Superposition of Second Harmonic Light Beam with Twin One-mode Subharmonic Light Beams

Sisay Abedla

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ABBREVITAIONS

SH = second harmonic

κ = damping constant

κ_a = damping constant for fundumental mode

κ_b = damping constant for second harmonic mode

ε = Driving mode for SH-light

Γ = Driving mode for twin one-mode subharmonic light beams

ω_0 = frequency for fundumental mode

$2\omega_0$ = frequency for SH-mode

λ = Coupling constant

Abstract

In this thesis, we have seen the Hamiltonian and the master equation for the second harmonic light beam and one-mode subharmonic light beams coupled to vacuum reservoir.

Employing the master equation, we have determined the c-number Langevin equation for the second harmonic light beam and one-mode subharmonic light beams. Using the solution of c-number Langevin equation we established the Q function and then we calculated the mean photon number, the variance of photon number and the quadrature fluctuation for both second harmonic light beam and one-mode subharmonic light beams.

Finally, we have determined the density operator for the superposition of second harmonic light beam and one-mode subharmonic light beams. Using the superposed density operator, we have calculated the mean photon number, the variance of photon number and the quadrature fluctuation for superposed light beams. We found that the mean photon number is a simple sum of the mean photon number of the separate lights at steady state and the degree of squeezing for superposed light beam is 55.57% at threshold and steady state.

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Chapter 1

Introduction

Non-linear optics has been a rapidly growing scientific field in recent decades. It is based on phenomenon related to the interaction of intense coherent light with matter. Non-linear optics is the study of the interaction of light with matter under conditions in which the non-linear response of the atoms plays an important role. During the past three decades optics has secured a good place in the application areas previously dominated by electronics. Developments in the field of non-linear optics holds promise for important application in the optical information processing, telecommunications and integrated optics. Because of the emergency of this field from solid-state physics in which inorganic semiconductors, insulators and crystals have constituted a major part of scientific base for the early experimental and theoretical investigations were primarily concerned with the material from these classes [1].

The non-linear terms in the interaction of light with atoms give rise to a variety of optical processes, in addition to multi-photon absorption (which is also a non-linear phenomenon arching because of the excitation of an atom by processes, in which two or more photons are absorbed). A beam of monochromatic light in interaction with atoms can be partially converted into light, whose frequencies are harmonics of the fundamental frequency. Similarly, beams of two or more frequencies can be combined to produce a light beam and can alter the refractive index of a medium through which it passes, by an amount proportional to the intensity of the beam [1].

Second harmonic generation was first demonstrated by peter Fran ken, A.E. Hill, C.W Peters, and G. Weinreich at the University of Michigan, Ann Arbor, in 1961 [2]. The demonstration was made possible by the invention of the laser, with a wavelength of 694 nm into a quartz sample. They sent the output light through a spectrometer, recording the spectrum on photographic paper, which indicated the production of light at 347nm. Famously, when published in the journal Physical Review Letters [2], the copy editor mistook the dim spot (at 347nm) on the photographic paper as a speck of dirt and removed it from the publication [2]. The formulation of second harmoonic generation was initially described by N. Bloembergen and P.S.Pershan at Harvard in

1962 [3]. In their extensive evaluation of Maxwell's equations at the planar interface between a linear and non linear medium, several riles for the interaction of light in non-linear mediums were elucidated.

S.F Piraere, M.Xiao, H.J. Kinnble and J.L. Hall, showed the squeezing properties of light experimentally [4]. Morover, Fesseha Kassahun theoreticaly studied the case for which the nonlinear crystal is placed inside a cavity driven by coherent light and coupled to two indipendent vacuum reservoirs via a single port-mirror. Employing the linearization scheme of approximation, he obtaind closed form expression for the quadrature variance, the mean photon number and the squeezing spectrum for the fundamental mode as well as the second harmonic mode [5].

Squeezing state of light has played a crucial role in the development of quantum optics. Squeezing is one of the nonclassical features of light that have been extensively studied by several authors [6,7]. In a squeezing state the quantum noise in one quadrature is below the vacuum-state level or the coherent-state level at the expense of enhanced fluctuation in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Because of the quantum noise reduction achievable below the vacuum level, squeezed light has potential applications in the detection of weak signal and low-noise communication [8-10].

