SUPERPOSED COHERENTLY DRIVEN DEGENERATE THREE-LEVEL LASER WITH PARAMETRIC AMPLIFIER IN A VACUUM RESERVOIR

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In Partial Fulfilment of the Requirements for the Degree of Masters of Science in

Physics (***Quantum Optics***)

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented in any other university, and that all sources of material used for the thesis have been dully acknowledged.

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Abstract

In this research, we investigate the squeezing and statistical properties of coherently driven degenerate three-level laser with parametric amplifier for the cavity mode coupled to a single-mode a vacuum reservoir via a single-port mirror. We first derived the master equation in the linear approximation scheme which is used to determine stochastic differential equations. We carry out analysis applying the solutions of c-number Langevin equations associated with the normal ordering, we determined the quadrature squeezing and squeezing spectrum. Using the antinormal order characteristic function, we obtain the Q-function to analyze the squeezing and statistical properties of the generated cavity radiation. Expanding the density operator in the normal order and applying the completeness relation for coherent states, we determined the Q-function of superposed light to analyze the squeezing and statistical properties. We have found that a single light is 47.9% squeezed below the coherent state level at steady state. The effect of parametric amplifier is to increase the intra-cavity squeezing by a maximum of 50%. The maximum intracavity squeezing is found to be 93.2% below coherent state level. The mean photon number of superposed light beam is twice of that of single light beam. The squeezing of the superposed light of single-mode light increases with linear gain coefficients with a squeezing of 95.8% below coherent state level.

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INTRODUCTION

Degenerate parametric amplifier is a typical source of squeezed light, with a maximum of 50% intracavity noise reduction [1-5]. Some authors have also established that a three-level laser under certain conditions generates squeezed light [6-7]. A squeezed state is now belonging to the selected technologies for detection of weak signals and in low noise communication [8-10]. We define a three-level laser as a quantum optical system in which three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a certain rate into a cavity coupled to a vacuum reservoir via a single-port mirror see Fig.1.1. The three-level laser in which a considerable role is played by the coherent superposition of the top and bottom level of the injected atoms have been studied by different authors [7, 8, 9, 10, 12, 13, 14, 16, 17].

The squeezing in such a laser is due to the coherent superposition of the top and bottom levels. It now appears that a highly squeezed light could be generated by a combination of these two quantum optical systems. The set of energy levels of an atom consists of an infinite number of discrete levels corresponding to the bound states of the electrons [11]. For a three-level atom, out of these set of energy lev-

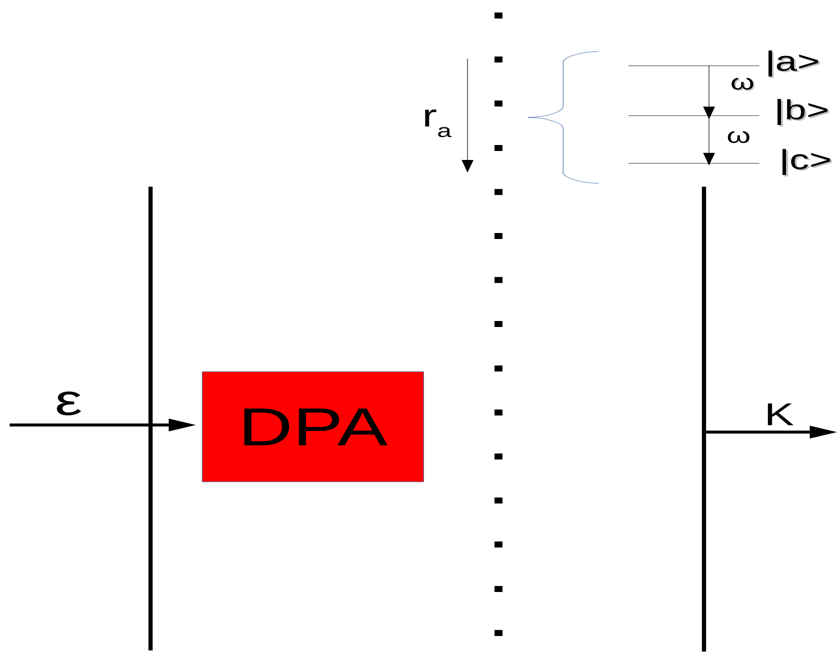

Figure 1.1: Schematic diagram of three-level laser with a degenerate parametric amplifier (DPA)

els only three-levels interact with electromagnetic radiation. When the three-level atom interacts with radiation, then it under goes a transition from top to bottom level via the intermediate level by emitting two photons. If the two photons generated have different frequencies, a two-mode light is generated. In this case the atom is called **non-degenerate three-level atom**. But, when the frequencies of these photons are equal, the atom generates a single-mode light. For this condition the atom is called **degenerate three-level atom**.

Ansari [7] has found the quadrature variance of degenerate three-level laser using the steady state solution of the expectation value of cavity mode variables. He found that the cavity mode is in squeezed state if the probability for the injected atoms to be in the bottom levels is larger than the probability to be in the top levels. And almost perfect squeezing can be achieved for slightly high probability for the atoms to be in the bottom levels and for large value of linear gain coefficient. Alebachew and Fesseha [12] have studied the squeezing properties of the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier by applying the solution of the stochastic differential equations, with the top and bottom levels of injected atoms coupled by the pump mode emerging from the parametric amplifier. In this study they showed that the optical system generates light in a squeezed state with a maximum interacavity squeezing of 93% below the coherent state level. Recently, Misrak [13] has studied the squeezing properties of cavity mode produced by degenerate three-level laser with parametric amplifier by applying the solution of stochastic differential equations. This study showed that the quantum optical system generates squeezed light and the degree of squeezing increases with the linear gain coefficient with maximum interacavity squeezing of 96.5% below the coherent state level. The injected coherent superposition creates a population transfer pathway which is the basis for the correlated two photon emission.

In this thesis, we seek to analyze the squeezing and statistical properties for degenerate three-level laser whose cavity contains parametric amplifier for single light and for superposition of light beams produced by pair of degenerate three-level lasers. By making use of stochastic differential equations, we carry out the analysis applying the solutions of c-number Langevin equations associated with the normal ordering. These equations are obtained using the master equation derived in the linear approximation scheme in the good cavity limit. This is used to analyze the squeezing and statistical properties of the generated cavity radiation with the aid of the pertinent stochastic differential equations associated with the normal ordering.

Imposing the requirement that the c-number equations of evolution for the first and second-order moments have the same forms as the corresponding operator [14], we obtain stochastic differential equations, associated with the normal ordering, for the dynamical variables of the cavity mode. The solutions of the resulting equations are then used to calculate the quadrature variance and the squeezing spectrum. Applying the same solutions, we also determine the antinormally ordered characteristic function with the aid of which the Q function is obtained. K. Fesseha [15] has shown that the effect of parametric amplifier is to increase the intracavity squeezing by a maximum of 50%. Finally, the Q function is used to calculate the mean photon number and the photon number distribution. Furthermore, using the Q-function, we obtain the Q-function for the superposition of two light beams produced by three-level lasers. Upon employing this Q-function, we determine the squeezing and statistical properties of the single-mode light obtained from superposition of two light beams produced by three-level lasers. We then calculate the mean photon number, the variance of photon number, the photon number distribution and the quadrature variance of the superposed light beams.

2

STOCHASTIC DIFFERENTIAL EQUATIONS

2.1 The Hamiltonian

A three-level laser consists of a cavity in which three-level atoms in a cascade configuration are injected at a constant rate r_a and removed from the cavity after a certain time τ. We represent the top, middle, and bottom levels by $|a\rangle$, $|b\rangle$ and $|c\rangle$ respectively. In addition, we assume that the cavity mode to be at resonance with the two transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$, and with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. The interaction of a three-level atom with the cavity mode can be described in the interaction picture by the Hamiltonian

$$\hat{H} \;=\; ig\Big[\hat{a}^{\dagger}(|b\rangle\langle a| + |c\rangle\langle b|) - \hat{a}(|a\rangle\langle b| + |b\rangle\ \langle c|)\Big], \tag{2.1}$$

where g is the coupling constant and $\hat{a}$ is the annihilation operator for the cavity mode. In this study we take the initial state of a three-level atom to be

$$\varphi(0)\rangle \;=\; C_a(0)|a\rangle + C_c(0)|c\rangle, \tag{2.2}$$

and hence the initial density operator for a single atom has the form

$$\hat{\rho}_A(0) \;=\; \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.3}$$

where $\rho_{aa}^{(0)} = |c_a|^2$, $\rho_{ac}^{(0)} = C_a C_c^*$, $\rho_{ca}^{(0)} = C_c C_a^*$ and $\rho_{cc}^{(0)} = |c_c|^2$, and

$$A = 2r_a g^2/\gamma^2 \tag{2.4}$$

is the linear gain coefficient. It can be readily established that the equation of the density operator for the cavity mode has in the linear approximation given as [18].

2.2 The Master Equation

The quantum analysis of the interaction of a system such as a cavity mode or a two-level atom with the external environment is a relatively complex problem. The external environment, usually referred to as a reservoir, can be thermal light, ordinary or squeezed vacuum. We are interested in the dynamics of the system and this is describable by the master equation, the Fokker-Planck equation, or quantum Langevin equations. In this section, we obtain the above set of dynamical equations for a cavity mode coupled to a squeezed vacuum reservoir via a single-port mirror. The resulting equations are easily adaptable to the case when the external environment is either a thermal or a vacuum reservoir. We then focus our study when the cavity mode is couple to a vacuum reservoir. A system coupled with a squeezed vacuum reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}. \tag{2.5}$$

The equation of evolution of density operator is given by

$$\frac{d}{dt}\hat{X}(t) = -i\left[\hat{H}_S(t) + \hat{H}_{SR}, \hat{X}(t)\right], \tag{2.6}$$

where $\hat{X}$(t) is the density operator for the system. Using equation (2.6) the reduced density operator

$$\hat{\rho}(t) \;\; = \;\; Tr\hat{X}(t) \tag{2.7}$$

evolves in time according to

$$\frac{d}{dt}\hat{\rho}(t) \;\; = \;\; -i\Big[\hat{H}(t), \hat{\rho}(t)\Big] - iTr\Big[\hat{H}_{SR}(t), \hat{X}(t)\Big], \tag{2.8}$$

in which Tr_R indicates the trace over the reservoirs variables only. On the other hand, a formal solution of Eq. (2.6) can be written as

$$\hat{X}(t) \;\; = \;\; \hat{X}(0) - i\int_0^t \Big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{X}(t')\Big]dt'. \tag{2.9}$$

In order to obtain mathematically manageable that $\hat{X}(t')$ by some approximately valid expression. Then, in the first place, we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. This can be achieved by letting a beam of thermal light (or light in a vacuum state) of constant intensity fall continuously on the system. Moreover, we decouple the system and reservoirs density operators, so that

$$\hat{X}(t') \;\; = \;\; \hat{\rho}(t')\hat{R}. \tag{2.10}$$

Therefore, with the aid of this, one can rewrite Eq. (2.9) as

$$\hat{X}(t') \;\; = \;\; \hat{\rho}(t')\hat{R} - \int_0^t \Big[\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\Big]dt'. \tag{2.11}$$

Now on substituting (2.11) in to (2.8) there follows

$$\frac{d}{dt}\hat{\rho}(t) = -i\Big[\hat{H}_{SR}(t), \hat{\rho}(t)\Big] - i\Big[\langle\hat{H}_{SR}(t)\rangle_R, \hat{\rho}(0)\Big]$$
$$- \int_0^t \Big[\langle\hat{\rho}_{SR}(t)\rangle_R, \Big[\hat{H}_S(t'), \hat{\rho}(t')\Big]\Big] dt'$$
$$- \int_0^t Tr_R\Big[\hat{H}_{SR}(t'), \Big[\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}\Big]\Big] dt', \tag{2.12}$$

where the subscript R indicates that the expectation value is to be calculated using the reservoirs density operator $\hat{R}$. A light mode confined in a cavity, usually formed by two mirrors, is called a cavity mode. A commonly used cavity has a single port-mirror. One side of each cavity is a mirror through which light can enter or leave the cavity. We now proceed to obtain the equation of evolution of the reduced density operator, in short the master equation, for the cavity mode coupled to a squeezed vacuum reservoir via a single port-mirror. We consider the reservoirs to be composed of large number of submodes. Thus, the interaction of a cavity mode with squeezed vacuum reservoirs can be described by

$$\hat{H}_{SR}(t) \quad = \quad i\Sigma_k\lambda_k(\hat{a}^\dagger\hat{b}_k \exp i(\omega_0 - \omega_k) - \hat{a}\hat{b}_k^\dagger \exp -i(\omega_0 - \omega_k)), \tag{2.13}$$

where $\hat{a}$ and $\hat{b}$ are annihilation operator for the cavity and the reservoir submode respectively. In view of this, we have

$$\langle\hat{H}_{SR}(t)\rangle_R \quad = \quad i\Sigma_k\lambda_k\Big[(\hat{a}^\dagger\langle\hat{b}_k\rangle \exp i(\omega_0 - \omega_k) - \hat{a}\langle\hat{b}_k^\dagger\rangle \exp -i(\omega_0 - \omega_k)t)\Big]. \tag{2.14}$$

For squeezed vacuum reservoir to good approximation, a strong pump (signal) mode can be treated classically. In this case the Hamiltonian take form

$$\hat{H} = \frac{i\varepsilon}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}), \tag{2.15}$$

with $\varepsilon = \lambda\beta_0$, where ε is the parametric amplifier, λ is the coupling constant between these two modes and β_0 is assumed to be real, positive and constant which is replaced in the position of operator $\hat{b}$. The state vector for the signal mode initially in coherent state $|\alpha\rangle$ is expressible as

$$|\psi(t)\rangle = e^{\frac{\varepsilon i}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})}|\alpha\rangle. \tag{2.16}$$

