STELLAR STRUCTURE AND EVOLUTION WITH POLYTROPIC APPROACH

By
Dereje Tulu

A THESIS SUBMITTED TO THE COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF PHYSICS IN FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MSC. IN PHYSICS (ASTROPHYSICS)
JIMMA UNIVERSITY
JIMMA, ETHIOPIA
OCTOBER, 2017

JIMMA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
PHYSICS DEPARTMENT

The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a thesis entitled "STELLAR STRUCTURE AND EVOLUTION WITH POLYTROPIC APPROACH" by Dereje Tulu in partial fulfillment of the requirements for the degree of MSc. in Physics (Astrophysics).

Dated: October, 2017

Supervisor: ______________________
Tolu Biressa (PhD.Fellow)

Cosupervisor: ______________________
Milkessa G. (MSc.)

External Examiner: ______________________
Dr. Anno Karre

Internal Examiner: ______________________

Chairperson: ______________________

JIMMA UNIVERSITY

Date: October, 2017

Author: Dereje Tulu

Title: STELLAR STRUCTURE AND EVOLUTION WITH POLYTROPIC APPROACH

Department: College of Natural Sciences
Physics Department

Degree: MSc.
Convocation: October
Year: 2017

____
Signature of Author

To My Family

Table of Contents

Abstract

It is generally believed that understanding of the formation, structure and evolution of stars and stellar systems remains a prerequisite for understanding the whole world more broadly speaking including the universe. Consequently, there is an overall progress in astronomy and astrophysics, over wide range, from observational work to complex theoretical developments. So, this encouraging progress in astronomy today has becoming the center of attention and research. However, the theory of stellar formation and evolution is incomplete in the very fundamental aspects. Theoretical works and observational searches are still raw. The full understanding of the dynamics of stellar structure and evolution requires a complete analysis of the dynamics and key parameters involving in the evolutionary process. Motivated by this scientific rationale, we have worked out on the structure of stellar evolution theoretically by polytropic approach, which works fine with main sequence stars as literatures reveals. The appropriate polytropic equation was being derived from Boltzmann transport equation where a series of integral homology transformations were used. The analytically derived equations were used to generate numerical solutions in analyzing the structures of the stars of this kind. The result is well fit with the present standard models.

Key words: Stellar-evolution, Stellar-structure, Hydrodynamics, Homology-transformation, Polytropy.

Acknowledgements

I would like to express my gratitude to my supervisor, Mr.Tolu Biressa (Phd.fellow) for his guidance, constructive comments, consistent supports and thanks for providing me an interesting topic. I am also very grateful to my co-supervisor Mr.Milkessa G.(M.Sc) and Miss Hiwot (M.Sc) for their constructive comments. Finally, my deepest goes to my family; specially my brother Gezu Tulu and my wife Shitaye Negash for their love and supports.

General Introduction

I. Background

As stellar astrophysics has developed, attention has increasingly become focused on the details and refinements that make the current models of stars so quantitatively accurate. Despite the fact that, there is an overall progress in astronomy and astrophysics, several problems ranging from observational limitations to theoretical developments have remained unresolved. For example, the origin, evolution and structure of stars, galaxies and interstellar media are not yet fully developed [8, 9]. However, according to the current astrophysical understanding, most of the substances that make up our world are formed in stars. It is generally believed that stars are formed from dust molecular clouds made up of mostly from hydrogen gas. While stars burning phase is believed to be in plasma state at large involving nuclear fusion at the core. Again, the development of a relatively complete picture of the structure and evolution of the stars has been one of the great conceptual accomplishments of the twentieth century. While questions still exist concerning the details of the birth and death of stars.

Furthermore, our understanding of stellar structure has progressed to the point where it can be studied within an axiomatic framework comparable to those of other branches of Physics[8]. It is within this axiomatic framework that we will study stellar structure stellar, spectra - the traditional source of virtually all information about stars. Thus, for providing insight into the structure and behavior of real stars, an understanding of polytropes is essential[11]. when the polytropic equation of state is coupled with the equation of hydrostatic equilibrium,it will provide a single relation for the run of pressure or density with position. The solution of this equation basically solves the fundamental problem of stellar structure and evolution insofar as the equation of state correctly represents the behavior of the stellar gas. Motivated by this introductory scientific background, we have interested to work on the particular topic 'Stellar Structure and Evolution with Polytropic Approach' within the stellar interiors in order to draw some useful structure of the relevant parameters as of the existing classical model about stellar evolution.

II. Literature Review

The formation, structure and evolution of stars [1], and stellar systems remains a prerequisite for understanding the whole world more broadly speaking including the universe. Consequently, there is an overall progress in astronomy and astrophysics, over wide range, from observational work to complex theoretical developments. So,this encouraging progress in astronomy today has becoming the center of attention and research in diversified issues, fields, etc. However, the theory of stellar formation and evolution is incomplete in the very fundamental aspects [8].Theoretical works and observational searches are still raw. There is no definitive established model that answers the issues. The evolution of structure and transport process and the dynamical controlling parameters within, from, and between stars such as variable magnetic fields, accretion, convection, shocks, pulsations, and winds responsible for are all not well established [4, 3]. So a full understanding of the dynamics of stellar structure and evolution therefore requires a complete analysis of the dynamics and key parameters are involving in the process. To make this true, in this thesis, we have analyzed the sources of energy of the stars and energy conversion principle that the stars undergo for their energy generation and transport of this generated energy to develop equations of stellar structure. Then, we focused on polytropes starting from homology transformation. That is, for providing insight into the structure and behavior of real stars, an understanding of polytropes is essential. Many astrophysicist feel that the study of polytropes is historical interest only. While it is true that the study of polytropes did developed early in the history of stellar structures; this is so because polytropes provide significant insight into the structure and evolution of stars [11]. The equation of hydrostatic equilibrium can be solved if the pressure is a known function of the density. The resulting stellar models are known as polytropic stellar models or simply polytropes [6]. Polytropic models have played an important role in the historical development of stellar structure theory. Although, nowadays their practical use has mostly been superseded by more realistic stellar models. In this situation the mechanical structure of the star is determined. A special case of such a relation between pressure P and density ρ is called the polytropic relation.

Moreover, the polytropic relation is a good approximation to the real equation of state. We have encountered an example of polytropic equations of state e.g. the case where pressure and density are related adiabatically [9]. If the equation of state can be written in polytropic form, the equations for mass continuity and for hydrostatic equilibrium can be combined to give a second-order differential equation for the density. Therefore, by making use of polytropic solutions, it is possible to represent stars with convective cores and radiative envelopes with some accuracy and to get a rough idea of the run of pressure, density, and temperature throughout the star [8].

III. Statement of the Problem

The problems of stars and planet formation are among the most important challenges facing modern astrophysical research . According to the current understanding stars evolve throughout their lifetime from formation to death. However, the underlying theoretical models on the dynamical processes and mechanisms of incorporating significant observable parameters are still fresh and raw. It needs accurate theoretical modeling and matching mechanisms with observation [11].According to the current understanding stars are formed from dust molecular clouds made up of mostly from hydrogen gas.The existing models about how internal stellar parameters play role are not yet concisely and concretely established [8]. So that a gap exist in understanding the stellar structure and evolution-polytropes of the main sequence stars. Therefore, this study focus on stellar structure and evolution with polytropic approach when stars evolving at the center.

Research Questions

- What is the appropriate equation of state of main sequence stars?
- What are the dynamical time scales of evolution process in the main sequence?
- How these dynamical systems affect or responsible for star-evolution?
- How do we construct polytropic model depending on homology transformation?

IV . Objectives

I. General objective

To study stellar structure and evolution of stars with polytropic approach.

II. Specific Objectives

- To derive the appropriate equation of state from hydrodynamic equation that determine the structure and evolution of a stars.
- To derive a simple equation of state, in particular a polytropic equation with homology transformation.
- To generate numerical solutions from the equation of state and interpret the results to describe the structure and evolution of main sequence stars.

V. Methodology

The general method is to derive the appropriate equation of state from Boltzmann transport equation that determines the structure and evolution of stars. Then, with the classical ideal gas law assumption we further derive simple equation of state, namely polytropic equation by way of a series of integral homology transformations. The analytically derived equations are being used to generate numerical values for some relevant paraments such as density, internal energy, potential energy, etc computationally with MATHEMATICA. Finally, the results be discussed and summarized to remark.

The organization of the work is: In chapter 1 we present the basic principles and the physics being used. In chapter 2 we derive the fundamental equations and describe their application being used. Chapter 3 focuses in deriving the polytropic equation we are interested to work with. Chapter 4 deals with the result and discussions. The final fifth chapter provides summary and conclusion.

Chapter 1

Basic Definitions, Assumptions and Theory of Stellar Evolution

1.1 Definition and description of Stars

A star is an object that (1) radiates energy from an internal source and (2) is bound by its own gravity. The first criterion excludes objects like planets, comets and brown dwarfs where both are not hot enough for nuclear fusion. The second criterion excludes trivial objects that radiate (e.g. glowing coals). A star is born out of an interstellar (molecular) gas cloud, lives for a certain amount of time on its internal energy supply, and eventually dies when this supply is exhausted. The definition imposes that stars can have only a limited range of masses, between $\sim 0.1M_{\odot}$ and $\sim 1000M_{\odot}$.

Stars are considered to be isolated in space, so that their structure and evolution depend only on intrinsic properties (mass and composition). For most single stars in the Galaxy this condition is satisfied to a high degree (compare for instance the radius of the Sun with the distance to its nearest neighbor Proxima Centauri. However, for stars in dense clusters, or in binary systems, the evolution can be influenced by interaction with neighboring stars. Also, stars are formed with a homogeneous composition, a reasonable assumption since the molecular clouds out of which they form are well-mixed. We will often assume a so-called quasi-solar composition (X = 0.70, Y = 0.28 and Z = 0.02), even though recent determinations of solar abundances have revised the solar metallicity down to Z = 0.014. In practice there is relatively little variation in composition from star to star, so that the initial mass is the most important parameter that determines the evolution of a star.

Moreover, spherical symmetry, which is promoted by self-gravity a good approximation for most stars. Deviations from spherical symmetry can arise if non-central forces become important relative to gravity, in particular rotation and magnetic fields. Although many stars are observed to have magnetic fields, the field strength (even in highly magnetized neutron stars) is always negligible compared to gravity.

Finally, understanding the structure and evolution of stars, and their observational properties, requires laws of physics involving different areas (e.g. thermodynamics, nuclear physics, electrodynamics, plasma physics) [7].

1.2 Statistical Assumptions

Since stars are made up of gases and radiation, the microphysics involves particles and photons where the steady macro-state equilibrium is basically obtained when the number of microstates W is a maximum. This is obtained by the condition $dW = 0$. Or equivalently, $dlnW = 0$, since W is a monotonic function.

According to the current understanding we have three statistics: Maxwell-Boltzmann (MB), Bose-Einstein (BE) and Fermi-Dirac (FD) statistics.

$$\ln W_{MB} = \ln N! - \Sigma_i \; lnN! \; Maxwell - Boltzmann \tag{1.2.1}$$

$$\ln W_{BE} = \Sigma_i \ln(n + N - 1)! - \ln N! - \ln(n - 1)! \; Bose - Einstein \tag{1.2.2}$$

$$\ln W_{FD} = \Sigma_i \ln(2n)! - (2n - N)! - lnN! \; Fermi - Dirac \tag{1.2.3}$$

All that remains is to develop a physical interpretation of the undetermined parameters α_j and β_j. Let us introduce Maxwell-Boltzmann statistics for an example of how this is done. Since we have not said what β_i is, let us call it $1/(kT)$. Then

$$Ni = \alpha_1 e^{-wi/(kT)} \tag{1.2.4}$$

For more description, refer [7]. If the cell volumes of phase space are not all the same size, it may be necessary to weight the number of particles to adjust for the different cell volumes. We call these weight functions g_i. Then,

$$N = \sum_i g_i N_i = \alpha_1 \sum_i g_i e^{-w_i/(kT)} = \alpha_1 U(T) \tag{1.2.5}$$

The parameter $U(T)$ is called the partition function and it depends on the composition of the gas and the parameter T alone. Now if the total energy of the gas is E, then

$$E = \sum_i g_i w_i N_i = \sum_i w_i g_i \alpha_1 e^{-wi/kT} = [\sum_i w_i g_i N e^{-wi/kT}]/U(T) = NkT[dlnU/dlnT] \tag{1.2.6}$$

For a free particle like that found in a monatomic gas, the partition function is

$$U(T) = \frac{(2mkT)^{3/2}}{h^3} V \tag{1.2.7}$$

where V is the specific volume of the gas, m is the mass of the particle, and T is the kinetic temperature. Replacing dlnU/dlnT in equation 1.2.6 by its value obtained from equation 1.2.7, we get the familiar relation,

$$E = \frac{3}{2} NkT \tag{1.2.8}$$

which is only correct if T is the kinetic temperature. Thus we arrive at a self-consistent solution if the parameter T is to be identified with the kinetic temperature.

