PROBLEM ISSUES IN THE UNIFICATION OF GRAVITY AND QUANTUM THEORIES

By

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The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a thesis entitled "**Problem issues in the unification of gravity and quantum theories**" by **Terefe Demessa Badada** in partial fulfillment of the requirements for the degree of **Master of Science in Physics(Astrophysics)**.

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Abstract

A theory of everything (TOE) that fully explains and links together all physical aspects of the universe is the interest of all scientific community. But, finding such unifying theory is the major unsolved problem in physics. However currently, all the laws of physics and aspects are closely framed in two theoretical groups General relativity (GR) and Quantum field theory (QFT) where GR focuses on gravity for understanding the large structure of the universe with high mass while QFT focuses on non-gravitational forces in regions of small scale and low mass system of the universe. Both theories are successful in their region of application while all attempts to unify them for a century has remained unsuccessful. So, here in this thesis we have reviewed the attempts so far made in the unification of the theories. In our review, we have tried to see how far the unification attempts have gone deep in connecting and accommodating the standard physics from both. Especially, We have worked out the GR field equations in that how it has being used to incorporate quantum fields and then checked the contents of the physics therein as of the quantum principles seek. On the other hand, we reviewed the attempts from quantum perspective how gravity is being fit in quantum theory without much mathematical frameworks without loss of the fundamental frames. The conclusion of the review work is that there is no single theory that fully unifies gravity and quantum theories. However, the attempts seem successful where to focuss in the resolution of the connection problem, the so called dark-matter and dark-energy sectors.

Key words: TOE; GR; QG; QFT.

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Chapter 1

Introduction

1.1 Background and Literature Review

Our present outlook in all respects is the whole sum progress knowledge of human beings through time. As modern history tells us the overall development of human beings mainly comes from understanding of nature through their physical principles. To day, the progress is reaching towards understanding of nature in its most unification phase of all physical laws under one physical theory. That means, different aspects of reality turn out to be manifestations of the same phenomena. Gravity was unified with astronomy, electricity with magnetism, electromagnetism with weak nuclear interactions, for detailed recent reviews refer. Nowadays, the most awaited is the unification of quantum theory and Einstein's gravity. Each step in the unification process requires the desertion of well-established schemes and gives rise to conceptual problems. Some of them result from theory's complexity and can be solved within it, others require a radical change of the paradigm. Both Einstein's and quantum theories suffer from conceptual problems: spacetime singularities, cosmological constant, wave function collapse. The hoped for unified theory of quantum gravity will necessarily lead to even more dramatic conceptual difficulties, as the very notions of point, time or causality become obscure. In the project, we will fathom out the conceptual problems lurking in unification theories with the help of sophisticated tools of mathematical physics enriched by a philosophical reflection. In particular, we hope to lift the veil on the following questions: Is dark energy, which according to the current cosmological model constitutes 68% of the Universe, only an artefact of Einstein's theory's complexity? What happens to the notions of time and space at the level of quantum gravity? One of the outstanding tasks in fundamental physics for theory of everything(TOE) fully explains and links together all physical aspects of the universe is the interest of all scientific community. The progressive physics for a theory of everything(TOE) focused two brands. Those are General Relativity(GR) and Quantum Field Theory(QFT). However currently, all the laws of physics and aspects are closely framed only in the two theory of everything(TOE) theoretical groups General relativity and Quantum field Theory. where General Relativity focus on gravity for under standing for the large scale structure of the universe and high mass, the field is continuum and also the Quantum Field Theory focus on the non-gravitational forces in the region of the small scale and low mass system of the universe. Quantum Field Theory (QFT) is the application of quantum mechanics to dynamical systems of fields, in the same sense that Quantum field theory is concerned mainly with the quantization of dynamical systems of particles. In QFT the field is discrete. The geometry is may be flat Minkowskian four-dimensional space time. But both a theory of everything(TOE) General relativity and Quantum field theories are successful in their region of application while all attempts to unify them for a century has remained unsuccessful. So, in this thesis we have revisited the attempts made earlier in the unification of the theories. The revolution in physics was so especially extent to in various disciplines that almost has brought the unification of the disciplines: Electromagnetism and special relativity, Quantum and electromagnetism, and fields that combine the three of these in more advanced and accurate way. Quantum theory is applicable to those particles which are so small and light that gravitational force between them can be neglected and General theory of relativity is applicable only to those objects which are so large and heavy that their quantum properties can be neglected [1]. In order to see where quantum gravity fits into the unification picture, it is helpful to briefly review the relations between some of the central theories of physics. Einstein after proposing general relativity thought that quantum effects must modify general relativity in his first paper on gravitational waves [2], although he switched to a different point of view working on the unification of electromagnetism and gravitation in the 1930. The derived wave equations are well in agreement to both theories both in their philosophical framework and their corresponding classical counterparts. However, we strongly recommend that the partial obscuration due to the dark components of the unverse still cast shadow on the completion of the unification. Klein argued that the quantum theory must ultimately modify the role of spatio-temporal concepts in fundamental physics [3-5] and his ideas were developed by Deser [6]. At the end of 1997, Isham pointed out several Structural Problems Facing Quantum Gravity Theory, at the beginning of this new century, the problem of quantizing the gravitational field was still open. In this work, we propose a new approach to Quantum Gravity. Starting from the generalization of the action function we have derived a theoretical background that leads to the quantization of gravity. However, recently there is much progress with the hope that tries to relate vacuum energy fluctuation in the quantum area and dark energy sector in the large scale area that involves cosmology, especially the cosmological constant is considered as the vacuum fluctuation energy source. Up to the present time, the introduced theories are so not yet refined and unripe that are awaiting further developments. Starting from the generalization of the action function we derive a theoretical background that leads to the quantization of gravity. Also, a complete description of the Electromagnetic Field, providing a consistent unification of gravity with electromagnetism [7]. Rosenfeld argued that there is no experimental need for quantizing gravity, it is better to stick with the so-called semi-classical gravity, which combines a classical description of the gravitational field with a quantum treatment of all other force fields and matter. Quantum gravity will imply that our usual classical notions of space and time are only approximate valid concepts, which some how emerge from the real quantum nature of space and time. So, the development of quantum theory is summarized, highlighting the contributions of Planck, Einstein, Schrodinger and Heisenberg. In this thesis,the main issues are how to the quantum gravity are unifies from the quantum field theory and General relativity. This does not yet lead to a unification of interactions; one arrives at a separate quantum theory for the gravitational field, in analogy to quantum electrodynamics. Dark energy and dark matter are a great mystery of the 20^{th} century physics, which has not been resolved yet within the paradigm of the contemporary physics. In retrospect, the resolution of a great puzzle requires the upheaval of a radical new physics. Quantum gravity, may be necessary to understand the nature of dark energy and dark matter [8]. However, recently there is much progress with the hope that tries to relate vacuum energy fluctuation in the quantum area and dark energy sector in the large scale area that involves cosmology, especially the cosmological constant is considered as the vacuum fluctuation energy source. Although the cosmological constant term Λ is a completely natural part of Einstein field equations, it encounters consistency or interpretation problems when particle physics, in its standard formulation, is taken into account [9-11]. In the usual Quantum Field Theory (QFT) approach, the Λ term cannot be distinguished from vacuum energy fluctuations.

The main objective of this thesis is to address current issues in the unification of gravity and quantum theories. The general introductory Chapter provides relevant review literature that addresses the issue of the thesis, objectives and Methodology. The out line of the this thesis is organized as follows: The thesis has five chapter. In the first chapter we deal with the theoretical back ground of General Theory of Relativity and the foundation to be implemented In chapter two we deal Quantum field Theory. In chapter three we deal with Hilbert-Lagrange Action To Develop The Hamiltonian Of Quantum Gravity.In chapter Four we deal Result and Discussion. In chapter Five we deal with Result and Discussion, the Summary and Future Plan.

1.2 Statement of The Problem

Since the beginning of the 20^{th} century the revolution in physics was so immense in various disciplines that almost has brought the unification of the disciplines: Electromagnetism and special relativity, Quantum and electromagnetism, and fields that combine the three of these in more advanced and accurate way. However, the large scale field that involves gravity and the small scale field that involves quantum theory has remained difficulties of unifying theory. However, recently there is much progress with the hope that tries to relate vacuum energy fluctuation in the quantum area and dark energy sector in the large scale area that involves cosmology, especially the cosmological constant is considered as the vacuum fluctuation energy source. Yet, the introduced theories are so not yet refined and immature that are awaiting further developments, debates, checks etc. So the issue is one of the most outstanding research problem, even probably, the one that needs more attention for the future of the generation.

1.2.1 Research Questions

- What are the major problems in the unification of all theories in physics?
- How will GR incorporate quantum fields?
- In what way will the unified theory be reduced to their corresponding classical limits?
- How much so far the unification attempts be progressed?

1.3 Objectives

1.3.1 General Objectives

- To address current issues in the unification of gravity and quantum theories.

1.3.2 Specific Objectives

- To identify the unification problems of gravity and quantum.
- To explain how Einstein field equations incorporate the vacuum energy fluctuation in quantum gravity as fundamental connection between quantum and gravity theories?
- To address the attempts made from quantum perspective in the unified theory and discuss their limitations.