It proves to be useful to introduce an ideal squeezed coherent state defined by

$$|\alpha, r\rangle = \hat{S}(r)|\alpha\rangle, \tag{2.17}$$

in which

$$\hat{S}(r) = e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} \tag{2.18}$$

is the squeeze operator and the squeeze parameter r is taken for convenience to be real and positive. For $\alpha = 0$ the squeezed coherent state reduces to a squeezed vacuum state or in short squeezed vacuum. We assume the reservoir submodes to be independent and consider the case for which the squeezed vacuum is incident on a system from one direction. Then on account of (2.17) with $\alpha = 0$ and (2.18), the density operator for a single submode has the form

$$\hat{\rho}_\kappa = S_k(r)|0_k\rangle\langle 0_k|S^\dagger(r), \tag{2.19}$$

where

$$S_\kappa = e^{\frac{r}{2}(\hat{b}_k^2 - \hat{b}_k^{\dagger 2})} \tag{2.20}$$

is the squeeze operator. The expectation value of the operator $\hat{b}_k$ can thus be expressed as

$$\langle\hat{b}_k\rangle = \langle 0_k|\hat{b}_k(r)|0_k\rangle, \tag{2.21}$$

in which

$$\hat{b}_k(r) = S_k^\dagger(r)\hat{b}_k S_k(r). \tag{2.22}$$

Now on comparing (2.22) with the ideal squeezed state, called displaced squeezed vacuum given in the form

$$\hat{a}(r) = S^\dagger \hat{a} S(r) \tag{2.23}$$

and

$$\hat{a} = a\cosh r - a^\dagger \sinh r, \tag{2.24}$$

we see that

$$\hat{b}_k(r) = \hat{b}_k \cosh r - \hat{b}_k^\dagger \sinh r. \tag{2.25}$$

Hence on introducing (2.25) in to (2.24), we have

$$\langle \hat{b}_k \rangle = \cosh r \langle 0_k | b_k | 0_k \rangle - \sinh r \langle 0_k | b_k^\dagger | 0_k \rangle. \tag{2.26}$$

It then follows that

$$\langle \hat{b}_k \rangle = 0 \tag{2.27}$$

and

$$\langle \hat{b}_k^\dagger \rangle_R = 0. \tag{2.28}$$

Employing Eqs. (2.27) and (2.28) in to (2.14) we can easily express

$$\left[\langle \hat{H}_{SR}(t) \rangle_R, \hat{\rho}(0) \right] \;\; = \;\; 0, \tag{2.29}$$

and

$$\left[\langle \hat{H}_{SR}(t)\rangle_R, \left[\hat{H}_S(t'), \hat{\rho}(t')\right]\right] = 0. \tag{2.30}$$

Since the submodes are assumed to be independent, we note that for $\kappa' \neq \kappa$

$$\langle \hat{b}_j^\dagger \hat{b}_{k'}\rangle = \langle \hat{b}_k^\dagger\rangle\langle \hat{b}_{k'}\rangle \tag{2.31}$$

and in view of (2.27), we see that

$$\langle \hat{b}_k^\dagger \hat{b}_{k'}\rangle = 0. \tag{2.32}$$

On the other hand, for $k' = k$ the expectation value of $\hat{b}_k^\dagger \hat{b}_{k'}$ is expressible in the form

$$\langle \hat{b}_k^\dagger \hat{b}_{k'}\rangle = \langle 0_k|S_k^\dagger(r)\hat{b}_k^\dagger \hat{b}_k S_\kappa(r)|0_k\rangle. \tag{2.33}$$

Using the unitary property of the squeezed operator, this can be written as

$$\langle \hat{b}_k^\dagger \hat{b}_{k'}\rangle = \langle 0_k|\hat{b}_k^\dagger(r)\hat{b}_k|0_k(r)\rangle, \tag{2.34}$$

where $\hat{b}_\kappa$ is defined by (2.22). Hence on introducing of Eq. (2.24) and its adjoint in to (2.34) leads to

$$\langle \hat{b}_k^\dagger \hat{b}_{k'}\rangle = \langle 0_k|\left[\hat{b}_k^\dagger \hat{b}_k \cosh^2(r) + \hat{b}_k^\dagger \sinh^2(r)\hat{b}_{k'}\hat{b}_k^\dagger - \cosh(r)\sinh(r)(\hat{b}_k^\dagger \hat{b}_k^\dagger + \hat{b}_k \hat{b}_k)\right]|0_k\rangle, \tag{2.35}$$

from which follows

$$\langle \hat{b}_k^\dagger \hat{b}_{k'}\rangle = \sinh^2 r\langle 0_k|\hat{b}_{k'}\hat{b}_k^\dagger|0_k\rangle, \tag{2.36}$$

we assume that the reservoir submode operator satisfy the commutation relation

$$\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}. \tag{2.37}$$

Now applying the commutation relation, we arrive at

$$\langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle = \sinh^2 r, \tag{2.38}$$

which holds for $k' = k$. Therefore, on account of (2.32) and (2.38)

$$\langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle = N \delta_{kk'}, \tag{2.39}$$

where $N = \sinh^2 r$. In addition, with the aid of (2.37) and (2.39), one easily gets

$$\langle \hat{b}_k \hat{b}_{k'}^\dagger \rangle = (N+1) \delta_{kk'}. \tag{2.40}$$

It can also be readily verified in a similar manner that

$$\langle \hat{b}_k \hat{b}_{k'} \rangle = -M \delta_{kk'}, \tag{2.41}$$

with$M = \cosh r \sinh r$, expression (2.41) can be written as

$$\langle \hat{b}_k \hat{b}_{k'} \rangle = -M \delta_{k', 2k_{0-k}}. \tag{2.42}$$

where $k \approx 2k_{0-k}$.

In view Eqs. (2.27)-(2.30), expression (2.12) reduces to

$$\begin{aligned}
\frac{d}{dt}\hat{\rho} = &-i\Big[\hat{H}_S(t), \hat{\rho}\Big] - \int_0^t Tr\Big[\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t')\Big]dt' \\
&- \int_0^t \hat{\rho}(t') Tr\Big[\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t)\Big]dt' \\
&+ \int_0^t Tr\Big[\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t')\Big]dt' \\
&+ \int_0^t Tr\Big[\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)\Big]dt'.
\end{aligned} \tag{2.43}$$

Furthermore, using Hamiltonian described by (2.13) we have

$$-Tr_R(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')) = I_1\hat{a}\hat{a}^\dagger + I_2\hat{a}^\dagger\hat{a} + I_3\hat{a}^2 + I_4\hat{a}^{\dagger 2}, \tag{2.44}$$

where

$$I_1 = -\sum_{j,k}\lambda_j\lambda_k\langle\hat{b}_j^\dagger\hat{b}_k\rangle_R \exp\Big[-i(\omega_0-\omega)t + i(\omega_0-\omega)t'\Big], \tag{2.45}$$

$$I_2 = -\sum_{j,k}\lambda_j\lambda_k\langle\hat{b}_j\hat{b}^\dagger\rangle_R \exp\Big[i(\omega_0-\omega_j) - i(\omega_0-\omega)t'\Big], \tag{2.46}$$

$$I_3 = -\sum_{j,k}\lambda_j\lambda_k\langle\hat{b}_j^\dagger\hat{b}^\dagger\rangle_R \exp\Big[-i(\omega_0-\omega_j) - i(\omega_0-\omega)t'\Big], \tag{2.47}$$

$$I_4 = -\sum_{j,k}\lambda_j\lambda_k\langle\hat{b}_j\hat{b}_k\rangle_R \exp\Big[-i(\omega_0-\omega_j) - i(\omega_0-\omega)t'\Big]. \tag{2.48}$$

In view of Eqs. (2.39), (2.40) and (2.42) one can easily write the relations in the form

$$\langle\hat{b}_j\hat{b}_k\rangle_R = N\delta_{jk}, \tag{2.49}$$

$$\langle\hat{b}_j\hat{b}_k^\dagger\rangle_R = (N+1)\delta_{jk}, \tag{2.50}$$

$$\langle\hat{b}_j\hat{b}_k\rangle_R = \langle\hat{b}_j^\dagger\hat{b}_k^\dagger\rangle_R = -M\delta_{j,2k_{0-k}}, \tag{2.51}$$

from these relations it is not difficult to obtain

$$I_1 = -N\sum_k \lambda_k^2 \exp -i(\omega_0-\omega_k)(t-t'), \tag{2.52}$$

$$I_2 = -(N+1)\sum_k \lambda_k^2 \exp i(\omega_0-\omega_k)(t-t'), \tag{2.53}$$

$$I_3 = -M\sum_k \lambda_{2k_0-k} \exp\Big[i(\omega_0-\omega_{2k_0-k})t - i(\omega_0-\omega_k)t'\Big], \tag{2.54}$$

$$I_4 = -M\sum_k \lambda_{2k_0-k}\exp\Big[i(\omega_0-\omega_{2k_0-k})t + i(\omega_0-\omega_k)t'\Big]. \quad (2.55)$$

Assuming the frequencies of the reservoirs submodes to be closely spaced summation over k can be converted in to an integration over ω

$$\sum_k \lambda_k^2 \exp i(\omega_0-\omega_k)(t-t') = \int_0^t g(\omega)\lambda_k^2 \exp i(\omega_0-\omega_k)(t-t')d\omega. \quad (2.56)$$

Replace $g(\omega)$ and $\lambda^2(\omega_0)$ by $g(\omega)\lambda\omega_0^2$ and extend the lower limit of integration to ∞

$$\sum_k \lambda_k^2 \exp i(\omega_0-\omega_k)(t-t') = g(\omega_0)\lambda^2(\omega_0)\int_0^t \exp i(\omega_0-\omega_k)(t-t')d\omega. \quad (2.57)$$

Let $\omega' = \omega$-ω_0, then

$$\sum_\kappa \lambda_\kappa^2 \exp i(\omega_0-\omega_\kappa)(t-t') = g(\omega_0)\lambda^2(\omega_0)\int_0^t \exp i\omega'(t-t')d\omega, \quad (2.58)$$

from which follows

$$\sum_k \lambda_k^2 \exp i(\omega_0-\omega_\kappa)(t-t') = \kappa\delta(t-t'), \quad (2.59)$$

where

$$\kappa = 2\Pi g(\omega_0)\lambda^2(\omega_0) \quad (2.60)$$

is defined to be the cavity damping constant. On account of (2.53) Eq. (2.52) and (2.53) take the form

$$I_1 = -\kappa N\delta(t-t'), \quad (2.61)$$

$$I_2 = -\kappa(N+1)\delta(t-t'). \quad (2.62)$$

Performing a similar procedure, one can also readily establish

$$I_3 = I_4 = -\kappa m\delta(t-t'). \quad (2.63)$$

Hence, the combination of (2.44), (2.61), (2.62), and (2.63) yields

$$Tr(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')) = \kappa\Big[(N+1)\hat{a}^\dagger\hat{a} + N\hat{a}\hat{a}^\dagger + M\hat{a}^2 + M\hat{a}^\dagger\Big]\delta(t-t'). \quad (2.64)$$

In view of this result

$$\int_0^t Tr(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t')dt' = \frac{\kappa}{2}\Big[(N+1)\hat{a}^\dagger\hat{a}\hat{\rho} + N\hat{a}\hat{a}^\dagger\hat{\rho} + M\hat{a}^2\hat{\rho} + M\hat{a}^{\dagger 2}\hat{\rho}\Big]. \quad (2.65)$$

With $\hat{\rho} = \hat{\rho}(t)$, it is not hard to see that

$$\int_0^t \hat{\rho}(t')Tr(\hat{R}\hat{H}_{SR}(t)\hat{H}_{SR}(t')dt' = \frac{\kappa}{2}\Big[(N+1)\hat{\rho}\hat{a}^\dagger\hat{a} + N\hat{\rho}\hat{a}\hat{a}^\dagger + M\hat{\rho}\hat{a}^2 + M\hat{\rho}\hat{a}^{\dagger 2}\Big]. \quad (2.66)$$

In addition, employing (2.13) one can write

$$Tr(\hat{R}\hat{H}_{SR}(t')\hat{\rho}(t')\hat{H}_{SR}(t)dt' = -\Big[I_1\hat{a}^\dagger\hat{\rho}(t)\hat{a} + I_2\hat{a}\hat{\rho}(t)\hat{a}^\dagger + I_3\hat{a}\hat{\rho}(t')\hat{a} + I_4\hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger\Big], \quad (2.67)$$

so that the application of the results described by (2.61), (2.62) and (2.63)

leads to

$$\int_0^t Tr_R\Big(\hat{H}_{SR}(t)\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)\Big)dt' = \frac{\kappa}{2}\Big[(N+1)\hat{a}\hat{\rho}\hat{a}^\dagger + N\hat{a}^\dagger\hat{\rho}\hat{a} + M\hat{a}\hat{\rho}\hat{a} + M\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\Big] \quad (2.68)$$

and

$$\int_0^t Tr_R\Big(\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)\Big)dt' = \frac{k}{2}\Big[(N+1)\hat{a}\hat{\rho}\hat{a}^\dagger + N\hat{a}^\dagger\hat{\rho}\hat{a} + M\hat{a}\hat{\rho}\hat{a}\Big]. \quad (2.69)$$

Therefore, on account of Eqs. (2.65), (2.66), (2.68) and (2.69), Eq. (2.43) for the squeezed vacuum reservoir, it takes the form

$$\frac{d}{dt}\hat{\rho} = -i\left[\hat{H}_S, \hat{\rho}\right] + \frac{\kappa}{2}(N+1)\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right]$$
$$+\frac{\kappa N}{2}\left[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right]$$
$$+ \frac{\kappa M}{2}\left[2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right]. \tag{2.70}$$

The effects of the reservoir are incorporated via the parameter N and M, with $M = \sqrt{N(N+1)}$. Note that, for vacuum reservoir, $N = M = 0$ then the equation given (2.70) reduces to

$$\frac{d}{dt}\hat{\rho} \;=\; -i\left[\hat{H}_S, \hat{\rho}\right] + \frac{\kappa}{2}\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right]. \tag{2.71}$$

