(\Delta\hat{a}_\pm)^2 = \langle \hat{a}_\pm^2\rangle - \langle \hat{a}_\pm\rangle^2. \tag{2.3.3}$$

The quadrature variance of plus and minus quadrature can be rewritten as

$$\begin{aligned}(\Delta\hat{a}_\pm)^2 &= 2 + 2\langle \hat{a}^\dagger a\rangle \pm \langle \hat{a}^{\dagger 2}\rangle \pm \langle \hat{a}^2\rangle \\ &\quad \mp\langle \hat{a}^\dagger\rangle^2 \mp \langle \hat{a}\rangle^2 - 2\langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle. \end{aligned} \tag{2.3.4}$$

Since $\hat{a_1}$ and $\hat{a_2}$ are Gaussian variables with zero mean, $\hat{a}$ is also Gaussian variable with zero mean, Eq. (2.3.4) thus becomes

$$(\Delta\hat{a_\pm})^2 = 2 + 2\langle\hat{a^\dagger\hat{a}}\rangle \pm \langle\hat{a^{\dagger 2}}\rangle \pm \langle\hat{a^2}\rangle, \tag{2.3.5}$$

now one can easily obtain

$$2\langle\hat{a^\dagger a}\rangle = 2\langle(\hat{a_1^\dagger} + \hat{a_2^\dagger})(\hat{a_1} + \hat{a_2})\rangle = 2\langle\hat{a_1^\dagger a_1}\rangle + 2\langle\hat{a}_2^\dagger\hat{a}_2\rangle. \tag{2.3.6}$$

Similarly,

$$\langle\hat{a^{\dagger 2}}\rangle = \langle(\hat{a_1^\dagger} + \hat{a_2^\dagger})(\hat{a_1^\dagger} + \hat{a_2^\dagger})\rangle = \langle\hat{a_1^\dagger a_2^\dagger}\rangle + \langle\hat{a_2^\dagger a_1^\dagger}\rangle, \tag{2.3.7}$$

$$\langle\hat{a^2}\rangle = \langle(\hat{a_1} + \hat{a_2})(\hat{a_1} + \hat{a_2})\rangle = \langle\hat{a_1}\hat{a_2}\rangle + \langle\hat{a_2}\hat{a_1}\rangle. \tag{2.3.8}$$

Form Eqs. (2.2.16) , (2.2.17) , (2.2.18) and (2.2.19) , we note that

$$\langle\hat{a}_1\hat{a}_2\rangle = \langle\hat{a}_2\hat{a}_1\rangle = \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle = \langle\hat{a}_2^\dagger\hat{a}_1^\dagger\rangle. \tag{2.3.9}$$

Substituting Eqs. (2.3.6) , (2.3.7) , (2.3.8) , and (2.3.9) into (2.3.5) , one finds

$$(\Delta\hat{a}_\pm)^2 = 2 + 2[\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle \pm 2\langle\hat{a}_1\hat{a}_2\rangle]. \tag{2.3.10}$$

One can easily recall that

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = \langle\hat{a}_2^\dagger\hat{a}_2\rangle = \sinh^2\varepsilon t \tag{2.3.11}$$

and

$$\langle\hat{a_1}\hat{a_2}\rangle = \langle\hat{a_2}\hat{a_1}\rangle = -\cosh\varepsilon t\sinh\varepsilon t. \tag{2.3.12}$$

Inserting Eq. (2.3.11) and (2.3.12) into (2.3.10) , one can easily obtain

$$(\Delta\hat{a_\pm})^2 = 2 + 2\bar{n} \pm (-2\bar{n}\cosh^2\varepsilon t). \tag{2.3.13}$$

From the definition of cosine and sine, we see that

$$\sinh^2\varepsilon t = (\frac{e^{\varepsilon t} - e^{-\varepsilon t}}{2})(\frac{e^{\varepsilon t} - e^{-\varepsilon t}}{2}) \tag{2.3.14}$$

and

$$\cosh\varepsilon t = \frac{e^{\varepsilon t} + e^{-\varepsilon t}}{2}. \tag{2.3.15}$$

Plagging Eqs. (2.3.14) , and (2.3.15) into (2.3.13) , we have

$$(\Delta\hat{a_{\pm}})^2 = [e^{2\varepsilon t} + e^{-2\varepsilon t}] \pm [e^{-2\varepsilon t} - e^{+2\varepsilon t}]. \tag{2.3.16}$$

With the aid of Eq. (2.3.16) , one easily obtains the plus and minus quadrature variance as

$$(\Delta a_+)^2 = e^{-2\varepsilon t} \tag{2.3.17}$$

and

$$(\Delta a_-)^2 = e^{2\varepsilon t}. \tag{2.3.18}$$

From this result we observe $(\Delta a_+)^2 = e^{-2\varepsilon t} < 2$ and $(\Delta a_-)^2 = e^{2\varepsilon t} > 2$. This shows that the signal-idler modes are in a squeezed state and squeezing occurs in the plus quadrature

2.3.2 Quadrature squeezing

The quadrature squeezing of the two mode light beams can be written as

$$S_+ = \frac{2 - \Delta(a_+)^2}{2}. \tag{2.3.19}$$

Substituting Eq. (2.3.17) into (2.3.19), we find

$$S_+ = \frac{2 - e^{-2\varepsilon t}}{2}, \tag{2.3.20}$$

$$S_+(t) = 1 - \frac{1}{2}e^{-2\varepsilon t}, \tag{2.3.21}$$

with εt being the squeezing parameter taken to be real and positive for convenience. We note that for t=0, there is a 50% quadrature squeezing below the two-mode vacuum state level

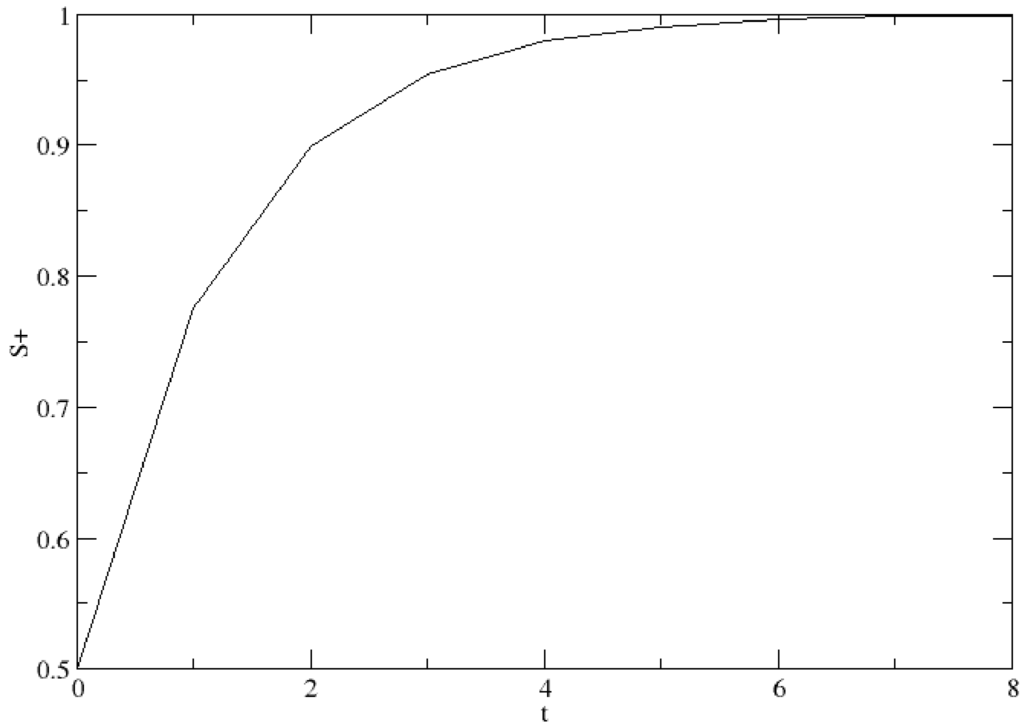

Figure 2.2: A plot of S+(t) Eq. (2.3.21) versus t for ε = 0.4

2.4 Entanglement analysis

In this section we seek to study the entanglement condition for twin beams of light. On the basis of this criteria, twin light beams is said to be entangled if the sum of the variance of the two EPR-like operators $\hat{s}$ and $\hat{t}$ satisfies the inequality[12].