The situation for a photon gas in the presence of material matter is somewhat simpler because the matter acts as a source and sinks for photons. Now we can no longer apply the constraint $dN = 0$. This is equivalent to adding $ln\alpha_2 = 0 (i.e., \alpha_2 = 1)$ to the equations of condition. If we let $\alpha_2 = 1/(kT)$ as we did with the Maxwell- Boltzmann statistics , then the appropriate solution to the Bose-Einstein formula 1.2.1 becomes

$$\frac{N_i}{n} = \frac{1}{e^{h\nu/(kT)} - 1} \tag{1.2.9}$$

where the photon energy wi has been replaced by $h\nu$. Since two photons in a volume h^3 can be distinguished by their state of polarization , the number of phase space compartments is

$$n = (2/h^3)dx_1dx_2dx_3dp_1dp_2dp_3 \tag{1.2.10}$$

We can replace the rectangular form of the momentum volume $dp_1dp_2dp_3$, by its spherical counterpart $4\pi p^2 dp$ and remembering that the momentum of a photon is $h\nu/c$, we get

$$\frac{dN}{V} = \frac{8\pi\nu^2}{C^3}\frac{1}{e^{h\nu/(kT)} - 1}dv \tag{1.2.11}$$

Here we have replaced N_i with dN. This assumes that the number of particles in any phase space volume is small compared to the total number of particles. Since the energy per unit volume dE_v is just $h\nu dN/V$, we get the relation known as Planck's law or sometimes as the black-body law:

$$\frac{dN}{V} = \frac{8\pi\nu^2}{C^3}\frac{1}{e^{h\nu/(kT)} - 1}dv = \frac{4\pi}{c}B_v(T) \tag{1.2.12}$$

The parameter $B_v(T)$ is known as the Planck function. This,is the distribution law for photons which are in strict thermodynamic equilibrium. If we were to consider the Bose-Einstein result for particles and let the number of Heisenberg compartments be much larger than the number of particles in any volume, we would recover the result for Maxwell-Boltzmann statistics. This is further justification for using the Maxwell-Boltzmann result for ordinary gases.

1.3 Theory of Stellar Evolution

The theory of stellar evolution tell us which parameters are related to various aspects of a star's life. Therefore we need a theory of stellar structure to derive the internal properties of a star[5].

The time span of any observations is much smaller than a stellar lifetime: observations are like snapshots in the life of a star. The observed properties of an individual star contain no (direct) information about its evolution[12]. The diversity of stellar properties (radii, luminosity, surface abundances) does, however, depend on how stars evolve, as well as on intrinsic properties (mass, initial composition). Properties that are common to a large number of stars must correspond to long-lived evolution phases, and vice versa. By studying samples of stars statistically we can infer the (relative) lifetimes of certain phases, which provides another important constraint on the theory of stellar evolution.

Furthermore , observations of samples of stars reveal certain correlations between stellar properties that the theory of stellar evolution must explain. Most important are relations between luminosity and effective temperature, as revealed by the Hertzsprung-Russell diagram[10], and relations between mass, luminosity and radius.

Chapter 2

Hydrodynamics Equations and its application to stellar evolution

Astrophysical fluids are complex, with a number of different components: neutral atoms and molecules, ions, dust grains (often charged), and cosmic rays. The magnetic fields generally ties all these fluids together except where gradients are very steep, as in shocks. It is the study of the motion of the fluids (liquids and gas). Although fluids are made of particles, it is sufficient to treat a fluid as a continuous substance in many situations. Moreover, in the fluid approximation, we treat the ensemble of particles as a single fluid. To describe an ensemble of particles precisely we need to know the position and velocity of each particle

2.1 Boltzmann Transport Equation (BTE)

In stellar astrophysical, modeling gas flows around stars or in interstellar space, the ideal gas assumption is very much accurate. Therefore, in our analysis of the stellar structure and evolution we apply the classical Boltzmann statistical distributions and derive the dynamic equations from Boltzmann transport equations.

The Boltzmann transport equation in six dimenstional postion-velocity phase space basically expresses the change in the phase density within a differential volume, in terms of the flow through these faces, and the creation or destruction of particles within that volume. In the canonical postion-momentum coordinate system, the Boltzmann transport equation (BTE) is given by

$$\boxed{\sum_{i=1}^{3}(\dot{x}_i\frac{\partial f}{\partial x_i}+\dot{p}_i\frac{\partial f}{\partial p_i})+\frac{\partial f}{\partial t}=S} \quad \text{- } BTE \tag{2.1.1}$$

where $f \equiv f(x, \dot{x}; t)$ is the number densty distribution function, S is the rate of particle creation/destruction, $\dot{x}_i = \frac{\partial x_i}{\partial t}$ and $\dot{p}_i = \frac{\partial p_i}{\partial t}$

This equation can be recast in vector notation as

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \vec{F}\cdot\vec{\nabla}_p f = S \tag{2.1.2}$$

where $\vec{F}$ is force and $\vec{\nabla}_p$ is the momentum gradient.

In conservative field system since $\vec{F} = -\vec{\nabla}\Phi$ where Φ is a scalar potential (eg. gravitational scalar potential), then, BTE will be given as

$$\frac{\partial f}{\partial t} + \vec{v}.\nabla f - \frac{1}{m}\nabla\Phi.\nabla_v f = S \tag{2.1.3}$$

The potential gradient $\nabla\Phi$ has replaced the momentum time derivative while ∇_v is a gradient with respect to velocity. The quantity m is the mass of a typical particle. It is also not unusual to find the BTE written in terms of the total stokes time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}.\nabla \tag{2.1.4}$$

where $\vec{v}$ is the flow velocity and $\frac{\partial}{\partial t}$ is the Eulerian time derivative.

If we take $\vec{v}$ to be a six-dimensional 'velocity' and ∇ to be a six-dimensional gradient the BTE becomes

$$\frac{Df}{Dt} = S \tag{2.1.5}$$

If the creation/destruction rate of particles is zero ($S = 0$), we will obtain the homogeneous Boltzmann Transport Equation (BTE) given as

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}.\nabla f - \frac{1}{m}\nabla\Phi.\nabla_v f = 0} \quad \textit{The Liouville's theorem} \tag{2.1.6}$$

In the case of classical statistical mechanics, the number of particles N is very large, (of the order of Avogrados number, for a laboratory-scale system). Setting $\frac{\partial\rho}{\partial t} = 0$ gives an equation for the stationary states of the system and can be used to find the density of microstates accessible in a given statistical ensemble. For eg. in an equilibrium of the Maxwell-Boltzmann statistical distribution ρ is given as

$$\rho \propto e^{H/(k_B T)}$$

where H is the Hamiltonian, T is the temperature and k_B is the Boltzmann constant. The equations of fluid dynamics can b e derived by calculating moments of the Boltzmann equation for quantities that are conserved in collisions of the particles.

The n^{th} moment of a function f with primary variable x is

$$M_n\left[f(x)\right] = \int x^n f(x)dx \tag{2.1.7}$$

2.1.1 The zeroth moment of BTE and the Continuity Equation

When $n = 0$ as in eq.(2.1.7) we derive the local spatial density given as

$$\boxed{\rho = m\int_{-\infty}^{+\infty} f(x,\vec{v})d\vec{v}} \tag{2.1.8}$$

The related BTE is

$$\int_{-\infty}^{+\infty}\left(\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i\frac{\partial f}{\partial x_i} + \sum_{i=1}^{3}\dot{v}_i\frac{\partial f}{\partial v_i}\right)d\vec{v_i} = \int_{-\infty}^{+\infty} S d\vec{v} \tag{2.1.9}$$

The integral of the creation rate S over all velocity space becomes the creation rate for particles in physical space, which we call $\Im$.

In the conservative field system eq.(2.1.9), the zeroth moment of BTE is given as

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} f d\vec{v} + \int_{-\infty}^{+\infty}\vec{v}.\nabla f d\vec{v} + \int_{-\infty}^{+\infty}\dot{\vec{v}}.\nabla_v f d\vec{v} = \Im \tag{2.1.10}$$

In view of eq.(2.1.8), the first integral of the left hand side of eq.(2.1.10) is given by

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} f d\vec{v} = \frac{1}{m}\frac{\partial\rho}{\partial t} \tag{2.1.11}$$

The second term is simplified to yield ([8])

$$\int_{-\infty}^{+\infty} \vec{v} \cdot \nabla f d\vec{v} = \int_{-\infty}^{+\infty} (\vec{v} \cdot f \vec{v} d\vec{v}) \tag{2.1.12}$$

The third term from equation (2.1.10) is

$$\dot{\vec{v}}.\vec{\nabla}_v f = -\frac{\vec{\nabla}\Phi}{m}.\vec{\nabla} f \tag{2.1.13}$$

Now using these equations (2.1.11, 2.1.12 and 2.1.13) eqn.(2.1.10) becomes

$$\frac{1}{m}\frac{\partial \rho}{\partial t} + \int_{-\infty}^{+\infty} (\vec{v}.f\vec{v}d\vec{v}) - \int_{-\infty}^{+\infty} (\frac{\nabla\Phi}{m}.\vec{\nabla}_v f)d\vec{v} = \overrightarrow{\Im}$$

$$\frac{\partial \rho}{\partial t} + m\vec{\nabla}.(\int_{-\infty}^{+\infty} \vec{v} f d\vec{v}) - \vec{\nabla}\Phi. \int_{-\infty}^{+\infty} \vec{\nabla}_v f d\vec{v} = \vec{\Im}$$

For realistic physical system with finite velocity the second integral of the left hand side equation has to vanish. Then,

$$\frac{\partial \rho}{\partial t} + m\nabla.(\int_{-\infty}^{+\infty} \vec{v} f d\vec{v}) = \vec{\Im}_m \tag{2.1.14}$$

Using the normalized mean flow velocity $\vec{u}$, a measure of the mean flow rate of the material defined as

$$\vec{u} = \frac{\int_{-\infty}^{+\infty} \vec{v} f(\vec{v}) d\vec{v}}{\int_{-\infty}^{+\infty} f(\vec{v}) d\vec{v}} \tag{2.1.15}$$

the zeroth moment of BTE yields the continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{u}) = \Im_m} \tag{2.1.16}$$

In the absence of creation field, the continuity equation gives the familiar **local matter conservation**.

2.1.2 The First Moment of BTE and the Euler-Lagrange Equations of Hydrodynamic Flow

Multiplying the BTE by the local particle velocity $\vec{v}$ and integrating over all velocity space will produce momentum like moments given as

$$\int_{-\infty}^{+\infty} \vec{v}\left(\frac{\partial f}{\partial t} + \int_{-\infty}^{+\infty} \vec{v}.\nabla f + \int_{-\infty}^{+\infty} (\dot{\vec{v}}.\vec{\nabla}_v f)\right) d\vec{v} = \int_{-\infty}^{+\infty} \vec{v} S d\vec{v} \tag{2.1.17}$$

The integral of this equation are not a simple scalars or vectors, but are the vector outer products called tensors.

Using the expressions of local spatial density ρ and the mean velocity $\vec{u}$, the first integral of the left hand side of this equation can be recast as

$$\int_{-\infty}^{+\infty} \vec{v}\frac{\partial f}{\partial t} d\vec{v} = \frac{\partial}{\partial t}(n\vec{u}) \tag{2.1.18}$$

where $n = \rho/m$, the number density.