Now employing interaction Hamiltonian given by Eq. (1) for the cavity mode in to Eq. (2.71) can be expressible as

$$\frac{d}{dt}\hat{\rho} = -iTr\left[\left(ig\hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|) - \hat{a}(|a\rangle\langle b| + |b\rangle\ \langle c|)\right), \hat{\rho}_{AR}(t)\right]$$
$$+ \frac{\kappa}{2}\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right]. \tag{2.72}$$

After performing the trace operation and the cyclic property, we have

$$\frac{d}{dt}\hat{\rho} = g\Big[\langle a|\hat{\rho}_{AR}\hat{a}^\dagger|b\rangle + \langle b|\hat{\rho}_{AR}\hat{a}^\dagger|c\rangle - \langle b|\hat{\rho}_{AR}\hat{a}|a\rangle - \langle c|\hat{\rho}_{AR}\hat{a}|b\rangle$$
$$-\hat{a}^\dagger\langle a|\hat{\rho}_{AR}|b\rangle - \hat{a}^\dagger\langle b|\hat{\rho}_{AR}|c\rangle + \hat{a}\langle b|\hat{\rho}_{AR}|a\rangle + \hat{a}\langle c|\hat{\rho}_{AR}|b\rangle\Big]$$
$$+ \frac{\kappa}{2}\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right], \tag{2.73}$$

in which the matrix element

$$\hat{\rho}_{\alpha\beta} \quad = \quad \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.74}$$

with α, β = a, b, c, so that Eq. (2.73) can be written as

$$\frac{d}{dt}\hat{\rho} = g\bigg(\hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{bc}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{bc} + \hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} + \hat{a}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{a}\bigg)$$

$$+ \frac{\kappa}{2}\bigg[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\bigg]. \tag{2.75}$$

Suppose $\hat{\rho}_{AR}(t,t')$ is the density operator for single atom plus the cavity mode at a time t, with the atom injected at time t_j such that

$$(t-\tau) \leq t_j \leq t. \tag{2.76}$$

The density operator for all atoms in the cavity mode at time t can be written as

$$\frac{d}{dt}\hat{\rho}(t) \quad = \quad r_a\sum_j \hat{\rho}_{AR}(t,t_j)\Delta t_j, \tag{2.77}$$

where $r_a\Delta t_j$ is the number of atoms injected in to the cavity at a time Δt_j. Integrating in the limit $\Delta t_j \rightarrow 0$

$$\hat{\rho}_{AR}(t) \quad = \quad r_a\int_{t-\tau}^{t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.78}$$

Differentiating with respect to time t, we get

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t,t-\tau) + r_a\int_{t-\tau}^{t}\frac{d}{dt}\hat{\rho}_{AR}(t,t')dt'. \tag{2.79}$$

One can write

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(t)\hat{\rho}(t), \tag{2.80}$$

where $\hat{\rho}(t)$ being the density operator for the cavity mode alone and $\hat{\rho}_{AR}(t, t-\tau)$ represents the density operator for an atom plus the cavity mode at a time t, with the atom being removed from the cavity at this time

$$\hat{\rho}_{AR}(t, t-\tau) \quad = \quad \hat{\rho}_{AR}(t, t-\tau)\hat{\rho}_{AR}(t). \tag{2.81}$$

Using Eqs. (2.80) and (2.81), one can write Eq. (2.79) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(t) - \hat{\rho}_{AR}(t, t-\tau)\hat{\rho}(t)) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt'. \tag{2.82}$$

In the absence of damping of the cavity mode by a vacuum reservoir, the density operator $\hat{\rho}_{AR}(t, t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t') \quad = \quad -i\Big[\hat{H}, \hat{\rho}_{AR}(t, t')\Big], \tag{2.83}$$

so that using this and taking in to account (2.78), one can put Eq. (2.82) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) \quad = \quad r_a\Big(\hat{\rho}_A(t) - \hat{\rho}_{AR}(t, t-\tau)\hat{\rho}(t)\Big) - i\Big[\hat{H}, \hat{\rho}_{AR}(t, t)\Big]. \tag{2.84}$$

Furthermore, tracing over the atomic variables and taking in to account the damping of the cavity mode by a vacuum reservoir and using the fact that

$$Tr\hat{\rho}_A(t) = Tr\hat{\rho}_{AR}(t-\tau) = 1,$$

we have

$$\frac{d}{dt}\hat{\rho}(t) \quad = \quad -iTr_A\Big[\hat{H}, \hat{\rho}_{AR}(t, t)\Big] + \frac{1}{2}\kappa\Big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\Big). \tag{2.85}$$

On the other hand, from Eq. (2.84) that

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta} = r_a\bigg(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}(t-\tau)|\beta\rangle\bigg)\hat{\rho}(t)$$

$$- i\bigg(\langle\alpha|\hat{H}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle\bigg) - \gamma\rho_{\alpha\beta}, \qquad (2.86)$$

where $\gamma\rho_{\alpha\beta}$ is included to account for the decay of the atom due to spontaneous emissions and γ is considered to be the same for all the three levels, is the atomic decay rate. We assume that the atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0, \qquad (2.87)$$

so that Eq. (2.86) reduces to

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta} = r_a(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i(\langle\alpha|\hat{H}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle) - \gamma\rho_{\alpha\beta}. \qquad (2.88)$$

Inserting Eqs. (2.1) and (2.3) in to Eq. (2.88) we have the following expressions

$$\frac{d}{dt}\hat{\rho}_{ab} = g\bigg(\hat{\rho}_{ac}\hat{a}^\dagger + \hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a}\bigg) - \gamma\hat{\rho}_{ab}, \qquad (2.89)$$

$$\frac{d}{dt}\hat{\rho}_{bc} = g\bigg(\hat{a}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{a} - \hat{a}^\dagger\rho_{ac}\bigg) - \gamma\hat{\rho}_{bc}, \qquad (2.90)$$

$$\frac{d}{dt}\hat{\rho}_{aa} = r_a\hat{\rho}^{(0)}_{aa}\hat{\rho} + g\bigg(\hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}\hat{\rho}_{ba}\bigg) - \gamma\hat{\rho}_{aa}, \qquad (2.91)$$

$$\frac{d}{dt}\hat{\rho}_{bb} = g\bigg(\hat{\rho}_{bc}\hat{a}^\dagger + \hat{a}\hat{\rho}_{cb} - \hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ba}\hat{a}\bigg) - \gamma\hat{\rho}_{bb}, \qquad (2.92)$$

$$\frac{d}{dt}\hat{\rho}_{ac} = r_a\hat{\rho}^{(0)}_{ac}\hat{\rho} + g\bigg(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{a}\bigg) - \gamma\hat{\rho}_{ac}, \qquad (2.93)$$

$$\frac{d}{dt}\hat{\rho}_{cc} = r_a\hat{\rho}^{(0)}_{cc}\hat{\rho} - g\bigg(\hat{a}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{a}\bigg) - \gamma\hat{\rho}_{cc}. \qquad (2.94)$$

By dropping the g terms in Eqs. (2.91)- (2.94) and imposing the condition that $\kappa \ll \gamma$ (the good-cavity limit) since the atomic variable reach steady state in the relatively short period of γ^{-1}, we can take the time derivative of such variable to be zero, keeping the zeroorder and cavity mode variables at time t. This is termed as adiabatic approximation scheme. Then we get

$$\hat{\rho}_{aa} = \frac{r_a \rho_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{2.95}$$

$$\hat{\rho}_{bb} = 0, \tag{2.96}$$

$$\hat{\rho}_{ac} = \frac{r_a \rho_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{2.97}$$

$$\hat{\rho}_{cc} = \frac{r_a \rho_{cc}^{(0)}}{\gamma}\hat{\rho}. \tag{2.98}$$

Moreover, substituting Eqs. (2.95)-(2.97) in to Eq. (2.89) yields

$$\frac{d}{dt}\hat{\rho}_{ab} = \frac{g r_a}{\gamma}\left(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^{\dagger} - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\right) - \gamma\hat{\rho}_{ab}. \tag{2.99}$$

Now on account of Eqs. $(2.96)-(2.98)$, Eq. (2.90) takes the form

$$\frac{d}{dt}\hat{\rho}_{bc} = \frac{g r_a}{\gamma}\left(\rho_{cc}^{(0)}\hat{a}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}\right) - \gamma\hat{\rho}_{bc}. \tag{2.100}$$

Using once more the adiabatic approximation, we easily find

$$\hat{\rho}_{ab} = \frac{g r_a}{\gamma}\left(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^{\dagger} - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\right), \tag{2.101}$$

$$\hat{\rho}_{bc} = \frac{g r_a}{\gamma}\left(\rho_{cc}^{(0)}\hat{a}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}\right), \tag{2.102}$$

On account of (2.101) and (2.102), the master equation for the cavity mode coupled with a vacuum reservoir given by Eq. (2.75) takes the form

$$\frac{d}{dt}\hat{\rho} = \frac{1}{2}A\rho_{aa}^{(0)}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\right)$$
$$+\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right)$$
$$+\frac{1}{2}A\rho_{ac}^{(0)}\left(\hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\right)$$
$$+\frac{1}{2}A\rho_{ca}^{(0)}\left(\hat{\rho}\hat{a}^2 + \hat{a}^2\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}\right), \tag{2.103}$$

where $A = 2r_a g^2/\gamma^2$ is the linear gain coefficient, κ is assumed to be the cavity damping constant and γ is the spontaneous atomic decay rate. It is worth mentioning that the quantum properties of the light generated by the three-level laser are determine by the master equation (2.103). It is easy to observe that with $\rho_{aa}^{(0)} = 1$ and $\rho_{ac}^{(0)} = \rho_{cc}^{(0)}$ = 0, this equation reduces to the master equation for the two-level laser operating bellow threshold.

Moreover, with the pump mode treated classically, a degenerate parametric amplifier is describable in the interaction picture by the Hamiltonian

$$\hat{H} = \frac{1}{2}i\varepsilon(\hat{a}^{\dagger 2} - \hat{a}^2), \tag{2.104}$$

where ε is real and constant and proportional to the amplitude of the pump mode. The master equation associated with this Hamiltonian $\hat{H}$ can be derived using the commutation relation

$$\frac{d\hat{\rho}}{dt} = -i\left[\hat{H}, \hat{\rho}\right]. \tag{2.105}$$

and on account of Eq. (2.104), we see that

$$\frac{d\hat{\rho}}{dt} = \frac{\varepsilon}{2}\bigg(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\bigg). \tag{2.106}$$

Therefore, Eq. (2.106) represents the master equation for the pump mode treated classically for degenerate parametric amplifier in the interaction picture. Now taking in to account Eqs. (2.103) and (2.106), the master equation for the cavity mode of a three-level laser containing parametric amplifier can be written as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = & \frac{\varepsilon}{2}\bigg(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\bigg) \\ & + \frac{1}{2}A\rho_{aa}^{(0)}\bigg(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\bigg) \\ & + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\bigg(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\bigg) \\ & + \frac{1}{2}A\rho_{ac}^{(0)}\bigg(\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\bigg) \\ & + \frac{1}{2}A\rho_{ca}^{(0)}\bigg(\hat{\rho}\hat{a}^2 + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}\bigg).\end{aligned} \tag{2.107}$$

2.3 Stochastic Differential Equations

We next seek to obtain stochastic differential equations for the cavity mode variables. To this end applying Eq. (2.107), we are able to find the time evolution of the expectation value of the annihilation operator

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\bigg(\frac{d\rho}{dt}\hat{a}\bigg), \tag{2.108}$$

from which follows

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\Bigg[\frac{\varepsilon}{2}\Bigg(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\Bigg)\hat{a}$$
$$+\frac{1}{2}A\rho_{aa}^{(0)}\Bigg(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger} - \hat{a}\hat{a}^{\dagger}\hat{\rho}\Bigg)\hat{a}$$
$$+\frac{1}{2}(A\rho_{cc}^{(0)} + k)\Bigg(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\Bigg)\hat{a}$$
$$+\frac{1}{2}A\rho_{ac}^{(0)}\Bigg(\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\Bigg)\hat{a}$$
$$+\frac{1}{2}A\rho_{ca}^{(0)}\Bigg(\hat{\rho}\hat{a}^{2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}\Bigg)\hat{a}\Bigg]. \tag{2.109}$$

Applying this master equation along with cyclic property of trace operation, and the commutation relations of

$$[a, a^{\dagger}] = 1 \qquad and \qquad [a, a] = [a^{\dagger}, a^{\dagger}] = 0, \tag{2.110}$$

one can readily obtains

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{1}{2}\mu\langle\hat{a}\rangle + \varepsilon\langle\hat{a}^{\dagger}\rangle. \tag{2.111}$$

Following similar procedures, we have

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\mu\langle\hat{a}^2\rangle + 2\varepsilon\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \varepsilon + A\hat{\rho}_{ac}^{(0)}, \tag{2.112}$$

and

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\mu\langle\hat{a}^{\dagger}\hat{a}\rangle + \varepsilon\langle\hat{a}^2(t)\rangle + \varepsilon\langle\hat{a}^{\dagger 2}(t)\rangle + A\hat{\rho}_{aa}^{(0)}, \tag{2.113}$$

in which

$$\mu = \frac{1}{2}A(\hat{\rho}_{cc}^{(0)} - \hat{\rho}_{aa}^{(0)}) + \kappa. \tag{2.114}$$

The corresponding c-number of Eqs. (2.111), (2.112) and (2.113) are given as

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu\langle\alpha(t)\rangle + \varepsilon\langle\alpha^*(t)\rangle, \tag{2.115}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle + A\rho_{ac}^{(0)}, \tag{2.116}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon\langle\alpha^2(t)\rangle + \varepsilon\langle\alpha^{*2}(t)\rangle + A\rho_{aa}^{(0)}. \tag{2.117}$$

On the bases of Eq. (2.115), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu\alpha(t) + \varepsilon\alpha^*(t) + f(t), \tag{2.118}$$

where $f(t)$ is a noise force, the properties of which remain to be determined. We see that Eq. (2.115) and the expectation value of Eq. (2.118), will have identical forms if

$$\langle f(t)\rangle = 0. \tag{2.119}$$

Now applying the relation

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = 2\langle\alpha(t)\frac{d}{dt}\alpha(t)\rangle \tag{2.120}$$

along with Eq.(2.118), one can readily get

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle + 2\langle\alpha(t)f(t)\rangle, \tag{2.121}$$

and using the partial differentiation of

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\alpha^*(t)\frac{d}{dt}\alpha(t)\rangle + \langle\alpha(t)\frac{d}{dt}\alpha^*(t)\rangle, \tag{2.122}$$

up on inserting Eq. (2.118) and its complex conjugate, in to Eq. (2.122) we see that