A one-mode sub-harmonic generation, consisting of a nonlinear crystal pumped by coherent light and placed in a cavity coupled to a vacuum reservoir, is a prototype source of a single-mode squeezed light. In this system a pump photon of frequency 2ω is down converted into a pair of highly correlated signal photons each of frequency ω. Subharmonic generator is one of the most reliable source of squeezed light. A theoretical analysis of the quantum fluctuation and photon statistics of the signal mode produced by a sub-harmonic generator has been made by a number of authors [11-13]. A maximum of 50% squeezing of the intracavity signal mode produced by the sub-harmonic generator has been predicted by a number of authors [14,15].

In this thesis, we study the squeezing and statistical properties of light produced by superposed second harmonic light beam and twin one-mode subharmonic light beams. We carry out our study of second harmonic light beam, twin one-mode subharmonic light beams and superposed light beams, using the solution of the c-number Langevin equations obtained with the aid of the master equation. Employing the solution of the resulting equation we calculated the antinormally orderd characterstic function and then Q function of the separate light beams. And with the help of the superposed density operator, we get the mean photon number, the variance of the photon number and the quadrature fluctuation for the superposed light beams.

Chapter 2

Second Harmonic Light

In second harmonic generation a light mode of frequency ω_0 (the fundamental mode) interacts with a nonlinear crystal and is up converted into a light mode of frequency $2\omega_0$ (the second harmonic mode). It so happens that the second harmonic mode is in a squeezed state.The fundamental mode which is initially in a coherent state also gets squeezed as time progresses[5].

We consider the case for which the nonlinear crystal is placed inside a cavity driven by coherent light and coupled to two independent vacuum reservoirs via a single port mirror. Employing the linearizion scheme of approximation, we obtain closed form expression of the Q function for the second harmonic light [5].

2.1 Linearizing differential equations

The process of second harmonic generation is described by the Hamiltonian[5]

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a}) + \frac{i\lambda}{2}(\hat{b}^\dagger \hat{a}^2 - \hat{b}\hat{a}^{\dagger 2}), \tag{2.1.1}$$

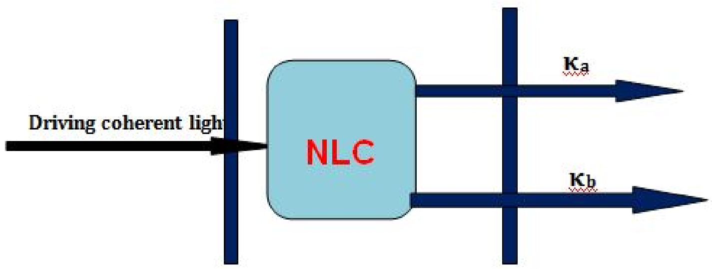

Figure 2.1: Schematic Diagram for SH-light

where $\hat{a}(\hat{b})$ is the anihilation operator for the fundamental (second harmonic) mode, λ is the coupling constant and ε is proportional to the amplitude of the driving coherent light [5]. Applying Eq.(2.1.1) and taking into account the interaction of the fundamental and second harmonic modes with two independent vacuum reserviors, the master equation for the cavity modes can be written as[5]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \varepsilon(\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a} - \hat{a}\hat{\rho}) + \frac{\lambda}{2}(\hat{b}^\dagger\hat{a}^2\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^2 + \hat{\rho}\hat{b}\hat{a}^{\dagger 2} - \hat{b}\hat{a}^{\dagger 2}\hat{\rho}) \\ &\quad + \frac{\kappa_a}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa_b}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}), \qquad (2.1.2)\end{aligned}$$

in which κ_a and κ_b are the cavity damping constants.

Now employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \qquad (2.1.3)$$

together with Eq.(2.1.2), we readily obtain

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = \varepsilon - \frac{\kappa_a}{2}\langle\hat{a}(t)\rangle - \lambda\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle, \qquad (2.1.4)$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = 2\varepsilon\langle\hat{a}(t)\rangle - \kappa_a\langle\hat{a}^2(t)\rangle - 2\lambda\langle\hat{b}(t)\hat{a}^\dagger(t)\hat{a}(t)\rangle - \lambda\langle\hat{b}(t)\rangle, \qquad (2.1.5)$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle &= \varepsilon(\langle\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\rangle - \kappa_a\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle \\ &\quad - \lambda\langle\hat{b}^\dagger(t)\hat{a}^2(t)\rangle - \lambda\langle\hat{b}(t)\hat{a}^{\dagger 2}(t)\rangle, \qquad (2.1.6)\end{aligned}$$