$$(\Delta s)^2 + (\Delta t)^2 < 2, \tag{2.4.1}$$

where

$$\hat{s} = \frac{1}{2}\left(\hat{a}_{1_+} + \hat{a}_{2_+}\right), \tag{2.4.2}$$

with

$$\hat{t} = \frac{1}{2}\left(\hat{a}_{1_-} + \hat{a}_{2_-}\right), \tag{2.4.3}$$

$$\hat{a}_{1_+}(t) = \hat{a}_1^\dagger(t) + \hat{a}_1(t) \tag{2.4.4}$$

and

$$\hat{a}_{1_-}(t) = i\left(\hat{a}_1^\dagger(t) - \hat{a}_1(t)\right), \tag{2.4.5}$$

$$\hat{a}_{2_+}(t) = \hat{a}_2^\dagger(t) + \hat{a}_2(t), \tag{2.4.6}$$

$$\hat{a}_{2_-}(t) = i\left(\hat{a}_2^\dagger(t) - \hat{a}_2(t)\right). \tag{2.4.7}$$

The variance of the operators $\hat{s}$ and $\hat{t}$ can be expressed as

$$(\Delta s)^2 = \langle \hat{s^2} \rangle - \langle \hat{s} \rangle^2 \tag{2.4.8}$$

and

$$(\Delta t)^2 = \langle \hat{t^2} \rangle - \langle \hat{t} \rangle^2. \tag{2.4.9}$$

In view of the fact that $\hat{a}(t)$ and $\hat{b}(t)$ are Gaussian variables with zero mean and employing Eqs. (2.4.4) , (2.4.6) , and (2.4.8) , one can readily obtains

$$\begin{aligned}(\Delta s)^2 \;=\; & \frac{1}{2}[1 + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_1^2 \rangle + \langle \hat{a}_1^{\dagger 2} \rangle \\ & + \langle \hat{a}_2^{\dagger 2} \rangle + \langle \hat{a}_2^2 \rangle - \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle - \langle \hat{a}_1 \hat{a}_2 \rangle - \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \\ & - \langle \hat{a}_1 \hat{a}_2^\dagger \rangle - \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle - \langle \hat{a}_2 \hat{a}_1 \rangle - \langle \hat{a}_2^\dagger \hat{a}_1 \rangle - \langle \hat{a}_2 \hat{a}_1^\dagger \rangle].\end{aligned} \tag{2.4.10}$$

Following the same procedure, we get

$$\begin{aligned}(\Delta t)^2 \;=\; & \frac{1}{2}[1 + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle - \langle \hat{a}_1^2 \rangle - \langle \hat{a}_1^{\dagger 2} \rangle \\ & - \langle \hat{a}_2^{\dagger 2} \rangle - \langle \hat{a}_2^2 \rangle - \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle - \langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \\ & + \langle \hat{a}_1 \hat{a}_2^\dagger \rangle - \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle - \langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2 \hat{a}_1^\dagger \rangle].\end{aligned} \tag{2.4.11}$$

On account of Eqs. (2.4.10) and (2.4.11) , we see that the sum of the variance of the two EPR-like operators is expressible as

$$\begin{aligned}(\Delta s)^2 + (\Delta t)^2 \;=\; & 1 + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle - \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \\ & - \langle \hat{a}_1 \hat{a}_2 \rangle - \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle - \langle \hat{a}_2 \hat{a}_1 \rangle.\end{aligned} \tag{2.4.12}$$

Upon introducing Eqs. (2.2.7) , (2.2.8) and (2.3.9) into (2.4.12) there follows

$$\Delta s)^2 + (\Delta t)^2 = 1 + 2\langle \hat{a}_1^\dagger \hat{a}_1 \rangle - 4\langle \hat{a}_1 \hat{a}_2 \rangle \tag{2.4.13}$$

Substituting Eqs. (2.2.7) and (2.2.19) into Eq. (2.4.13) , we get

$$\Delta s)^2 + (\Delta t)^2 = 1 + 2\sinh^2 \varepsilon t - 4 \sinh \varepsilon t \cosh \varepsilon t. \tag{2.4.14}$$

At steady-state the sum of the variance of the two EPR-like operators to be

$$(\Delta s)^2 + (\Delta t)^2 = 1. \tag{2.4.15}$$

On the basis of the criteria Eq. (2.4.1) , we clearly see that twin beams of light are entangled at steady-state.

Chapter 3

A Pair of Superposed Twin Beams Squeezed State

Any two or more beam of light can be added together and the resultant beam is said to be superposed beams of light. A beam splitter, used in many quantum optical measurements, is a mirror that partly reflects and partly transmits a light beam incident on it[2]. And we intended to use the horizontal polarizing beam splitter to get uni-directional beam of light that might be attain the superposed squeezed light beams which to be measured by the detector.

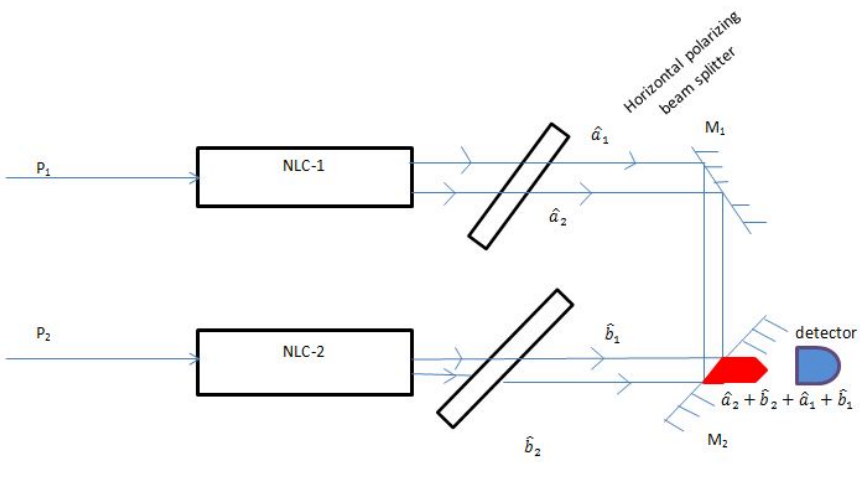

Figure 3.1: A pair of superposed twin beams of squeezed light.