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon\langle\alpha^2(t)\rangle + \varepsilon\langle\alpha^{*2}(t)\rangle$$

$$+ \langle\alpha(t)f^*(t)\rangle + \langle\alpha^*(t)f(t)\rangle. \tag{2.123}$$

We note that the c-number Eqs. (2.112) and (2.121) will have the same forms if

$$\langle\alpha(t)f(t)\rangle = \frac{1}{2}(\varepsilon + A\rho_{ac}^{(0)}), \tag{2.124}$$

and similarly in view of Eqs. (2.113) and (2.123), we have

$$\langle\alpha(t)f^*(t)\rangle + \langle\alpha^*(t)f(t)\rangle = A\rho_{ac}^{(0)}. \tag{2.125}$$

A formal solution of Eq. (2.118) can be written as

$$\alpha(t) = \alpha(0)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\Big[\varepsilon\alpha^*(t') + f(t')\Big]dt'. \tag{2.126}$$

We then see that

$$\langle\alpha(t)f(t)\rangle = \langle\alpha(0)f(t)\rangle e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\varepsilon\Big(\langle\alpha^*(t')f(t)\rangle\Big)dt'$$

$$+ \int_0^t e^{-\mu(t-t')/2}\langle f(t)f(t')\rangle dt'. \tag{2.127}$$

Assuming that the noise force f at a time t does not affect the cavity mode variables at earlier times

$$\langle\alpha^*(t')f(t)\rangle = 0 \tag{2.128}$$

and taking in to account Eq. (2.124), we have

$$\int_0^t e^{-\mu(t-t')}\langle f(t)f(t')\rangle dt' = \frac{1}{2}(\varepsilon + A\rho_{ac}^{(0)}). \tag{2.129}$$

One can then write on the bases of this result

$$\langle f(t)f(t')\rangle \quad = \quad (\varepsilon + A\rho_{ac}^{(0)})\delta(t-t'). \tag{2.130}$$

It can also be established in a similar manner that

$$\langle f^*(t)f(t')\rangle \quad = \quad A\rho_{aa}^{(0)}\delta(t-t'). \tag{2.131}$$

It is worth mentioning that Eqs. (2.130) and (2.131) describe the correlation properties of the noise force $f(t)$ associated with the normal ordering. Now introducing a new variable defined by

$$\alpha_{\pm}(t) \quad = \quad \alpha^*(t) \pm \alpha(t), \tag{2.132}$$

on account of Eq. (2.118) one can readily write

$$\frac{d}{dt}\alpha^*(t) \quad = \quad -\frac{1}{2}\mu\alpha^*(t) + \varepsilon\alpha(t) + f^*(t). \tag{2.133}$$

Differentiating Eq. (2.132), one can obtain

$$\frac{d\alpha_{\pm}}{dt}(t) \quad = \quad \frac{d}{dt}\alpha^*(t) \pm \frac{d}{dt}\alpha(t). \tag{2.134}$$

Upon substituting Eq. (2.118) and (2.133) in to Eq. (2.134), we can readily get

$$\frac{d\alpha_{\pm}}{dt} \quad = \quad -\frac{1}{2}\lambda_{\mp}\alpha_{\pm} + f^*(t) \pm \ f(t), \tag{2.135}$$

where

$$\lambda_{\mp} \quad = \quad \mu \mp 2\varepsilon. \tag{2.136}$$

The solution of Eq. (2.135) can be written as

$$\alpha_{\pm}(t) \quad = \quad \alpha_{\pm}(0)e^{-\lambda_{\mp}t/2} + \int_0^t e^{-\lambda_{\mp}(t-t')/2}\Big[f^*(t') \pm f(t')\Big]dt'. \tag{2.137}$$

It then follows that

$$\alpha_+(t) \;=\; \alpha_+(0)e^{-\lambda_- t'/2} + \int_0^t e^{-\lambda_- t/2}\Big[f^*(t') + f(t')\Big]dt' \tag{2.138}$$

and

$$\alpha_-(t) \;=\; \alpha_-(0)e^{-\lambda_+ t'/2} + \int_0^t e^{-\lambda_+ t/2}\Big[f^*(t') - f(t')\Big]dt'. \tag{2.139}$$

Combining with

$$\alpha_\pm(t') \;=\; \alpha^*(t') \pm \alpha(t'), \tag{2.140}$$

which then follows

$$\alpha(t) \;=\; A(t)\alpha(0) + B(t)\alpha^*(0) + F(t), \tag{2.141}$$

in which

$$A(t) \;=\; \frac{1}{2}(e^{-\lambda_- t/2} + e^{-\lambda_+ t/2}), \tag{2.142}$$

$$B(t) \;=\; \frac{1}{2}(e^{-\lambda_- t/2} - e^{-\lambda_+ t/2}), \tag{2.143}$$

and

$$F(t) \;=\; F_+(t) + F_-(t), \tag{2.144}$$

with

$$F_\mp(t) \;=\; \int_0^1 e^{-\lambda_\mp (t-t')/2}\Big[f(t') \pm f^*(t')\Big]dt'. \tag{2.145}$$

Upon substituting Eqs. (2.142), (2.143) and (2.145) in to Eq. (2.141) we readily see that

$$\alpha(t) = \frac{1}{2}\alpha(0)\left(e^{-\lambda_- t/2} + e^{-\lambda_+ t/2}\right) + \frac{1}{2}\alpha^*(0)\left(e^{-\lambda_- t/2} - e^{-\lambda_+ t/2}\right) + \frac{1}{2}\int_0^1 e^{-\lambda_\mp (t-t')/2}\left[f(t') \pm f^*(t')\right]dt'. \quad (2.146)$$

Rewriting Eq. (2.146), we readily get

$$\alpha(t) = \frac{1}{2}\alpha(0)e^{-\lambda_\mp t/2} + \frac{1}{2}\alpha^*(0)e^{-\lambda_\mp t/2} + \frac{1}{2}\left[\int_0^1 e^{-\lambda_\mp (t-t')/2} f(t') \pm f^*(t')\right]dt'. \quad (2.147)$$

3

THE QUADRATURE FLUCTUATIONS

In this chapter, we seek to calculate the quadrature variances of the cavity modes as well as the squeezing spectrum of the output mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier driven by coherent light and coupled to a vacuum reservoir, using the solutions of the stochastic differential equations and the correlation properties of the noise forces.

3.1 Quadrature Variance

The squeezing properties of single mode light are described by two quadrature operators is defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{3.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{3.2}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures, respectively, while $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for light mode a respectively. With the help of Eqs. (3.1) and (3.2), we can show that the two quadrature operators satisfy the commutation relation. It is possible to express in terms of c-number variables associated with the normal ordering as

$$\Delta\alpha_{\pm}^{2} = 1 \pm \langle \alpha_{\pm}(t), \alpha_{\pm}(t)\rangle, \tag{3.3}$$

in which $\alpha_{\pm}(t)$ is given by Eq. (2.132). We consider here the case for which the cavity mode is initially in the vacuum state. Hence on account of Eq. (2.137) along with Eq. (2.119), we see that

$$\langle \alpha_{\pm}(t)\rangle = 0, \tag{3.4}$$

and expression (3.3) takes the form of

$$\triangle\alpha_{\pm}^{2} = 1 \pm \langle \alpha_{\pm}^{2}(t)\rangle. \tag{3.5}$$

Furthermore, one easily gets with the aid of Eq. (2.135) that

$$\frac{d}{dt}\langle \alpha_{\pm}^{2}(t)\rangle = -\lambda_{\mp}\langle \alpha_{\pm}^{2}(t)\rangle + 2\langle \alpha_{\pm}(t) f^{*}(t)\rangle \pm 2\langle \alpha_{\pm}(t) f(t)\rangle. \tag{3.6}$$

On account of Eq. (2.132) along with Eqs. (2.124) and (2.125), we note that

$$\langle \alpha_{\pm}(t) f^{*}(t)\rangle = \frac{1}{2}[\varepsilon + A(\rho_{ca}^{(0)} \pm \rho_{aa}^{(0)})], \tag{3.7}$$

$$\langle \alpha_{\pm}(t) f(t)\rangle = \frac{1}{2}[A\rho_{aa}^{(0)} \pm (\varepsilon + \rho_{ac}^{(0)})]. \tag{3.8}$$

Therefore, in view of this result, Eq. (3.6) can be rewritten as

$$\frac{d}{dt}\langle \alpha_{\pm}^{2}(t)\rangle = -\lambda_{\mp}\langle \alpha_{\pm}^{2}(t)\rangle + 2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ca}^{(0)} \pm 2\rho_{aa}^{(0)}). \tag{3.9}$$

With the cavity mode initially in a vacuum state, the solution of this equation has the form

$$\langle \alpha_{\pm}^{2}(t)\rangle = \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ca}^{(0)} \pm 2\rho_{aa}^{(0)})}{\lambda_{\mp}}[1 - e^{-\lambda_{\mp}t}]. \tag{3.10}$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{3.11}$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1 \tag{3.12}$$

and

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{3.13}$$

one easily finds that

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2} \tag{3.14}$$

and

$$|\rho_{ac}^{(0)}| = \frac{1}{2}(1-\eta^2)^{1/2}. \tag{3.15}$$

Up on setting

$$\rho_{ac}^{(0)} = |\rho_{ac}^{(0)}|e^{i\theta} \tag{3.16}$$

and taking in to account of Eq. (2.136), along with Eq. (2.114), expression (3.10) can thus be put in the form

$$\langle \alpha_{\pm}^2(t)\rangle = \frac{2\varepsilon + A[(1-\eta^2)\cos\theta \pm (1-\eta)]}{A\eta + \kappa \mp 2\varepsilon}[1 - e^{-(A\eta+\kappa\mp 2\varepsilon)t}]. \tag{3.17}$$

Now a combination of Eqs. (3.5) and (3.17) yields

$$\Delta\alpha_{\pm}^2(t) = 1 \pm \frac{2\varepsilon + A[(1-\eta^2)\cos\theta \pm (1-\eta)]}{A\eta + \kappa \mp 2\varepsilon}[1 - e^{-(A\eta+\kappa\mp 2\varepsilon)t}], \tag{3.18}$$

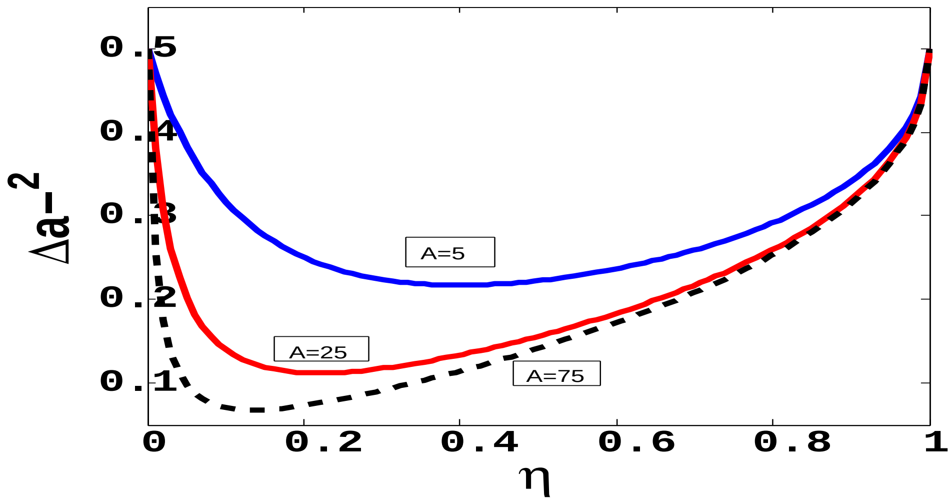

Figure 3.1: Plots of quadrature variance $(\Delta a_{-})^2$ Vs η [Eq. (3.22)] for $\kappa = 0.8$, $\theta = 0$,and $2\varepsilon = A\eta + \kappa$ and for different values of linear gain coefficient.

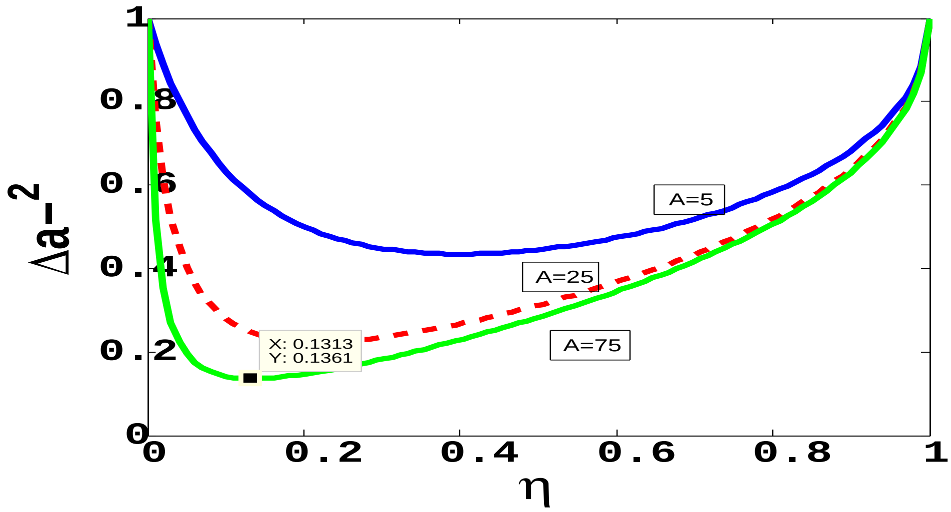

Figure 3.2: Plots of $(\Delta a_{-})^2$ for single mode [Eq. (3.23)] versus η of for $\kappa = 0.8$ and $\varepsilon = 0$ and for different values of linear gain coefficient.

so that at steady state

$$\Delta\alpha_+^2(t) \;=\; \frac{\kappa + A[(1 + (1-\eta^2)^{1/2}\cos\theta]}{A\eta + \kappa - 2\varepsilon}. \tag{3.19}$$

and

$$\Delta\alpha_-^2(t) \;=\; \frac{\kappa + A[(1 - (1-\eta^2)^{1/2}\cos\theta]}{A\eta + \kappa + 2\varepsilon}. \tag{3.20}$$

Since no well-behaved solution of Eq. (2.135) exists for (Aη + κ)$<$2ε, we interpret (Aη + κ) = 2ε as the threshold condition. Hence the solution of of this equation given by Eq. (2.137) is valid for 2$\varepsilon$$<$(A$\eta$+$\kappa$).