1.4 Methodology

Critical literature reviewed is used to address in the current progress and issues the unifications. Especially, GR field equations be considered where the most general Hilbert-Lagrange action is used to explain its Hamiltonian content both for gravity and quantum theories. On the other hand, reviews on the attempts from quantum perspective that how gravity is being fit in quantum theory without much mathematical framework is given, without loss of the fundamental frameworks. Finally, we provide our concluding remark on the progress and future direction.

Chapter 2

Einstein Theory of General Relativity

2.1 The Framework Of General Theory Of Relativity

The general theory of relativity is Einstein's theory of gravity. The frame work of the General theory of relativity are: special theory of relativity,gravitation,general coordinate system. The geometry may be curved space time. In General Theory of Relativity there are two principles. Those are Einstein equivalency principle which implies the foundation of the General theory of relativity. The Einstein equivalency principle has been thoroughly tested with standard matter, the question of its validity in the Dark sector remains open. In this talk we will discuss the constraints of the Einstein equivalency principle in the Dark sector. We will place particular emphasis to the constrain an Einstein equivalency principle violation in the Dark sector. The framework of a general tensor-scalar theory with two different conformal couplings to standard matter and to Dark matter and the second principle is the Weak equivalency principle which implies the property of a body(inertial) that regulates it response to an external applied force be equal to its weight, the property that regulates it response to gravity.The strong one states that the outcome of any local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime. In general relativity, the gravitational force of Newtons theory that accelerates particles in an Euclidean space is replaced by a curved space-time in which particles move force-free along geodesic lines. In particular, as in special relativity along curves satisfying $ds^2 = 0$, while all effects of gravity are now encoded in the form of the line-element ds. Thus all information about the geometry of a space-time is contained in the metric $g_{\mu\nu}$. Einstein thought about the consequences of these principles for many years using many thought experiments. He then realized the importance of Riemannian geometry to construct a new theory in which the gravitational force was a result of the curvature of space-time. General covariant principle is the law of physics are invariant under general coordinate transformation system. In Newtonian gravity, the source of gravity is the mass. In general theory of relativity, the mass turns out to be part of a more general quantity called the energy-momentum tensor ($T_{\mu\nu}$), which includes both energy and momentum densities as well as stress. It is natural to assume that the field equation for gravity involves this tensor. The energy-momentum tensor is divergence free where its covariant derivative in the curved spacetime is zero ($\nabla^{\mu} T_{\mu\nu} = 0$) and the field is continuum. By finding a tensor on other side which is divergence free, this yields the simplest set of equations which are called Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.1.1}$$

where G is is the Newton gravitational constant, and c is the speed of light. $g_{\mu\nu}$ is defined as the spacetime metric. The spacetime metric captures all the geometric and causal structure of spacetime.

2.2 Metric Tensors

Flat Euclidian space Our common sense has taught us to think in terms of a at space metric (Euclidian), where parallel lines never cross and angles in a triangle always sum up to

180^o, thus strongly reinforcing our Newtonian notion of absolute space. In this formulation, the invariant line element in Cartesian coordinates of space $(x^1; x^2; x^3)$ is:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{2.2.1}$$

and space is assumed to be flat. Another way to write this is

$$ds^2 = \delta_{ij} dx^i dx^j \tag{2.2.2}$$

where δ_{ij} is the Kronecker delta function

$$\delta_{ij} = \begin{cases} 1, & \text{if i = j ;} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the Euclidian flat space metric tensor for Cartesian coordinates is given by:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2.3}$$

The geometry of curved spaces was studied in the 19^{th} century by Gauss, Riemann and others. Riemann realized that Euclidean geometry was just a particular choice suited to flat space and Mach realized that one had to abandon the concept of absolute space altogether. Einstein learned about tensors from his friend Marcel Gross-man, and used these key quantities to go from flat Euclidean three-dimensional space to curved Minkowskian four-dimensional space in which physical quantities are described by invariants. Tensors are quantities which provide generally valid relations between different four-vectors. The quantities that consists contra variant indices with corresponding Lorentz transformations are tensor. In tensor notation the Minkowski metric includes the coordinate $dx^0 = cdt$ and so that the invariant line element can be written as;

$$ds^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{2.2.4}$$

where the metric:-

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2.5}$$

where ds^2 is the Minkowsiki line element,and $g_{\mu\nu}$ is the metric tensor that represents position. The components of $g_{\mu\nu}$ in flat Minkowski space-time are given by the diagonal matrix $\eta_{\mu\nu}$, a generalization of the Kronecker delta function to four-space-time in which all non diagonal components are vanished.The basic objects of a metric are the Christoffel symbols, the Riemann and Ricci tensors as well as the Ricci scalars which are defined as follows:

2.3 The Christoffel Tensor

One of an invariant rank three Tensor derived from the metric $g_{\mu\nu}(x)$ its first derivative is the so called Christoffel Tensor which plays the role of gravitation. Affine connection is the field that determines the gravitational force and used as to represent the gravitational field. It also call as Christoffel second symbol which denoted as $\Gamma^{\lambda}_{\mu\nu}$. The metric tensor is use to determine the proper time interval between two events with a given infinitesimal coordinate separation and also the gravitational potential. It is given as:

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^{\alpha}\partial\xi^{\beta}}{\partial x^{\mu}\partial x^{\nu}} \tag{2.3.1}$$

and also we have the relation:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\tau}\left[\partial_{\nu}g_{\tau\nu} + \partial_{\mu}g_{\tau\nu} - \partial_{\tau}g_{\mu\nu}\right] \tag{2.3.2}$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial\xi^{\beta}}\frac{\partial^{2}x^{\beta}}{\partial x^{\mu}\partial x^{\nu}} \tag{2.3.3}$$

where; $\partial_{\nu}g_{\tau\nu} = \frac{\partial g_{\tau\nu}}{\partial x^{\mu}}$; ...

2.4 The Riemann-Christoffel Curvature Tensor

The invariant Tensor derived from the metric $g_{\mu\nu}$ is the 4-rank Tensor called the Riemann curvature tensor from the metric itself, first and second derivative. The Riemann curvature tensor plays an important role in specifying the geometrical properties of space time. The space time is considered flat, if the Riemann tensor vanishes everywhere. It is also possible to write the Riemann curvature tensor in its fully covariant form as

$$R^{\lambda}_{\mu\lambda\nu} = \Gamma^{\lambda}_{\mu\nu}, \lambda - \Gamma^{\lambda}_{\lambda\nu}, \mu + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\eta\nu} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\lambda} \qquad (2.4.1)$$

where $\Gamma^{\lambda}_{\mu\nu}$-is the Christoffel symbol.

2.5 Einstein Field Equations

The Einstein field equations were initially formulated in the context of a four-dimensional theory, some theorists have explored their consequences in N dimensions. The equations in contexts outside of general relativity are still referred to as the Einstein field equations. The vacuum field equations obtained when $T_{\mu\nu}$ is identically zero define Einstein manifolds. Dynamical dark energy can explain the size of the dark energy density. Despite the simple appearance of the equations they are, in fact, quite complicated. Given a specified distribution of matter and energy in the form of a stressenergy tensor, the Einstein Field Equation(EFE) are understood to be equations for the metric tensor, as both the Ricci tensor and scalar curvature depend on the metric in a complicated nonlinear manner.

Ricci Tensor:-is an important tool related to curvature , the second rank Ricci tensor $R_{\mu\nu}$, obtained from the Riemann tensor by summing operation over repeated indices, called contraction:

$$R^{\lambda}_{\mu\lambda\kappa} = R_{\mu\kappa} = g_{\alpha\beta}g^{\alpha\lambda}R^{\beta}_{\mu\lambda\kappa} \qquad (2.5.1)$$

Ricci tensor is symmetric. These implies that;

$$R_{\mu\kappa} = R_{\kappa\mu} \tag{2.5.2}$$

That is:-

$$g^{\lambda\nu}(R_{\lambda\mu\nu\kappa}) = R_{\nu\kappa\lambda\mu} \tag{2.5.3}$$

$$g^{\lambda\nu}R_{\lambda\mu\nu\kappa} = g^{\lambda\nu}R_{\nu\kappa\lambda\mu} \tag{2.5.4}$$

$$\Rightarrow R_{\mu\kappa} = R_{\kappa\mu} \tag{2.5.5}$$

In four-space the 10 components of the Ricci tensor lead to Einsteins system of ten gravitational equations.