The second integral of the left hand side of this moment like equation as discussed earlier(eqn. 2.1.10) is

$$\int_{-\infty}^{+\infty} \vec{v}.\vec{\nabla} f d\vec{v} = \int_{-\infty}^{+\infty} \vec{v}(\vec{v}.\vec{\nabla} f) d\vec{v} \tag{2.1.19}$$

Since $\dot{\vec{v}} = -\frac{\vec{\nabla}\Phi}{m}$ and $\vec{v}$ are independent orthogonal phase space coordinates, then the third integral of the left hand side of the moment like equation under discussion is equivalent to

$$\int_{\infty}^{+\infty} \vec{v}(\vec{v}.\vec{\nabla}_v f) d\vec{v} = -\frac{\vec{\nabla}\Phi}{m} \cdot \int_{-\infty}^{+\infty} (\vec{\nabla}_v f)\vec{v} d\vec{v} \tag{2.1.20}$$

Using the vector algebra

$(\vec{\nabla}_v f)\vec{v} = \vec{\nabla}_v(f\vec{v}) - f(\vec{\nabla}_v \vec{v}) = \vec{\nabla}_v(f\vec{v}) - f\mathrm{I}$,

where I is the identity matrix and the relation

$n = \int_{-\infty}^{+\infty} f d\vec{v}$

$$\int_{-\infty}^{+\infty} \vec{v}(\vec{v}.\vec{\nabla}_v f) d\vec{v} = -\frac{\vec{\nabla}\Phi}{m}. \int_{-\infty}^{+\infty} \vec{\nabla}_v(fv)\vec{d}\vec{v} = \frac{\vec{\nabla}\Phi}{m}. \int_{-\infty}^{+\infty} f d\vec{v}$$

$$\int_{-\infty}^{+\infty} \vec{v}(\vec{v}.\vec{\nabla}_v f)d\vec{v} = -\frac{\vec{\nabla}\Phi}{m}.\int_{-\infty}^{+\infty} f d\vec{v} = -n\frac{\vec{\nabla}\Phi}{m}$$

$$\frac{\partial}{\partial t}(n\vec{u}) + \int_{-\infty}^{+\infty} \vec{v}(\vec{\nabla}.(\vec{v}f))d\vec{v} + n\frac{\vec{\nabla}\Phi}{m} = \int_{-\infty}^{+\infty} S\vec{v}d\vec{v} \tag{2.1.21}$$

$$\frac{\partial}{\partial t}(n\vec{u}) = \vec{u}\frac{\partial n}{\partial t} + n\frac{\partial \vec{u}}{\partial t} \tag{2.1.22}$$

$$\frac{\partial n}{\partial t} = -\nabla.(n\vec{u}) + \int_{-\infty}^{+\infty} S d\vec{v} = -(\vec{u}.\vec{\nabla}_n + \vec{\nabla}_n.\vec{u}) + \int_{-\infty}^{+\infty} S d\vec{v}$$

From the continuity equation

$$\frac{\partial n}{\partial t} + \nabla.(n\vec{u}) = \int_{-\infty}^{+\infty} S d\vec{v} \tag{2.1.23}$$

Then combining equations (2.1.21),(2.1.22) and (2.1.23) we get

$$\frac{\partial}{\partial t}(n\vec{u}) = \vec{u}\frac{\partial n}{\partial t} + n\frac{\partial \vec{u}}{\partial t} - (\vec{u}.\nabla n + n\nabla.\vec{u})\vec{u} + \int_{-\infty}^{+\infty} \vec{u}S d\vec{v} \tag{2.1.24}$$

Using equations (2.1.24)

$$\frac{\partial}{\partial t}(n\vec{u}) = n\frac{\partial \vec{u}}{\partial t} - (\vec{u}.\vec{\nabla}n + \vec{\nabla}n.\vec{u})\vec{u} + \int_{-\infty}^{+\infty} \vec{u}S d\vec{v} \tag{2.1.25}$$

$$n\frac{\partial \vec{u}}{\partial t} - (\vec{u}.\vec{\nabla}_n + \vec{\nabla}_n.\vec{u})\vec{u} + \int_{-\infty}^{+\infty} \vec{v}(\vec{\nabla}.(\vec{v}f)d\vec{v} + n\frac{\vec{\nabla}\Phi}{m} = \int_{-\infty}^{+\infty} S(\vec{v}-\vec{u})d\vec{v} \tag{2.1.26}$$

Defining the velocity tensor $\overleftrightarrow{u}$ as

$$\overleftrightarrow{u} = \frac{\int_{-\infty}^{+\infty} \vec{v}\vec{v}f(\vec{v})d\vec{v}}{\int_{-\infty}^{+\infty} f(\vec{v})d\vec{v}} \tag{2.1.27}$$

$$\rho\frac{\partial u}{\partial t} + \rho(\vec{u}.\vec{\nabla})\vec{u} + \vec{\nabla}.(\rho(\vec{\vec{u}} - \vec{u}\vec{u})) + n\vec{\nabla}\Phi = \int_{-\infty}^{+\infty} mS(\vec{v}-\vec{u})d\vec{v} \tag{2.1.28}$$

The quantity $\rho(\vec{u} - \vec{u}\vec{u})$ is the pressure tensor.The pressure tensor the second moment of f(v) as $\vec{\vec{\rho}}$ equal to

$$\frac{\int_{-\infty}^{+\infty} f(v)(\vec{v}-\vec{u})(\vec{v}-\vec{u})d\vec{v}}{\int_{-\infty}^{+\infty} f(v)d\vec{v}} \tag{2.1.29}$$

It describes the different between the local flow $\vec{v}$ and the mean flow $\vec{u}$. The first velocity moment of the BTE becomes

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\nabla\Phi - \frac{1}{\rho}\nabla P + \frac{1}{\rho}\int_{-\infty}^{+\infty} mS(\vec{v} - \vec{u})d\vec{v} \tag{2.1.30}$$

This set of vector equations are called **Euler-Lagrange equations of hydrodynamic flow**.

On the other hand the assumption of excessive collisions where $\vec{v}$ is considered to be random and the assumption S to be symmetrical implies the integral over all velocity space vanishes. Then Euler-Lagrange equations of hydrodynamic flow of BTE is given by

$$\boxed{\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\nabla\Phi - \frac{\nabla P}{\rho}} \tag{2.1.31}$$

Under the assumption of a nearly isotropic velocity field, **P** will be $\mathbf{P}(\rho)$ and an expression known as an equation of state. From equation (2.1.31)the left-hand side is zero. The Euler-Lagrange equations of hydrodynamic flow is

$$\nabla P = -\rho\nabla\Phi \tag{2.1.32}$$

Which is known as the **equation of hydrostatic equilibrium**. This equation is usually an expression of the c**onservation of linear momentum**. The zeroth moment of the BTE results in the conservation of matter, where as the first velocity moment equations which represent the conservation of linear momentum. The second velocity moment represent an expression for the conservation of energy.

2.2 Boltzmann Transport Equation and the Virial Theorem

The Euler-Lagrange equations of hydrodynamic flow are vector equations and represent vectors. We can obtain a scalar result by taking the scalar product of a position vector with the flow equations and integrating over all space with the system. The origin of the position vector is important only in the interpretation of some of the terms which will arise in the expression.

So now the spatial first moment of the Euler-Lagrange equations of hydrodynamic flow eq.(2.1.30).

$$\int_V \vec{r} \cdot \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} + \nabla\Phi + \frac{\nabla P}{\rho} \right) dV = 0 \tag{2.2.1}$$

Working out this equation gives us

$$\frac{1}{2}\frac{d^2 I}{dt^2} - 2T - 2U + \Omega = 0 \tag{2.2.2}$$

Where I is the moment of inertia, T is kinetic energy in bulk motion and U is the internal energy and Ω is the total potential energy of the system.

This equation is known as the Non-averaged form of the virial theorem. For a system in equilibrium, the time average of eq.(2.2.2) removes the accelerative changes of the moment of inertia($(< \frac{d^2 I}{dt^2} >= 0)$) so that

$$2 < T > +2 < U > + < \Omega >= 0 \tag{2.2.3}$$

The theorem which permits is the Ergodic theorem.

2.3 Application of BTE in stellar evolution

2.3.1 Equation of State for an Ideal Gas

In thermodynamics, an equation of state provides the mathematical relation among variables such as temperature, pressure, density, and internal energy. Equations of state (EOS) are useful in describing the properties of fluids, mixtures of fluids, solids, and even the interiors of stars. For stars, the of state usually describe the relation among pressure(P), temperature(T),density (n: number of density of particles or (ρ) :mass density). Formulation of the Boltzmann Transport Equation (BTE) also provides an ideal setting for the formulation of the equation of state for a gas under wide-ranging conditions. The relationship between the pressure as given by the pressure tensor and the state variables (p,T,ρ) of the distribution function. The pressure tensor is $\mathbf{p}(\vec{u} - \vec{u}\vec{u})$. If f($\vec{v}$) is symmetric in $\vec{v}$, then $\vec{u}$ must be zero (or there exist an inertial coordinate system in which $\vec{u}$ is zero), and the divergence of the pressure can be replaced by the gradient of a scalar, which call the gas pressure, and will be given by

$$\vec{p} = \rho \frac{\int_{-\infty}^{+\infty} v^2 f(v) d\vec{v}}{\int_{-\infty}^{+\infty} f(v) d\vec{v}} \tag{2.3.1}$$

From the Maxwell-Boltzmann statistics, the distribution function of particles, in terms of their velocity, is given by

$$f(v) = constant. \exp(\frac{-mv^2}{2kT}) \tag{2.3.2}$$

The mean pressure is

$$\begin{aligned} \vec{p} &= c\rho \frac{\int_{-\infty}^{+\infty} v^2 \exp(\frac{-mv^2}{2kT})}{c \int_{-\infty}^{+\infty} \exp(\frac{-mv^2}{2kT})} \\ &= \rho \frac{\int_{-\infty}^{+\infty} v^2 \exp(-\alpha v^2) dv}{\int_{-\infty}^{+\infty} \exp(-\alpha v^2) dv} \end{aligned}$$

Where $\alpha = \frac{m}{2kT}$ The integral of the function is

$$\int_{-\infty}^{+\infty} v^2 \exp(-\alpha v^2) dv = \frac{1}{4} \sqrt{\pi} \alpha^{\frac{-3}{2}}$$

and the integral of the denominator is given as

$$\int_{-\infty}^{+\infty} \exp(-\alpha v^2) dv = \sqrt{\frac{\pi}{\alpha}}$$

Then,

$$\bar{p} = \frac{\rho \frac{1}{4} \sqrt{\pi} \alpha^{\frac{-3}{2}}}{\sqrt{\frac{\pi}{\alpha}}}$$

$$\bar{p} = \frac{\rho \frac{\sqrt{\pi} \alpha^{\frac{-3}{2}}}{4}}{2 \sqrt{\pi} \alpha^{\frac{-1}{2}}}$$

$$= \rho\frac{\alpha^{-1}}{2} = \frac{\rho}{2\alpha}$$

But, $\alpha = \frac{m}{2kT}$ then, the mean pressure is

$$\bar{p} = \frac{\rho}{\frac{2m}{2kT}} = \frac{\rho kT}{m} \tag{2.3.3}$$

2.4 Energies of Stars

One of the great mysteries of the late nineteenth and early twentieth centuries was the source of the energy required to sustain the luminosity of the sun. By then, the defining solar parameters of mass, radius, and luminosity were known with sufficient precision to attempt to relate them. For instance, it was clear that if the sun derived its energy from chemical processes typically yielding less that $10^{12} erg/g$, it could shine no longer than about 10,000 years at its current luminosity. It is said that Lord Kelvin, in noting that the liberation of gravitational energy could only keep the sun shining for about 10 million years, found it necessary to reject Charles Darwin's theory of evolution because there would have been insufficient time for natural selection to provide the observed diversity of species.

2.4.1 Gravitational Energy

It is generally conceded that the sun has shone at roughly its present luminosity for at least the past 2 billion years and has been in existence for nearly 5 billion years. Perhaps the most obvious source of energy is that suggested by Lord Kelvin , namely gravitation. From the integral theorems of hydrostatic equilibrium, we may place a limit on the gravitational energy of the sun by remembering that $I_{1,1}(R)$ is related to the total gravitational potential energy. Thus, we have:

$$\Omega \leq \frac{3}{5}\frac{GM^2}{R} \tag{2.4.1}$$

The right-hand side of the inequality is the gravitational potential energy for a uniform density sphere , which provides a sensible upper limit for the energy. Remember that the gravitational energy is considered negative by convention; a rather larger magnitude of energy may be available for a star that is more centrally concentrated than a uniform-density sphere. We may acquire a better estimate of the gravitational potential energy by using the results for a polytrope. Chandrasekhar obtains the following result for the gravitational potential energy of a polytrope:

$$\Omega \leq -\frac{3}{5-n}\frac{GM^2}{R} \tag{2.4.2}$$

For a star in convective equilibrium (that is, $n = 3/2$) the factor multiplying GM^2/R becomes $6/7$ or nearly unity. Note that for a polytrope of index 5, $\Omega \rightarrow -\infty$ implying an infinite central concentration of material. This is also one of the polytropes for which there exists an analytic solution and $\xi = \infty$. Thus, the picture of a mass point surrounded by a massless envelope of infinite extent. Which implies that the polytropic index increases, so does the central concentration. It is not at all obvious that the total gravitational energy would be available to permit the star to shine. Some energy must be provided in the form of heat, to provide the pressure which supports the star. We may use the Virial theorem to estimate how much of the gravitational energy can be utilized by the luminosity. Consider a star with no mass motions, so that the macroscopic kinetic energy $\mathbf{T}$ is zero. Assume that the equilibrium state is good enough that we can replace the time averages by the instantaneous values. Then the Virial theorem becomes

$$2\mathbf{U} + \Omega = 0 \tag{2.4.3}$$

Where $\mathbf{U}$ is the total internal kinetic energy of the gas which includes all motions of the particles making up the gas. Now from thermodynamics that not all the internal kinetic energy is available to do work, and it is therefore not counted in the internal energy of the gas. The internal kinetic energy density of a differential mass element of the gas is

$$dU = (3/2)RTdm = (3/2)(CP - CV)Tdm \tag{2.4.4}$$

where the relationship of the gas constant R to the specific heat C , then the internal heat energy of a differential mass element is

$$dU = CVTdm \tag{2.4.5}$$

Eliminating Tdm from equations 2.4.4 and 2.4.5 and integrating the energy densities of the entire star , we get

$$U = (3/2) < \gamma - 1 > U \tag{2.4.6}$$

where U is the total internal heat energy or just the total internal energy. The quantity $< \gamma - 1 >$ is the value of $\gamma - 1$ averaged over the star. For simplicity, let us assume that γ is constant through out the star. Then the Virial theorem becomes

$$3(\gamma - 1)U + \Omega = 0 \tag{2.4.7}$$

Remembering that the total energy E is the sum of the internal energy and the gravitational energy, we can express the Virial theorem in the following ways:

$$\begin{aligned} U &= \frac{-\Omega}{3(\gamma - 1)} \\ E &= -(3\gamma - 4)U \\ E &= \frac{3\gamma - 4}{3(\gamma - 1)}\Omega \end{aligned} \tag{2.4.8}$$

For $\gamma > 4/3$ (that is, $n < 3$, the total energy of the star will be negative. This simply says that the star is gravitationally bound and can be in equilibrium. So we can look for the physically reasonable polytropes to have indices less than or equal to 3. The case of n = 3, represents radiation dominated gas. In the limit of complete radiation dominance, the total energy of the configuration will be zero.