On the other hand, we note that from Eq. (2.104) that ε is the only parameter representing the parametric amplifier. And inspection of Eq. (3.20) shows that the effect of this parameter is to decrease the value of the quadrature variance $\Delta\ a_-^2$. In addition, we see that expressions in Eqs. (3.19) and (3.20) take at threshold the form

$$\Delta\alpha_+^2 \rightarrow \infty \tag{3.21}$$

and

$$\Delta\alpha_-^2(t) \;=\; \frac{\kappa + A[(1 - (1-\eta^2)^{1/2}\cos\theta]}{2(A\eta + \kappa)}. \tag{3.22}$$

Now upon setting ε = 0 in Eq. (3.20) it reduces to

$$\Delta\alpha_-^2(t) \;=\; \frac{\kappa + A[(1 - (1-\eta^2)^{1/2}\cos\theta]}{(A\eta + \kappa)}. \tag{3.23}$$

Comparing the resulting expression (3.23) with Eq. (3.22) along with Fig.3.2, we observe that the effect of the parametric amplifier is to increase the intracavity squeez-

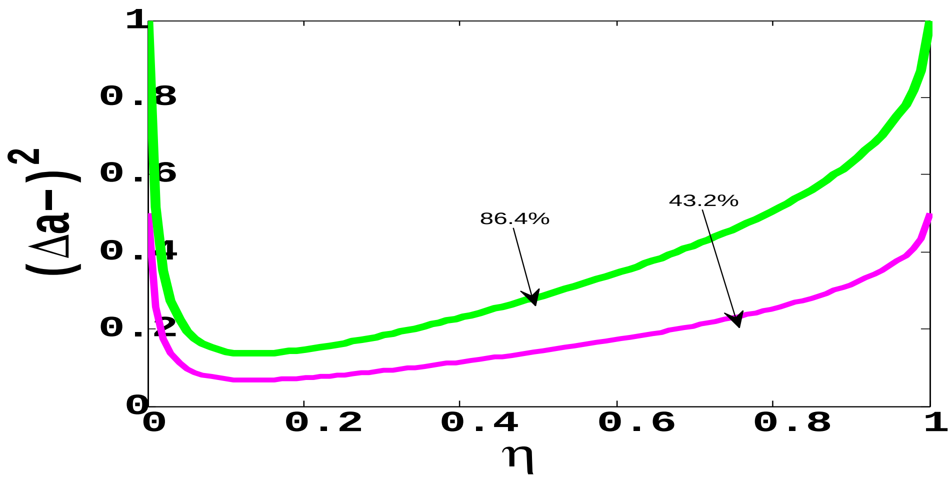

Figure 3.3: Plots of intra-cavity quadrature variance $(\Delta a_{-})^2$ for single mode [Eq. (3.22) and (3.23)] versus η of for $\kappa = 0.8$, $A = 75$, and in the absence of the parametric amplifier with $\varepsilon = 0$, (green curve) and in the presence of the parametric amplifier with $2\varepsilon = A\eta + \kappa$, (pink curve)

ing by a maximum of 50%. The degree of squeezing increases with linear gain coefficients and it appears that almost perfect squeezing can be achieved for sufficiently large values of the linear gain coefficient.

Fig.3.3 indicates that the squeezing vanishes for η =0 and η =1 which corresponds to maximum injected atomic coherence, $\rho_{ac}^{(0)}$=1/2, and no injected atomic coherence, $\rho_{ac}^{(0)}$=0, respectively. However, as can be seen from the pink curve, the presence of the parametric amplifier leads to some degree of squeezing for η=0.

3.2 Squeezing Spectrum

The squeezing spectrum of a single-mode light is expressible in terms of c-number variables associated with the normal ordering as

$$S_{\pm}^{out}(\omega) \quad = \quad 1 + 2Re\int_{0}^{\infty} \langle \alpha_{\pm}^{out}(t), \alpha_{\pm}^{out}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau, \tag{3.24}$$

where the subscript ”ss” stands for steady state and

$$\alpha_{\pm}^{out}(t) = \alpha_{out}^{*}(t) \pm \alpha_{out}(t). \quad (3.25)$$

We note that for a cavity mode coupled to a vacuum reservoir, the output and intra-cavity variables are related by

$$\alpha_{\pm}^{out}(t) = \sqrt{\kappa}\alpha_{\pm}(t). \quad (3.26)$$

In view of Eqs. (3.4) and (3.26), the squeezing spectrum can be put in the form

$$S_{\pm}^{out}(\omega) = 1 + 2\kappa Re\int_{0}^{\infty}\langle\alpha_{\pm}(t)\alpha_{\pm}(t+\tau)\rangle_{ss}e^{i\omega\tau}d\tau. \quad (3.27)$$

Furthermore, the solution of the expectation value of of Eq. (2.135)can be written as

$$\langle\alpha_{\pm}(t+\tau)\rangle = \langle\alpha_{\pm}(t)\rangle e^{-\lambda_{\mp}\tau/2}\rangle, \quad (3.28)$$

so that on account of the quantum regression theorem, have

$$\langle\alpha_{\pm}(t)\alpha_{\pm}(t+\tau)\rangle = \langle\alpha_{\pm}^{2}(t)\rangle e^{-\lambda_{\mp}\tau/2}. \quad (3.29)$$

Now with the aid of Eq. (3.29) together with Eq. (3.17), the squeezing spectrum is found to be

$$S_{\pm}^{out}(\omega) = 1 \pm \frac{2\kappa\varepsilon + \kappa A[(1-\eta^{2})^{1/2}\cos\theta \pm (1-\eta)]}{\omega^{2} + [\frac{1}{2}(A\eta + \kappa \mp 2\varepsilon)]^{2}}, \quad (3.30)$$

It is easy to see that at threshold

$$S_{+}^{out}(\omega) = \frac{\omega^{2} + \kappa^{2} + \kappa A[(1 + (1-\eta^{2})\cos\theta]}{\omega^{2}} \quad (3.31)$$

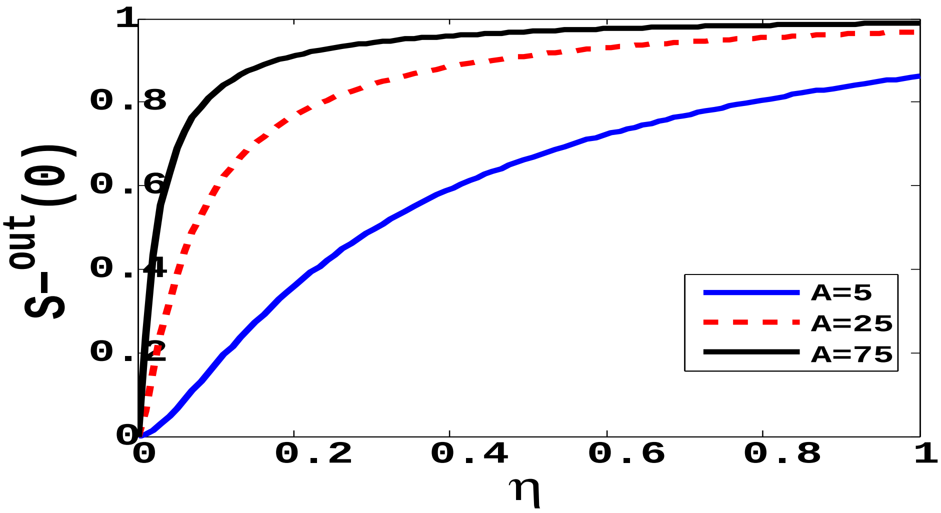

Figure 3.4: Plots of squeezing spectrum $(S_{-}^{out}(0)$ [Eq. (3.32)] versus η of for $\kappa = 0.8$, $\theta = 0.\omega = 0$ and for different values of linear gain coefficient.

and

$$S_{-}^{out}(\omega) \quad = \quad \frac{\omega^2 + A^2\eta^2 + \kappa A[(1-(1-\eta^2)\cos\theta]}{\omega^2 + [A\eta + \kappa]^2}. \tag{3.32}$$

As shown in Fig.3.4, the squeezing spectrum of the out put cavity variable $\langle S_{-}^{out(0)}\rangle$ increases as the linear gain coefficients, A, increases. For A=75, k=0.8, θ=0 and ω=0, the maximum squeezing spectrum gives 98.9%. For A=5 and the same values of k, θ and ω, the squeezing spectrum is 86.2% at maximum value of η=1.

4

PHOTON STATISTICS

In this chapter, with the aid of the antinormally-ordered characteristic function, we obtain the Q function. Using the Q function, we obtain the mean photon number, the photon number variance and the photon number distribution for the cavity mode.

4.1 The Q Function

The Q function is expressible in the form

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2}\int d^2z\phi(z^*, z, t)exp(z^*\alpha - z\alpha^*), \tag{4.1}$$

where the antinormally ordered characteristic function $\phi(z^*,z,t)$ is defined in the Heisenberg picture by

$$\phi(z^*, z, t) = Tr(\hat{\rho}(0)e^{z^*\hat{a}(t)}e^{-z^*\hat{a}^\dagger(t)}). \tag{4.2}$$

Applying the identity

$$e^A e^B = e^B e^A e^{[A,B]}, \tag{4.3}$$

the expression for the characteristic function can be written in terms of c-number variables associated with the normal ordering as

$$\phi(z^*, z, t) = e^{-z^*z}\langle \exp(z\alpha^* - z^*\alpha)\rangle, \tag{4.4}$$

so that employing Eq. (2.141) and assuming that $\alpha(0)$is independent of the noise force $F(t)$, we get

$$\phi(z^*, z, t) = e^{-z^*z}\langle \exp[(zA - z^*B)\alpha^*(0) + (zB - z^*A)\alpha(0)]\rangle$$
$$\langle \exp(zF^* - z^*F)\rangle. \tag{4.5}$$

Considering the cavity mode to be initially in a vacuum state, we see that

$$\langle \exp[(zA - z^*B)\alpha^*(0) + (zB - z^*A)\alpha(0)]\rangle = 1 \tag{4.6}$$

and hence

$$\phi(z^*, z, t) = e^{-z^*z}\langle \exp(-zF^* - z^*F)\rangle. \tag{4.7}$$

On account of the fact that F is Gaussian random variable, one can express Eq. (4.7) in the form [18]

$$\phi(z^*, z, t) = e^{-z^*z}\exp(\frac{1}{2}\langle [zF^* - z^*F]^2\rangle). \tag{4.8}$$

It then follows that

$$\phi(z^*, z, t) = e^{-z^*z}\exp(\frac{1}{2}\langle [z^2F^{*2} + z^{*2}F^2 - 2z^*zF^*F]\rangle). \tag{4.9}$$

Furthermore, from Eq. (2.144) and (2.145) one easily gets

$$\langle F^2\rangle = \langle F_+^2\rangle + \langle F_-^2\rangle + 2\langle F_+F_-\rangle, \tag{4.10}$$

$$\langle F^*F\rangle = \langle F_+^2\rangle - \langle F_-^2\rangle. \tag{4.11}$$

Applying Eq. (2.145) along with Eqs. (2.130) and (2.131), it can be easily established that

$$\langle F_+^2\rangle = \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ca}^{(0)} + 2\rho_{aa}^{(0)})}{4\lambda_-}[1 - e^{-\lambda_- t}], \tag{4.12}$$

$$\langle F_-^2\rangle = \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ca}^{(0)} - 2\rho_{aa}^{(0)})}{4\lambda_+}[1 - e^{-\lambda_+ t}], \tag{4.13}$$

$$\langle F_+F_-\rangle = \frac{2\varepsilon + A(\rho_{ac}^{(0)} - \rho_{ca}^{(0)})}{4\lambda_+}[1 - e^{-\mu t}], \tag{4.14}$$

so that in view of these results, there follows

$$\begin{aligned}\langle F^2\rangle &= \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ca}^{(0)} + 2\rho_{aa}^{(0)})}{4\lambda_-}[1 - e^{-\lambda_- t}] \\ &\quad + \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ac}^{(0)} - 2\rho_{aa}^{(0)})}{4\lambda_+}[1 - e^{-\lambda_+ t}] \\ &\quad + \frac{2\varepsilon + A(\rho_{ac}^{(0)} - \rho_{ca}^{(0)})}{4\lambda_+}[1 - e^{-\mu t}],\end{aligned} \tag{4.15}$$

$$\begin{aligned}\langle F^*F\rangle &= \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ca}^{(0)} + 2\rho_{aa}^{(0)})}{4\lambda_-}[1 - e^{-\lambda_- t}] \\ &\quad - \frac{2\varepsilon + A(\rho_{ac}^{(0)} + \rho_{ac}^{(0)} - 2\rho_{aa}^{(0)})}{4\lambda_+}[1 - e^{-\lambda_+ t}].\end{aligned} \tag{4.16}$$

Now on account of Eqs. (4.15) and (4.16), the characteristic function (4.9) can be written as

$$\phi(z^*, z, t) = \exp[-az^*z + (bz^2 + b^*z^{*2})/2], \tag{4.17}$$

where the coefficients are expressible in terms of the parameter η as

$$a = 1+\frac{2\varepsilon + A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)}[1-e^{-(A\eta+k-2\varepsilon)t}]$$

$$-\frac{2\varepsilon + A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)}[1-e^{-(A\eta+k+2\varepsilon)t}] \quad (4.18)$$

$$b = \frac{2\varepsilon + A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)}[1-e^{-(A\eta+k-2\varepsilon)t}]$$

$$+\frac{2\varepsilon + A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)}[1-e^{-(A\eta+k+2\varepsilon)t}]$$

$$+\frac{iA(1-\eta^2)^{1/2}\sin\theta}{2(A\eta+\kappa)}[1-e^{-(A\eta+k)t}]. \quad (4.19)$$