3.1 The density operator

Here we seek to determine the density operator for a pair of superposed twin beams squeezed state. Suppose $\hat{\rho}\left(\hat{a^{\dagger}}_1, \hat{a^{\dagger}}_2, t\right)$ is the density operator for a certain two mode light beams. Then upon expanding this density operator in normal order [12], we see that

$$\hat{\rho}_1\left(\hat{a_1^{\dagger}}, \hat{a_2^{\dagger}}, t\right) = \sum_{klmn} C_{klmn} \hat{a_1^{\dagger k}}(t)\hat{a_2^{\dagger l}}(t)\hat{a_1^{m}}(t)\hat{a_2^{n}}(t). \tag{3.1.1}$$

Now employing completeness relation

$$I = \frac{1}{\pi^2}\int d^2\alpha_1 d^2\alpha_2 |\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|, \tag{3.1.2}$$

for a two-mode coherent light, one easily finds

$$\hat{\rho}_1 = \frac{1}{\pi^2}\int d^2\alpha_1 d^2\alpha_2 \sum_{klmn} C_{klmn}|\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|\hat{a_1^{\dagger k}}(t)\hat{a_2^{\dagger l}}(t)\hat{a_1^{m}}(t)\hat{a_2^{n}}(t), \tag{3.1.3}$$

in which $\hat{a_1}(t)$ ($\hat{a_2}(t)$) is the annihilation operators for the first signal(idler) light modes, respectively. This expression can be written as

$$\hat{\rho_1} = \frac{1}{\pi^2}\int d^2\alpha_1 d^2\alpha_2 \sum_{klmn} C_{klmn}\alpha_1^{*k}\alpha_2^{*l}|\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|\hat{a_1^{m}}(t)\hat{a_2^{n}}(t). \tag{3.1.4}$$

Applying the relation

$$|\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|\hat{a_1}(t)\hat{a_2}(t) = \left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}\right)\left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}\right)|\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|, \tag{3.1.5}$$

one easily obtains

$$\begin{aligned}\hat{\rho_1} &= \frac{1}{\pi^2}\int d^2\alpha_1 d^2\alpha_2 \sum_{klmn} C_{klmn}\alpha_1^{*k}\alpha_2^{*l}\left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)^m \\ &\left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)^n |\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|.\end{aligned} \tag{3.1.6}$$

Then density operator for the first light beam in terms of the displacement operator can be rewritten as

$$\begin{aligned}\hat{\rho_1} &= \frac{1}{\pi^2}\int d^2\alpha_1 d^2\alpha_2 \sum_{klmn} C_{klmn}\alpha_1^{*k}\alpha_2^{*l}\left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)^m \\ &\left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)^n \hat{D}(\alpha_1)\hat{D}(\alpha_2)\hat{\rho}_0\hat{D}(-\alpha_2)\hat{D}(-\alpha_1),\end{aligned} \tag{3.1.7}$$

in which

$$\hat{\rho}_0 = |0\alpha_1, 0\alpha_2\rangle\langle 0\alpha_2, 0\alpha_1|,$$

represents the density operator for the light initially in a two-mode vacuum state. Now we realize that upon expanding the density operator for the superposition of the first beam and another one is expressible as

$$\hat{\rho}_2\left(\hat{b_1}, \hat{b_2}, t\right) = \sum_{k'l'm'n'} C_{k'l'm'n'} \hat{b}_1^{\dagger k'}(t) \hat{b}_2^{\dagger l'}(t) \hat{b}_1^{m'}(t) \hat{b}_2^{n'}(t), \tag{3.1.8}$$

from which follows

$$\begin{aligned} \hat{\rho_2} \;=\; & \frac{1}{\pi^2}\int d^2\beta_1 d^2\beta_2 \sum_{k'l'm'n'} C_{k'l'm'n'} \beta_1^{*k'} \beta_2^{*l'} \left(\beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)^{m'} \\ & \left(\beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)^{n'} \hat{D}(\beta_1)\hat{D}(\beta_2)\hat{\rho}_1\hat{D}(-\beta_2)\hat{D}(-\beta_1). \end{aligned} \tag{3.1.9}$$

Moreover, on account of Eqs. (3.1.7) and (3.1.9) , the density operator for the pair of superposed twin light beams can be put in the form

$$\begin{aligned} \hat{\rho}_2 \;=\; & \frac{1}{\pi^4}\int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 \sum_{klmn} C_{klmn} \alpha_1^{*k} \alpha_2^{*l} \left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)^m \\ & \left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)^n \sum_{k'l'm'n'} C_{k'l'm'n'} \beta_1^{*k'} \beta_2^{*l'} \left(\beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)^{m'} \\ & \left(\beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)^{n'} \hat{D}(\beta_1)\hat{D}(\beta_2)\hat{D}(\alpha_1)\hat{D}(\alpha_2)\hat{\rho}_0 \\ & \hat{D}(-\alpha_2)\hat{D}(-\alpha_1)\hat{D}(-\beta_2)\hat{D}(-\beta_1). \end{aligned} \tag{3.1.10}$$

Now one can easily write Eq. (3.1.10) as follows

$$\begin{aligned} \hat{\rho}_2 \;=\; & \frac{1}{\pi^4}\int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 \sum_{klmn} C_{klmn} \alpha_1^{*k} \alpha_2^{*l} \left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)^m \\ & \left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)^n \sum_{k'l'm'n'} C_{k'l'm'n'} \beta_1^{*k'} \beta_2^{*l'} \left(\beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)^{m'} \\ & \left(\beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)^{n'} \hat{D}(\beta_1)\hat{D}(\beta_2)\hat{D}(\alpha_1)\hat{D}(\alpha_2)|0\alpha_1, 0\alpha_2\rangle\langle 0\alpha_2, \alpha_1| \\ & \hat{D}(-\alpha_2)\hat{D}(-\alpha_1)\hat{D}(-\beta_2)\hat{D}(-\beta_1). \end{aligned} \tag{3.1.11}$$

Employing the relations

$$\hat{D}(\alpha_1)\hat{D}(\alpha_2)|0\alpha_1, 0\alpha_2\rangle = |\alpha_1, \alpha_2\rangle \quad (3.1.12)$$

and

$$\langle 0\alpha_2, 0\alpha_1|\hat{D}(-\alpha_2)\hat{D}(-\alpha_1) = \langle \alpha_2, \alpha_1|, \quad (3.1.13)$$

we can express Eq. (3.1.11) as

$$\begin{aligned}\hat{\rho}_2 \;=\; & \frac{1}{\pi^4}\int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 \sum_{klmn} C_{klmn}\alpha_1^{*k}\alpha_2^{*l}\left(\alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)^m \\ & \left(\alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)^n \sum_{k'l'm'n'} C_{k'l'm'n'}\beta_1^{*k'}\beta_2^{*l'}\left(\beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)^{m'} \\ & \left(\beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)^{n'} \hat{D}(\beta_1)\hat{D}(\beta_2)|\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|\hat{D}(-\beta_2)\hat{D}(-\beta_1). \quad (3.1.14)\end{aligned}$$