2.4.2 Rotational Energy

Using the Virial theorem to estimate the gravitational energy, we set the mass motions of the star to zero so that the macroscopic kinetic energy T was zero. However, stars do rotate, and we should not forget to count the rotational energy in the inventory of energies. We may place a reasonable upper limit on the magnitude of the rotational energy that we can expect by noting that :

- the moment of inertia of the star will always be less than that of a sphere of uniform density and

- there is a limit to the angular velocity ω_c at which the star can rotate.

Thus, for a centrally condensed star ,

$$\omega^2 \leq \frac{8GM}{27R_p^3}, I_z \leq \frac{2}{5}MR^2 \tag{2.4.9}$$

where G is gravitational constant , M is mass of star , I_z is moment of inertia of spherical star along z-axis and R is radius of star.

which implies that the rotational energy must be bounded by

$$E_{rot} = \frac{1}{2}I_z\omega^2 = \frac{8}{135}\frac{GM^2}{R} \tag{2.4.10}$$

we have used the angular velocity limit for a centrally condensed star and the moment of inertia for a uniform-density star, but the fact remains that it is extremely difficult for a star to have more than about 10 percent of the magnitude of its gravitational energy stored in the form of rotational energy.

2.4.3 Nuclear Energy

The ultimate upper limit for stored energy is the energy associated with the rest mass. It is also the common way of estimating the energy available from nuclear sources. Indeed, that fraction of the rest mass which becomes energy when four hydrogen atoms are converted to one helium atom provides the energy to sustain the solar luminosity.

Clearly most of the energy to be gained from nuclear fusion occurs by the conversion of hydrogen to helium and less than one-half of that energy can be obtained by all other fusion processes that carry helium to iron. Nevertheless, percent of Mc^2 is a formidable supply of energy.

In any event, only nuclear processes hold the promise of providing the solar luminosity for the time required to bring about agreement with the age of the solar system as derived from rocks and meteorites. However, the time scales are interesting because they provide an estimate of how long the various energy sources could be expected to maintain some sort of equilibrium configuration.

Stellar Timescales

One of the most useful notions in stellar astrophysics for establishing an intuitive feel for the significance of various physical processes is the time required for those processes to make a significant change in the structure of the star.To enable us to estimate the relative importance of these processes, we shall estimate the time scales for several of them.

There are three important basic stellar timescales: the nuclear time scale τ_n, the thermal time scale τ_t and the dynamical or free-fall time scale τ_d.

The Nuclear(evolutionary)Time Scale

The time in which a star radiates away all the energy that can be released by nuclear reactions. An estimate of this time can be obtained if one calculates the time in which all available hydrogen is turned into helium. On the basis of theoretical considerations and evolutionary computations it is known that only just over 10 percent of the total mass of hydrogen in the star can be consumed before other, more rapid evolutionary mechanisms set in. Since 0.7 percent of the rest mass is turned into energy in hydrogen burning, the nuclear time scale will be

$$\tau_n \sim \frac{K_n M C^2}{L} \tag{2.4.11}$$

where K_n is just the fraction of the rest mass available to a particular nuclear process, M is rest mass, L is stellar luminosity and c is speed of light.

$\Rightarrow \tau_n = \frac{E_{nuclear}}{L}$

$\tau_n \approx \frac{0.007X0.01Mc^2}{L}$

For the Sun one obtains the nuclear time scale 10^{10} years, and thus

$\tau_n \approx \frac{M/M_{sun}}{L/L_{sun}} X10^{10}a$

The Kelvin-Helmholtz (Thermal) Time Scale

The time in which a star would radiate away all its thermal energy if the nuclear energy production were suddenly turned off. This is also the time it takes for radiation from the centre to reach the surface /or how long can a star keep up its radiation if nuclear fusion stops and thermal energy is the only energy source left.

Thermal Timescale = Kelvin-Helmholtz Timescale

$$\tau_{KH} \approx \frac{GM^2}{RL} \tag{2.4.12}$$

i.e $\tau_{KH} \approx \frac{E_{th}}{L}$

but from Virial theorem, $E_{th} = -\frac{1}{2}E_{pot}$

Then, $\tau_{KH} = \frac{GM^2}{RL}$

The thermal time scale may be estimated as

$\tau \approx 0.5GM^2/RL$

$\approx \frac{(M/M\odot)^2}{(R/R\odot)(L/L\odot)} X2X10^7 a$

where G is the constant of gravity and R the stellar radius.

For the Sun the thermal time scale is about 20 million years or 1/500 of the nuclear time scale.

Dynamical(Free-fall) Time Scale

The third and shortest time scale is the time it would take a star to collapse if the pressure supporting it against gravity were suddenly removed. It can be estimated from the time it would take for a particle to fall freely from the stellar surface to the centre.

$$\tau_{ff} \approx \frac{1}{\sqrt{G\rho_{mean}}} \tag{2.4.13}$$

For Sun: $1hr << 3.10^7 yr << 10^{10} yr$

So except for explosive phases, stars are always in quasi-hydrostatic equilibrium, and contraction phases last about 1 percent of nuclear phases.

Nuclear processes in stars

For a star in thermal equilibrium, an internal energy source is required to balance the radiative energy loss from the surface. This energy source is provided by nuclear reactions that take place in the deep interior, where the temperature and density are sufficiently high [10]. In ordinary stars, where the ideal-gas law holds, this stellar nuclear reactor is very stable: the rate of nuclear reactions adapts itself to produce exactly the amount of energy that the star radiates away from its surface. Another important effect of nuclear reactions is that they change the composition by transmutations of chemical elements into other, usually heavier, elements. In this way stars produces all the elements in the Universe heavier than helium by a process called stellar nucleosynthesis.

Consider a reaction whereby a nucleus X reacts with a particle a, producing a nucleus Y and a particle b. This can be denoted as

$$X + a \rightarrow Y + b \; or \; X(a, b)Y \qquad (2.4.14)$$

The particle a is generally another nucleus, while the particle b could also be a nucleus, a photon or perhaps another kind of particle. Some reactions produce more than two particles (e.g. when a weak interaction is involved, an electron and anti-neutrino can be produced in addition to nucleus Y).

Each nucleus is characterized by two integers , the charge Zi (representing the number of protons in the nucleus) and the baryon number or mass number Ai (equal to the total number of protons plus neutrons). Charges and baryon numbers must be conserved during a reaction, i.e. for the example

$$Z_X + Z_a = Z_Y + Z_b \; and \; A_X + A_a = A_Y + A_b \qquad (2.4.15)$$

If a or b are non-nuclear particles then $A_i = 0$, while for reactions involving weak interactions the lepton number must also be conserved during the reaction. Therefore any three of the reactants uniquely determine the fourth.

The main nuclear burning cycles

Many different nuclear reactions can occur simultaneously in a stellar interior. If structural changes occur on a very short timescale, a large network of reactions has to be calculated.

However, for the calculation of the structure and evolution of a star usually a much simpler procedure is sufficient, for the following reasons:

• The very strong dependence of nuclear reaction rates on the temperature, combined with the sensitivity to the Coulomb barrier $Z_1 Z_2$, implies that nuclear fusions of different possible fuels - hydrogen, helium, carbon, etc, are well separated by substantial temperature differences.The evolution of a star therefore proceeds through several distinct nuclear burning cycles.

• For each nuclear burning cycle, only a handful of reactions contribute significantly to energy production and/or cause major changes to the overall composition.

• In a chain of subsequent reactions, often one reaction is by far the slowest and determines the rate of the whole chain. Then only the rate of this bottleneck reaction needs to be taken into account.

When the temperature becomes higher than about $10^9 K$, the energy of the photons becomes large enough to destroy certain nuclei. Such reactions are called photonuclear reactions or photodissociations. The production of elements heavier than iron requires an input of energy, and therefore such elements cannot be produced by thermonuclear reactions. Elements heavier than iron are almost exclusively produced by neutron capture during the final violent stages of stellar evolution.

Chapter 3

Homology Transformation , Polytropes and the Structures of the Stars

3.1 Homology Transformation and Basic assumptions

Any rational structure must have a beginning, a set of axioms, upon which to build.In addition to the known laws of physics, we have to assume a few things about stars to describe them. A self-gravitating plasma will assume a spherical shape. This fact can be rigorously demonstrated from the nature of an attractive central force so it does not fall under the category of an axiom. A less obvious axiom, but one which is essential for the construction of the stellar interior , is that the density is a monotonically decreasing function of the radius. Mathematically, this can be expressed as

$$\rho(r) \leq <\rho>(r),\ for\ r>0, \tag{3.1.1}$$

$$where <\rho> = M(r)/[\frac{4}{3}\pi r^3] \tag{3.1.2}$$

and $M(r)$ is the mass interior to a sphere of radius r and is just $\int 4\pi r^2 \rho dr$.

The conservation of mass basically requires that the total mass interior to r be accounted for by summing over the density interior to r. i.e considering a spherical shell of thickness dr at the distance r from the centre, it is mass distribution is

$dM(r) = 4\pi r^2\rho$, giving the mass continuity equation

$\Rightarrow \frac{dM(r)}{dr} = 4\pi r^2\rho$

$\Rightarrow M(r) = \int 4\pi r^2\rho dr$

we assume as a working hypothesis that the appropriate equation of state is the ideal-gas law. Although this is expressed as an assumption , to estimate the conditions which exist inside a star and that they are fully compatible with the assumption. Thus, the assumption of hydrostatic equilibrium is an excellent one for virtually all aspects of stellar structure. Most stars have dynamical time scales ranging from fractions of a second to several months , but in all cases this time is a small fraction of the typical evolutionary time scale. Thus, the assumption of hydrostatic equilibrium is an excellent one for virtually all aspects of stellar structure. For spherical stars , we may take advantage of the simple form of the gradient operator and the source equation for the gravitational potential to obtain a single expression relating the pressure gradient to $M(r)$ and ρ . The source equation for the gravitational potential field is also known as **Poisson's** equation and in general it is

$$\nabla^2\Omega = 4\pi G\rho \tag{3.1.3}$$

which in spherical coordinates becomes

$$\frac{d}{dr}(r^2\frac{d\Omega}{dr}) = 4\pi G\rho r^2 \tag{3.1.4}$$

Integrating equation (3.1.4) over r, we get

$$\frac{d\Omega}{dr} = \frac{G}{r^2}\int_0^r 4\pi r^2\rho dr = \frac{dM(r)}{r^2} \tag{3.1.5}$$

But the potential gradient is given by: $\nabla P = -\rho\nabla\Phi$

Replacing this potential gradient equation in equation (3.1.5), we have

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{3.1.6}$$

This is the equation of hydrostatic equilibrium for spherical stars. Because of its generality and the fact that virtually no assumptions are required to obtain it, we can use its integral to place fairly narrow limits on the conditions that must prevail inside a star.