Finally, introducing Eq. (4.17) in to Eq. (4.1) and carrying out the integration, the Q function for the cavity mode is found to be

$$Q(\alpha^*,\alpha,t) = \frac{[u^2-vv^*]^{1/2}}{\pi}\exp[-u\alpha^*\alpha+(u\alpha^2+v^*\alpha^{*2})/2], \quad (4.20)$$

in which

$$u = \frac{a}{a^2-bb^*}, \quad (4.21)$$

$$v = \frac{b}{a^2-bb^*}. \quad (4.22)$$

4.2 The Mean Photon Number

The mean photon number can be written employing the Q function (4.20) as

$$\langle\hat{a}^\dagger\hat{a}\rangle = -\frac{1}{\pi}[u^2-vv^*]^{1/2}\int d^2\alpha\exp[-u\alpha^*\alpha+(v^*\alpha^{*2}+v\alpha^2)]/2-1, \quad (4.23)$$

so that on performing the integration, there follows

$$\langle \hat{a}^\dagger \hat{a} \rangle = -[u^2 - vv^*]^{1/2} \frac{d}{du} [\frac{1}{u^2 - uu^*}]^{1/2} - 1. \tag{4.24}$$

Therefore, carrying out differentiation and taking into account Eqs. (4.21) and (4.22) along with (4.18) and (4.19), one readily obtains

$$\begin{aligned} \langle \hat{a}^\dagger \hat{a} \rangle &= \frac{2\varepsilon + A[1 - \eta + (1 - \eta^2)^{1/2} \cos\theta]}{4(A\eta + \kappa - 2\varepsilon)} [1 - e^{-(A\eta + k - 2\varepsilon)t}] \\ &- \frac{2\varepsilon + A[\eta - 1 + (1 - \eta^2)^{1/2} \cos\theta]}{4(A\eta + \kappa + 2\varepsilon)} [1 - e^{-(A\eta + k + 2\varepsilon)t}]. \end{aligned} \tag{4.25}$$

At steady state, it is not hard to observe that the parametric amplifier contributes significantly to the mean photon number when the system is operating particularly near threshold. Up on direct use of steady state solutions of c-number Langevine equations,

$$\bar{n} = \frac{A(1 - \eta)}{2(A\eta + \kappa)}, \tag{4.26}$$

which is identical with the expression obtained by Fesseha [8].

4.3 Photon Number Distribution

Furthermore, the photon number distribution for a single mode light is expressible in terms of the Q function as [4,5]

$$p(n, t) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} [Q(\alpha^*, \alpha, t) e^{\alpha^* \alpha}]_{\alpha = \alpha^* = 0}. \tag{4.27}$$

Thus with the aid of Eqs. (4.20) and (4.27) the photon number distribution for the cavity mode can be written in the form

$$p(n, t) = \frac{1}{n!} [u^2 - vv^*]^{1/2} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} [\exp(1 - u)\alpha^* \alpha + (v^* \alpha^{*2} + v\alpha^2)/2]. \tag{4.28}$$

Now expanding the exponential function in power series, we have

$$p(n,t) = \frac{1}{n!}[u^2 - vv^*]^{1/2} \sum_{klm} \frac{(1-u)^k v^{*l} v^m}{2^{l+m} k! l! m!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n}$$

$$\times \left[(\alpha^*)^{k+2l} \alpha^{k+2m}\right]_{\alpha^*=\alpha=0}, \tag{4.29}$$

so that on carrying out the differentiation and applying the condition α=$\alpha^* = 0$, there follows

$$p(n,t) = \frac{1}{n!}[u^2 - vv^*]^{1/2} \sum_{klm} \frac{(1-u)^k v^{*l} v^m (k+2l)!(k+2m)!}{2^{l+m} k! l! m! (k+2l-n)!(k+2m-n)!} \delta_{k+2l,n} \delta_{k+2m,n}. \tag{4.30}$$

Finally, on account of the result that $m = l$ and $k = n - 2l$, the photon number distribution can be written as

$$p(n,t) = \frac{1}{n!}[u^2 - vv^*]^{1/2} \sum_{l=0}^{[n]} n! \frac{(1-u)^{n-2l}(vv^*)^l}{2^{2l} l!^2 (n-2l)!}, \tag{4.31}$$

where $[n]$=n/2 for even n and $[n]$=$(n-1)$/2 for odd n. The probability of finding an even number of photons is greater than the probability of finding an odd number of photons; whether the light is produced by a thee-level laser with or without a parametric amplifier. This is because the photons are always generated in pairs and the existence of some finite probability to find an odd number of photons is due to damping of the cavity mode. We also see that the probability of finding n photons, with n$\leq$4, is smaller for the light generated by the three-level laser with a parametric amplifier than for that produced without parametric amplifier, and the opposite of this holds n$\geq$5.

5

SUPERPOSITION OF TWO LASER BEAMS

We first seek to calculate the Q function to describe the superposition of the two light beams, with the same frequency produced by a pair of degenerate three-level lasers. Applying the Q function, we calculate the mean photon number, the variance of photon number, the photon number distribution and the quadrature variance.

5.1 The Q Function

The source of light emitted by a three-level atom in a cavity coupled to a squeezed vacuum reservoir via a single port mirror is known to be a degenerate three-level laser. In this section, we first calculate the Q functions for the superposition of two light beams produced by three-level lasers. The Q function is used to describe the superposition of two light beams with the same frequency with in the same or different states. Suppose $\rho(\hat{a}^{\dagger}, \hat{a})$ is the density operator for a certain light beam. Then upon expanding the density operator in the normal order and applying the completeness relation for coherent states, one can easily establish that

$$\hat{\rho}^{'} = \int d^2\beta Q(\beta^*, \beta + \frac{\partial}{\partial\beta})|\beta\rangle\langle\beta|, \tag{5.1}$$

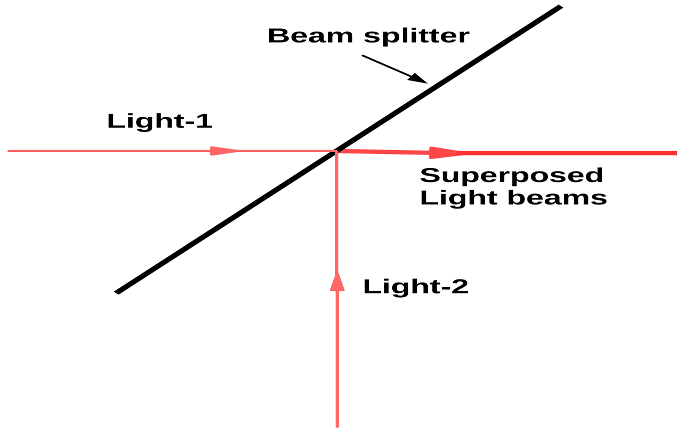

Figure 5.1: Schematic diagram of superposition of light beams emitted from two degenerate three-level atoms injected in to a cavity at a rate of r_a

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Where $\hat{\rho}'$ is the density operator for the first light beam by injecting a light beams in to the cavity which initially contains a light beams of the same frequency Fig. 5.1, the density operator for superposed light beams is expressible as

$$\hat{\rho} = \int d^2\gamma Q\left(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}\right)\hat{D}(\gamma)\hat{\rho}'\hat{D}^+(\gamma). \tag{5.2}$$

By employing the value of $\hat{\rho}'$ of Eq. (5.1) in to (5.2) which turns out to be

$$\hat{\rho} = \int d^2\gamma d^2\beta Q\left(\gamma^*, \gamma + \frac{\partial}{\partial\gamma}\right)Q\left(\beta^*, \beta + \frac{\partial}{\partial\beta}\right)|\beta + \gamma\rangle\langle\beta + \gamma|. \tag{5.3}$$

Here, the Q function for the superposition of the two light beams become

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{5.4}$$

Up on inserting the value of $\hat{\rho}$ we readily obtain

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\int d^2\beta d^2\gamma Q\Big(\beta^*, \beta + \frac{\partial}{\partial\beta}^*\Big) Q\Big(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}\Big)$$

$$\times \exp\Big[-\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \alpha^*\beta + \alpha\beta^* + \alpha^*\gamma + \alpha\gamma^* - \beta^*\gamma - \beta\gamma^*\Big]. \quad (5.5)$$

It then follows that

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\int d^2\beta d^2\gamma \exp\Big[-\alpha^*(\alpha - \beta - \gamma)\Big]$$

$$\times Q(\beta^*, \beta + \frac{\partial}{\partial\beta}) \exp\Big[\beta^*(\alpha - \beta - \gamma)\Big]$$

$$\times Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma}) \exp\Big[\gamma^*(\alpha - \beta - \gamma)\Big]. \quad (5.6)$$

Using the binomial expansion theorem

$$(x + \frac{d}{dy})^l = \sum_j^l (x)^{l-j} (\frac{d}{dy})^j, \quad (5.7)$$

we readily find

$$Q(\beta^*, \beta + \frac{\partial}{\partial\beta}) \times \exp\Big[\beta^*(\alpha - \beta - \gamma)\Big] = Q(\beta^*, \alpha - \gamma) \exp\Big[\beta^*(\alpha - \beta - \gamma)\Big], (5.8)$$

and

$$Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma}) \exp\Big[\gamma^*(\alpha - \beta - \gamma)\Big] = Q(\gamma^*, \alpha - \beta) \exp\Big[\gamma^*(\alpha - \beta - \gamma)\Big]. \quad (5.9)$$

In view of Eqs.(5.8) and (5.9), Eq. (5.6) takes the form

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\int d^2\beta d^2\gamma Q(\beta^*, \alpha - \gamma) Q(\gamma^*, \alpha - \beta)$$

$$\times \exp\Big[-\alpha^*\alpha - \beta^*\beta - \gamma^*\gamma + \alpha^*\beta + \alpha\beta^* + \alpha^*\gamma + \alpha\gamma^* - \beta^*\gamma - \beta\gamma^*\Big]. \quad (5.10)$$

Let $Q(\gamma^*,\gamma)$ and $Q(\beta^*,\beta)$ be the Q function of the first and second light beams, respectively. On account of Eq. (4.20), the Q function of the first light beam is given as

$$Q(\gamma^*,\gamma,t) \;=\; \frac{(u_1^2-v_1^2)^{1/2}}{\pi}\exp\bigg(-u_1\gamma\gamma^*+v_1(\gamma^2+\gamma^{*2})/2\bigg), \tag{5.11}$$

then

$$Q(\gamma^*,\alpha-\beta)=\frac{(u_1^2-v_1^2)^{1/2}}{\pi}\exp\bigg(-u_1(\alpha\gamma^*-\gamma^*\beta)+v_1(\alpha^2+\gamma^*+\beta^2-2\alpha\beta)/2\bigg), \tag{5.12}$$

where $u_1=\frac{a_1}{(a_1^2-b_1^2)}$ *and* $v_1=\frac{b_1}{(a_1^2-b_1^2)}$ for the first light beam and $u_2=\frac{a_2}{(a_2^2-b_2^2)}$ *and* $v_2=\frac{b_2}{(a_2^2-b_2^2)}$ for the the second light beam. Similarly, the Q function for the second light beam is given as

$$Q(\beta^*,\alpha-\gamma)=\frac{(u_2^2-v_2^2)^{1/2}}{\pi}\exp\bigg(-u_2(\alpha\beta^*-\beta^*\beta)+v_2(\alpha^2+\gamma^2+\beta^{*2}-2\alpha\gamma)/2\bigg). \tag{5.13}$$

In view of Eqs. (5.12) and (4.13) together with taking the two identical light beams, then u_1=u_2=u and v_1=v_2=v, Eq. (5.10) can be written as

$$\begin{aligned}Q(\alpha^*,\alpha,t)=\frac{(u^2-v^2)^{1/2}}{\pi}&\exp(-\alpha^*\alpha+(2v)\alpha^2/2)\times\int\frac{d^2\beta}{\pi}\exp\\&\bigg[-\beta^*\beta+(\alpha^*-v\alpha)\beta+(\alpha-u\alpha)\beta^*+v\beta^{*2/2+v\beta^2/2}\bigg]\\&\times\int\frac{d^2\gamma}{\pi}\exp\bigg[-\gamma^*\gamma+(\alpha^*-\beta^*+u\beta^*-v\alpha\alpha)\\&+(\alpha-\beta-u(\alpha-\beta))\gamma^*+v\gamma^{*2}/2+v\gamma^2/2\bigg].\end{aligned} \tag{5.14}$$

Integrating Eq. (5.14) over γ, we readily get

$$Q(\alpha^*,\alpha,t) = \frac{1}{\pi}\left[\frac{(u^2-v^2)^{1/2}}{1-v^2}\right]^{1/2} \exp\left[\frac{-u\alpha^*\alpha}{1-v^2} + (\frac{-v^3+vu^2+v}{1-v^2})\frac{\alpha^2}{2} + \frac{v}{1-v^2}\frac{\alpha^{*2}}{2}\right]$$

$$\times \int \frac{d^2\beta}{\pi} \exp\Bigg[-\beta^*\beta(\frac{2u-u^2-v^2}{1-v^2}) + \beta\frac{\alpha^*(u-v^2)+\alpha(u-u^2)}{1-v^2}$$

$$+\beta^*\frac{\alpha^*(vu-v)+vu^2-2vu}{1-v^2}(\frac{\beta^{*2}}{2}+\frac{\beta^2}{2})\Bigg]. \tag{5.15}$$

Similarly, integrating Eq. (5.15) over β gives the Q function for the superposition of two light beams can be expressed as

$$Q(\alpha^*,\alpha,t) = \frac{R}{\pi}\exp\left[-U\alpha\alpha^* + V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\right], \tag{5.16}$$

in which

$$R = \sqrt{\frac{u^2-v^2}{4-4u+u^2-v^2}} \tag{5.17}$$

$$U = \frac{2u-u^2+v^2}{4-4u+u^2-v^2}, \tag{5.18}$$

and

$$V = \frac{2v}{4-4u-u^2-v^2}. \tag{5.19}$$

By integrating Eq. (5.16) over α

$$\int d^2\alpha Q(\alpha,t) = R\int\frac{d^2\alpha}{\pi}\exp\left[-U\alpha\alpha^* + V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\right] \tag{5.20}$$

using the identity

$$\int\frac{d^2z}{\pi}\exp(-azz^*+bz+cz^*+A'z^2+B'z^{*2}) = \left[\frac{1}{a^2-4A'B}\right]^{1/2}$$

$$\times\exp\left[\frac{abc+A'c^2+B'b^2}{a^{2-4A'B'}}\right], \tag{5.21}$$

where $a > 0$. Up on inserting Eq. (5.21) in to Eq. (5.20) and carrying out integration it reduces to

$$\int d^2\alpha Q(\alpha,t) \quad = \quad 1. \tag{5.22}$$

Hence, the Q function for the superposition of the light beams produced a pair of degenerate three-level light beams is normalized.