Ricci Scalar: By further contracting the Ricci tensor with the contravariant components of the metric, one can express curvature scalar as:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\kappa}R_{\mu\kappa} \tag{2.5.6}$$

Where $R_{\mu\nu}$ is Ricci Tensor and the Ricci scalar R together with the metric $g_{\mu\nu}$ are used to construct an invariant, divergence free tensor that completely determine the geometry of spacetime. So it contains only terms which are either quadratic in the first derivatives of the metric tensor or linear in the second derivatives. Thus the Einstein's Field Equations were dynamic. Latter on Einstein brought an additional term with introduction of Cosmological constant Λ is considered as the vacuum fluctuation energy source or to keep the Universe static. However this modified field equation was solved by De-sitter to empty an evolving Universe. As the consequence De-sitter, established an empty Universe whose dynamism is related to the cosmological constant [12][13]. And also the Einsteins equation has the following form in the presence of the cosmological constant given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.5.7}$$

where $\Lambda > 0$, $R_{\mu\nu}$ is the Ricci tensor for $g_{\mu\nu}$, R is the Ricci scalar, $\frac{8\pi G}{c^4}T_{\mu\nu}$ is the source term and $T_{\mu\nu}$ is the stress-energy tensor. From the Einstein field equation we obtain $D_\nu T^{\mu\nu} = 0$ which are the equations of motion of the matter which creates the gravitational field. but do not admit a at spacetime as a possible matter-vacuum ($T^{\mu\nu};\nu = 0$) solution. For $T^{\mu\nu};\nu = 0$ and thus at least one component of the curvature tensor is nonzero. By transferring this term to right hand side one can interpret $\frac{c^4}{8\pi G}\Lambda g_{\mu\nu}$ as the energy-momentum of the vacuum, the so called Dark Energy favored in recent years by cosmological observations that indicate an accelerated expansion of the universe. So,One can always write the gravitational field equation in this form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.5.8}$$

Chapter 3

Quantum Field Theory

Introduction

As the term quantum field theory(QFT) suggests, QFT is the application of quantum mechanics to dynamical systems of fields, in the same sense that it concerned mainly with the quantization of dynamical systems of particles. Since about a century, the relation between quantum physics and gravitation is not fully understood. Quantum theory is very successful in nonrelativistic physics where precise mathematical results can be compared with experiments, somewhat less successful in elementary particle physics. General relativity, as the widely accepted theory of gravity, is excellently confirmed by astronomical data and deviations can be explained by plausible assumptions (dark matter, dark energy). However, finding a consistent theory which combines both general relativity and quantum physics is still an open problem. First steps for investigating the relation between these two fundamental aspects of nature are experiments with slow neutrons in the gravitational field of the earth. This can be treated as a problem in quantum mechanics with the Newtonian gravitational potential, and the experiments are in very good agreement with the theory. A more ambitious problem is the fluctuations of the cosmological microwave background which are explained by quantum fluctuations of the gravitational field in the inflationary era. On the theoretical side, the last decades were dominated by attempts to unify general relativity with quantum theory by rather radical new concepts, the best known being string theory and loop quantum gravity. In this report, however, we will concentrate on a much more modest goal to provide a consistent interpretation of existing experimental data in the situation when gravitational forces are weak. In such circumstances one can neglect the back reaction of quantum fields on the gravitational field. In this review article, we want to expose a systematic development of quantum field theory on curved spacetimes. The leading principle is the emphasis on local properties. It turns out that this requires a reformulation of the QFT framework which also yields a new perspective for the theories on Minkowski space.

3.1 Quantum Field Theory

The term quantum Field Theory (QFT) is the application of quantum mechanics to dynamical systems of fields, in the same sense that Quantum field theory is concerned mainly with the quantization of dynamical systems of particles. In QFT the field is discrete. Quantum Field Theory focus on non gravitational forces in the region of the small scale and low mass system of the universe. The geometry is may be flat Minkowskian four-dimensional space time. It has been suggested that such a locally finite version of Quantum Field Theory should be implemented by the Hilbert-Einstein action, schrodinger ,klein-Gordon space theory. Although the two theories are quite different in the way they approach the technical and physical problems that emerge when building a quantum picture of gravity and space-time. In particular, they differ in the way they handle the mathematical infinities that naturally occur in quantum descriptions of gravitational fields. The quantum field theory for electrodynamics is called quantum electrodynamics. If the weak interaction is included, it is called the electro weak theory. Quantum field theory (QFT), provides an account of all the known fundamental forces of nature. On the other hand, QFT is a theory of fields which are defined on a static, background spacetime. Basically, the theory states that all matter is composed of particles, which are understood as local excitations of quantum fields; the fundamental forces are themselves represented by quantum fields, whose corresponding excitations interact locally with the other particles, depending on their type [14]. Quantum gravity is a domain of research that, in some sense, unifies GR and quantum field theory. There are many different ways in which this may be interpreted. For instance, a quantum field theory focus on the non gravitation system, but general relativity focus on the gravitational system. A central role in the quantization of the gravitational field is played by the graviton a massless particle of spin-2, which is the mediator of the gravitational interaction. It is analogous to the photon in quantum electrodynamics.

3.2 Quantum Gravity

Quantum theory is a general theoretical framework to describe states and interactions in Nature. It does so successfully for the strong, weak, and electromagnetic interactions. Gravity is, however, still described by a classical theory - Einstein's theory of general relativity, also called geometrodynamics. So far, general relativity seems to accommodate all observations which include gravity; there exist some phenomena which could in principle need a more general theory for their explanation (Dark Matter, Dark Energy), but this is an open issue. So Quantum gravity would ultimately be a physical theory, both mathematically consistent and experimentally tested, that accommodates the gravitational interaction into the quantum framework. Such a theory is not yet available. Therefore, one calls quantum gravity all approaches which are candidates for such a theory or suitable approximations thereof. The following sections will first focus on the general motivation for constructing such a theory, and then introduce the approaches which at the moment look most promising. The meaning of quantum gravity Quantum gravity has been conjectured for almost

80 years since the introduction of the graviton. It is commonly believed that gravity is a fundamental interaction and as such, it would obey quantization similar to electrodynamics. However, it is significant to point out that there is not a single observational evidence so far showing the need of a quantum theory of gravity. On the theoretical side, despite enormous efforts in the past decades, there is still no consistent quantum gravity theory with any predictive power. The only great result from quantum gravity efforts over the last 50 years is the renormalization of Yang-Mills theory by t' Hooft and Veltman, using the techniques developed by Feynman and Dewitt for perturbative calculations in general relativity. It may be helpful to look at the problem of quantum gravity in a different perspective by asking what is not quantum gravity. At once, we see it is not about motion in spacetime, for motion with a precise trajectory is a classical concept. It is not about translation in spacetime. It is not about Lorentz transformation. It is not about elementary particles, or necessarily about unification of forces. Thus, quantizing general relativity with special relativity as a limit is a contradiction. We recognize that the quantum gravity domain is naturally of Planck size $10^{33}cm$. Furthermore, that quantum gravity domain must be physical and contain real degrees of freedom, and that those degrees of freedom should also carry energy, momentum, spin and other attributes like any other excellent degrees of freedom in physics. We should also point out that the term quantum gravity has come to acquire different meaning to different researcher. The proper case is that quantum gravity should start only with Einstein's equation and not some modification of it by adding extra terms to its Lagrangian as in modified gravity theories. We can illustrate this situation with electrodynamics. In quantum electrodynamics, the equations are exactly the same Maxwell's equations as in classical electrodynamics and not some modification of Maxwell's equations. It is well known that a remarkable modification of Maxwell's equations has become the Yang-Mills theory. Although the modified equations contain the original equations

in form, the theory is completely different and it is no longer electrodynamics. Quantizing a modified theory does not lead to a successful quantum theory of electrodynamics and its spectacular experimental confirmation. Therefore, current modified gravity theories are not what they intend to accomplish, namely, a quantum theory of gravitation based on general relativity. They may end up having nothing to do with ordinary gravity in the $\hbar \rightarrow 0$ limit. Quantum gravity changes radically space and time at the Planck scale, which is 10^{-35} of a meter. We used to think this was impossible to do experiments to probe this scale: "An accelerator powerful enough to study Planckian objects would have to be as large as the entire galaxy." Background dependent ordinary quantum mechanics the standard model assume the properties of space and time are fixed and unchanging background independent General relativity tells us that space and time are dynamical, The physics community is now familiar with a picture relying upon four fundamentals interactions: electromagnetic, weak, strong and gravitational. The large-scale structure of the universe, however, is ruled by gravity only. All unifications beyond Maxwell involve non-Abelian gauge groups (either Yang-Mills or Diffeomorphism group). At least three extreme views have been developed along the years, That is,

(i) Gravity arose first, temporally, in the very early Universe, then all other fundamental interactions.

(ii)Gravity might result from Quantum Field Theory (this was the Sakharov idea

(iii) The vacuum of particle physics is regarded as a cold quantum liquid in equilibrium. Protons, gravitons and gluons are viewed as collective excitations of this liquid

3.2.1 Unification of All Fundamental Interactions

- The fully established unifications of modern physics are as follows:

  Maxwell: electricity and magnetism are unified into electromagnetism. All related phenomena can be described by an antisymmetric rank-two tensor field, and derived from a one-form, called the potential.

Einstein: space and time are unified into the spacetime manifold. Moreover, inertial and gravitational mass, conceptually different,are actually unified as well.

Standard model of particle physics: electromagnetic, weak and strong forces are unified by a non-Abelian gauge theory, normally considered in Minkowski spacetime.

- All unifications beyond Maxwell involve non-Abelian gauge groups (either Yang-Mills or Diffeomorphism group).

- At least three extreme views have been developed along the years, That is,

  Gravity arose first, temporally, in the very early Universe, then all other fundamental interactions.

  Gravity might result from Quantum Field Theory (this was theSakharov). The vacuum of particle physics is regarded as a cold quantum liquid in equilibrium.