3.2 Integral Theorems from Hydrostatic Equilibrium

3.2.1 Limits on State Variables

Following Chandrasekhar [13], we wish to define a quantity $I_{\sigma,v}(r)$ which is effectively the σ^{th} moment of the mass distribution further weighted by r^{-v}.

Specifically

$$I_{\sigma,v}(r) = \frac{G}{4\pi}\int_0^r \frac{[M(r)]^\sigma}{r^v} dM(r) \tag{3.2.1}$$

There are quite a variety of physical quantities which can be related to $I_{\sigma,v}$. For example,

$$4\pi I_{11} = G\int_0^r M(r)\rho 4\pi r^2 \frac{dr}{r} \tag{3.2.2}$$

is just the absolute value of the total gravitational energy of the star.

We can use this integral quantity to place limits on physical quantities of interest if we replace ρ by $< \rho >$ as defined by equation (3.1.2). Since

$$r^v = [\frac{M(r)}{\frac{4}{3}\pi < \rho > (r)}]^{v/3} \tag{3.2.3}$$

We may rewrite I_ρ, V as,

$$I_{\rho,v}(r) = \frac{G}{4\pi}\int_0^r [M(r)]^{\sigma-v/3}(4\pi)^{v/3} < \rho >^{v/3} dM(r) \tag{3.2.4}$$

Now since our assumption of the monotonicity of ρ requires $\rho_c \geq < \rho > \geq \rho(r)$, we can obtain an inequality to set limits on $I_{\rho,v}$. Namely,

$$\frac{G}{4\pi}[\frac{4\pi}{3}]^{v/3}\rho_c^{v/3}\frac{M^{(\sigma+1-v/3)}(r)}{(\sigma+1-v/3)} \geq I_{\sigma,v}(r) \geq \frac{G}{4\pi}[\frac{4\pi}{3}]^{v/3} < \rho >^{v/3} (r)\frac{M^{(\sigma+1-v/3)}(r)}{(\sigma+1-v/3)} \tag{3.2.5}$$

Now let us relate $< P >$, $< T >$, and $< g >$ to $I_{\sigma,v}$, where these quantities are defined as

$$\begin{aligned}
< P > &\equiv \int_0^M P(r)\frac{dM(r)}{M} \\
< T > &\equiv \int_0^M T(r)\frac{dM(r)}{M} \\
< g > &\equiv \int_0^M T(r)\frac{dM(r)}{M} \qquad (3.2.6)
\end{aligned}$$

Making use of the result that the surface pressure and temperature are effectively zero compared to their internal values, we can eliminate the temperature by using the ideal-gas law, integrate the first two members of equations (3.2.6) by parts and eliminate the pressure gradient by utilizing hydrostatic equilibrium. We obtain

$$\begin{aligned}
< P > &= \frac{I_{2,4}(R)}{M} \\
< T > &= \frac{4\pi\mu m_h}{3k}\frac{I_{1,1}(R)}{M} \\
< g > &= \frac{4\pi I_{1,2}(R)}{M} \qquad (3.2.7)
\end{aligned}$$

The last of these expressions comes immediately from the definition of g. Applying the inequality [(equation (3.2.5)], we can immediately obtain lower limits for these quantities of

$$\begin{aligned}
< P > &\geq \frac{3GM}{20\pi R^4} = 5.4X10^8(\frac{M}{M_\odot})^2(\frac{R_\odot}{R})^4 atm \\
< T > &\geq \frac{G\mu m_h M}{5kR} = 4.61X10^6\mu(\frac{M_\odot}{M})(\frac{R_\odot}{R})K \\
< g > &\geq \frac{3GM}{4R^2} = 2.05X10^4(\frac{M_\odot}{M})(\frac{R_\odot}{R})cm/s^2 \qquad (3.2.8)
\end{aligned}$$

Since these theorems apply for any gas sphere in hydrostatic equilibrium where the ideal-gas law applies, we can use them for establishing the range of values to be expected in stars in general. In addition, it is possible to use the other half of the inequality to place upper limits on the values of these quantities at the center of the star.

3.2.2 β^* Theorem and Effects of Radiation Pressure

A theorem that places limits on the effects of radiation pressure; is generally known as the β^* theorem [13]. Let us define β as the ratio of the gas pressure to the total pressure which includes the radiation pressure.

i.e $\beta = P_{gas}/P_T$, but $P_T = \frac{k}{\mu m_H}\rho T + \frac{1}{3}aT^4$,

and the radiation pressure for a photon gas in equilibrium is $P_r = aT^{4/3}$, where a is the radiation constant. Combining these definitions with the ideal-gas law, we can write

$$\begin{aligned} P_g &= \beta P_T = [\frac{3}{a}(\frac{k}{\mu m_h})^4\frac{1-\beta}{\beta}]^{1/3}\rho^{4/3} \\ P_r &= (1-\beta)P_T = \frac{aT^4}{3} \\ T &= [\frac{3k(1-\beta)}{\mu a m_h \beta}]^{1/3}\rho^{1/3} \\ P_c &= \frac{P_{g,c}}{\beta_c}\frac{1}{\beta_c}[\frac{3}{a}(\frac{k}{\mu m_h})^4\frac{1-\beta_c}{\beta_c}]^{1/3}\rho_c^{4/3} \end{aligned} \tag{3.2.9}$$

Using the integral theorems to place an upper limit on the central pressure, we get

$$P_c \le \frac{1}{2}G(\frac{4\pi}{3})^{1/3}\rho_c^{4/3}M^{2/3} \tag{3.2.10}$$

Equation (3.2.10), when combined with the last of equations (3.2.9) and solved for M, yields

$$M \ge (\frac{6}{\pi})^{1/2}[\frac{1-\beta_c}{\beta_c^4}(\frac{k}{km_h})^4\frac{3}{a}]^{1/2}G^{-3/2} \tag{3.2.11}$$

Now we define $\beta*$ to be the value of β which makes Equation (3.2.11) an equality, and then we obtain the standard result that

$$\frac{1-\beta^*}{(\beta_*)^4} \ge \frac{1-\beta_c}{(\beta_c)^4} \tag{3.2.12}$$

Since $(1-\beta)/\beta^4$ is a monotone increasing function of $(1-\beta)$,

$$1-\beta^* \ge 1-\beta_c = \frac{P_{r,c}}{P_r} \tag{3.2.13}$$

3.3 Homology Transformations

The term homology means "proportional to" and is denoted by the symbol $\sim$. In astronomy, the term homology has been used almost exclusively to relate one stellar structure to another in a special way. Thus a homology transformation is a mapping which relates the elements of one set to those of another.

One can characterize the structure of a star by means of the five variables $P(r)$, $T(r)$, $M(r)$, $\mu(r)$, and $\rho(r)$ which are all dependent on the position coordinate r.we have produced three constraints on these variables, the ideal-gas law, hydrostatic equilibrium, and the definition of $M(r)$.Thus specifying the transformation of any two of the five dependent variables and of the independent variable r specifies the remaining three. If the transformations can be written as simple proportionalities, then the two stars are said to be homologous to each other.

For example, if

$$\begin{aligned} r' &= C_1 r \\ \rho'(r') &= C_2 \rho(r) \\ \xi'(r') &= C_3 \xi(r) \end{aligned} \tag{3.3.1}$$

then

$$\begin{aligned} \zeta'(r') &= C_4 \zeta(r) \\ \eta'(r') &= C_5 \eta(r) \\ \chi'(r') &= C_6 \chi(r) \end{aligned} \tag{3.3.2}$$

where ξ, ζ, η, and χ stand for any of the remaining structure variables. However, because of the constraints C4, C5, and C6 are not linearly independent but are specified in terms of the remaining C's. Consider the definition of $M(r)$ and a homology transformation from

$r \to r'$. Then

$$\frac{M'(r')}{M(r)} = \frac{\int_0^{r'} 4\pi(x'^2)\rho'(x')dx}{\int_0^{r} \rho(x)dx} = C_2 C_1^3 \tag{3.3.3}$$

Similarly,the homology transformation for pressure P,(equation of hydrostatic equilibrium) is given as

$$\frac{P'(r')}{P(r)} = \frac{\int_0^{r'} [GM'(x')/x'^2]dx'}{\int_0^{r} [GM(x)\rho(x)/x'^2]dx} = C_2^2 C_1^2 \tag{3.3.4}$$

If we take μ to be the chemical composition m, then the remaining structure variable is the temperature whose homology transformation is specified by the ideal-gas law as

$$\frac{P'(r')}{P(r)} = \frac{\rho'(r')kT'(r')/\mu'(r')}{\rho(r)kT(r)/\mu(r)} = C_2^2 C_1^2 \tag{3.3.5}$$

Take ξ to be T, then the homology transform for μ is specified as

$$\frac{\mu'(r)}{\mu(r)} = \frac{C^3}{C_1^2 C_2} \tag{3.3.6}$$

We can use the constraints specified by equations (3.3.3), (3.3.4), and (3.3.6) and the initial homology relations [equation (3.3.1)] to find how the structure variables transform in terms of observables such as the total mass M and radius R. Thus,

$$\begin{aligned} \frac{\rho'(r)}{\rho(r)} &= \frac{M'}{M}(\frac{R}{R'})^3 \\ \frac{P(r')}{P(r)} &= (\frac{M'}{M})^2(\frac{R}{R'})^4 \\ \frac{T'(r')}{T(r)} &= \frac{\mu' M' R}{\mu M R} \end{aligned} \tag{3.3.7}$$

The primary utility of homology transformations is that they provide a feel for how the physical structure variables change given a simple change in the defining parameters of the star, all other things being equal. An intuitive feel for the behavior of the state variables P, T, and p which result from the scaling of the mass and radius is essential if one is to understand stellar evolution. Consider the homologous contraction of a homogeneous uniform density mass configuration. Here the total mass and composition remain constant, and we obtain a very specific homology transformation

$$\begin{aligned} \frac{\rho}{\rho_o} &= (\frac{R_o}{R})^3 \\ \frac{P}{P_o} &= (\frac{R_o}{R})^4 \\ \frac{T}{T_o} &= \frac{R_o}{R} \end{aligned} \tag{3.3.8}$$

which is known as Lane's Law and has been thought to play a role in star formation and phases of stellar collapse.

3.4 Polytropes

Historically, the idea of polytropes arose through modelling of fully convective gases, in which the gas is completely turned over hence polytropic (poly, many, much; tropos, to turn). convection may be assumed to occur rapidly enough that a cell of gas does not exchange heat with its surroundings, i.e., is adiabatic, so that PV^{γ} is constant.

We can generalize this adiabatic equation of state to a polytropic equation of state, such that the pressure is assumed to be proportional to density to some power (equivalent to $PV^{\gamma} = constant$, since $V \alpha 1/\rho$ for fixed mass). This relationship between pressure and density has its origins in thermodynamics and results from the notion of polytropic change. This gives rise to the polytropic equation of state

$$P(r) = K\rho^{\gamma}(r) \equiv P(r) = K\rho(r)^{(1+n)/n} \tag{3.4.1}$$

where K is the polytropic constant (of proportionality),

$\gamma = (n+1)/n$ is the polytropic exponent, and

$n = 1/(\gamma - 1)$ is the polytropic index (not number-density)

Clearly, an equation of state of this form, when coupled with the equation of hydrostatic equilibrium, will provide a single relation for the run of pressure or density with position. The solution of this equation basically solves the fundamental problem of stellar structure insofar as the equation of state correctly represents the behavior of the stellar gas. Such solutions are called polytropes of a particular index n.

3.4.1 Polytropic Change and the Lane-Emden Equation

From basic thermodynamics,the infinitesimal change in the heat of a gas $\eth Q$ can be related to the change in the internal energy dU.The work done on the gas so that

$$\eth Q = dU + PdV = \frac{\partial U}{\partial T} dT + PdV \tag{3.4.2}$$

The strange-looking derivative $\eth$ is known as a Pfaffian derivative, and its most prominent property is that it is not an exact differential. A complete discussion of the mathematical properties is given by [2]. The ideal-gas law can be stated in its earliest form as $PV = RT$, which leads to

$$PdV + VdP = RdT \tag{3.4.3}$$

where R is the gas constant and V is the specific volume (i.e., the volume per unit mass). Now let us define the specific heat at constant a C_α as

$$[\frac{\eth Q}{dT}]|_{\alpha=constant} \equiv C_\alpha \tag{3.4.4}$$

Here,the differentiation is done in such a way that α remains constant. Thus $(\frac{dU}{dT})|_{V=constant}$ is the specific heat, C_V , at constant volume.
Using equation (3.4.3) to eliminate PdV in equation (3.4.2), we get

$$C_P = C_V + R \tag{3.4.5}$$

where C_P is the specific heat at constant pressure.
With this notion that $(\eth Q/dT)|_\alpha$ is the specific heat at constant α , we make the generalized definition of polytropic change to be

$$\frac{\eth Q}{dT} = C \tag{3.4.6}$$

where C is some constant.