5.2 Photon Statistics

We now proceed to calculate the photon number distribution, the mean photon number and variance photon number for the superposition of two light beams by employing the Q function.

5.2.1 The Mean Photon Number

For the superposition of two light beams the mean number of photons obtained from superposition of two identical degenerate three-level laser is represented by an operator $\hat{n}$=$\langle\hat{a}^{\dagger}\hat{a}\rangle$. But $\langle\hat{a}^{n}\hat{a}^{\dagger m}\rangle$ is the product of operators in the antinormal order (antinormal moments) which can be evaluated using Q function as

$$\langle\hat{a}^{n}\hat{a}^{\dagger m}\rangle \quad = \quad Tr(\rho\hat{a}^{n}\hat{a}^{\dagger m}). \tag{5.23}$$

From the cyclic property of trace operation

$$\langle\hat{a}^{n}\hat{a}^{\dagger m}\rangle \quad = \quad Tr(\hat{a}^{\dagger m}\rho\hat{a}^{n}), \tag{5.24}$$

the completeness relation for coherent state of light leads to

$$\langle \hat{a}^n \hat{a}^{\dagger m} \rangle = Tr\bigg(\int \frac{d^2\alpha}{\pi} \hat{a}^{\dagger m} \rho \hat{a}^n |\alpha\rangle\langle\alpha| \bigg)$$

$$= \int \frac{d^2\alpha}{\pi} Tr\bigg(\hat{a}^{\dagger m} \rho \hat{a}^n |\alpha\rangle\langle\alpha| \bigg)$$

$$= \int \frac{d^2\alpha}{\pi} \langle\alpha| \hat{a}^{\dagger m} \rho \hat{a}^n |\alpha\rangle$$

$$= \int d^2\alpha \frac{\langle\alpha|\rho|\alpha\rangle}{\pi} \alpha^{*m} \alpha^n$$

$$= \int d^2\alpha Q(\alpha, \alpha^*) \alpha^{*m} \alpha^n, \tag{5.25}$$

in which

$$Q = \frac{\langle\alpha|\rho|\alpha\rangle}{\pi} \tag{5.26}$$

is the Q function and then follows

$$\bar{n} = \int d^2\alpha Q(\alpha, \alpha^*)(\alpha^*\alpha - 1), \tag{5.27}$$

where $\alpha^*\alpha - 1$ is the c-number function corresponding to the operator $\hat{n}$ in the anti-normal order. Using Eq. (5.27), we get

$$\bar{n} = \int d^2\alpha Q(\alpha, \alpha^*)\alpha^*\alpha - \int d^2\alpha Q(\alpha, \alpha^*). \tag{5.28}$$

Integrating Eq. (5.15) over β and substituting for U, V and R in to Eq. (5.28), we can write the mean photon number as

$$\bar{n} = R \int \frac{d^2\alpha}{\pi} \exp\bigg[- U\alpha\alpha^* + V(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})\alpha^*\alpha \bigg] - 1. \tag{5.29}$$

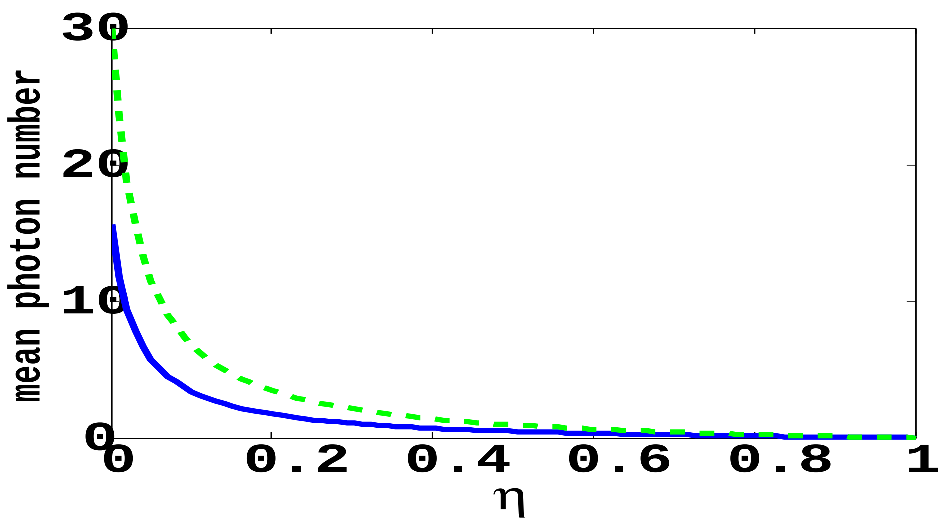

Figure 5.2: Plots of steady state mean number of photons for superposed light (dashed-line) and for single light (solid line) verses η [Eqs. (5.34) and (4.26)] for $\kappa = 0.8$, $A = 25$.

Now applying the relation

$$\int d^2\alpha e^{-\alpha^*\alpha}\alpha^n\alpha^{*m} = \frac{\partial^n}{\partial a^n}\frac{\partial^m}{\partial b^m}\int d^2\alpha e^{-\alpha^*\alpha+a\alpha+b\alpha^*}, \tag{5.30}$$

Eq. (5.29) is expressible as

$$\bar{n} = R\frac{-\partial}{\partial m}\Big(\int \frac{d^2\alpha}{\pi}\exp\Big[-U\alpha\alpha^* + V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Big]\Big) - 1. \tag{5.31}$$

In view of Eq. (5.21), expression (5.31) reduces to

$$\bar{n} = \frac{RU}{\sqrt{(U^2-V^2)^3}} - 1. \tag{5.32}$$

But

$$R = \sqrt{(U^2-V^2)},$$

then

$$\bar{n} = \frac{U}{U^2 - V^2} - 1 \tag{5.33}$$

which is the mean of photons for superposition of two light beams obtained from two identical degenerate three-level lasers with steady state value

$$\bar{n} = \frac{A(1-\eta)}{A\eta + k} = 2\bar{n}, \tag{5.34}$$

which is a simple sum of the mean photon numbers of two light beams as it can also be seen in Fig. 5.2.

5.2.2 The Variance of Photon Number

We now seek to evaluate the variance of the photon number of the light beams expressed as

$$(\Delta n)^2 = \langle n^2 \rangle - \bar{n}^2. \tag{5.35}$$

Since the mean photon number is already found, we then need to find $\langle n^2 \rangle$ which needs the c-number function corresponding to the operator in the antinormal order. The number operator is expressed in terms of the annihilation and creation operators in the antinormal order using the commutation relation $a^\dagger a$=$aa^\dagger$-1 as follows:

$$\begin{aligned} \langle n^2 \rangle &= \langle a^\dagger a a^\dagger a \rangle \\ &= \langle (aa^\dagger - 1)(aa^\dagger - 1) \rangle \\ &= \langle a^2 a^{\dagger 2} - 3aa^\dagger + 1 \rangle \\ &= \langle a^2 a^{\dagger 2} \rangle - 3\bar{n} - 2 \end{aligned} \tag{5.36}$$

and

$$\begin{aligned}\langle a^2 a^{\dagger 2}\rangle &= \int d^2\alpha Q(\alpha,\alpha^*,t)\alpha^2\alpha^{*2} \\ &= R\int \frac{d^2\alpha}{\pi}\exp\Bigg(-U\alpha\alpha^* + V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Bigg)\alpha^2\alpha^{*2}\end{aligned} \tag{5.37}$$

which can be rewritten as

$$\langle a^2 a^{\dagger 2}\rangle = R\frac{\partial^2}{\partial m^2}\Bigg(\int \frac{d^2\alpha}{\pi}\exp(-U\alpha\alpha^* + V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Bigg). \tag{5.38}$$

Up on substituting Eq. (5.21), this reduces to

$$\langle a^2 a^{\dagger 2}\rangle = R\frac{\partial^2}{\partial m^2}\Bigg(\frac{1}{\sqrt{U^2-V^2}}\Bigg), \tag{5.39}$$

carrying out differentiation this gives

$$\langle a^2 a^{\dagger 2}\rangle = \frac{R(2U^2+V^2)}{(U^2-V^2)^{5/2}}. \tag{5.40}$$

From Eqs. (5.35), (5.36) and (5.40) the variation of photon number is

$$(\Delta n)^2 = \frac{R(2U^2+V^2)}{(U^2-V^2)^{5/2}} - \bar{n}^2 - 3\bar{n} - 2, \tag{5.41}$$

which follows

$$(\Delta n)^2_{ss} = \frac{2A\eta\kappa + \kappa^2 - A^2(-2+\eta^2)}{(A\eta+\kappa)^2} - \bar{n}^2 - 3\bar{n} - 2. \tag{5.42}$$

Using Eq. (5.34), one can readily put in steady state

$$(\Delta n)^2_{ss} = \frac{A(1-\eta)(A(2+\eta)+\kappa)}{(A\eta+\kappa)^2}. \tag{5.43}$$

Eq. (5.43) can be written as

$$(\Delta n)^2_{ss} = \bar{n}(1+\frac{2A}{A\eta+\kappa}), \tag{5.44}$$

where $0 < \eta < 1$, is the photon statistic is super-Poissonian which can easily be observed clearly in Fig.5.2

5.2.3 The Photon Number Distribution

We seek to study the photon number distribution for the light obtained from the superposition of two light beams generated by identical degenerate three-level lasers employing the Q function (5.16). The photon number distribution for light is expressible in terms of the Q function as

$$\begin{aligned} P(n,t) &= \frac{\Pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\left[Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}\right]_{\alpha=\alpha^*=0} \\ &= \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\left[\exp(1-U)\alpha\alpha^* + V(\tfrac{\alpha^2}{2}+\tfrac{\alpha^{*2}}{2})\right]_{\alpha=\alpha^*=0} \end{aligned} \quad (5.45)$$

Up on using the power series expansion, one can find

$$e^{(1-U)\alpha^*\alpha} = \sum_i \frac{(1-U)^l\alpha^l}{\alpha^{*l}}, \quad (5.46)$$

$$e^{V\alpha^2/2} = \sum_j \frac{(\frac{V}{2})^j\alpha^{2j}}{j!}, \quad (5.47)$$

$$e^{\frac{V}{2}\alpha^{*2}} = \sum_r \frac{\frac{V}{2}\alpha^{*2r}}{r!}, \quad (5.48)$$

so that

$$p(n,t) = \frac{R}{n!}\sum_{ljr}\frac{\frac{V}{2}^{r+j}(1-U)^l}{l!r!j!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\left[\alpha^{2j+l}\alpha^{*2r+l}\right]_{\alpha=\alpha^*=0}. \quad (5.49)$$

By the help of the relation

$$\frac{\partial^n x^m}{\partial\alpha^n} = \frac{m!}{(m-n)!}x^{m-n}, \quad (5.50)$$

we get

$$p(n,t) = \frac{R}{n!}\sum_{ljr}\frac{\frac{N}{2}^{r+j}(1-U)^l(2j+l)!(2r+l)!}{l!r!j!(2j+l-n)!}\left[\alpha^{2j+l-m}\alpha^{*2r+l-n}\right]_{\alpha=\alpha^*=0}. \quad (5.51)$$

If we apply the condition, the photon number distribution function under Eq. (5.49) vanishes. This function will have non-zero value only for the condition $l = n - 2r$, $l = n - 2j$ and $j = r$

$$p(n,t) = R\sum_{ljr} n!\frac{(\frac{V}{2})^{r+j}(1-U)^{l}}{r!j!(n-2j)!}\delta_{2j+l,n}\delta_{2r+l,n}. \tag{5.52}$$

From the property of keronecker delta function, we get

$$p(n,t) = R\sum_{j}^{[n]} n!\frac{(\frac{V}{2})^{j}(1-U)^{n-2j}}{j!(n-2j)!}\sum_{r}^{[n]}\frac{(\frac{V}{2})^{r}}{r!}. \tag{5.53}$$

To avoid the factorial of a negative number we get $n - 2j \geq 0$, then

$$r = j \leq n/2 \tag{5.54}$$

hence

$$p(n,t) = R\sum_{r=0}^{[n]}\frac{n!(1-U)^{n-2r}V^{2r}}{2^{2r}(r!)(n-2r)!}. \tag{5.55}$$

Applying the values of R, U and V, we arrive at

$$p(n,t) = \left[\frac{u^2 - v^2}{4 - 4u + u^2 - v^2}\right]^{1/2}$$

$$\times\sum_{r=0}^{[n]} n!\frac{\left(\frac{4-6u+2u^2-2v^2}{4-4u+u^2-v^2}\right)^{n-2r}\left(\frac{2v}{4-4u+u^2-v^2}\right)^{2r}}{2^{2r}(r!)^2(n-2r)!}. \tag{5.56}$$

Along with steady state value we get

$$p(n,t) = \left[\frac{(A\eta+\kappa)^2}{A^2\eta^2 + 2A\kappa + 2A\kappa + \kappa^2}\right]^{1/2}$$

$$\times\sum_{r=0}^{[n]}\frac{n!\left(\frac{A(1-\eta)(-A\eta+\kappa)}{A^2\eta^2+2A\kappa+2A\kappa+\kappa^2}\right)^{n-2r}\left(\frac{A(A\eta+\kappa)\sqrt{1-\eta^2}}{A^2\eta^2+2A\kappa+2A\kappa+\kappa^2}\right)^{2r}}{2^{2r}(r!)^2(n-2r)!} \tag{5.57}$$

where $[n]=\frac{n-1}{2}$ for odd n and $[n]=\frac{n}{2}$ for even n. This expression represents the photon number distribution for superposed two similar light beams produced by degenerate three level lasers having the same form as the case of light generated by three-level laser coupled to a squeezed vacuum reservoir in chapter four. Then it is possible to check that the photon number distribution decreases with number of photons.