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa_b}{2}\langle\hat{b}(t)\rangle + \frac{\lambda}{2}\langle\hat{a}^2(t)\rangle, \qquad (2.1.7)$$

$$\frac{d}{dt}\langle\hat{b}(t)\hat{b}(t)\rangle = -\kappa_b\langle\hat{b}^2(t)\rangle + \frac{\lambda}{2}\langle\hat{b}(t)\hat{a}^2(t)\rangle \qquad (2.1.8)$$

and

$$\frac{d}{dt}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = -\kappa_b\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{\lambda}{2}\langle\hat{b}^\dagger(t)\hat{a}^2(t)\rangle + \frac{\lambda}{2}\langle\hat{b}(t)\hat{a}^{\dagger 2}(t)\rangle. \qquad (2.1.9)$$

Next we note that the c-number equations corresponding to Eqs. (2.1.4), (2.1.5), (2.1.6), (2.1.7), (2.1.8) and (2.1.9) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = \varepsilon - \frac{\kappa_a}{2}\langle\alpha(t)\rangle - \lambda\langle\beta(t)\alpha^*(t)\rangle, \qquad (2.1.10)$$

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = 2\varepsilon\langle\alpha(t)\rangle - \kappa_a\langle\alpha^2(t)\rangle - 2\lambda\langle\beta(t)\alpha^*(t)\alpha(t)\rangle - \lambda\langle\beta(t)\rangle, \quad (2.1.11)$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle \; &= \; \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle) - \kappa_a\langle\alpha^*(t)\alpha(t)\rangle \\ &\quad -\lambda\langle\beta^*(t)\alpha^2(t)\rangle - \lambda\langle\beta(t)\alpha^{*2}(t)\rangle, \end{aligned} \quad (2.1.12)$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{\kappa_b}{2}\langle\beta(t)\rangle + \frac{\lambda}{2}\langle\alpha^2(t)\rangle, \quad (2.1.13)$$

$$\frac{d}{dt}\langle\beta(t)\beta(t)\rangle = -\kappa_b\langle\beta^2(t)\rangle + \frac{\lambda}{2}\langle\beta(t)\alpha^2(t)\rangle \quad (2.1.14)$$

and

$$\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle = -\kappa_b\langle\beta^*(t)\beta(t)\rangle + \frac{\lambda}{2}\langle\beta^*(t)\alpha^2(t)\rangle + \frac{\lambda}{2}\langle\beta(t)\alpha^{*2}(t)\rangle. \quad (2.1.15)$$

On the basis of Eqs.(2.1.10) and (2.1.13), one can write

$$\frac{d\alpha}{dt} = \varepsilon - \frac{\kappa_a}{2}\alpha - \lambda\beta\alpha^* + g(t) \quad (2.1.16)$$

and

$$\frac{d\beta}{dt} = -\frac{\kappa_b}{2}\beta + \frac{\lambda}{2}\alpha^2 + f(t), \quad (2.1.17)$$

where $g(t)$ and $f(t)$ are noise forces associated with the normal ordering. When taking the expectation value of Eq.(2.1.16) and comparing with Eq.(2.1.10) and taking the expectation value of Eq.(2.1.17) and comparing with Eq.(2.1.13), we see that

$$\langle g(t)\rangle = 0, \quad (2.1.18)$$

$$\langle f(t)\rangle = 0. \quad (2.1.19)$$

We next proceed to determine the correlation properties of the noise forces.To this end, the solution of Eq.(2.1.16) can be written as

$$\alpha(t) \; = \; \alpha(0)e^{-\frac{1}{2}\kappa_a t} + \int_0^t e^{-(\frac{1}{2}\kappa_a)(t-t')}[g(t') - \lambda\beta(t')\alpha^*(t') + \varepsilon]dt', \quad (2.1.20)$$

Multiplying this equation by $g(t)$ on the left and taking the expectation values with the condition that the noise force at a time should not affect cavity mode variables at an earlier time, we see that

$$\langle g(t)\alpha(t)\rangle \;=\; \int_0^t e^{-\kappa_a(t-t')}\langle g(t)g(t')\rangle dt'. \tag{2.1.21}$$