3.3 Space-Time Singularities

Now we revert to the geometric side. In Riemannian or pseudo-Riemannian geometry, geodesics are curves whose tangent vector x moves by parallel transport, so that eventually

$$\frac{dx^2}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0 \tag{3.3.1}$$

where $d\tau$ is proper time and $\Gamma^{\lambda}_{\mu\nu}$ are the affine connection. In general relativity, timelike geodesics correspond to the trajectories of freely moving observers, while null geodesics describe the trajectories of zero-restmass particles [15]. At a spacetime singularity in general relativity, all laws of classical physics would break down, because one would witness very pathological events such as the sudden disappearance of freely moving observers, and one would be completely unable to predict what came out of the singularity. Hawking and Penrose proved that spacetime singularities are generic properties of general relativity, provided that physically realistic energy conditions hold. Very little analytic use of the Einstein equations is made, whereas the key role emerges of topological and global methods in general relativity. Interestingly, near the singularity the spatial points essentially decouple, that is the evolution of the spatial metric at each spatial point is asymptotically governed by a set of second-order, non-linear ordinary differential equations in the time variable [16]. Moreover, the use of qualitative Hamiltonian methods leads naturally to a billiard description of the asymptotic evolution, where the logarithms of spatial scale factors define a geodesic motion in a region of the plane, interrupted by geometric reflections against the walls bounding the region.

3.4 Boundary Conditions

The ambiguity in the solutions of Einstein Field equation can be removed by choosing a particular gauge. We generally eliminate the ambiguity in the metric tensor by adopting some particular coordinate system. The choice of a coordinate system can be expressed as a coordinate conditions. One particularly convenient choice of a coordinate system is harmonic coordinate conditions. .

3.5 Weak Field Limit

Since astrophysical observations are made in the radiation zone it is sufficient to consider far field approximation with Quasi-minkowiskian coordinate system. Einsteins theory of general relativity leads to Newtonian gravity in the limit when the gravitational field is weak, static and the particles in the gravitational field move slowly. We now consider a less restrictive situation where the gravitational field is weak but not static, and there are no restrictions on the motion of particles in the gravitational field. In the absence of gravity, space-time is characterized by the Minkowiski metric in the limit of weak and slowly varying fields for which all time derivatives of $g_{\mu\nu}$ vanish and the velocity components $\frac{dx^0}{dt}$ are negligible compared with $\frac{dx^0}{dt} = c\frac{dt}{d\tau}$. Where $d\tau$ is the proper time, so the Geodesic equation becomes to

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}(\frac{dt}{d\tau})^2 = 0 \tag{3.5.1}$$

Also the Affine connection $\Gamma^\mu_{\nu\rho}$ is given by:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\lambda}[\partial_\rho g_{\nu\lambda} + \partial_\nu g_{\lambda\rho} - \partial_\lambda g_{\nu\rho}] \tag{3.5.2}$$

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}[\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00}] \tag{3.5.3}$$

$$= -\frac{1}{2}g^{\mu\lambda}[\partial_\lambda g_{00}] \tag{3.5.4}$$

Where g_{00} is the $\nu = \rho = 0$ or time time component of $g_{\mu\nu}$ and the sum over λ is implied. In a weak static field the metric is almost that of at space-time, so we can approximate the metric component of $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.5.5}$$

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} \tag{3.5.6}$$

$$= (\eta_{\mu\nu} + h_{\mu\nu})^{-1} \tag{3.5.7}$$

$$= (\eta_{\mu\nu})^{-1}(1 + \eta^{-1}_{\mu\nu}h_{\mu\nu}) \tag{3.5.8}$$

$$= (\eta^{\mu\nu})^{-1}(1 - \eta^{\mu\nu}h_{\mu\nu}) \tag{3.5.9}$$

$$= (\eta^{\mu\nu})(1 - \eta^{\mu\nu}h_{\mu\nu}) \tag{3.5.10}$$

$$= (\eta^{\mu\nu})(1 - h^\sigma_\sigma) \tag{3.5.11}$$

$$\Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{3.5.12}$$

Where $h_{\mu\nu}$ is small increment to $\eta_{\mu\nu}$, to lowest order in $h_{\mu\nu}$ we can then write affine connection as follows

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\frac{\eta^{\mu\lambda}\partial h_{00}}{\partial x^{\lambda}} \tag{3.5.13}$$

Now inserting (3.5.13) equation into equation (3.5.1), then the equation of motion we obtain is become to

$$\frac{d^2x^{\mu}}{dt} = -\frac{1}{2}(\frac{dt}{d\tau})^2\nabla^2 h_{00} \tag{3.5.14}$$

Now we have $dt = d\tau$

$$\Rightarrow \frac{dt}{d\tau} = \frac{dt}{dt} = 1 \tag{3.5.15}$$

and also we obtain

$$\frac{d^2x^{\mu}}{dt} = -\frac{1}{2}\nabla^2 h_{00} \tag{3.5.16}$$

$\Rightarrow h_{00} = -2\Phi$ where Φ is the Newtonian potential.so it follows that

$$g^{00} \approx 1 + 2\Phi = 1 - 2\frac{GM}{r} \tag{3.5.17}$$

So for c=1, poisson equation: $\nabla^2\Phi = 4\pi G\rho$;In fact $T^{00} = \rho$, This implies that

$$\nabla^2\Phi = -8\pi G T_{00} \tag{3.5.18}$$

Equation (3.5.18) generalized

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{3.5.19}$$

Where $G_{\mu\nu}$ is the general field tensor formed from linear combination of the metric it self, and its first and second derivative. In fact the weak field case,it reduces to that of the classical Newtonian poisson equation.

3.6 The Lorentz Gauge Transformation

Some theories are distinguished by being gauge invariant, which means that gauge transformations of certain terms do not change any observable quantities. Requiring gauge invariance provides an elegant and systematic way of introducing terms for interacting fields. Moreover, gauge invariance plays an important role in selecting theories. The prime example of an intrinsically gauge invariant theory is electrodynamics. In the potential formulation of Maxwells equations one introduces the vector potential A and the scalar potential, which are linked to the magnetic field and the electric field. We must choose a coordinate gauge. As it stands is completely general, and therefore does not impose any coordinate system. To find the form of $h_{\mu\nu}$, we must first choose a coordinate gauge. This coordinate gauge will impose four conditions on $h_{\mu\nu}$. We observe that it is possible to choose coordinates where the form of the background spacetime, $\eta_{\mu\nu}$, is conserved, but the (still undetermined) perturbative field is changed. Perturbation theory is a set of approximation schemes directly related to mathematical perturbation for describing a complicated quantum system in terms of a simple way. If one considers a coordinate transformation

$$x^\mu \rightarrow x'^\mu \quad = \quad x^\mu + \epsilon^\mu(x) \qquad (3.6.1)$$

Where the parameter of translation $\epsilon^\mu(x)$ is assumed to be infinitesimal and constant (global). Then that leaves the background spacetime untouched, but transforms the perturbative field as

$$h'_{\mu\nu} \quad = \quad h_{\mu\nu} - \varepsilon_{\nu,\mu} - \epsilon_{\mu,\nu} \qquad (3.6.2)$$

Choosing four coordinate conditions, $V_\mu(x) = V'_\mu(x) = 0$, can now be done using the arbitrary, but small, functions, $\epsilon^\mu(x)$. Since the background spacetime is unperturbed, we know that the change in $h_{\mu\nu} \rightarrow h'_{\mu\nu}$ is the same change as the one in $g_{\mu\nu} \rightarrow g_{\mu\nu}$. We may choose the Lorentz gauge

$$\partial^\lambda h_{\mu\lambda} = \frac{1}{2}\partial_\mu h \qquad (3.6.3)$$

This can be further transformed to a trace reversed version is given the following ways:

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad (3.6.4)$$

3.7 The Scalar Field

It is straightforward to derive the equations of motion for a real scalar field ϕ from the lagrangian densities,

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \qquad (3.7.1) \\ &= -\frac{1}{2}\phi(\partial_\mu\partial^\mu - m^2)\phi \end{aligned}$$

which differ only by surface terms, leading to

$$(\Box + m^2)\phi(x) = 0 \qquad (3.7.2)$$

For the complex scalar field one conventionally uses

$$L = \partial_\mu\phi^*\partial^\mu\phi - \frac{1}{2}m^2\phi^*\phi$$

which can be considered as the sum of the lagrangian densities for two real scalar fields ϕ_1 and ϕ_2 with $\phi = (\phi_1 + i\phi_2)2$. One easily obtains

$$(\Box + m^2)\phi(x) = 0 \qquad (3.7.3)$$

$$(\Box + m^2)\phi^*(x) = 0 \qquad (3.7.4)$$

Chapter 4

Hilbert-Lagrange Action To Develop The Hamiltonian For Gravity and Quantum Theories

4.1 The Hilbert-Einstein Action

To derive the equation that yields the Einstein field equations as its equations of motion, we initially consider the form that the action must take, namely the integral of a scalar Lagrange density. Since derivatives lower the order on the field upon which it acts by one, this Lagrange density should contain at least two derivatives of the metric to ensure that the equation of motion for the metric field which is what we are ultimately interested in when trying to find the dynamics of spacetime curvature is at least linear. Since any nontrivial tensor made from the metric and its derivatives can be expressed in terms of the metric and the Riemann tensor, the only independent scalar that can be constructed from the metric that is no higher than second order in its derivatives is the Ricci scalar (as this is the unique scalar that we can construct from the Riemann tensor that is itself made from second derivatives of the metric). The Einstein field equation can be derived by varying the