Using the relation[2,14,15]

$$\hat{D}(\alpha)|\beta\rangle\langle\beta|\hat{D}(-\alpha) = |\alpha+\beta\rangle\langle\alpha+\beta|, \quad (3.1.15)$$

we easily adopt an expression that holds true for the two mode light beams as

$$\begin{aligned}\hat{D}(\beta_1)\hat{D}(\beta_2)|\alpha_1, \alpha_2\rangle\langle\alpha_2, \alpha_1|\hat{D}(-\beta_2)\hat{D}(-\beta_1) \;=\; & |\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\ & \langle\beta_2+\beta_1+\alpha_2+\alpha_1|. \quad (3.1.16)\end{aligned}$$

Applying Eq. (3.1.16) in (3.1.14), the density operator for the superposition can be put in the form

$$\begin{aligned}\hat{\rho} \;=\; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1'\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) \\ & Q_2'\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & |\alpha_1+\alpha_2+\beta_1+\beta_2\rangle\langle\beta_2+\beta_1+\alpha_2+\alpha_1|, \quad (3.1.17)\end{aligned}$$

where

$$Q_1\left(\alpha_1^*,\alpha_1+\frac{\partial}{\partial\alpha_1^*},t\right)=\frac{1}{\pi}\sum_{km}\alpha_1^{*k}\left(\alpha_1+\frac{\partial}{\partial\alpha_1^*},t\right)^m \tag{3.1.18}$$

$$Q_2\left(\alpha_2^*,\alpha_2+\frac{\partial}{\partial\alpha_2^*},t\right)=\frac{1}{\pi}\sum_{ln}\alpha_2^{*l}\left(\alpha_2+\frac{\partial}{\partial\alpha_2^*},t\right)^n \tag{3.1.19}$$

$$Q_1^{'}\left(\beta_1^*,\beta_1+\frac{\partial}{\partial\beta_1^*},t\right)=\frac{1}{\pi}\sum_{k'm'}\beta_1^{*k'}\left(\beta_1+\frac{\partial}{\partial\beta_1^*},t\right)^{m'} \tag{3.1.20}$$

$$Q_2^{'}\left(\beta_2^*,\beta_2+\frac{\partial}{\partial\beta_2^*},t\right)=\frac{1}{\pi}\sum_{l'n'}\beta_2^{*l'}\left(\beta_2+\frac{\partial}{\partial\beta_2^*},t\right)^{n'}. \tag{3.1.21}$$

We see that Eqs. (3.1.18)-(3.1.21) represent the Q-functions for the first signal beam, the first idler beam, the second signal beam and the second idler beam, respectively. Now introducing the Q function that we obtained in chapter two, the Q function for the first signal-idler modes can be written as

$$Q_1(\alpha_1,\alpha_2,t)=\frac{sech^2\varepsilon t}{\pi^2}\exp[-\alpha_1\alpha_1^*-\alpha_2\alpha_2^*-\tanh\varepsilon t(\alpha_1\alpha_2+\alpha_1^*\alpha_2^*)]. \tag{3.1.22}$$

And the Q function for the second signal-idler modes can be written as

$$Q_2(\beta_1,\beta_2,t)=\frac{sech^2\varepsilon t}{\pi^2}\exp[-\beta_1\beta_1^*-\beta_2\beta_2^*-\tanh\varepsilon t(\beta_1\beta_2+\beta_1^*\beta_2^*)]. \tag{3.1.23}$$

3.2 Photon statistics

In this section we wish to calculate the mean and variance of the photon number for the superposed light beams.

3.2.1 The mean photon number

The mean photon number for a pair of superposed two-mode light beams in terms of density operator can be written as

$$\bar{n}=Tr\left(\hat{\rho}(t)\hat{c}^\dagger(0)\hat{c}(0)\right), \tag{3.2.1}$$

where

$$\hat{c} = \hat{a} + \hat{b}, \tag{3.2.2}$$

with

$$\hat{a} = \hat{a_1} + \hat{a_2} \tag{3.2.3}$$

and

$$\hat{b} = \hat{b_1} + \hat{b_2}. \tag{3.2.4}$$

Using Eq. (3.2.2) , Eq. (3.2.1) cab be written as

$$\bar{n} = Tr\left(\hat{\rho}(t)(\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})\right). \tag{3.2.5}$$

On account of Eq. (3.1.17) , we find

$$\begin{aligned}\bar{n} \;=\; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & Tr(|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1| \\ & (\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})).\end{aligned} \tag{3.2.6}$$

Applying the cyclic property of trace, we obtain

$$\begin{aligned}\bar{n} \;=\; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & [\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}^\dagger\hat{a}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}^\dagger\hat{b}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{b}^\dagger\hat{a}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{b}^\dagger\hat{b}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle].\end{aligned} \tag{3.2.7}$$

Introducing Eqs. (3.2.3) and (3.2.4) into (3.2.7) , we get

$$\begin{aligned}\bar{n} \;=\; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & [\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}_1^\dagger\hat{b}_1 + \hat{a}_1^\dagger\hat{b}_2 + \hat{a}_2^\dagger\hat{b}_1 + \hat{a}_2^\dagger\hat{b}_2|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{b}_1^\dagger\hat{a}_1 + \hat{b}_1^\dagger\hat{a}_2 + \hat{b}_2^\dagger\hat{a}_1 + \hat{b}_2^\dagger\hat{a}_2|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{b}_1^\dagger\hat{b}_1 + \hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle]\end{aligned} \tag{3.2.8}$$

This can be re-written as

$$\begin{aligned}
\bar{n} &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\
& Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\
& [\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_1^\dagger\hat{a}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_1^\dagger\hat{a}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_2^\dagger\hat{a}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_2^\dagger\hat{a}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_1^\dagger\hat{b}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_1^\dagger\hat{b}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_2^\dagger\hat{b}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_2^\dagger\hat{b}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_1^\dagger\hat{a}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_1^\dagger\hat{a}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_2^\dagger\hat{a}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_2^\dagger\hat{a}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_1^\dagger\hat{b}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_1^\dagger\hat{b}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_2^\dagger\hat{b}_1|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle \\
& +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_2^\dagger\hat{b}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle]. \qquad (3.2.9)
\end{aligned}$$

Applying the annihilation and creation operator appeared in Eq. (3.2.9) on the state vectors $|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle$ and $|\beta_2+\beta_1+\alpha_2+\alpha_1$, respectively, we obtain

$$\begin{aligned}
\bar{n} &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\
& Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\
& [\alpha_1^*\alpha_1+\alpha_1^*\alpha_2+\alpha_2^*\alpha_1+\alpha_2^*\alpha_2+\alpha_1^*\beta_1+\alpha_1^*\beta_2+\alpha_2^*\beta_1+\alpha_2^*\beta_2 \\
& +\beta_1^*\alpha_1+\beta_1^*\alpha_2+\beta_2^*\alpha_1+\beta_2^*\alpha_2+\beta_1^*\beta_1+\beta_1^*\beta_2+\beta_2^*\beta_1+\beta_2^*\beta_2] \qquad (3.2.10)
\end{aligned}$$