5.3 Quadrature Variance

In this section, we seek to study and evaluate the squeezing property of a single-mode light produced as a result of superposition of two light beams generated by three-level lasers. We next evaluate the quadrature variance of this light. The squeezing property of single mode light is described by two quadrature operators.

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger \tag{5.58}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{5.59}$$

where $\hat{a}_+$ and $\hat{a}_-$ are hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{a}^\dagger$, $\hat{a}$ are the creation and annihilation operators of the light obtained from the superposition of two light beams. The quadrature variance can be expressed in terms of the quadrature operators as

$$(\Delta\hat{a}_\pm)^2 = \langle\Delta\hat{a}_\pm^2\rangle - \langle\Delta\hat{a}_\pm\rangle^2. \tag{5.60}$$

The explicit form of quadrature variance for the plus quadrature can be expressed in terms of the creation and annihilation operators as

$$(\Delta\hat{a}_+)^2 = 1 + \langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle + 2\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle. \tag{5.61}$$

In the same way quadrature variance of the minus quadrature will be given as

$$(\Delta\hat{a}_-)^2 \;=\; 1+2\langle\hat{a}^\dagger\hat{a}\rangle+\langle\hat{a}\rangle^2+\langle\hat{a}^\dagger\rangle^2-\langle\hat{a}^2\rangle-\langle\hat{a}^{\dagger 2}\rangle-2\langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle. \tag{5.62}$$

But

$$\langle\hat{a}^\dagger\hat{a}\rangle \;=\; \bar{n}.$$

Then we calculate the remaining expectation values using the Q function of light beam and the c-number variable corresponding each operator or product of operators as

$$\langle\hat{a}\rangle \;=\; \int d^2\alpha Q(\alpha,\alpha^*)\alpha, \tag{5.63}$$

in which α is the c-number variable corresponding to the annihilation operator $\hat{a}$. Up on using Eq. (5.22), we have

$$\begin{aligned}\langle\hat{a}\rangle &= R\int\frac{d^2\alpha}{\pi}\exp\Big[-U\alpha\alpha^*+V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Big]\alpha\\ &= R\frac{\partial}{\partial p}\bigg(\int\frac{d^2\alpha}{\Pi}\exp\Big[-U\alpha\alpha^*+p\alpha+V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Big]\bigg)_{p=0}\\ &= \frac{R}{\sqrt{U^2-V^2}}\frac{\partial}{\partial p}\bigg[\exp\bigg(\frac{Vp^2}{2(U^2-V^2)}\bigg)\bigg]_{p=0}\\ &= 0 \qquad .\end{aligned} \tag{5.64}$$

Similarly

$$\langle\hat{a^\dagger}\rangle = 0, \tag{5.65}$$

$$\begin{aligned}\langle\hat{a}^2\rangle &= R\int\frac{d^2\alpha}{\pi}\exp\Big[-U\alpha\alpha^*+V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Big]\alpha^2\\ &= R\frac{\partial^2}{\partial q^2}\bigg(\int\frac{d^2\alpha}{\pi}\exp\Big[-U\alpha\alpha^*+q\alpha+V(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})\Big]\bigg)_{q=0}\\ &= \frac{R}{\sqrt{U^2-V^2}}\frac{\partial}{\partial q}\bigg[\exp\bigg(\frac{Vq^2}{2(U^2-V^2)}\bigg)\bigg]_{q=0}\\ &= \frac{R}{\sqrt{U^2-V^2}}\frac{\partial}{\partial q}\bigg[\frac{Vq}{U^2-V^2}\exp\bigg(\frac{Vq^2}{2(U^2-V^2)}\bigg)\bigg]_{q=0}\end{aligned}$$

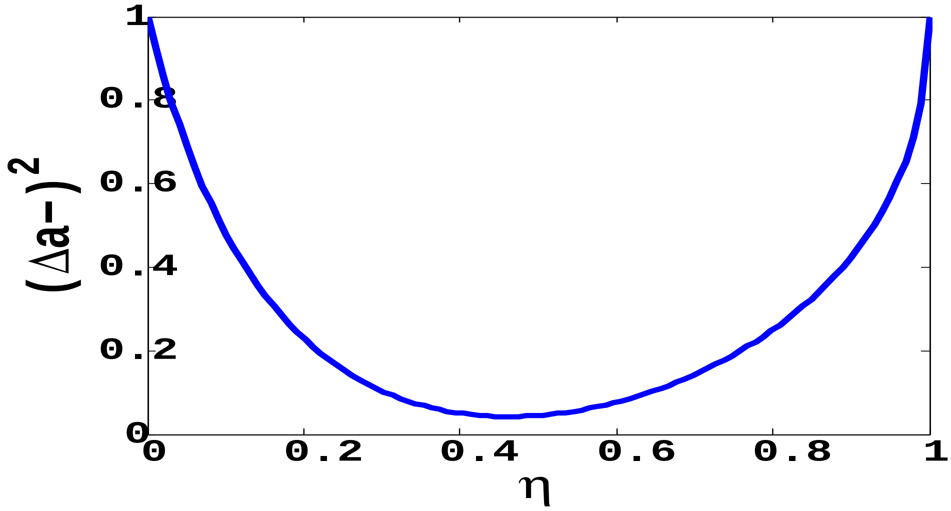

Figure 5.3: Plots of quadrature variance $(\Delta a_{-})^2$ for superposed light [Eq. (5.71)] verses η for $\kappa = 0.8$, $A = 3$.

$$\langle \hat{a}^2 \rangle = \frac{VR}{(U^2 - V^2)^{3/2}}. \tag{5.66}$$

In the same way

$$\langle \hat{a}^{\dagger 2} \rangle = \frac{VR}{(U^2 - V^2)^{3/2}}. \tag{5.67}$$

Applying Eqs. (5.64), (5.65), (5.66) and (5.67) in Eq. (5.61), the quadrature variance for the plus quadrature becomes

$$(\Delta \hat{a}_{+})^2 = 1 + 2\bar{n} + \frac{2VR}{(U^2 - V^2)^{3/2}}, \tag{5.68}$$

with steady state value

$$(\Delta \hat{a}_{+})^2 = \frac{A\eta + 2A(1 - \eta + \sqrt{1 - \eta^2}) + \kappa}{A\eta + \kappa}. \tag{5.69}$$

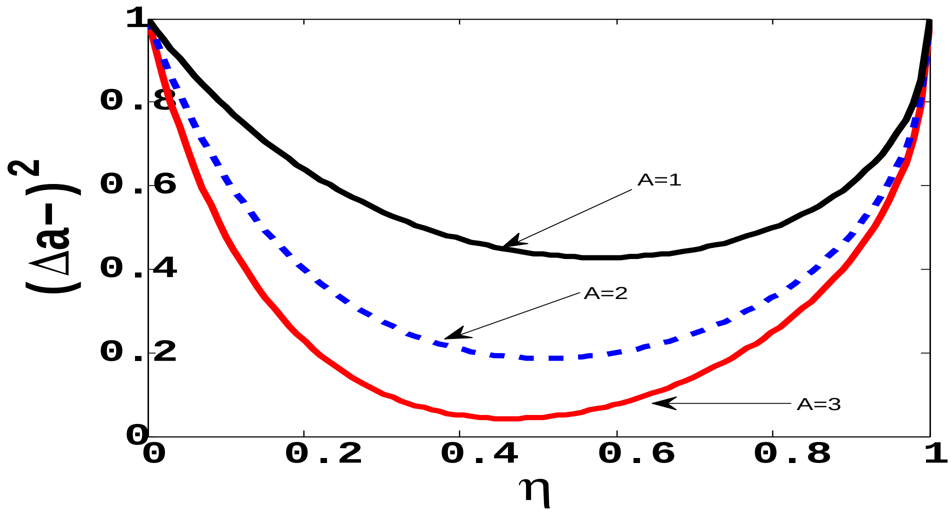

Figure 5.4: Plots of quadrature variance $(\Delta a_-)^2$ for superposed light [Eq. (5.71)] verses η for $\kappa = 0.8$, $A = 1, 2, 3$.

Applying Eqs. (5.64), (5.66), (5.67) and (5.62) in Eq. (5.63), the quadrature variance for the minus quadrature

$$(\Delta \hat{a}_-)^2 = 1 + 2\bar{n} - \frac{2VR}{(U^2 - V^2)^{3/2}}, \tag{5.70}$$

with steady state value

$$(\Delta \hat{a}_-)^2 = \frac{A\eta + 2A(1 - \eta - \sqrt{1-\eta^2}) + \kappa}{A\eta + \kappa}. \tag{5.71}$$

From the direct look at the Eqs. (5.69) and (5.71), we could not judge the squeezing property of the light. However, we can draw the graph of quadrature variance against η for some values of A and κ. It is observed that the light mode is in squeezed state. Fig.5.3 of course, the squeezing occurs in the minus quadrature. We can see Fig.5.4 is the quadrature variance for the superposition of the light beams produced by a pair of degenerate three-level lasers for different values of A. It shows that the

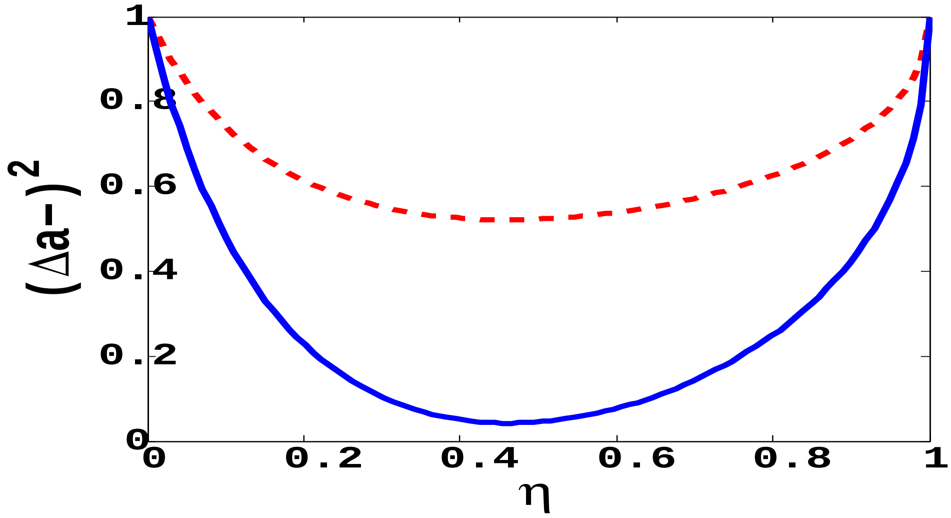

Figure 5.5: Plots of quadrature variance $(\Delta a_-)^2$ for single three-level laser (dashed line) and the light beam produced by superposed (pair of) three-level laser (solid line) verses η for $\kappa = 0.8$, $A = 3$.

degree of squeezing increases with the linear gain coefficient. It is seen that almost perfect squeezing could be achieved by taking large values of A with maximum value of $A = 3$ and for small values of η. Moreover, the minimum value of quadrature variance for $A = 3$ and $\kappa = 0.8$ is found to be 0.0425 which occurs at $\eta = 0.4545$. This implies that the maximum squeezing is 95.8% bellow the coherent state level. Fig.5.4 shows that for A=3 and k=0.8 the quadrature variance of the minus quadrature is 0.5213 which occurs at $\eta = 0.4545$. In other words, the degenerate three-level laser generate squeezed light with a squeezing of 47.9%. Besides, the superposition of two light beams generate a squeezed light with quadrature squeezing of 95.8% for the same value of A, κ and η. From this we can see that the superposition two light beams changes the quadrature squeezing. For our case, it is found that when we superpose two identical single mode light beams, the quadrature squeezing doubles, which can be seen in Fig.5.5

6

CONCLUSION

In this thesis, we have studied the squeezing and statistical properties of the light generated by degenerate three-level laser in which degenerate three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity coupled to vacuum reservoir via a single port-mirror. Applying the linear approximation scheme we found the master equation for a light produced by degenerate three-level laser from which we obtained the stochastic differential equations and the corresponding c-number langevine equations. Employing these solutions, we found antinormally ordered characteristic function which is used to find the Q function of the light beams. Using the Q function, we calculated the photon statistics of the light and it appears that the photon statistic is super-Poissonian while the photon number distribution decreases with photon number. We have calculated quadrature variance for $A = 3$ and $\kappa = 0.8$ at steady state to be 0.5213 with a squeezing of 47.9% which occurs at $\eta = 0.4545$ and the quadrature squeezing increases with the linear gain coefficient. We have shown that the effect of parametric amplifier is to increase the intracavity squeezing by maximum of 50%. Our study showed that the quantum optical system generates squeezed light and the degree of squeezing increases with the linear gain coefficient with maximum interacavity squeezing of 93.2% bellow the coherent state level.

Applying the Q function derived for superposed light, we calculated the mean photon number, which becomes twice that of single light beam at steady state. This shows that the superposition of two light beams is super-Poissonian. The quadrature variance for superposed of two identical light beams decreases with linear gain coefficient having minimum value of 0.0425 for $A = 3$, $\kappa = 0.8$ and $\eta = 0.4545$. For $A > 3$ the quadrature variance has negative values which requires further investigation. The result also indicate that the superposed light mode is in squeezed state with maximum squeezing of 95.8% bellow coherent state level for the same values of A, κ and η.

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