On account of Eq.(2.1.16) and $\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \langle\alpha(t)\frac{d\alpha}{dt}\rangle + \langle\frac{d\alpha}{dt}\rangle$, we have

$$\begin{aligned}\langle\frac{d}{dt}\alpha(t)\rangle \;&=\; 2\varepsilon\langle\alpha(t)\rangle - \varepsilon\langle\alpha(t)^2\rangle - 2\lambda\langle\beta(t)\alpha^*(t)\alpha(t)\rangle \\ &\quad +2\langle g(t)\alpha(t)\rangle.\end{aligned} \tag{2.1.22}$$

Inspection of Eq.(2.1.11) and (2.1.22) indicates that

$$2\langle g(t)\alpha(t)\rangle = -\lambda\langle\beta(t)\rangle \tag{2.1.23}$$

Making use of Eqs.(2.1.21) and (2.1.23), we have

$$-\frac{1}{2}\lambda\langle\beta(t)\rangle \;=\; \int_0^t e^{-(\frac{1}{2}\kappa_a)(t-t')}\langle g(t)g(t')\rangle dt' \tag{2.1.24}$$

and in view of Eqs.(2.1.16) and (2.1.18), we obtain

$$\langle g(t)g(t')\rangle = -\lambda\langle\beta(t)\rangle\delta(t-t'). \tag{2.1.25}$$

Following the same procedure,we readily obtain

$$\langle g(t)g(t')\rangle = \langle g^*(t)g(t')\rangle = \langle g(t)g^*(t')\rangle = 0. \tag{2.1.26}$$

$$\langle f(t)f(t')\rangle = \langle f^*(t)f(t')\rangle = 0 \tag{2.1.27}$$

Dropping the noise force in Eq.(2.1.17), so that we have

$$\frac{d}{dt}\beta = -\frac{\kappa_b}{2} + \frac{\lambda}{2}\alpha^2. \tag{2.1.28}$$

Since Eqs.(2.1.16) and (2.1.28) are nonlinear differential equations, it is not possibel to obtain exact solution of these equations. We carried out our analysis using linearized differential equations. Here in order to obtain the steady state values of the pertinent variables, we set the time derivatives in Eqs.(2.1.16) and (2.1.28) equal to zero and we get[5]

$$\frac{1}{2}\kappa_a\langle\alpha\rangle_{ss} + \lambda\langle\beta\alpha^*\rangle_{ss} = \varepsilon, \tag{2.1.29}$$

$$\kappa_b\langle\beta\rangle_{ss} - \lambda\langle\alpha^2\rangle_{ss} = 0. \tag{2.1.30}$$

Now Applying the semiclassical approximations

$$\langle\beta\alpha^*\rangle_{ss} = \langle\beta\rangle_{ss}\langle\alpha^*\rangle_{ss} \tag{2.1.31}$$

and

$$\langle\alpha^2\rangle_{ss} = \langle\alpha\rangle_{ss}^2, \tag{2.1.32}$$

we have

$$\frac{1}{2}\kappa_a\alpha_0 - \lambda\beta_0\alpha_0^* = \varepsilon, \tag{2.1.33}$$

$$\kappa_b\beta_0 = \lambda\alpha_0^2, \tag{2.1.34}$$

in which $\alpha_0 = \langle\alpha\rangle_{ss}$ and $\beta_0 = \langle\beta\rangle_{ss}$. Moreover, upon multiplying Eqs.(2.1.29) and (2.1.30) by λ we see that

$$\varepsilon_1^*\varepsilon_2 + \frac{1}{2}\kappa_a\varepsilon_1 = \lambda\varepsilon, \tag{2.1.35}$$

$$\varepsilon_1^2 = \kappa_b\varepsilon_2, \tag{2.1.36}$$

where

$$\varepsilon_1 = \lambda\alpha_0, \tag{2.1.37}$$

$$\varepsilon_2 = \lambda\beta_0. \tag{2.1.38}$$

Multiplying Eq.(2.1.35) by ε_1^* and taking into account Eq.(2.1.36), we have

$$\kappa_b\varepsilon_2^*\varepsilon_2 + \frac{1}{2}\kappa_a\varepsilon_1^*\varepsilon_1 = \lambda\varepsilon\varepsilon_1^*. \tag{2.1.39}$$