Hilbert-Einstein action as

$$S_{HE} = -\int d^4x\sqrt{-g}\frac{c^4}{16\pi G_N}(R + 2\frac{\Lambda}{c^2}) \quad (4.1.1)$$

$$S = S_{HE} + S_{matter} \quad (4.1.2)$$

$$S_{matter} = \int d^4x\sqrt{-g}L_{matter} \quad (4.1.3)$$

Where $g = det[g_{\mu\nu}]$ is the determinant of the metric tensor, **R** is the Ricci curvature scalar, Λ is the cosmological term, and $\sqrt{-g}L_{matter}$ is the matter field Lagrangian. Note that the Hilbert-Einstein action equation (4.1.1) is the most general action which transforms as a scalar under general coordinate transformations, and which contains terms up to second order in derivatives of the metric tensor. There are two unspecified constants in the action (4.1.1), which are not determined by the symmetry (general covariance). The second constant is proportional to the cosmological term Λ, and it can be determined by considering the dynamics of gravitating bodies on very large (cosmological) scales. This illustrates how powerful the principle of general covariance can be when constructing the gravitational action.In order to calculate the variation δS of the action (4.1.1), we first observe that

$$\delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu} \quad (4.1.4)$$

which immediately implies

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \quad (4.1.5)$$

Recalling that $R = g^{\mu\nu}R_{\mu\nu}$ yields the following intermediate result for the variation of the Hilbert- Einstein action

$$\begin{aligned}
\delta S_{HE} &= \delta(-\int d^4x\sqrt{-g}\frac{c^4}{16\pi G_N}(R+2\frac{\Lambda}{c^2})) \\
&= -\int d^4x\delta\sqrt{-g}(\frac{c^4}{16\pi G_N})(R+2\frac{\Lambda}{c^2}) + (-\int d^4x\sqrt{-g}\frac{c^4}{16\pi G_N}\delta R) \\
&= \int d^4x(\frac{-c^4}{16\pi G_N})[\delta\sqrt{-g}(R+2\frac{\Lambda}{c^2}+\delta R)] \\
&= \int d^4x(-\frac{c^4}{16\pi G_N})[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}(R+2\frac{\Lambda}{c^2}+\delta R)] \\
&= \int d^4x(-\frac{c^4}{16\pi G_N})[\frac{-1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}(g^{\mu\nu}R_{\mu\nu}+2\frac{\Lambda}{c^2}+\delta(g^{\mu\nu}R_{\mu\nu}))] \\
&= \int d^4x\sqrt{-g}(-\frac{c^4}{16\pi G_N})[\delta g^{\mu\nu}-\frac{1}{2}(R_{\mu\nu}+2\frac{\Lambda}{c^2}+\delta(g^{\mu\nu})R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu})] \\
&= \int d^4x\sqrt{-g}(-\frac{c^4}{16\pi G_N}\delta g^{\mu\nu}(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{\Lambda}{c^2}g_{\mu\nu})+\frac{c^4}{16\pi G_N}g^{\mu\nu}\delta R_{\mu\nu})
\end{aligned} \tag{4.1.6}$$

The variation of the Ricci tensor $\delta R_{\mu\nu}$ can be easily found by transforming to a local Minkowski frame, in which $g_{\mu\nu} \to \eta^{\mu\nu} + O(\partial_\alpha g_{\mu\nu})$ such that we have

$$\delta R_{\mu\nu} = \delta R^{\alpha}_{\mu\alpha\nu} \simeq \delta\partial_\alpha\Gamma^{\alpha}_{\mu\nu} - \delta\partial_\nu\Gamma^{\alpha}_{\mu\alpha} \tag{4.1.7}$$

This then implies

$$g^{\mu\nu}\delta R_{\mu\nu} = \simeq \partial_\alpha(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}) - \partial_\nu(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\alpha}) \tag{4.1.8}$$

where we inserted $g^{\mu\nu}$ inside the derivatives, which is legitimate in the local Minkowski frame. Since the left-hand side of Equation(4.1.8) is a scalar, the right-hand side must also be a scalar, which implies that the covariant form of Equation (4.1.8) must read,

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\beta}_{\mu\beta}) \tag{4.1.9}$$

This has the form of a covariant divergence, such that upon integration over an invariant measure in Equation (4.1.6), the variation of the Ricci curvature tensor does not contribute to the Einstein field equation,

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \;=\; \int d^4x\sqrt{-g}\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta_{\mu\beta}) = 0 \qquad (4.1.10)$$

The last equality follows from the simple observation that the covariant divergence of the contravariant vector appearing in (4.1.10) can be also written as

$$\nabla A \;\equiv\; \nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \Gamma^\alpha_{\beta\alpha}A^\beta \qquad (4.1.11)$$

$$= \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}A^\alpha); \qquad (4.1.12)$$

$$\Gamma^\alpha_{\beta\alpha} = \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g} \qquad (4.1.13)$$

By taking account of the intermediate results equation(4.1.6) and equation (4.1.10),we arrive at the following form for the variation of the action of equation (4.1.1) up to (4.1.3),

$$\delta S \;=\; \int d^4x\sqrt{-g}\delta g^{\mu\nu}[-\frac{c^4}{16\pi G_N}(G_{\mu\nu} - g_{\mu\nu}\frac{\Lambda}{c^2}) + \frac{1}{2}T_{\mu\nu}] \qquad (4.1.14)$$

Now requiring that δS vanishes for an arbitrary variation of the metric tensor $\delta g^{\mu\nu}$ yields the Einstein field equation.

$$\delta S \;=\; \int(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d^4x + \int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g} \qquad (4.1.15)$$

Reim tensor is: Contracting to the Ricci tensor it becomes, Then varying it gives,

$$\delta R_{\mu\nu} \;=\; \partial_\alpha\delta\Gamma^\alpha_{\mu\beta} - \partial_\beta\delta\Gamma^\alpha_{\mu\alpha} + \delta\delta\Gamma^\alpha_{\gamma\alpha}\Gamma^\gamma_{\mu\beta} + \Gamma^\alpha_{\gamma\alpha}\delta\Gamma^\gamma_{\mu\beta} - \delta\Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\alpha} - \Gamma^\alpha_{\gamma\beta}\delta\Gamma^\gamma_{\mu\alpha} \qquad (4.1.16)$$

The first two terms of this expression suggest the difference between two covariant derivatives, which is the case given

$$\nabla_\alpha(\delta\Gamma^\alpha_{\mu\beta}) \;=\; \partial_\alpha(\delta\Gamma^\alpha_{\mu\beta}) + \delta\Gamma^\gamma_{\mu\beta}\Gamma^\alpha_{\gamma\alpha} - \delta\Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\alpha} - \delta\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\alpha} \qquad (4.1.17)$$

$$\nabla_\beta(\delta\Gamma^\alpha_{\mu\alpha}) \;=\; \partial_\alpha(\delta\Gamma^\alpha_{\mu\alpha}) + \delta\Gamma^\gamma_{\mu\alpha}\Gamma^\alpha_{\gamma\beta} - \delta\Gamma^\alpha_{\gamma\alpha}\Gamma^\gamma_{\mu\beta} - \delta\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\alpha\beta} \qquad (4.1.18)$$

Then

$$\begin{aligned}\nabla_\alpha(\delta\Gamma^\alpha_{\mu\beta}) - \nabla_\beta(\delta\Gamma^\alpha_{\mu\alpha}) &= \partial_\alpha(\delta\Gamma^\alpha_{\mu\beta}) - \partial_\alpha(\delta\Gamma^\alpha_{\mu\alpha}) + \delta\Gamma^\alpha_{\gamma\alpha}\Gamma^\gamma_{\mu\beta} + \delta\Gamma^\alpha_{\alpha}\Gamma^\gamma_{\mu\beta} - \delta\Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\alpha} - \Gamma^\alpha_{\gamma\beta}\delta\Gamma^\gamma_{\mu\alpha} \\ &= \delta R_{\mu\beta}\end{aligned}$$

so that we have

$$\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\beta}) - \nabla_\beta(\delta\Gamma^\alpha_{\mu\beta}) \quad (4.1.19)$$

Reballing and index gives the term needed for the action integral given earlier

$$\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha}) \quad (4.1.20)$$

How did we get this result? let's begin with the following expressions;

$$\nabla = e^\lambda\partial_\lambda \quad (4.1.21)$$

$$\partial_\beta e_\alpha = \Gamma^\sigma_{\alpha\beta\delta}e_\sigma \quad (4.1.22)$$

$$\partial_\beta e^\alpha = -\Gamma^\alpha_{\sigma\beta}e^\sigma \quad (4.1.23)$$

Now let's express the argument of the derivative in tensor basis form and carryout appropriate operations.