Using equations (3.4.2), and (3.4.3) and the definition of C we can write

$$T^{C_v - C} V^{C_p - C_v} = constant \tag{3.4.7}$$

For an ordinary gas the ratio of specific heats is defined as γ. i.e $\gamma = C_P/C_V$

In that same spirit, we can define a polytropic gamma as

$$\gamma' = \frac{C_P - C}{C_V - C} \tag{3.4.8}$$

Using of the ideal-gas law, we can write

$$P = const \times V^{-\gamma'} = (const)(\frac{k}{\mu m_h})P^{\gamma'} = K\rho^{(n+1)/n} \tag{3.4.9}$$

Thus, we can relate the specific heat C associated with polytropic change to the polytropic index n to be found in the polytropic equation of state , equation (3.4.1). So that

$$n = \frac{1}{(\gamma' - 1)} \tag{3.4.10}$$

If C = 0, then the general relation describes where the change in the internal energy is equal to the work done on the gas [see equation (3.4.2)], which means the gas behaves adiabatically. If C = 4, then the gas is isothermal.

The polytropic equation of state provides us with a highly specific relationship between P and ρ . However, hydrostatic equilibrium also provides us with a specific relationship between P and ρ , and we may use the two to eliminate the pressure P, thereby obtaining an equation in ρ alone which describes the run of density throughout the configuration. Then, differentiating equation (3.1.6) with respect to r and eliminating P by means of the polytropic equation of state, we get

$$\frac{d}{dr}[\frac{Kr^2(n+1)}{n\rho^{(n-1)/n}}\frac{dP}{dr}] \tag{3.4.11}$$

This nonlinear second-order differential equation for the density distribution is subject to the boundary conditions:

$\rho_0 = \rho_c$ and $\rho_R = 0$. Or to put it another way, the radius of the configuration is defined to be that value of r for which $\rho = 0$.

The only free parameters in the equation are the polytropic index (n) and the parameter K and any solution to such an equation is called a polytrope. The parameter K is related to the total mass of the configuration. In addition, the equation is generally known as the Lane-Emden Equation.If one is going to investigate the general solution-set of any equation, it is usually a good idea to express the equation in a dimensionless form. This can be done to equation (3.4.11) by transformation to the so-called Emden variables given by,

$$\begin{aligned} \rho &= \lambda\theta^n \\ r &= \alpha\xi \end{aligned} \tag{3.4.12}$$

where $\alpha = [(n+1)K\lambda^{\frac{1}{n}-1}]^{1/2}$

Here λ is just a scaling parameter useful for keeping track of the units of ρ ,ξ is just a scaled dimensionless radius , and θ is dimensionless by virtue of using λ to absorb the units of ρ , it does vary as $\rho^{(1/n)}$ and is the normalized ratio of P/ρ. Substituting for r , α and ρ in equation (3.4.11) we obtain an equation which is the standard form of the LaneEmden equation :

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \tag{3.4.13}$$

By picking K and n we can transform any solutions of eq (3.4.13) and obtain the solution for the polytrope of a given mass M and index n in terms of the run of physical density with position.The non-linear nature of the transformation has had the advantage that the boundary conditions of the physical equation can easily be written as initial conditions at

$\xi = 0$. The utility of λ now becomes clear as we can scale $\theta(0)$ to be 1 so that

$$\begin{aligned} \theta(\xi = 0) &= 1 \\ \lambda &= \rho_c \\ \frac{d\theta}{d\xi}|_{\xi=0} &= o \end{aligned} \tag{3.4.14}$$

The last initial condition comes from hydrostatic equilibrium. As r $\rightarrow$ 0, M(r) $\rightarrow$ 0, as r^3 and $\rho \rightarrow \rho_c$. Thus $dP/dr \rightarrow 0$ as well. This implies that $\frac{d\theta}{d\xi} \rightarrow 0$ as $\xi \rightarrow 0$.

we can use the Lane-Emden Equation for any polytropic index n. Unfortunately, only three analytic solutions exist, i.e for n = 0, 1 and 5, for which the solutions are:

$$\begin{aligned} n = 0: \ \theta_0(\xi) &= 1 - \frac{\xi^2}{6}, \ \xi_0 = \sqrt{6}, \\ n = 1: \ \theta_1(\xi) &= \frac{sin\xi}{\xi}, \ \xi_1 = \pi, \\ n = 5: \ \theta_5(\xi) &= (1 + \frac{\xi^2}{3})^{-1/2}, \ \xi_5 = \infty. \end{aligned} \tag{3.4.15}$$

The case $n = 0(\gamma' = \infty)$ corresponds to a homogeneous gas sphere with constant density ρ_c, following eqn. (3.4.12),/or the solution is monotonically decreasing toward the surface which is physically reasonable. This is also true for n = 1, and n = 5 although the rate of decline is slower.Indeed, the n = 5 case only, θ asymptotically approaches zero from arbitrarily large ξ.If we denote the value of ξ for which θ goes to zero as ξ_1,then

$$\xi_1 = \begin{cases} \sqrt{6} & \text{n=0,} \\ \pi & \text{n=1,} \\ \infty & \text{n=5} \end{cases}$$

For other values of the polytropic index n it is possible to develop a series solution which is useful for starting many numerical methods for the solution.

The first few terms of the solution:

$$\theta_n = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - (\frac{8n^2 - 5n}{15,120})\xi^6 + ...+ \tag{3.4.16}$$

3.4.2 Mass-Radius Relationship for Polytropes

Once the solution $\theta(\xi)$ of the Lane-Emden equation is found, eq. (3.4.12 and 3.4.14; $\rho = \rho_c\theta^n$, fixes the relative density distribution of the model,which is thus uniquely determined by the polytropic index n. Given the solution for a certain n, the physical properties of a polytropic stellar model, such as its mass and radius, are then determined by the parameters K and ρ_c The radius of a polytropic model follows from eq. (3.4.12)

$$R = \alpha\xi_n = [\frac{(n+1)K}{4\pi G}]^{1/2}\rho_c^{(1-n)/2n}\xi_n. \tag{3.4.17}$$

The mass $M(\xi)$ interior to ξ can be obtained from integrating mass continuity equation, $dM(r) = 4\pi r^2\rho dr$

So we can write for the total mass,

$$M(\xi_1) = \int_0^{R/\alpha} 4\pi r^2\rho dr = 4\pi\alpha^3\lambda\int_0^{\xi_1}\xi^2\theta^n d\xi = -4\pi\alpha^3\rho_c(\xi^2\frac{d\theta}{d\xi}\mid_{\xi_1}) \tag{3.4.18}$$

Using $R = \alpha\xi$, we can obtain a mass-radius relation for any polytrope ,

$$GM^{(n-1)/n}R^{(3-n)/n} = -K(n+1)[(4\pi)^{-1/2}][\xi^{(n+1)/(n-1)}(\frac{d\theta}{d\xi})|_{\xi_1}]^{(n-1)/n} \tag{3.4.19}$$

This equation (3.4.20) can be used to determine K since everything else on the right-hand side depends on only the polytropic index n. so that,

$$K = \frac{G}{n+1}[\frac{4\pi}{\xi_1^{(n+1)(\frac{-d\theta}{d\xi}|_{\xi_1})^{(1-n)}}}]^{1/n}M^{(n-1)/n}R^{(3-n)/n} \tag{3.4.20}$$

Thus , for a collection of polytropic model stars we can write the mass-radius relation as

$$M^{(n-1)/n}R^{(3-n)/n} = (const)(n) \tag{3.4.21}$$

The mean stellar density ,

$$\bar{\rho} = \frac{3M}{4\pi R^3}\rho_c\frac{3}{\xi^3}\{-\xi^2\frac{d\theta}{d\xi}_1\}$$

The central pressure can be expressed trivially as

$$P_c = K\rho_c^{(n+1)/n} = \{4\pi(n+1)(\frac{d\theta}{d\xi}|_{\xi_1})^2\}^{-1}\frac{GM^2}{R^4}$$

The core temperature can be estimated from the central pressure and density, if we know the equation of state.

$$T_C \backsimeq \frac{\mu m(H)}{k}\frac{P_c}{\rho_c}$$

3.4.3 Homology Invariants

Here, we can apply homology transformations to polytropes.In general, if $\theta_n(\xi)$ is a solution of the Lane-Emden equation, then $A^{2/(n-1)}\theta_n(\xi)$ is also a solution (for a proof see [13].Here A is an arbitrary constant, so Aξ is clearly a homology transformation of ξ.This produces an entire family of solutions to the Lane-Emden equation,and it would be useful if we could obtain a set of solutions which contained all the homology solutions. To do this, we must find a set of variables which are invariant to homology transformations. Following Chandrasekhar [13], we have

$$u = \frac{dln[M(r)]}{dlnr} = \frac{3\rho(r)}{<\rho(r)>}$$

$$(n+1)v = -\frac{dln[P(r)]}{dlnr} = +\frac{3}{2}\frac{[GM(r)/r]}{[(\frac{3}{2})kT/\mu m_h]} = -(n+1)\xi\theta^{-1}\frac{d\theta}{d\xi} \qquad (3.4.22)$$

Where u is 3 times the ratio of the local density to the local mean density, while (n+1)v is simply 1.5 times the ratio of the local gravitational energy to the local internal energy.In general, these quantities will remain invariant to any change in the structure which can be described by a homology transformation. We can use these variables to rewrite the Lane-Emden equation so as to obtain all solutions which are homologous to each other.

$$\frac{u}{v}\frac{dv}{du} = -\frac{u+v-1}{u+nv-3} \tag{3.4.23}$$

Not all solutions to this equation are physically reasonable.At the center of the polytrope the values [u=3, v=0] set the initial conditions for the unique solution meeting the minimal requirements for being a physical solution. These solutions are known as the E-solutions and we have already given a series expansion for the θ_E solution in equation (3.4.16). By substituting this series into the equations for u and v, and expanding by the binomial theorem we obtain the following series solutions for u and v:

$$\begin{aligned} v &= \frac{\xi^2}{3}(1 - \frac{3n-5}{30}\xi^2 + \frac{12n^2-39n+35}{1260}\xi^4 + ...+) \\ u &= 3 - \frac{n\xi^2}{5} + (\frac{19n^2-25}{1050})\xi^4 - (\frac{472n^2-125n^2+875n}{283,5000})\xi^6 + ...+ \end{aligned} \tag{3.4.24}$$

These solutions do not satisfy the condition of hydrostatic equilibrium at the center of the polytrope, they may represent valid solutions for stars composed of multiple polytropes joined in the interior.The polytrope with n = 1.5 represents the solution for a star in convective equilibrium, while the n = 3 polytrope solution is what is expected for a star dominated by radiation pressure.

Stars with convective energy transport

The first law of thermodynamics states that the change in internal energy of a system, dU, is given by the heat added to the system, dQ, less the work done by the system, dW:

i.e $dU = dQ - dW$

For fully convective stars, all the convective cells are supposed to be adiabatic, so $dQ \equiv 0$; and for a quasistatic process dW = PdV, whence dU = -PdV

$$P = nkT = \frac{N}{V}kT \text{ , } whereN = nV$$

$$U = \frac{3}{2}NkT = \frac{3}{2}VP$$

so that dU = -PdV $\Rightarrow d(\frac{3}{2}PV) = -PdV$

$\Rightarrow \frac{3}{2}(PdV + VdP) = -PdV$

$\Rightarrow \frac{5}{2}PdV = -\frac{3}{2}VdP$

$\Rightarrow \frac{dP}{P} = -\frac{5}{3}\frac{dV}{V}$

$\Rightarrow P \propto V^{-5/3}$

So $P \propto \rho^{5/3}$, since $V \propto \rho^{-1}$

That is, fully convective stars are approximately polytropic, with $n = 1/(\gamma - 1) = 3/2$.

Stars with radiative energy transport

The total pressure P is the sum of gas pressure P_{gas} and radiation pressure P_R, where

$$Pgas = \frac{\rho}{\mu m(H)}kT \; and \; PR = \frac{1}{3}aT^4$$

We define $P_{gas} \equiv \eta P$ (or, equivalently, $P_R \equiv (1-\eta)P$)

we can therefore write $P = \frac{P_{gas}}{\eta} = \frac{\rho kT}{\eta\mu m(H)}$

Rearranging for T, substituting the result into $\frac{1}{3}aT^4$ and inserting the result into eqn. $P_R = (1-\eta)P)$ gives

$P^4(\frac{\eta\mu m(H)}{\rho k})^4 = \frac{3(1-\eta)}{a}P$

i.e., $P = \{\frac{k}{\mu m(H})\}^{4/3}\{\frac{3(1-\eta)}{a\eta^4}^{1/3}\}\rho^{4/3}$,

$\equiv K\rho^{4/3}$ if η (and μ) are constant.