Then recall the normalization condition of the Q function, Eq. (3.2.10) reduces

$$\begin{aligned}
\bar{n} \;=\; & \int d^2\alpha_1 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1^*\alpha_1 \\
& + \int d^2\alpha_1 d^2\alpha_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1^* Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2 \\
& + \int d^2\alpha_1 d^2\alpha_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1 Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2^* \\
& + \int d^2\alpha_2 Q_1\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2^*\alpha_2 \\
& + \int d^2\alpha_1 d^2\beta_1 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1^* Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1 \\
& + \int d^2\alpha_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1^* Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2 \\
& + \int d^2\alpha_2 d^2\beta_1 Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2^* Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1 \\
& + \int d^2\alpha_2 d^2\beta_2 Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2^* Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2 \\
& + \int d^2\alpha_1 d^2\beta_1 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1 Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1^* \\
& + \int d^2\alpha_2 d^2\beta_1 Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2 Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1^* \\
& + \int d^2\alpha_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right)\alpha_1 Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2^* \\
& + \int d^2\alpha_2 d^2\beta_2 Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right)\alpha_2 Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2^* \\
& + \int d^2\beta_1 Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1^*\beta_1 \\
& + \int d^2\beta_1\beta_2 Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1^* Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2 \\
& + \int d^2\beta_1\beta_2 Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right)\beta_1 Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2^* \\
& + \int d^2\beta_2 Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right)\beta_2^*\beta_2
\end{aligned} \tag{3.2.11}$$

We know that

$$\langle \hat{A}^\dagger \hat{A} \rangle = \int d^2\alpha Q\left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t\right)\alpha^*\alpha, \tag{3.2.12}$$

in which $\alpha^*\alpha$ is c- number variable associated with the operators $\hat{A}^\dagger\hat{A}$ in normal order In view of Eq. (3.2.12), Eq. (3.2.11) can be put in the form

$$\begin{aligned}
\bar{n} \;=\; & \langle\hat{a_1}^\dagger(t)\hat{a_1}(t)\rangle + \langle\hat{b_1}^\dagger(t)\hat{b_1}(t)\rangle + \langle\hat{a_2}^\dagger(t)\hat{a_2}(t)\rangle + \langle\hat{b_2}^\dagger(t)\hat{b_2}(t)\rangle \\
& +\langle\hat{a_1}^\dagger(t)\rangle\langle\hat{a_2}(t)\rangle + \langle\hat{b_1}^\dagger(t)\rangle\langle\hat{b_2}(t)\rangle + \langle\hat{a_1}(t)\rangle\langle\hat{a_2}^\dagger(t)\rangle \\
& +\langle\hat{b_1}(t)\rangle\langle\hat{b_2}^\dagger(t)\rangle + \langle\hat{a_1}^\dagger(t)\rangle\langle\hat{b_1}(t)\rangle + \langle\hat{a_1}^\dagger(t)\rangle\langle\hat{b_2}(t)\rangle \\
& +\langle\hat{a_2}^\dagger(t)\rangle\langle\hat{b_1}(t)\rangle + \langle\hat{a_2}^\dagger(t)\rangle\langle\hat{b_2}(t)\rangle + \langle\hat{b_1}^\dagger(t)\rangle\langle\hat{a_1}(t)\rangle \\
& +\langle\hat{b_2}^\dagger(t)\rangle\langle\hat{a_1}(t)\rangle + \langle\hat{b_2}^\dagger(t)\rangle\langle\hat{a_2}(t)\rangle + \langle\hat{b_1}^\dagger(t)\rangle\langle\hat{a_2}(t)\rangle. \qquad (3.2.13)
\end{aligned}$$

Since, $\hat{a_1}, \hat{b_1}, \hat{a_2}$, and $\hat{b_2}$ are Gaussian operators with zero mean, we see that

$$\bar{n} = \langle\hat{a_1}^\dagger(t)\hat{a_1}(t)\rangle + \langle\hat{b_1}^\dagger(t)\hat{b_1}(t)\rangle + \langle\hat{a_2}^\dagger(t)\hat{a_2}(t)\rangle + \langle\hat{b_2}^\dagger(t)\hat{b_2}(t)\rangle. \qquad (3.2.14)$$

One can easily establish the relation

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = \langle\hat{b}_1^\dagger\hat{b}_1\rangle = \langle\hat{a}_2^\dagger\hat{a}_2\rangle = \langle\hat{b}_2^\dagger\hat{b}_2\rangle. \qquad (3.2.15)$$

Then on account of Eqs. (2.2.7) and (2.2.8) along with (3.2.15) , Eq. (3.2.14) we can write as

$$\bar{n}_{ss} = 4\sinh^2\varepsilon t. \qquad (3.2.16)$$

This result indicates that the mean photon number of a pair of two-mode superposed light beams is the sum of the mean photon number of the separate light beams.

3.2.2 The variance of the photon number

We next proceed to determine the variance of the photon number for a pair of two-mode superposed light beams. Then we define the photon number variance for a pair of superposed light beams as

$$(\Delta n)^2 = \langle(\hat{c}^\dagger(t)\hat{c}(t))^2\rangle - \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle^2. \qquad (3.2.17)$$

Now using the commutation relation

$$[\hat{c}, \hat{c}^\dagger] = 4, \qquad (3.2.18)$$

which holds true for a pair of superposed light beams, we find

$$(\Delta n)^2 = \langle\hat{c}^{\dagger 2}(t)\hat{c}^2(t)\rangle + 4\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle - \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle^2. \qquad (3.2.19)$$

We note that $\hat{c}(t)$ is a Gaussian operator with zero mean. Hence we see that

$$\langle \hat{c}^{\dagger 2}(t)\hat{c}^2(t)\rangle = 2\langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle^2 + \langle \hat{c}^{\dagger 2}(t)\rangle\langle \hat{c}^2(t)\rangle. \tag{3.2.20}$$

Thus one can put Eq. (3.2.19) in the form

$$(\Delta n)^2 = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle^2 + 4\langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle + \langle \hat{c}^2(t)\rangle^2. \tag{3.2.21}$$

Then the commutation relation for the annihilation operator representing two-mode light beams can be written as

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 2. \tag{3.2.22}$$

Now we can calculate the expectation value of $\hat{c}^2$ .Then using the density operator, the expectation value of the operator $\hat{c}^2$ can be written as

$$\langle \hat{c}^2\rangle = Tr(\hat{\rho}(t)\hat{c}^2). \tag{3.2.23}$$

In view of Eq. (3.1.17), we see that

$$\begin{aligned}\langle \hat{c}^2\rangle \; = \; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & Tr(|\alpha_2 + \beta_1 + \beta_2\rangle\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{c}^2).\end{aligned} \tag{3.2.24}$$

Employing Eq. (3.2.2), we find

$$\begin{aligned}\langle \hat{c}^2\rangle \; = \; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & Tr(|\alpha_2 + \beta_1 + \beta_2\rangle\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}^2 + \hat{a}\hat{b} + \hat{b}\hat{a} + \hat{b}^2).\end{aligned} \tag{3.2.25}$$

Applying cyclic properties of trace, we obtain

$$\begin{aligned}\langle \hat{c}^2\rangle \; = \; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial\beta_2^*}, t\right) \\ & [\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}^2|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}\hat{b}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{b}\hat{a}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle \\ & +\langle\beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{b}^2|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle].\end{aligned} \tag{3.2.26}$$