Upon subtracting this equation from its complex conjugate,we arrive at

$$\varepsilon_1^* = \varepsilon_1. \tag{2.1.40}$$

In additon, with the aide of Eq.(2.1.36) along with Eq.(2.1.40), it can be easly verified that

$$\varepsilon_2^* = \varepsilon_2. \tag{2.1.41}$$

We next proceed to linearize Eqs.(2.1.16) and (2.1.28), about the steady state values α_0 and β_0. To this end, we express $\alpha(t)$ and $\beta(t)$ as [16]

$$\alpha(t) = \alpha_0(t) + \alpha'(t), \tag{2.1.42}$$

$$\beta(t) = \beta_0(t) + \beta'(t), \tag{2.1.43}$$

in which $\alpha'(t)$ and $\beta'(t)$ repersents small variations about the steady state variables. Now substituting Eqs.(2.1.42) and (2.1.43) into Eqs.(2.1.16) and (2.1.28) and neglecting second-order terms in $\alpha'(t)$ and $\beta'(t)$, one can obtain

$$\frac{d}{dt}\alpha' = -\frac{1}{2}\kappa_a\alpha' - \varepsilon_2\alpha'^* - \varepsilon_1\beta' + g(t), \tag{2.1.44}$$

$$\frac{d}{dt}\beta' = -\frac{1}{2}\kappa_b\beta' + \varepsilon_1\alpha', \tag{2.1.45}$$

where we have taken into account Eqs.(2.1.33) and (2.1.34).

It proves to be convenient to introduce new variables defined by

$$\alpha_\pm = \alpha'^* \pm \alpha', \tag{2.1.46}$$

$$\beta_\pm = \beta'^* \pm \beta', \tag{2.1.47}$$

so that on account of Eqs.(2.1.44), (2.1.45), (2.1.46) and (2.1.47), one obtains

$$\frac{d}{dt}\alpha_\pm = -(\frac{1}{2}\kappa_a \pm \varepsilon_2)\alpha_\pm - \varepsilon_1\beta_\pm + g_\pm(t), \tag{2.1.48}$$

$$\frac{d}{dt}\beta_\pm = -\frac{1}{2}\kappa_b\beta_\pm + \varepsilon_1\alpha_\pm, \tag{2.1.49}$$

in which

$$g_\pm(t) = g^*(t) \pm g(t). \tag{2.1.50}$$

Then Eq.(2.1.48) and Eq.(2.1.49) can be represented in the form

$$\frac{d}{dt}U(t) = -AU(t) + N(t), \tag{2.1.51}$$

where

$$U(t) = \begin{bmatrix} \alpha_{\pm}(t) \\ \beta_{\pm}(t) \end{bmatrix}, \tag{2.1.52}$$

$$N(t) = \begin{bmatrix} g_{\pm}(t) \\ 0 \end{bmatrix}, \tag{2.1.53}$$

$$A(t) = \begin{bmatrix} a_{\pm}(t) & \varepsilon_1 \\ -\varepsilon_1 & b \end{bmatrix}, \tag{2.1.54}$$

with

$$a_{\pm} = \frac{\kappa_a}{2} \pm \varepsilon_2, \tag{2.1.55}$$

$$b = \frac{\kappa_b}{2}. \tag{2.1.56}$$

In order to solve Eq.(2.1.51), we need to introduce the eigenvalues and eigenvectors of matrix A. We apply, the eigenvector equation [19]

$$AV_i = \lambda_i V_i, \tag{2.1.57}$$

with i = 1 and the eigenvectors

$$V_i(t) = \begin{bmatrix} X_i \\ Y_i \end{bmatrix}, \tag{2.1.58}$$

subject to the normalization condition

$$X_i^2 + Y_i^2 = 1. \tag{2.1.59}$$

Eq.(2.1.57) has nontrivial solution, only if

$$det(A - \lambda I) = 0. \tag{2.1.60}$$

Employing Eq.(2.1.60) togther with Eq.(2.1.54), we have

$$\begin{bmatrix} a \pm (t) - \lambda & \varepsilon_1 \\ \varepsilon_1 & b - \lambda \end{bmatrix} = 0, \tag{2.1.61}$$

so that the characterstic equation takes the form

$$\lambda^2 - \lambda(a_{\pm} + b) + a_{\pm}b + \varepsilon_1^2 = 0. \tag{2.1.62}$$