That is, for constant η and constant μ, $P \propto \rho^{4/3}$; a polytropic equation of state with polytropic index n = 3.

3.4.4 Isothermal Sphere

It is when the temperature remains constant throughout the configuration.For example, if the thermal conductivity is very high, the energy will be carried away rapidly from any point where an excess should develop. Such a configuration is known as an isothermal sphere, and it has a characteristic structure all its own.An isothermal gas may be characterized by a polytropic C = 4. This leads to a polytropic index of n = 4 and some problems with the Emden variables (by eqns 3.4.8 and 3.4.10).Certainly the Lane-Emden equation in physical variables [equation (3.4.11)] is still valid since it involves only the hydrostatic equilibrium and the polytropic equation of state.

$$\frac{d}{dr}(\frac{r^2}{\rho}\frac{d\rho}{dr}) = -\frac{4r^2G\rho}{K} \tag{3.4.25}$$

However,transforming to the dimensionless Emden variables since the earlier transformation will no longer work. The traditional transformation is

$$r = \alpha\xi,\ \rho = \lambda e^{-\psi},\ \alpha = (\frac{k}{4\pi G\lambda}) \tag{3.4.26}$$

which leads to the Lane-Emden equation for the isothermal sphere:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\psi}{d\xi}) = e^{-\psi} \tag{3.4.27}$$

The initial conditions for the corresponding E solution are $\psi(0) = 0$ and $d\psi/d\xi = 0$ at $\xi = 0$. All the homology theorems hold, and the homology invariant variables u and v in terms of these new Emden variables,are

$$u = \frac{\xi e^{-\psi}}{d\psi/d\xi}, \ v = \xi\frac{d\psi}{d\xi} \tag{3.4.28}$$

Lane-Emden equation in u and v is only slightly modified to account for the isothermal condition.

$$\frac{u}{v}\frac{dv}{du} = -\frac{u-1}{u+v-3} \tag{3.4.29}$$

The solution to this equation in the u-v plane is unique; in the vicinity of $\xi = 0$, ψ can be expressed as

$$\psi = \frac{\xi^2}{6} - \frac{\xi^4}{120} + \frac{\xi^6}{1890} - \frac{61\xi^8}{16,32,960} + ...+ \tag{3.4.30}$$

which leads to the following expansions for the homology invariants u and v as given by equations (3.4.28),

$$\begin{aligned} u &= 3 - \frac{\xi^2}{3} + \frac{19\xi^4}{1050} - \frac{118\xi^6}{70,875} + ... + \\ v &= (\frac{\xi^2}{3})(1 - \frac{\xi^2}{10} + \frac{\xi^4}{105} - ...+) \end{aligned} \tag{3.4.31}$$

Direct substitution of a density dependence with the form $\rho(r)\ r^{-2}$ into equation (3.4.25) shows that such a density law will satisfy hydrostatic equilibrium at all points within an isothermal sphere.

3.4.5 Fitting Polytropes Together

Many stars, including those on the main sequence, can be reasonably represented by a combination of polytropes where the local value of the polytropic index is chosen to reflect the physical constraints placed on the star by the mode of energy transport or possibly the equation of state. Thus, it is useful to understand what conditions must hold where the polytropes meet.Thus, it is useful to understand what conditions must hold where the polytropes meet.consider a simple star composed of a core, and an envelope having different polytropic indices and let q be the fraction of the total mass in the core, n the polytropic index of the core, and m the total mass of the core. Physically, we must require that the pressure and density be continuous across the boundary. This implies that u and v are continuous across the boundary between the two polytropes. Since the initial conditions at the center of the core must be u = 3, v = 0, the core solution must be an E solution for the core index n. The envelope solution will not, in general, be an E solution; but as long as the central point (u=3, v=0) is not encountered, there is no violation of hydrostatic equilibrium by such a solution. Thus one can construct a reasonable model by proceeding outward along the core solution until the mass of the core is reached. This defines the fitting point in the u-v plane.From the ideal-gas law, the ratio of the density in the envelope to that of the core is

$$\frac{\rho_e}{\rho_c} = \frac{\mu_e}{\mu_c}$$

Chapter 4

Result and Discussion

We had worked out on stellar structure and evolution with polytropic approach to develop a simple stellar polytropic model. For this reason, the energy source of star (the nuclear reaction in it's core) and transport phenomena in the stellar interiors were well discussed with basic assumptions and theory of stellar evolution. Then,the basic equations of state which so called stellar structure equations were derived in terms of polytropic concepts. Further more the following points were summarized as result and their interpretations were given:

4.1 Summarized Hydrodynamics Equations

4.1.1 The virial theorem and its consequences

The virial theorem ,

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2T + \Omega \tag{4.1.1}$$

For steady state,

$$2T + \Omega = 0 \tag{4.1.2}$$

For perfect gas configuration in statistical equilibrium:

$$dT = \frac{3}{2}kTdN \Rightarrow \frac{3}{2}(C_p - C_v)Tdm \tag{4.1.3}$$

$$dV = C_v T dm \tag{4.1.4}$$

$$\Rightarrow T = \frac{3}{2}(\gamma - 1)U \tag{4.1.5}$$

But from eqn.(4.1.1) we have $2T = -\Omega$, then eqn(4.1.5) implies that

$$3(\gamma - 1)U + \Omega = 0 \tag{4.1.6}$$

The total energy of the system,

$$U + \Omega = E \tag{4.1.7}$$

Combining eqn.(4.1.6) and eqn.(4.1.7) we have,

$$E = (4 - 3\gamma)U = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega \tag{4.1.8}$$

Cases on eqn(4.1.8) a) for $\gamma = 4/3 \Rightarrow$ E = 0

This implies, the system of equilibrium dynamics for the γ to $\gamma = 4/3$ is changed to instability.

b) For $\gamma = 1 \Rightarrow \Omega = 0$ no stable configuration. Only $\gamma > 4/3$ is stable configuration.

c) For $\gamma > 4/3 \Rightarrow$ E is negative.

4.1.2 Radiation Stefan-Boltzmann law

Consider a perfect black body

Initial energy density $u = \alpha T^4$; $P = \frac{1}{3}\alpha T^4$

Adiabatic changes in electrons containing matter and radiation

a) For electrons containing only radiation

$dQ = du + PdV$

$= d(uV) + \frac{1}{3}udV = Vdu + \frac{4}{3}udV$

Adiabatic: $dQ = 0$

$\Rightarrow uV^{4/3} = constant \Rightarrow = TV^{4/3} = constant\ (u = \alpha T^4)$

$\Rightarrow PV = const$

This implies a radiation body perfect gas with $\gamma = 4/3$.

b) For electrons containing both matter and radiation,

$u = \alpha_v T^4 + C_v T$

; $P = P_r + P_{gas}$

$P = \frac{1}{3}aT^4 + \frac{R}{V}T$

4.1.3 Integral theorem on the equilibrium of state

1)**Hydrostatic equilibrium**

$\Rightarrow$ The total force acting on the volume element should be zero.

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho \tag{4.1.9}$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}\frac{dp}{dr}) = -4\pi G\rho \tag{4.1.10}$$

2)**The potential gradient Φ and Ω**

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2} \Rightarrow \frac{1}{\rho}\frac{dP}{dr} = -\frac{d\Phi}{dr} \tag{4.1.11}$$

for $r \geq R, M(r) = M = const.$

4.1.4 Homology transformation and polytropic equation

Homology transformation

Two stars are said to be homologous to each other, if proportionalities of transformations are given as:

$$\frac{r'}{r} = C_1, \frac{\rho'(r')}{\rho(r)} = C_2, \frac{\xi'(r')}{\xi(r)} = C_3 \tag{4.1.12}$$

and

$$\frac{\zeta'(r')}{\zeta(r)} = C_4, \frac{\eta'(r')}{\eta(r)} = C_5, \frac{\chi'(r')}{\chi(r)} = C_6 \tag{4.1.13}$$

$\Rightarrow$ Homology transformation is from $r \rightarrow r'$ i.e. $\frac{r'}{r} = constant$. Then,

a) Homology transformation of M(r),

$\frac{M(r')}{M(r)} = const.$

$\Rightarrow \frac{M(r')}{M(r)} = \frac{\int_0^{r'} 4\pi r'^2 \rho'(r')dr'}{\int_0^{r} 4\pi r^2 \rho(r)dr} = \frac{r'^3}{r^3}\frac{r'}{r}\frac{\rho'}{\rho} = C_1^3 C_2$

Therefore,

$$\frac{M(r')}{M(r)} = \frac{\int_0^{r'} 4\pi r'^2 \rho'(r')dr'}{\int_0^{r} 4\pi r^2 \rho(r)dr} = C_1^3 C_2 \tag{4.1.14}$$

b) Homology transformation for eqn. of hydrostatic equilibrium:

$$\frac{P'(r')}{P(r)} = \frac{\int_0^{r'}[GM'(r')\rho'(r')/r'^2]dr'}{\int_0^{r}[GM(r)\rho(r)/r^2]dr} = \frac{M'(r')}{M(r)}\frac{\rho'(r')}{\rho(r)}\frac{r}{r'} = C_1^3 C_2 . C_2 . 1/C1 = C_1^2 C_2^2 \tag{4.1.15}$$

c) Homology transformation of temperature:

$T(r) = \frac{\mu m_H}{k} \frac{P}{\rho}$

$$\frac{T'(r')}{T(r)} = \frac{[\mu'(r')m_H/k]P'(r')/\rho'(r')}{[\mu(r)m_H/k]P(r)/\rho(r)} = \frac{\mu'(r')}{\mu(r)} \frac{P'(r')}{P(r)} \frac{\rho(r)}{\rho'(r')} = C_3.C_2^2C_1^2.1/C_2 = C_3C_2C_1^2 \quad (4.1.16)$$

Take ξ (initial homology relation) to be T, then the homology transformation of μ:

$$\mu = \frac{\mu'(r')}{\mu(r)} = [\frac{P(r)}{P'(r')} \frac{\rho'(r')}{\rho(r)}]T = 1/C_2^2C_1^2.C_2.C_3 = \frac{C_3}{C_2C_1^2}$$

Polytropic equation

The polytropic equation of state,

$$P(r) = K\rho(r)^{(n+1)/n} \quad (4.1.17)$$

where $n = 1/(\gamma' - 1)$

From equation of hydrostatic equilibrium; $\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$

$$\frac{d}{dr}[\frac{Kr^2(n+1)}{n\rho^{(n-1)/n}} \frac{dP}{dr}] = -4\pi r^2 G\rho \quad (4.1.18)$$

This equation is transformed considering:

$$\begin{aligned} \rho &= \lambda\theta^n \\ r &= \alpha\xi \end{aligned} \quad (4.1.19)$$

where $\alpha = [(n+1)K\lambda^{\frac{1}{n}-1}]^{1/2}$

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \quad (4.1.20)$$

this governs ρ distribution in any region.

i.e.$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n$

$\Rightarrow \frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\xi^2\theta$

$\Rightarrow \xi^2\frac{d\theta}{d\xi} + 2\xi\theta = --\xi^2\theta$

Then it gives a series of expansions,

$$\frac{d\theta}{d\xi} = -\frac{\xi}{3} + \frac{C_1}{\xi^2} + ... + \frac{C_n}{\xi^n} \tag{4.1.21}$$

Now integrating this equation both sides,we get the first few terms of the solution:

$$\theta_n = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - (\frac{8n^2 - 5n}{15,120})\xi^6 + ...+ \tag{4.1.22}$$

i.e for n = 0,1 and 5, for which the solutions are:

$$\begin{aligned} n = 0: \ \theta_0(\xi) &= 1 - \frac{\xi^2}{6}, \ \xi_0 = \sqrt{6}, \\ n = 1: \ \theta_1(\xi) &= \frac{sin\xi}{\xi}, \ \xi_1 = \pi, \\ n = 5: \ \theta_5(\xi) &= (1 + \frac{\xi^2}{3})^{-1/2}, \ \xi_5 = \infty. \end{aligned} \tag{4.1.23}$$

Some numerical solutions of Lane-Emden equation is generated computationally by Mathematica and the roots of the equation for a range of polytropic indices(n=0 , 1 , ... , 6) are tabulated and the graph is plotted as the seen below.

n	ξ	n(ξ)
	0	1.
	1	0.833333
0	2	0.333333
	3	-0.5
	4	-1.66667
	0	1.
	1	0.841471
1	2	0.454649
	3	0.04704
	4	-0.189201
	0	1.
	1	0.848654
2	2	0.529836
	3	0.241824
	4	0.0488402
	0	1.
	1	0.855058
3	2	0.582851
	3	0.359227
	4	0.209282
	0	1.
	1	0.860814
4	2	0.622941
	3	0.440051
	4	0.318042
	0	1.
	1	0.866025
5	2	0.654654
	3	0.5
	4	0.39736
	0	1.
	1	0.870773
6	2	0.680548
	3	0.546667

Figure 4.1: Solution for some polytropic indices

Interpretation of the table: For $n < 5$ polytropes, the solution for θ drops below zero at a finite value of ξ and hence the radius of the polytrope ξ can be determined at this point. In the numerically integrated solutions, a linear interpolation between the points immediately before and after θ becomes negative will give the value for ξ at θ=0.