With the aid of Eqs. (3.2.3) and (3.2.4) , we get

$$\begin{aligned}\langle \hat{c}^2\rangle &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1+\frac{\partial}{\partial\alpha_1^*}, t\right)\\ &\quad Q_2\left(\alpha_2^*, \alpha_2+\frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1+\frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2+\frac{\partial}{\partial\beta_2^*}, t\right)\\ &\quad [\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_1^2+\hat{a}_1\hat{a}_2+\hat{a}_2\hat{a}_1+\hat{a}_2^2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle\\ &\quad +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{a}_1\hat{b}_1+\hat{a}_1\hat{b}_2+\hat{a}_2\hat{b}_1+\hat{a}_2\hat{b}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle\\ &\quad +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_1\hat{a}_1+\hat{b}_1\hat{a}_2+\hat{b}_2\hat{a}_1+\hat{b}_2\hat{a}_2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle\\ &\quad +\langle\beta_2+\beta_1+\alpha_2+\alpha_1|\hat{b}_1^2+\hat{b}_1\hat{b}_2+\hat{b}_2\hat{b}_1+\hat{b}_2^2|\alpha_1+\alpha_2+\beta_1+\beta_2\rangle].\end{aligned} \quad (3.2.27)$$

Then it follows

$$\begin{aligned}\langle \hat{c}^2\rangle &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1+\frac{\partial}{\partial\alpha_1^*}, t\right)\\ &\quad Q_2\left(\alpha_2^*, \alpha_2+\frac{\partial}{\partial\alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1+\frac{\partial}{\partial\beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2+\frac{\partial}{\partial\beta_2^*}, t\right)\\ &\quad [(\alpha_1+\alpha_2)^2+(\beta_1+\beta_2)^2\\ &\quad +(\alpha_1+\alpha_2)(\beta_1+\beta_2)+(\beta_1+\beta_2)(\alpha_1+\alpha_2)].\end{aligned} \quad (3.2.28)$$

Then on account of Eq. (3.2.12) , Eq. (3.2.28) can be put in the form

$$\begin{aligned}\langle \hat{c}^2\rangle &= \langle\hat{a}_1^2(t)\rangle+\langle\hat{a}_1(t)\hat{a}_2(t)\rangle+\langle\hat{a}_2(t)\hat{a}_1(t)\rangle+\langle\hat{a}_2^2(t)\rangle\\ &\quad +\langle\hat{b}_1^2(t)\rangle+\langle\hat{b}_1(t)\hat{b}_2(t)\rangle+\langle\hat{b}_2(t)\hat{b}_1(t)\rangle+\langle\hat{b}_2^2(t)\rangle\\ &\quad +\langle\hat{a}_1(t)\rangle\langle\hat{b}_1(t)\rangle+\langle\hat{a}_1(t)\rangle\langle\hat{b}_2(t)\rangle+\langle\hat{a}_2(t)\rangle\langle\hat{b}_1(t)\rangle\\ &\quad +\langle\hat{a}_2(t)\rangle\langle\hat{b}_2(t)\rangle\langle\hat{b}_1^2(t)\rangle\langle\hat{a}_1(t)\rangle+\langle\hat{b}_1(t)\rangle\langle\hat{a}_2(t)\rangle\\ &\quad +\langle\hat{b}_2^2(t)\rangle\langle\hat{a}_2(t)\rangle+\langle\hat{b}_2^2(t)\rangle\langle\hat{a}_1(t)\rangle.\end{aligned} \quad (3.2.29)$$

For the case in which $\hat{a}_1, \hat{b}_1, \hat{a}_2$ and $\hat{b}_2$ are Gaussian operators with zero mean, we see that

$$\begin{aligned}\langle \hat{c}^2\rangle &= \langle\hat{a}_1(t)\hat{a}_2(t)\rangle+\langle\hat{a}_2(t)\hat{a}_1(t)\rangle\\ &\quad +\langle\hat{b}_1(t)\hat{b}_2(t)\rangle+\langle\hat{b}_2(t)\hat{b}_1(t)\rangle\end{aligned} \quad (3.2.30)$$

Substituting Eqs. (2.2.19) , (2.2.20) and (3.2.15) into (3.2.30) , we have

$$\langle \hat{c}^2\rangle = 0. \quad (3.2.31)$$

Now squaring the expectation value of $\hat{c}^2$, we find

$$\langle \hat{c}^2\rangle^2 = 0. \quad (3.2.32)$$

Finally, with the aid of Eqs.(3.2.21) and (3.2.32) the variance of the photon number for a pair of superposed twin two mode light beams

$$(\Delta n)^2 = 16\sinh^4 \varepsilon t + 16\sinh^2 \varepsilon t. \tag{3.2.33}$$

Eq. (3.2.33) can be rewritten as

$$(\Delta n)^2 = 4\bar{n} + \bar{n}^2. \tag{3.2.34}$$

We see from the above result, unlike mean photon number, the variance of the photon number for a pair of superposed twin light beams is not the sum of the constituent light beams and the photon statistics is super-poissonian

3.3 Quadrature fluctuation

In this section, we seek to determine the quadrature variance and quadrature squeezing for the superposed light beams.

3.3.1 Quadrature variance

Here we determine the quadrature variance for a pair of two-mode superposed light beams. We define the quadrature variance for a pair of superposed light beams by

$$(\Delta c\pm)^2 = \langle \hat{c_\pm}(t), \hat{c_\pm}(t)\rangle, \tag{3.3.1}$$

where

$$\hat{c_+} = \hat{c}^\dagger + \hat{c} \tag{3.3.2}$$

and

$$\hat{c_-} = i\left(\hat{c}^\dagger - \hat{c}\right), \tag{3.3.3}$$

are the plus and minus quadrature operators for the superposed light beams. With the aid of the commutation relation described by Eq. (3.2.18) , Eq. (3.3.1) can be put in the form

$$(\Delta c_\pm)^2 = 4 + \langle : \hat{c_\pm}(t), \hat{c_\pm}(t) :\rangle. \tag{3.3.4}$$

We note that 4 is the quadrature variance of a pair of superposed two-mode vacuum states. Then employing Eq. (3.3.2) and Eq. (3.3.3) , we can express Eq. (3.3.1) as

$$\begin{aligned}(\Delta c_\pm)^2 \;=\; & 4 + [2\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle \pm \langle \hat{c}^{\dagger 2}(t)\rangle \pm \langle \hat{c}^2(t)\rangle \\ & \mp\langle \hat{c}^\dagger(t)\rangle^2 \mp \langle \hat{c}(t)\rangle^2 - 2\langle \hat{c}^\dagger(t)\rangle\langle \hat{c}(t)\rangle]. \end{aligned} \tag{3.3.5}$$

In view of the fact that $\hat{c}(t)$ is Gaussian operator with zero mean, Eq. (3.3.5) reduces to

$$(\Delta c_{\pm})^2 = 4 + 2[\langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle \pm \langle \hat{c}^2(t)\rangle]. \tag{3.3.6}$$

Introducing Eq. (3.2.2) into (3.3.6) , we have

$$\begin{aligned}(\Delta c_{\pm})^2 &= 4 + 2[\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle \pm \langle \hat{a}^2(t)\rangle \\ &\pm \langle \hat{b}^2(t)\rangle \pm \langle \hat{a}(t)\hat{b}(t)\rangle \pm \langle \hat{b}(t)\hat{a}(t)\rangle]. \end{aligned} \tag{3.3.7}$$

We next calculate the expectation value of operator $\hat{a}\hat{b}$, employing the density operator for a pair of two-mode superposed signal-idler beams. We thus see that

$$\left\langle \hat{a}\hat{b} \right\rangle = Tr\left(\hat{\rho}\hat{a}\hat{b}\right). \tag{3.3.8}$$