$$\begin{aligned}\nabla(\delta\Gamma^\rho_{\nu\mu}) &= e^\lambda\partial_\lambda \otimes (\delta\Gamma^\rho_{\nu\mu})e_\rho \otimes e^\nu \otimes e^\mu[tosavespacewewilldrop\otimes] \\ &= e^\lambda\partial_\lambda(\delta\Gamma^\rho_{\nu\mu})e_\rho e^\nu e^\mu + e^\lambda\delta\Gamma^\rho_{\nu\mu}\partial_\lambda(e_\rho)e^\nu e^\mu + e^\lambda\delta\Gamma^\rho_{\nu\mu}e_\rho\partial_\lambda(e^\nu)e^\mu + e^\lambda\delta\Gamma^\rho_{\nu\mu}e_\rho e^\nu\partial_\lambda(e^\mu) \\ &= e^\lambda\partial_\lambda(\delta\Gamma^\rho_{\nu\mu})e_\rho e^\nu e^\mu + e^\lambda\delta\Gamma^\rho_{\nu\mu}\Gamma^\alpha_{\rho\lambda}e_\alpha e^\nu e^\mu - e^\lambda\delta\Gamma^\rho_{\nu\mu}\Gamma^\nu_{\alpha\lambda}e^\alpha e_\rho e^\mu - e^\lambda\delta\Gamma^\rho_{\nu\mu}\Gamma^\mu_{\alpha\lambda}e^\alpha e_\rho e^\mu \\ &= \partial_\lambda(\delta\Gamma^\rho_{\nu\mu})e^\lambda e_\rho e^\nu e^\mu + \delta\Gamma^\alpha_{\nu\mu}\Gamma^\rho_{\alpha\lambda}e^\lambda e_\rho e^\nu e^\mu - \delta\Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\nu\lambda}e^\lambda e^\nu e_\rho e^\mu - \delta\Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\lambda}e^\lambda e^\mu e_\rho e^\nu \\ &= [\partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \delta\Gamma^\alpha_{\nu\mu}\Gamma^\rho_{\alpha\lambda} - \delta\Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\nu\lambda} - \delta\Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\lambda}]e^\lambda e_\rho e^\mu e^\nu\end{aligned}$$

Now the component form is,

$$\nabla(\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \delta\Gamma^\alpha_{\nu\mu}\Gamma^\rho_{\alpha\lambda} - \delta\Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\nu\lambda} - \delta\Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} \quad (4.1.24)$$

From which we can get by substitution of indices

$$\nabla_\alpha(\delta\Gamma^\alpha_{\mu\beta}) = \partial_\alpha(\delta\Gamma^\alpha_{\mu\beta}) + \delta\Gamma^\gamma_{\mu\beta}\Gamma^\alpha_{\gamma\alpha} - \delta\Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\alpha} - \delta\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\alpha} \quad (4.1.25)$$

$$\nabla_\beta(\delta\Gamma^\alpha_{\mu\alpha}) = \partial_\alpha(\delta\Gamma^\alpha_{\mu\alpha}) + \delta\Gamma^\gamma_{\mu\alpha}\Gamma^\alpha_{\gamma\beta} - \delta\Gamma^\alpha_{\gamma\alpha}\Gamma^\gamma_{\mu\beta} - \delta\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\alpha\beta} \quad (4.1.26)$$

$$\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x = \int g^{\mu\nu}(\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha}))\sqrt{-g}d^4x \quad (4.1.27)$$

$$= \int(\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha}))\sqrt{-g}d^4x \quad (4.1.28)$$

$$= \int(\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\alpha(g^{\mu\alpha}\delta\Gamma^\nu_{\mu\nu}))\sqrt{-g}d^4x \quad (4.1.29)$$

$$= \int \nabla_\alpha[(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - (g^{\mu\alpha}\delta\Gamma^\nu_{\mu\nu})]\sqrt{-g}d^4x \quad (4.1.30)$$

$$= \int \nabla_\alpha A^\alpha\sqrt{-g}d^4x \quad (4.1.31)$$

This look like the the divergence theorem is needed here to evaluate this integral. The vector form the theorem is expressed as

$$\int\int\int_v \nabla A dv = \oint_s A\widehat{n}ds \quad (4.1.32)$$

In generalized coordinates it is

$$\int_v \nabla_\alpha A^\alpha\sqrt{-g}d^4x = \int_s A^\alpha n_\alpha\sqrt{-h}d^3x \quad (4.1.33)$$

From the above equation $\sqrt{-h}d^3x$ -is induced arbitrary metric. So the last surface integral is taken over the boundary of space-time where we have set the variation to be zero. So we have

$$\int_v \nabla_\alpha A^\alpha\sqrt{-g}d^4x = \int_s A^\alpha n_\alpha\sqrt{-h}d^3x = 0 \quad (4.1.34)$$

$$\Rightarrow \int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x = 0 \quad (4.1.35)$$

This gives us the following variation in the action,

$$\delta S = \int(R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d^4x \quad (4.1.36)$$

We derive the resulting field equations for the metric tensor $g_{\mu\nu}$ directly from the action principle

$$\delta S_{EH} = \delta \int_\Omega d^4x\sqrt{-g}(R - 2\Lambda) = \delta \int d^4x\sqrt{-g}(g^{\mu\nu}R_{\mu\nu} - 2\Lambda) = 0 \quad (4.1.37)$$

It is worthwhile pointing out that in general relativity geometry is the central idea and the theory is covariant in its description of nature. We allow for variations of the metric gab restricted by the condition that the variation of $g_{\mu\nu}$ and its first derivatives vanish on the boundary ∂_Ω .

$$\delta S_{EH} = \delta \int_\Omega d^4x\sqrt{-g}(g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + (R - 2\Lambda)\delta\sqrt{-g} \quad (4.1.38)$$

Our task is to rewrite the first and third term as variations of $\delta g_{\mu\nu}$ or to show that they are equivalent to boundary terms. Finally we obtain

$$\delta S_{EH} = \int_\Omega d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu})\delta g^{\mu\nu} = 0 \quad (4.1.39)$$

Hence the metric tensor fulfils in vacuum the equation

$$\frac{1}{2}\sqrt{-g}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \quad (4.1.40)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (4.1.41)$$

Where we introduced the Einstein tensor $G_{\mu\nu}$. The constant Λ is called cosmological constant.

4.2 Derivation Of The Klein-Gordon Equation

In the mid-1920s, physicists were intrigued by the idea of being able to mathematically describe a particle using relativistic quantum mechanics. After Erwin Schrodinger derived his famous wave equation, the Klein-Gordon equation surfaced within the community. The equation was named after the physicists Oskar Klein and Walter Gordon, who in 1926 proposed that it describes relativistic electrons. Although it turned out that modeling the electron's spin required the Klein-Gordon equation correctly describes the spinless relativistic composite particles. The Klein-Gordon equation was first considered as a quantum wave equation by Schrodinger in his search for an equation describing de Broglie waves. Apparently, he fell under the wrong spectrum and abandoned his work and reverted to a classical argument in terms of the energy For a particle moving at a relativistic speed, the nonrelativistic equation for fractional kinetic energy, $E = \frac{p^2}{2m}$, is no longer valid, and one must use instead an alternative equation for energy. We begin with the relativistic connection between energy and momentum, often called the Einstein's energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4 \tag{4.2.1}$$

Now by using de Broglie and Einstein relations $p = \hbar k$; $E = \hbar\omega$ implies the quantum operator interpretation as

$$P \rightarrow -i\hbar\nabla \quad ; \quad E \rightarrow i\hbar\frac{\partial}{\partial t} \tag{4.2.2}$$

Then we substituting equation (4.2.2) into equation (4.2.1) then we obtained

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi = \hbar^2c^2\nabla^2\psi + m^2c^4\psi \tag{4.2.3}$$

and also In covariant notation

$$[\partial_\mu\partial^\mu + (\frac{mc}{\hbar})^2] = 0 \tag{4.2.4}$$

where $\partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$; $\psi = \psi(x,t)$ Rearranging the terms and simplifying gives

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = (\frac{mc}{\hbar})^2 \tag{4.2.5}$$

Notice that for massless particles, this is reduced down to the regular wave equation. It would simply be the time component subtracted from the spatial component, where c^2 is the wave number corresponding to electromagnetic waves. We can write down the characteristics of relativity such that $x^2 - c^2t^2 = 0$ which describes traveling waves that are invariant under Lorentz transformation. That is to say, the Klein-Gordon equation is covariant. We now let $k = \frac{m^2c^2}{\hbar}$ and introduce the D'Alembertian Operator as

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Collectin all the terms gives

$$\Box\psi - k^2\psi = 0, \tag{4.2.6}$$

This is the famous Klein-Gordon equation for spinless free particles. Such particle is said to be on its mass shell. So Klein-Gordon equation is second order in time derivative and the norm of a state is not in general time independent. So it is perhaps convenient to physically interpret the Klein-Gordon equation is a relativistic wave equation, meaning that it is a quantized version of the energy-momentum relation. We expect solutions to include a quantum oscillation, precisely for particles of spin-zero.Now we seek for solution for ψ as a product in which the dependence on x, y, z, and t are separated, that is:

$$\psi(x,t) = X(x)Y(y)Z(z)T(t) \tag{4.2.7}$$

Finally get a plane wave solution to the Klein-Gordon equation that corresponds to

$$\psi(x,t) = \Lambda exp[i(\pm k.x \mp \omega t)]; \tag{4.2.8}$$

which corresponds to waves traveling right and left.