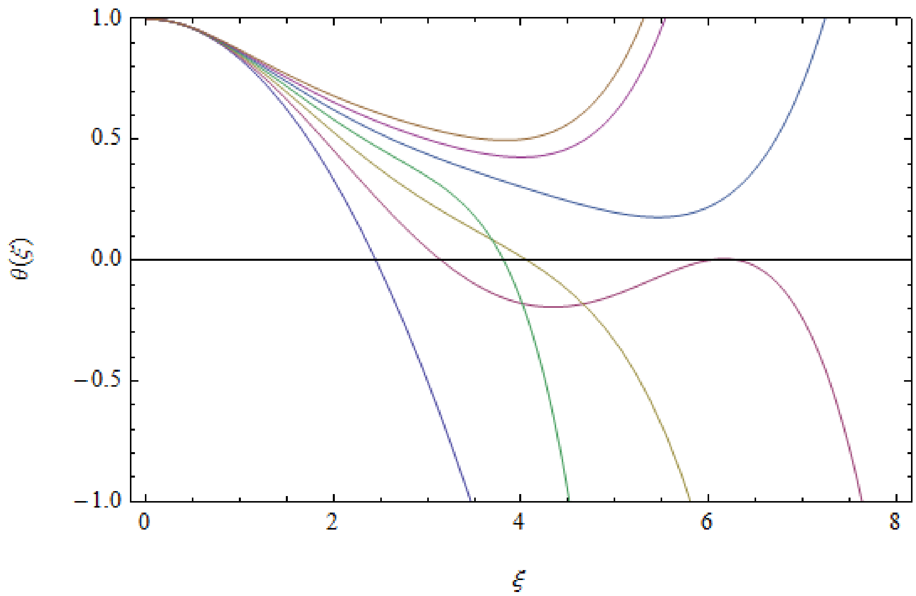

Figure 4.2: Graphical representation of polytropic solutions

This graph shows numerical solutions to the Lane-Emden equation for (left-to-right) n = 0,1,2,3,4,5,6. where $\xi = \frac{r}{\alpha}$ and $\theta(\xi) = \frac{\rho}{\lambda}$, but $\lambda = \rho_c$ then $\theta(\xi) = \frac{\rho}{\rho_c}$

4.2 Physical Interpretation of the result

From Virial theorem:

The virial theorem links the gravitational potential energy to the internal (kinetic) energy of star as whole. It tells us that a more tightly bound star have a higher internal energy, i.e. it must be hotter.In other words, a star that contracts quasi-statically must get hotter in the process.

The virial theorem,

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2T + \Omega \tag{4.2.1}$$

For steady state,

$$2T + \Omega = 0 \tag{4.2.2}$$

Transport with both radiation and matter in stars

When adiabatic changes in electrons containing both radiation and matter, the total pressure P is the sum of gas pressure Pg and radiation pressure Pr . That is, for stars where energy transport is radiative, there is a constant ratio of gas pressure to radiation pressure.

Polytropes and its implicity

Polytropes - allow simple solutions to the equations of stellar structure and describes, in principle, how P, ρ vary with radius r in a star.

$PV^{\gamma} = constant$

$\Rightarrow$ Adiabatic equation of state to a polytropic equation of state, such that the pressure is assumed to be proportional to density to some power (equivalent to $PV^{\gamma} = constant$, since $V \propto 1/\rho$ for fixed mass). So that,

$$P = K\rho^{\gamma} = K\rho^{(n+1)/n} \tag{4.2.3}$$

LaneEmden equation is the solution of the equations of hydrostatic equilibrium and mass continuity, for a polytropic equation of state, expressed in dimensionless form.

i.e,

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \tag{4.2.4}$$

A polytrope with index n = 0 has a uniform density (recall $\rho/\rho_c = \theta^n$), while a polytrope with index n = 5 has an infinite radius. In general, the larger the polytropic index, the more centrally condensed the density distribution; and only polytropes with $n \leq 5$ are bound systems.A polytrope with index$n = \infty$ has $P(= K\rho^{(1+1/n)} = K\rho$ and is a so-called isothermal sphere; a self-gravitating, isothermal body.

E-solutions

For isothermal system, temperature is constant throughout configuration and its characterized by a polytropic index of n=4 .

From hydrostatic equilibrium and polytropic equation of state,

$$\frac{d}{dr}(\frac{r^2}{\rho}\frac{dP}{dr}) = -\frac{4r^2G\rho}{K}$$

Considering homology transformation :

$r = \alpha\xi$, $\rho = \lambda = \lambda e^{-\psi}$, $\alpha = (\frac{K}{4\pi G\lambda})$ this is equation of isothermal sphere.

Isothermal conditions and E-solutions

⇒ The solution reaching the center of polytropes.

Initial conditions

$\psi(0) = 0$ and $\frac{d\psi}{d\xi} = 0$ at $\xi = 0$

Homology invariant variables u and v;

$u = \frac{\xi e^{-\psi}}{d\psi/d\xi}$, $v = \xi\frac{d\psi}{d\xi}$

where u is that it is 3 times the ratio of the local density to the local mean density and v is 1.5 times the ratio of the local gravitational energy to the local internal energy. In general, these quantities will remain invariant to any change in the structure which can be described by a homology transformation. And for the isothermal condition.

$$\frac{u}{v}\frac{dv}{du} = -\frac{u-1}{u+v-3}$$

For the ideal-gas law , the ratio of the density in the envelope to that of the core is

$$\frac{\rho_e}{\rho_c} = \frac{\mu_e}{\mu_c}$$

which is equivalent to specifying a jump in u and $(n+1)v$ by the ratio of the mean molecular weights of the core and envelope. Thus the fitting point, when it is reached, is displaced toward the origin in u and $(n + 1)v$ by the ratio of the mean molecular weights of the envelope and core.

Many stars, including those on the main sequence, can be reasonably represented by a combination of polytropes where the local value of the polytropic index is chosen to reflect the physical constraints placed on the star by the mode of energy transport or possibly the equation of state. Thus, it is useful to understand what conditions must hold where the polytropes meet. This is a simple star composed of a core and an envelope having different polytropic indices(See below graph).

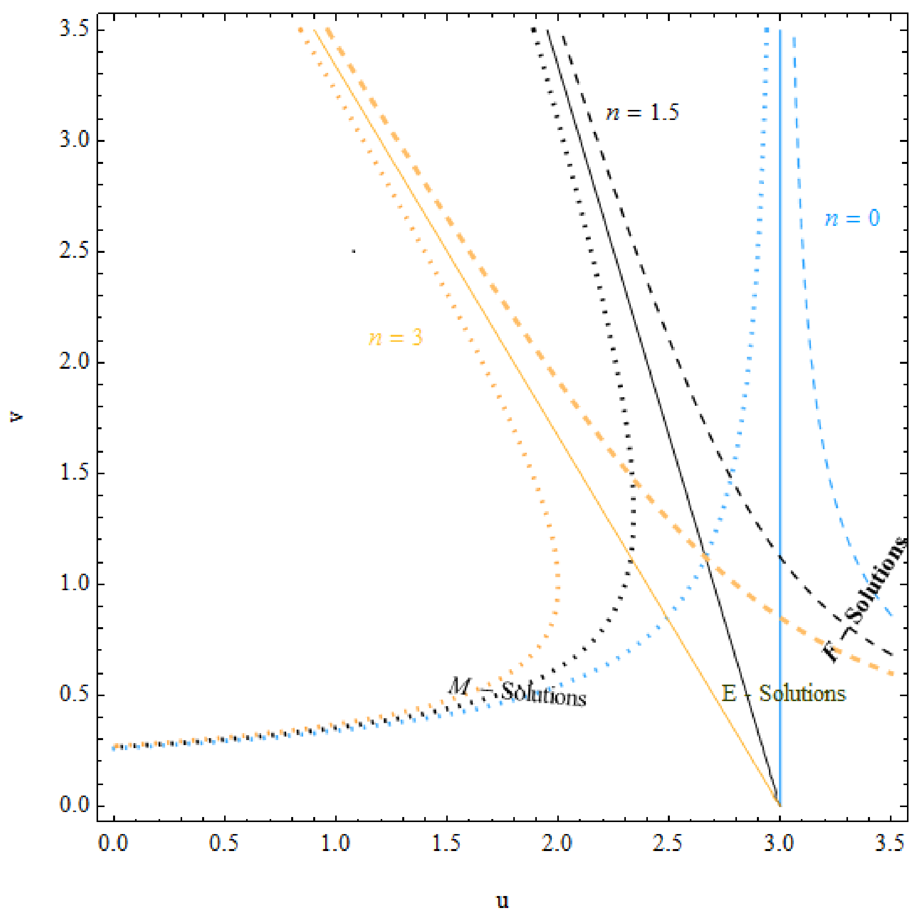

Figure 4.3: Represents a stellar model composed of two polytropes

Figure 4.3 Shows the solution for two common polytropes with physical interpretations: The solid lines represent the E-solutions which satisfy hydrostatic equilibrium at the origin. The dashed and dotted lines depict samples of the F- and M- solutions respectively. These solutions do not satisfy the condition of hydrostatic equilibrium at the center of the polytrope, they may represent valid solutions for stars composed of multiple polytropes joined in the interior. The solution reaching the center must always be an E-solution. The polytrope with n = 1.5 represents the solution for a star in convective equilibrium, the n = 3 polytropic solution is for a star dominated by radiation pressure and a polytrope with index n = 0 has a uniform density.

Chapter 5

Summary and Conclusion

Summary

For a star in thermal equilibrium, an internal energy source is required to balance the radiative energy loss from the surface. This energy source is provided by nuclear reactions that take place in the deep interior, where the temperature and density are sufficiently high. In ordinary stars, where the ideal gas law holds, this stellar nuclear reactor is very stable: the rate of nuclear reactions adapts itself to produce exactly the amount of energy that the star radiates away from its surface. Another important effect of nuclear reactions is that they change the composition by transmutations of chemical elements into other, usually heavier, elements. In this way stars produces all the elements in the Universe heavier than helium-a process called stellar nucleosynthesis.
The most important physical processes taking place in stellar interiors, and the differential equations of state that determine the structure and evolution of a star were derived. By putting these ingredients together we can construct simple polytropic model of star.

Conclusion

polytropic equation was derived from homology transformation and combined with hydrostatic equilibrium equation resulted in Lane-Emden equation. Then, the solutions of

Lane-Emden equation with some initial conditions and polytropic index give E-solutions ,F and M solutions plotted graphically using Mathematica [see fig.5.3].

Generally, by making use of polytropic solutions, it is possible to represent stars with convective cores and radiative envelopes with some accuracy and to get a rough idea of the run of pressure, density, and temperature throughout the star. Polytropes are useful in determining the effects of the buildup of chemical discontinuities as a result of nuclear burning. Polytropes often can be used as an initial model which is then perturbed to approximate a given physical situation.

Bibliography

[1] A.Maeder, *Formation and evolution of rotating stars*, 2009.

[2] Warrick Ball, *Stellar structure and evolution*, (2014).

[3] Eric Joseph Bubar, *Stellar atmosphere/interiors.*

[4] Carpenter.K;et.al., *Stellar astrophysicist*, 2003.

[5] D.Prianlnik., *An introduction to the theory of stellar structure and evolution*, 2nd ed., Cambridge University Press, 2009.

[6] A.S. Eddington, *The internal constitution of the stars dover*, New York, 1959.

[7] Tadele Guta, *Stellar evolution: The case of accreting main sequence stars*, Master's thesis, october 2016.

[8] George W.Collins II, *The fundamental of stellar astrophysics*, 2nd ed., January 2003.

[9] Hannu Karttunen, *Fundamental astronomy*, 2007.

[10] Henny Lamers, *Understanding stellar structure and evolution*, (2014), lecture notes.

[11] Onno Pols, *Stellar structure and evolution*, September 2011.

[12] R.Kippenhahn and A.Weight., *Stellar structure and evolution*, Springer, 1990.

[13] Chandrasekhar S., *An introduction to the study of stellar structure dover*, (2013).