Introducing Eq. (3.1.17) into (3.3.8) , we get

$$\begin{aligned}\left\langle \hat{a}\hat{b} \right\rangle &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \\ & Tr\left(|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle\langle \beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}\hat{b}\right). \end{aligned} \tag{3.3.9}$$

Applying the cyclic properties of trace, one can get

$$\begin{aligned}\left\langle \hat{a}\hat{b} \right\rangle &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \\ & \langle \beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a}\hat{b}|\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle, \end{aligned} \tag{3.3.10}$$

which can be re-written as

$$\begin{aligned}\left\langle \hat{a}\hat{b} \right\rangle &= \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\ & Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \\ & \langle \beta_2 + \beta_1 + \alpha_2 + \alpha_1|\hat{a_1}\hat{b_1} + \hat{a_1}\hat{b_2} + \hat{a_2}\hat{b_1} + \hat{a_2}\hat{b_2} \\ & |\alpha_1 + \alpha_2 + \beta_1 + \beta_2\rangle. \end{aligned} \tag{3.3.11}$$

Then it follows

$$\begin{aligned}
\left\langle \hat{a}\hat{b} \right\rangle \;=\; & \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\
& Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \alpha_1\beta_1 \\
& + \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\
& Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \alpha_1\beta_2 \\
& + \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\
& Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \alpha_2\beta_1 \\
& + \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 Q_1\left(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial \alpha_1^*}, t\right) \\
& Q_2\left(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial \alpha_2^*}, t\right) Q_1^{'}\left(\beta_1^*, \beta_1 + \frac{\partial}{\partial \beta_1^*}, t\right) \\
& Q_2^{'}\left(\beta_2^*, \beta_2 + \frac{\partial}{\partial \beta_2^*}, t\right) \alpha_2\beta_2 \qquad (3.3.12)
\end{aligned}$$

Then on account of Eq. (3.2.12) , we can put Eq. (3.3.12) in the form

$$\left\langle \hat{a}\hat{b} \right\rangle = \langle \hat{a_1}\hat{b_1} \rangle + \langle \hat{a_1}\hat{b_2} \rangle + \langle \hat{a_2}\hat{b_1} \rangle + \langle \hat{a_2}\hat{b_2} \rangle. \qquad (3.3.13)$$

Finally, we get

$$\left\langle \hat{a}\hat{b} \right\rangle = 0. \qquad (3.3.14)$$

Following similar procedure, one finds

$$\left\langle \hat{b}\hat{a} \right\rangle = 0. \qquad (3.3.15)$$

In view of Eqs. (3.3.14) and (3.3.15) , Eq. (3.3.7) tends to

$$(\Delta c_{\pm}(t))^2 = 2 + 2\langle \hat{a}^\dagger \hat{a} \rangle \pm 2\langle \hat{a}^2 \rangle + 2 + 2\langle \hat{b}^\dagger \hat{b} \rangle \pm 2\langle \hat{b}^2 \rangle. \qquad (3.3.16)$$

This can be re-written as

$$(\Delta c_{\pm}(t))^2 = (\Delta a_{\pm}(t))^2 + (\Delta b_{\pm}(t))^2. \qquad (3.3.17)$$

We easily observe that the quadrature variance of a pair of two-mode superposed light beams is 2 times that of the separate light beams. Substituting Eqs. (2.3.13) into Eq. (3.3.17) , we get

$$(\Delta c_{\pm}(t))^2 = 4 + 4\bar{n} \pm (-4\bar{n}\cosh^2 \varepsilon t). \tag{3.3.18}$$

From Eq. (3.3.18), we get

$$(\Delta c_{\pm}(t))^2 = [e^{2\varepsilon t} + e^{-2\varepsilon t}] \pm [e^{-2\varepsilon t} - e^{+2\varepsilon t}]. \tag{3.3.19}$$

In view of Eq. (3.3.19) , the plus and minus quadrature variances become

$$(\Delta c_{+})^2 = 2e^{-2\varepsilon t} \tag{3.3.20}$$

and

$$(\Delta c_{-})^2 = 2e^{2\varepsilon t}. \tag{3.3.21}$$

We clearly see that squeezing occurs in the plus quadrature

3.3.2 Quadrature squeezing

We finally proceed to calculate the quadrature squeezing for a pair of superposed light beams. The quadrature squeezing of a pair of two-mode superposed light beams is defined as [12]

$$S_{\pm} = \frac{4 - (\Delta c_{\pm})^2}{4}. \tag{3.3.22}$$

Substituting Eq. (3.3.20) into (3.3.22) , we obtain

$$S_{+} = \frac{4 - 2e^{-2\varepsilon t}}{4}, \tag{3.3.23}$$

$$S_{+}(t) = 1 - \frac{1}{2}e^{-2\varepsilon t}. \tag{3.3.24}$$

The quadrature squeezing of the superposed light beam is the same as that of the separate light beams. We note that for t=0, there is a 50% quadrature squeezing below the two-mode vacuum state level

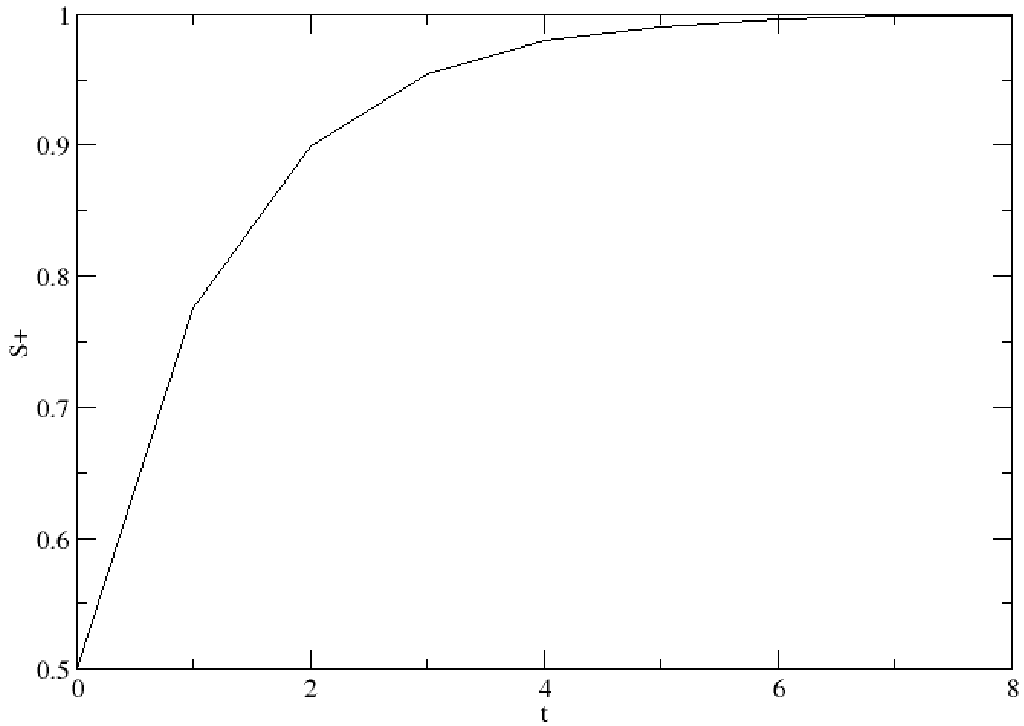

Figure 3.2: A plot of S+(t) Eq. (3.3.24) versus t for ε = 0.4

3.4 Entanglement analysis

In this section we seek to study the entanglement condition for a pair of superposed beams of light. Hence in order to show the entanglement of a pair of superposed beams of light . On the basis of this criteria, a pair of superposed beams of light is said to be entangled if the sum of the variance of the two EPR-like operators $\hat{s}$ and $\hat{t}$ satisfies the inequality[12].