- **Scalar Field**

A scalar field is a smoothly varying mathematical function that assigns a value to every point in space. An example of a scalar field in classical physics is the gravitational field that describes the gravitational potential of a massive object. Examples include: Potential fields, such as the Newtonian gravitational potential, or the electric potential in electrostatics, are scalar fields which describe the more familiar forces. Scalar field changes its value even in the present cosmological epoch Scalar field changes its value even in the present cosmological epoch Dynamical dark energy, generated by scalar field. Now let we consider the scalar as; We calculate first the dynamical energy-momentum tensor for a scalar field with potential $V(\phi)$

$$\delta S_{KG} = \frac{1}{2}\int d^4x\sqrt{|\, g\,|}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu} + [g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)]\delta\sqrt{|\, g\,|} \quad (4.2.9)$$

$$= \int d^4x\sqrt{|\, g\,|}\delta g^{\mu\nu}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}L \quad (4.2.10)$$

Remember that $\nabla_\mu = \partial_\mu$ for a scalar field. Varying the action gives and thus

$$T_{\mu\nu} = \frac{2}{\sqrt{|\, g\,|}}\frac{\delta s_m}{\delta g^{\mu\nu}} \quad (4.2.11)$$

$$= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}L \quad (4.2.12)$$

Let's now consider time that is not empty but contains field.All we need to do is add a second action term S_ϕ representing the matter to the Einstin-Hilbert action(S_{EH}). So our new action is

$$S = \frac{c^4}{2(8\pi G)}S_{EH} + S_m \quad (4.2.13)$$

The expression of the energy-momentum tensor $T_{\mu\nu}$ and is defined as

$$T_{\mu\nu} = 2\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \quad (4.2.14)$$

If the source of the gravitational field is some matter field, or some other field (for instance an electromagnetic field), the corresponding equations can be found by varying the total action with respect to the metric tensor, so

$$g\delta S_{EH} + \delta S_{field} = 0 : \forall\delta g^{\mu\nu} \quad (4.2.15)$$

Now We consider we combined action of gravity and field, as the sum of the Einstein- Hilbert Lagrange density $L_{EH}/2k$ and the Lagrange density L_m including all relevant fields,

$$L = \frac{1}{2k}L_{EH} + L_\phi \quad (4.2.16)$$

$$= \frac{1}{2k}\sqrt{-g}(R - 2\Lambda) + L_\phi \quad (4.2.17)$$

In L_ϕ, the effects of gravity are accounted for by the replacements $\partial_\mu, \eta_{\mu\nu} \to \nabla_\mu, g_{\mu\nu}$, while we have to adjust later the constant k such that we reproduce Newtonian dynamics in the weak-field limit. We expect that the source of the gravitational field is the energy-momentum tensor. More precisely, the Einstein tensor should be determined by the matter, $G_{\mu\nu} = kT$. Since we know already the result of the variation of S_{EH}, we conclude that the variation of S_ϕ should given as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} \quad (4.2.18)$$

Here we have defined $\rho_{\Lambda vac}$, the energy density The tensor $T_{\mu\nu}$ defined by this equation is called dynamical energy-momentum tensor. In order to show that this rather bold definition makes sense, and we have to convince ourselves that this definition reproduces the standard results we know already. So, the Einstein field equation is given as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + kT_{\mu\nu} = 0 \quad (4.2.19)$$

The matter and geometry of spacetime and cosmological constant (Λ) are related by the Einstein field equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = KT_{\mu\nu}$, where $G_{\mu\nu}$ is Einsteins field tensor that tales geometry of spacetime and $T_{\mu\nu}$ is energy-momentum tensor of matter. Recall that the field equations in the presence of cosmological constant given as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (4.2.20)$$

Or rearranged as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G(T_{\mu\nu} - T_{\mu\nu}^{vacuum}) \quad (4.2.21)$$

The cosmological constant Λ. On the right side the former term is due to matter and the latter to the vacuum fields, both stress-energy tensors, $T_{\mu\nu}$ and $T_{\mu\nu}^{vacuum}$, depending on spacetime coordinates. The variation of the vacuum tensor takes into account that vacuum fields fluctuate, the energy of the vacuum fluctuations would be of order the vacuum energy itself. It seems more plausible that the cut-off momentum is far smaller, of order the typical particle scales.

4.3 Quantum Theory Of Gravitation

At present there does not exist any complete and self-consistent quantum theory of gravitation, and it would be out of place in this book to describe in detail the attempts that have been made to construct such theory. However, it will be possible and it may be useful to give the reader some taste of what a quantum theory of gravitation would be like. To start at the simplest level, we would interpret a gravitational plane wave, with wave k_μ and helicity ± 2, as consisting of gravitons: quanta with energy-momentum vector $p^\mu = \hbar k^\mu$ and spin component in the direction of motion $\pm 2\hbar$. (Here $\hbar = 1.054 \times 10^{-27} ergsec$). Since $k_\mu k^\mu = 0$, the graviton is a particle of zero mass, like the photon and neutrino [17].

Recall the Energy-momentum tensor is given by

$$T^{\alpha\beta}(x,t) = \sum_n \frac{p_n^\alpha p_n^\beta}{E} \delta^3(x - x_n(t)) \tag{4.3.1}$$

Then energy-momentum tensor of an assembly of gravitons, all of which have four-momentum $p^\mu = \hbar k^\mu$, is

$$\Rightarrow T_{\mu\nu} = \sum_n \frac{(\hbar k_\mu)(\hbar k_\nu)}{E} \delta^3(x - x_n(t)) \tag{4.3.2}$$

$$= \sum_n \frac{(\hbar k_\mu)(\hbar k_\nu)}{\hbar\omega} \delta^3(x - x_n(t)) \tag{4.3.3}$$

$$\Rightarrow T_{\mu\nu} = \frac{\hbar k_\mu k_\nu N}{\omega} \tag{4.3.4}$$

Where N is the number of gravitations per unit volume. Comparing this with our result for a gravitational plane wave,

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{16\pi G}(|e_+|^2 + |e_-|^2) \tag{4.3.5}$$

We conclude that the number density of gravitations with helicity ± 2 in a plane wave is

$$N_\pm = \frac{\omega}{16\pi\hbar G}(|e_\pm|^2 \tag{4.3.6}$$

The total number density is

$$N = N_- + N_+ \tag{4.3.7}$$

$$= \frac{\omega}{16\pi\hbar G}(e^{\lambda\nu} * e_{\lambda\nu} - \frac{1}{2}|e^\lambda_\lambda|^2) \tag{4.3.8}$$

4.4 Energy and Momentum of Plane Wave

Let's the Energy-momentum tensor is given by

$$T^{\alpha\beta}(x,t) = \sum_n \frac{p_n^\alpha p_n^\beta}{E}\delta^3(x - x_n(t)) \tag{4.4.1}$$

Then energy-momentum tensor of an assembly of gravitons, all of which have four-momentum $p^\mu = \hbar k^\mu$, is However,the energy-momentum tensor $T^{\lambda\nu}(k,\omega)$ must now be interpreted as a matrix element of an energy-momentum tensor operator between final and initial state, and also from the physical significance of the the plane-wave solution

$$h_{\mu\nu}(x) = e_{\mu\nu} exp(ik_\lambda x^\lambda) + e^*_\mu exp(-ik_\lambda x^\lambda) \tag{4.4.2}$$

is brought forward by calculating the energy and momentum it carries. According to

$$t_{\mu\nu} = \frac{1}{8\pi G}\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\lambda_\lambda - R^{(1)}_{\mu\kappa} + \frac{1}{2}\eta_{\mu\kappa}R^{(1)\lambda}_\lambda \tag{4.4.3}$$

Then energy-momentum tensor of gravitation is given to order of h^2 by

$$t_{\mu\nu} \simeq \frac{1}{8\pi G}\big(- \frac{1}{2}h_{\mu\nu}\eta^{\lambda\rho}R^{(1)}_{\lambda\rho} + \frac{1}{2}\eta_{\mu\nu}h^{\lambda\rho}R^{(1)}_{\lambda\rho} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho}\big) \tag{4.4.4}$$

where $R_{\mu\nu}^{(N)}$ is the term in the Ricci tensor of order N in $h_{\mu\nu}$. The metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies the first-order Einstein equation $R_{\mu\nu}^{(1)} = 0$, so we can drop these terms in $t_{\mu\nu}$ and use

$$t_{\mu\nu} \simeq \frac{1}{8\pi G}[R_{\mu\nu}^{(2)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R_{\lambda\rho}^{(2)}] \tag{4.4.5}$$

For the actual metric it is $R_{\mu\nu}$ rather than $R_{\mu\nu}^{(1)}$ that vanishes,and $t_{\mu\nu}$ arises only from the first-order terms. Here it is $R_{\mu\nu}^{(1)}$ that vanishes, because $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ satisfies the first-order Einstein equations rather than the exact equations. The difference is only of the order $h^{(3)}$. To calculate $R_{\mu\nu}^{(2)}$ we must use the following equation

$$h_{\mu\nu}(x) = e_{\mu\nu}exp(ik_\lambda x^\lambda) + e_\mu^* exp(-ik_\lambda x^\lambda) \tag{4.4.6}$$

in to equation of the second-order part of the Ricci tensor

$$\begin{aligned} R_{\mu\kappa}^{(2)} &= -\frac{1}{2}h^{\lambda\nu}[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}] \\ &+ \frac{1}{4}[2\frac{\partial h_\sigma^\nu}{\partial x^\nu} - \frac{\partial h_\nu^\nu}{\partial x^\sigma}][\frac{\partial h_\mu^\sigma}{\partial x^\kappa} + \frac{\partial h_\kappa^\sigma}{\partial x^\mu} - \frac{\partial h_{\mu\kappa}}{\partial x_\sigma}] - \frac{1}{4}[\frac{\partial h_{\sigma\kappa}}{\partial x_\lambda} \\ &+ \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}][\frac{\partial h_\mu^\sigma}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x_\mu} - \frac{\partial h_\mu^\lambda}{\partial x_\sigma}] \end{aligned} \tag{4.4.7}$$