Solving for λ , the eigenvalues are found to be

$$\lambda_{1\pm} = \frac{1}{2}[a_\pm + b + \sqrt{(a_\pm + b)^2 - 4\varepsilon_1^2}], \tag{2.1.63}$$

$$\lambda_{2\pm} = \frac{1}{2}[a_\pm + b - \sqrt{(a_\pm + b)^2 - 4\varepsilon_1^2}]. \tag{2.1.64}$$

We next seek to obtain the eigenvectors of matrix A. The eigenvector corresponding to $\lambda_\pm$ is expressible as

$$V_1 = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}. \tag{2.1.65}$$

Employing Eq.(2.1.57) and taking into account Eq.(2.1.64), one can write

$$\begin{bmatrix} a_\pm(t) & \varepsilon_1 \\ -\varepsilon_1 & b \end{bmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \lambda_\pm \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}. \tag{2.1.66}$$

It then follows

$$a_\pm X_1 + \varepsilon_1 Y_1 = \lambda_\pm X_1, \tag{2.1.67}$$

$$-\varepsilon_1 X_1 + bY_1 = \lambda_\pm Y_1. \tag{2.1.68}$$

Using Eq.(2.1.67) along with (2.1.59), we see that

$$X_1 = \frac{\varepsilon_1}{\sqrt{(\lambda_{1\pm} - a_\pm)^2 + \varepsilon_1^2}}, \tag{2.1.69}$$

$$Y_1 = (\frac{\lambda_{1\pm} - a_\pm}{\varepsilon_1})X_1. \tag{2.1.70}$$

In view of Eq.(2.1.67) and Eq.(2.1.68), the eigenvectors corresponding to $\lambda_\pm$ can be written as

$$V_1 = \frac{1}{\sqrt{(\lambda_{1\pm} - a_\pm)^2 + \varepsilon_1^2}} \begin{bmatrix} \varepsilon_1 \\ \lambda_{1\pm} - a_\pm \end{bmatrix}. \tag{2.1.71}$$

Follwing a similar procedure, the eigenvectors corresponding to $\lambda_{2\pm}$ can also be established as

$$V_2 = \frac{1}{\sqrt{(\lambda_{1\pm} - a_\pm)^2 + \varepsilon_1^2}} \begin{bmatrix} \varepsilon_1 \\ \lambda_{2\pm} - a_\pm \end{bmatrix}. \tag{2.1.72}$$

From Eqs.(2.1.69) and (2.1.70), we can construct a matrix V of the eigenvectors of matrix A as column matrices as

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \varepsilon_1/E_{1\pm} & \varepsilon_1/E_{2\pm} \\ (\lambda_{1\pm} - a_\pm)/E_{1\pm} & (\lambda_{2\pm} - a_\pm)/E_{2\pm} \end{bmatrix}, \qquad (2.1.73)$$

in which

$$E_{1\pm} = \sqrt{(\lambda_{1\pm} - a_\pm)^2 + \varepsilon_1^2}, \qquad (2.1.74)$$

$$E_{2\pm} = \sqrt{(\lambda_{2\pm} - a_\pm)^2 + \varepsilon_1^2}. \qquad (2.1.75)$$

We next proceed to determaine the inverse of matrix V.

Applying the relation

$$det(V - \lambda I) = 0 \qquad (2.1.76)$$

and taking into account Eq.(2.1.71), one can readily obtain the characterstic equation as

$$\lambda^2 - (\frac{\varepsilon_1}{E_{1\pm}} + \frac{\lambda_{2\pm} - a_\pm}{E_{2\pm}})\lambda - \frac{\varepsilon_1(\lambda_{1\pm} - \lambda_{2\pm})}{E_{1\pm}E_{2\pm}} I = 0. \qquad (2.1.77)$$

Hence applying the Cayley-Hamilton theorem that a matrix satisfies its own characterstic equation, we see that

$$V^2 - (\frac{\varepsilon_1}{E_{1\pm}} + \frac{\lambda_{2\pm} - a_\pm}{E_{2\pm}})V - \frac{\varepsilon_1(\lambda_{1\pm} - \lambda_{2\pm})}{E_{1\pm}E_{2\pm}} I = 0. \qquad (2.1.78)$$