$$(\Delta s)^2 + (\Delta t)^2 < 4, \tag{3.4.1}$$

where

$$\hat{s} = \frac{1}{2}\left(\hat{a_+} - \hat{b_+}\right), \tag{3.4.2}$$

$$\hat{t} = \frac{1}{2}\left(\hat{a_-} + \hat{b_-}\right), \tag{3.4.3}$$

with

$$\hat{a_+}(t) = \hat{a}^\dagger(t) + \hat{a}(t), \tag{3.4.4}$$

$$\hat{a_-}(t) = i\left(\hat{a}^\dagger(t) - \hat{a}(t)\right) \tag{3.4.5}$$

and

$$\hat{b_+}(t) = \hat{b}^\dagger(t) + \hat{b}(t), \tag{3.4.6}$$

$$\hat{b_-}(t) = i\left(\hat{b}^\dagger(t) - \hat{b}(t)\right). \tag{3.4.7}$$

The variance of the operators $\hat{s}$ and $\hat{t}$ can be expressed as

$$(\Delta s)^2 = \langle \hat{s^2}\rangle - \langle \hat{s}\rangle^2 \tag{3.4.8}$$

and

$$(\Delta t)^2 = \langle \hat{t^2}\rangle - \langle \hat{t}\rangle^2. \tag{3.4.9}$$

In view of the fact that $\hat{a}(t)$ and $\hat{b}(t)$ are Gaussian variables with zero mean and employing Eqs. (3.4.4) , (3.4.6) , and (3.4.8) , one can readily obtains

$$\begin{aligned}(\Delta s)^2 \;=\; & \frac{1}{2}[2 + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle \\ & + \langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b}^2\rangle - \langle\hat{a}^\dagger\hat{b}^\dagger\rangle - \langle\hat{a}\hat{b}\rangle - \langle\hat{a}^\dagger\hat{b}\rangle \\ & - \langle\hat{a}\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{a}^\dagger\rangle - \langle\hat{b}\hat{a}\rangle - \langle\hat{b}^\dagger\hat{a}\rangle - \langle\hat{b}\hat{a}^\dagger\rangle].\end{aligned} \tag{3.4.10}$$

Following the same procedure, we get

$$\begin{aligned}(\Delta t)^2 \;=\; & \frac{1}{2}[2 + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle - \langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle \\ & - \langle\hat{b}^{\dagger 2}\rangle - \langle\hat{b}^2\rangle - \langle\hat{a}^\dagger\hat{b}^\dagger\rangle - \langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}\rangle \\ & + \langle\hat{a}\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\hat{a}^\dagger\rangle - \langle\hat{b}\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}\rangle + \langle\hat{b}\hat{a}^\dagger\rangle].\end{aligned} \tag{3.4.11}$$

On account of Eqs. (3.4.10) and (3.4.11) , we see that the sum of the variance of the two EPR-like operators to be

$$(\Delta s)^2 + (\Delta t)^2 = 2 + 2\langle\hat{a}^\dagger\hat{a}\rangle + 2\langle\hat{b}^\dagger\hat{b}\rangle - \langle\hat{a}^\dagger\hat{b}^\dagger\rangle - \langle\hat{a}\hat{b}\rangle - \langle\hat{b}^\dagger\hat{a}^\dagger\rangle - \langle\hat{b}\hat{a}\rangle. \tag{3.4.12}$$

One can easily obtain

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a_1}^\dagger \hat{a_1} \rangle + \langle \hat{a_1}^\dagger \hat{a_2} \rangle + \langle \hat{a_2}^\dagger \hat{a_1} \rangle + \langle \hat{a_2}^\dagger \hat{a_2} \rangle \tag{3.4.13}$$

and

$$\langle \hat{b}^\dagger \hat{b} \rangle = \langle \hat{b_1}^\dagger \hat{b_1} \rangle + \langle \hat{b_1}^\dagger \hat{b_2} \rangle + \langle \hat{b_2}^\dagger \hat{b_1} \rangle + \langle \hat{b_2}^\dagger \hat{b_2} \rangle. \tag{3.4.14}$$

Substituting Eqs. (3.4.13) and (3.4.14) into (3.4.12), we get

$$(\Delta s)^2 + (\Delta t)^2 = 2 + \langle \hat{a_1}^\dagger \hat{a_1} \rangle + \langle \hat{a_2}^\dagger \hat{a_2} \rangle + \langle \hat{b_1}^\dagger \hat{b_1} \rangle + \langle \hat{b_2}^\dagger \hat{b_2} \rangle \tag{3.4.15}$$

Applying Eq. (3.2.14) Eq. (3.4.15), we get

$$(\Delta s)^2 + (\Delta t)^2 = 2 + 4\langle \hat{a_1}^\dagger \hat{a_1} \rangle \tag{3.4.16}$$

Substituting Eq. (2.2.7) into (3.4.16)

$$(\Delta s)^2 + (\Delta t)^2 = 2 + 4\sinh^2 \varepsilon t \tag{3.4.17}$$

The sum of the variance of the two EPR-like operators for a pair of superposed light beams is found

$$(\Delta s)^2 + (\Delta t)^2 = 2. \tag{3.4.18}$$

On the basis of the criteria Eq. (3.4.1) , we clearly see that a pair of superposed light beams are entangled at steady-state.

Chapter 4

Conclusion

In this thesis, we have analyzed the twin beams squeezed state with the same or different frequency must be represented in the Hamiltonian by the annihilation and creation operators. We first calculated the equation of evolution for the density operator for the twin light beams. Using this result we have found the operator dynamics for the twin signal(idler) light beams. Moreover, with the help of the operator dynamics, we have determined the Q function for the twin light beams. Finally, applying the Q function for twin light beams, we have calculated the photon statistics, quadrature fluctuation and entanglement for twin light beams.

Moreover, we have evaluated the density operator, Q function, photon statistics, quadrature variance, quadrature squeezing and entanglement for a pair of superposed twin squeezed states. With the aid of the resulting density operator we have calculated the mean photon number, the variance of the photon number, the quadrature variance, quadrature squeezing and photon entanglement.

Furthermore, we observe that the mean photon number for a pair of superposed twin light beams is the sum of that of the separate light beams. However, the variance of the photon number for a pair of superposed twin light beams does not happens to be the sum of that of the constituent light beams. And the photon statistics is super-Poissonian. On the other hand, the quadrature variance of superposed light beams is 2 times that of the separate light beams and we have found that the maximum quadrature squeezing for both separate and superposed light beams is 50% below the vacuum state level along with the squeezing occurs in the plus quadrature

Finally, the entanglement analysis reveals that photons in the superposed states are entangled and highly correlated.

Chapter 5

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