The result is extremely complicated,but simplified if we average $t_{\mu\nu}$ over a region of space and time much larger than $|k|^{-1}$. This is the way the energy and momentum of any wave are usually evaluated. The averaging kills all terms proportional to $exp(\pm 2ik_\lambda x^\lambda)$, and we are left with only the x^μ independent cross-term:

$$\begin{aligned} \langle R_{\mu\nu}^{(2)} \rangle &= Ree^{\lambda\nu *}[k_\mu k_\nu e_{\lambda\rho} - k_\mu k_\lambda e_{\nu\rho} - k_\nu k_\rho e_{\mu\lambda} + k_\lambda k_\rho e_{\mu\nu}] + [e_\rho^\lambda k_\lambda \\ &- \frac{1}{2}e_\lambda^\lambda k_\rho]^*[k_\mu e_\nu^\rho + k_\nu e_\mu^\rho - k^\rho e_{\mu\nu}] \\ &- \frac{1}{2}[k_\lambda e_{\rho\nu} + k_\nu e_{\rho\lambda} - k_\rho e_{\lambda\nu}]^*[k^\lambda e_\mu^\rho + k_\mu e^{\rho\lambda} - k^\rho e_\mu^\lambda] \end{aligned} \tag{4.4.8}$$

We have not yet made use of the conditions $k_\mu k^\mu = 0$ and $(k_\mu e_\nu^\mu = \frac{1}{2}k_\nu e_\mu^\mu$ appropriate to harmonic coordinates, so suppose for a moment that we leave the harmonic coordinate system by adding to $h_{\mu\nu}(x)$ a term

$$i(q_\mu \varepsilon_\nu + q_\nu \varepsilon_\mu) exp(iq_\lambda x^\lambda) - i(q_\mu \varepsilon_\nu^* + q_\nu \varepsilon_\mu^*) exp(-iq_\lambda x^\lambda) \qquad (4.4.9)$$

Where $q_\mu q^\mu \neq 0$. After averaging over space-time distances large compared with $\mid q-k \mid^{-1}$ and we find for $\langle R^{(2)}_{\mu\nu} \rangle$ term, plus another term obtained by replacing k with this second terms vanishes, so $\langle R^{(2)}_{\mu\nu} \rangle$ and hence $\langle t_{\mu\kappa} \rangle$ may be calculated in harmonic coordinates with no loss off generality.)

If we now use the harmonic coordinate conditions $(k_\mu k^\mu = 0)$ and $(k_\mu e^\mu_\nu = \frac{1}{2} k_\nu e^\mu_\mu)$, in to equation (4.4.8), then we obtain

$$\langle R^{(2)}_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{2}(e^{\lambda\rho *} e_{\lambda\rho} - \frac{1}{2} \mid e^\lambda_\lambda \mid^2) \qquad (4.4.10)$$

The quantity $\eta^{\lambda\rho} \langle R^{(2)}_{\lambda\rho} \rangle$ vanishes because $k^\mu k_\rho = 0$, So now gives the average energy-momentum tensor of a plane wave as

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{16\pi G}(e^{\lambda\rho *} e_{\lambda\rho} - \frac{1}{2} \mid e^\lambda_\lambda \mid^2) \qquad (4.4.11)$$

Note that a "gauge transformation" will change the terms in $\langle t_{\mu\kappa} \rangle$ into

$$e'^{\lambda\rho *} e'_{\lambda\rho} = e^{\lambda\rho *} e_{\lambda\rho} + 2Re\varepsilon^*_\rho k^\rho e^\lambda_\lambda + 2 \mid \varepsilon_\rho k^\rho \mid^2 \qquad (4.4.12)$$

$$e'^\lambda_\lambda = e^\lambda_\lambda + 2k^\lambda \varepsilon_\lambda \qquad (4.4.13)$$

but $\langle t_{\mu\kappa} \rangle$ is gauge-invariant. Thus,as far as energy and momentum are conserved, the polarizations $e_{\mu\nu}$ and $e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$ represent the same physical wave, and I see again that there are not six but only to physically significant polarization parameters. In particular, a wave travelling in the z-direction, with wave vector and polarization tensor given by $(k^1 = k^2 = 0; k^3 = k^0 = k > 0)$ and physical situation of the plane wave arbitrary value $(e_{\mu\nu})$, has the energy-momentum tensor

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{8\pi G}(\mid e_{11} \mid^2 + \mid e_{12} \mid^2) \qquad (4.4.14)$$

or in terms of the helicity amplitudes ($e_{\pm} = e_{11} \mp ie_{12} = -e_{12} \mp ie_{12}$),

$$\langle t_{\mu\nu} \rangle \quad = \quad \frac{k_\mu k_\nu}{16\pi G}(\mid e_+ \mid^2 + \mid e_- \mid^2) \tag{4.4.15}$$

As a result the solution of retarded potential for a source $\Lambda T_{\mu\nu}$ confined to a finite volume will be satisfies the harmonic coordinate condition. In a vacuum, the only energy-momentum tensor is the vacuum energy-momentum tensor $T_{\mu\nu}^{vacuum}$. Now the linearize Einstein field wave equation in a vacuum become to:

$$\Box h_{\mu\nu}^{(\omega)} \quad = \quad 16\pi G T_{\mu\nu}^{vacuum} \tag{4.4.16}$$

Finally, the field wave equation can be written as,

$$\Box h_{\mu\nu}^{(\omega)} \quad = \quad 2\Lambda \eta_{\mu\nu} \tag{4.4.17}$$

Chapter 5

Result and Discussion

A theory of everything (TOE), a single, all encompassing, coherent theoretical framework of physics that fully explains and links together all physical aspects of the universe is the interest of all scientific community. But, finding such unifying theory was the major unsolved problems in physics until recent. However, thanks to the effort of many scholars, currently all the laws of physics and aspects are closely framed only in two theoretical groups, most closely resemble to the TOE; General relativity (GR)and Quantum field theory (QFT). GR only focuses on gravity for understanding the universe in regions of both large scale and high mass while QFT only focuses on non-gravitational forces system in understanding the universe in regions of both small scale and low mass. Both theories are successful in their region of application. However, for a century ago, since from their birth all attempts to unify these theories were unsuccessful.

So, here in this thesis we have revisited the attempts made earlier in the unification of the theories.Knowing the unsuccessful/attempts to wards the unification, I need to revisit unsuccessful theory to just give an additional comment based on the review, and also identifying some critical discrepancy among the the existing theory and so supplementary comment and feature perspective is being supplied. In doing so, the most general Hilbert-Lagrange action is used to develop the Hamiltonian of quantum gravity from matter-energy tensor and scalar field tensors including quantum electrodynamics is used. Then the Hamiltonian is implemented in the Schrödinger equation and as well as in Klein-Gordon equation where the form of the dark energy sector is partially obscured in the development.

On the other hand, when simplifying boundary conditions being imposed on the derived wave equations their reductions produce their corresponding classical limits. So, with this regard the unification seems viable.

However, the main problems of the unification are:

• The theory of dark sectors of the formalism are still incomplete.

• The two theories, from which the unification is beng carried out are within their own theoretical limitations: both in their frameworks and perception of geometry.

• Implementation of boundary conditions are still not well behaved; Open and ambiguous.

General theory of relativity is the theory of gravitation and geometry of spacetime.The matter and geometry of spacetime and cosmological constant (Λ) are related by the Einstein field equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = KT_{\mu\nu}$, where $G_{\mu\nu}$ is Einsteins field tensor that tales geometry of spacetime and $T_{\mu\nu}$ is energy-momentum tensor of matter.In a vacuum, the only energy-momentum tensor is the vacuum energy-momentum tensor $T^{(vacuum)}_{\mu\nu}$. Now the linearize Einstein field wave equation in a vacuum become to:

$$\Box h^{(\omega)}_{\mu\nu} = 16\pi T^{(vacuum)}_{\mu\nu} \tag{5.0.1}$$

Finally, the field wave equation can be written as,

$$\Box h^{(\omega)}_{\mu\nu} = 2\Lambda \eta_{\mu\nu} \tag{5.0.2}$$

Chapter 6

Summary and conclusion

A theory of everything (TOE) that fully explains and links together all physical aspects of the universe is the interest of all scientific community. It is one of the current and difficult problems in physics. Currently it seems that all the laws of physics and aspects are closely framed only in two theoretical groups General relativity(GR) and Quantum field theory(QFT) where GR only focuses on gravity for understanding the large structure of the universe with high mass while QFT only focuses on non-gravitational forces in regions of small scale and low mass system of the universe. Both theories are successful in their region of application but the unification remained unsuccessful. The current attempts of unifying them are interesting and yet not converged. Hence, the problem is still open.

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DECLARATION

I hereby declare that this M.Sc thesis is my original work and has not been presented for a degree in any other University and that all source of materials used for the dissertation have been duly acknowledged.

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