Multiplying Eq.(2.1.74) by V^{-1} and rearranging, we have

$$V^{-1} = \frac{E_{1\pm}E_{2\pm}}{\varepsilon_1(\lambda_{1\pm} - \lambda_{2\pm})}(V - (\frac{\varepsilon_1}{E_{1\pm}} + \frac{(\lambda_{2\pm} - a_\pm)}{E_{2\pm}})I). \qquad (2.1.79)$$

Inserting Eq.(2.1.73) into (2.1.79), we get

$$V^{-1} = \frac{1}{\lambda_{1\pm} - \lambda_{2\pm}} \begin{bmatrix} -E_{1\pm}(\lambda_{2\pm} - a_\pm)/\varepsilon_1 E_{1\pm} & E_{1\pm} \\ E_\pm(\lambda_{2\pm} - a_\pm)/\varepsilon_1 & -E_{2\pm} \end{bmatrix}. \qquad (2.1.80)$$

In view of the fact that $VV^{-1} = I$, Eq.(2.1.51) can be written as

$$\frac{d}{dt}U(t) = -VV^{-1}AVV^{-1}U(t) + N(t). \qquad (2.1.81)$$

Multiplying this equation by V^{-1} from the left, we see that

$$\frac{d}{dt}V^{-1}U(t) = -V^{-1}VV^{-1}AVV^{-1}U(t) + V^{-1}N(t). \qquad (2.1.82)$$

We note that the matrix $V^{-1}AV$ is the diagonal matrix D having the eigenvalues of A along its main diagonal and hence Eq.(2.1.82) can be put in the form

$$\frac{d}{dt}V^{-1}U(t) = -DV^{-1}U(t) + V^{-1}N(t), \tag{2.1.83}$$

where

$$D = \begin{bmatrix} \lambda_{\pm 1} & 0 \\ 0 & \lambda_{\pm 2} \end{bmatrix}. \tag{2.1.84}$$

The formal solution of Eq.(2.1.83) can be written as

$$U(t) = Ve^{-Dt}V^{-1}U(0) + \int_0^t Ve^{D(t-t')}V^{-1}N(t')dt'. \tag{2.1.85}$$

Using Eq.(2.1.84), one can write

$$e^{-Dt} = \begin{bmatrix} e^{-\lambda_{1\pm}t} & 0 \\ 0 & e^{-\lambda_{2\pm}t} \end{bmatrix} \tag{2.1.86}$$

and

$$e^{-D(t-t')} = \begin{bmatrix} e^{-\lambda_{1\pm}(t-t')} & 0 \\ 0 & e^{-\lambda_{2\pm}(t-t')} \end{bmatrix}. \tag{2.1.87}$$

On account of Eqs.(2.1.73), (2.1.79), (2.1.86) and (2.1.52), we have

$$\begin{aligned} Ve^{-Dt}V^{-1}U(0) \;=\; & \begin{bmatrix} \varepsilon_1/E_{1\pm} & \varepsilon_2/E_{2\pm} \\ (\lambda_{1\pm} - a_\pm)/E1\pm & (\lambda_{2\pm} - a_\pm)/E2\pm \end{bmatrix} \begin{bmatrix} e^{-\lambda_{1\pm}t} & 0 \\ 0 & e^{-\lambda_{2\pm}t} \end{bmatrix} \\ & (\frac{1}{\lambda_{1\pm} - \lambda_{2\pm}}) \begin{bmatrix} -E_{1\pm}(\lambda_{2\pm} - a_\pm)/\varepsilon_1 E_{1\pm} & E_{1\pm} \\ E_\pm(\lambda_{2\pm} - a_\pm)/\varepsilon_1 & -E_{2\pm} \end{bmatrix} \begin{bmatrix} \alpha \pm (0) \\ \beta \pm (0) \end{bmatrix}, \end{aligned} \tag{2.1.88}$$

$$Ve^{-Dt}V^{-1}U(0) = \frac{1}{\lambda_{1\pm} - \lambda_{2\pm}} \begin{bmatrix} ((\lambda_{1\pm} - a_\pm)e^{-\lambda_{2\pm}t} - (\lambda_{2\pm} - a_\pm)e^{-\lambda_{1\pm}t})\alpha(0) \\ +\varepsilon_1(e^{-\lambda_{1\pm}t} - e^{-\lambda_{2\pm}t})\beta_\pm(0) \\ \frac{(\lambda_{1\pm}-a_\pm)(\lambda_{2\pm}-a_\pm)}{\varepsilon_1}(e^{-\lambda_{2\pm}t} - e^{-\lambda_{1\pm}t})\alpha(0) \\ +((\lambda_{1\pm} - a_\pm)e^{-\lambda_{1\pm}t} - (\lambda_{2\pm} - a_\pm)e^{-\lambda_{2\pm}t})\beta(0) \end{bmatrix}. \tag{2.1.89}$$

Moreover, using Eqs.(2.1.53), (2.1.73), (2.1.80) and (2.1.87), we see that

$$\begin{aligned} & \int_0^t Ve^{-D(t-t')}V^{-1}N(t')dt' = \\ & \frac{1}{\lambda_{1\pm} - \lambda_{2\pm}} \begin{bmatrix} \int_0^t ((\lambda_{1\pm} - a_\pm)e^{-\lambda_{2\pm}(t-t')} - (\lambda_{2\pm} - a_\pm)e^{-\lambda_{1\pm}(t-t')})g_\pm(t')dt' \\ \int_0^t \frac{(\lambda_{1\pm}-a_\pm)(\lambda_{2\pm}-a_\pm)}{\varepsilon_1}(e^{-\lambda_{2\pm}(t-t')} - e^{-\lambda_{1\pm}(t-t')})g_\pm(t')dt' \end{bmatrix}. \end{aligned} \tag{2.1.90}$$

Employing Eqs.(2.1.52), (2.1.81) and (2.1.86), one can readily obtain

$$\begin{aligned}\alpha_\pm(0) &= \frac{1}{\lambda_{1\pm}-\lambda_{2\pm}}[(\lambda_{1\pm}-a_\pm)e^{-\lambda_{2\pm}t} \\ &-(\lambda_{2\pm}-a_\pm)e^{-\lambda_{1\pm}t})a_\pm(0)+\varepsilon_1(e^{-\lambda_{1\pm}t}-e^{-\lambda_{2\pm}t})\beta_\pm(0) \\ &+\int_o^t((\lambda_{1\pm}-a_\pm)e^{-\lambda_{2\pm}(t-t')}-(\lambda_{2\pm}-a_\pm)e^{-\lambda_{1\pm}(t-t')})g_\pm(t')dt'] \quad (2.1.91)\end{aligned}$$

and

$$\begin{aligned}\beta_\pm(0) &= \frac{1}{\lambda_{1\pm}-\lambda_{2\pm}}[\frac{(\lambda_{1\pm}-a_\pm)(\lambda_{2\pm}-a_\pm)}{\varepsilon_1} \\ &\times e^{-\lambda_{1\pm}t}-e^{-\lambda_{2\pm}t})\alpha_\pm(0)+(\lambda_{1\pm}-a_\pm)e^{-\lambda_{1\pm}t}-(\lambda_{2\pm}-a_\pm)e^{-\lambda_{2\pm}t})\beta_\pm(0) \\ &+\int_o^t\frac{(\lambda_{1\pm}-a_\pm)(\lambda_{2\pm}-a_\pm)}{\varepsilon_1}(e^{-\lambda_{1\pm}(t-t')}-e^{-\lambda_{2\pm}(t-t')})g_\pm(t')dt'], \quad (2.1.92)\end{aligned}$$

In view of Eqs.(2.1.46) and (2.1.47), we note that

$$\alpha'(t)=\frac{1}{2}(\alpha_+(t)-\alpha_-(t)), \quad (2.1.93)$$

$$\beta'(t)=\frac{1}{2}(\beta_+(t)-\beta_-(t)). \quad (2.1.94)$$

Applying Eqs.(2.1.89) and (2.1.90) and taking into account (2.1.85) and (2.1.93) and (2.1.52), we can see that

$$\alpha'(t)=P_+(t)\alpha_+ + R_+(t)+E_+ - P_-(t)\alpha_- + R_-(t)+E_+ \quad (2.1.95)$$

and

$$\beta'(t)=S_+(t)\alpha_+ + T_+(t)+F_+ - S_-(t)\alpha_- + T_-(t)+F_+, \quad (2.1.96)$$

in which

$$P_\pm(t)=\frac{1}{2(\lambda_{1\pm}-\lambda_{2\pm})}(\lambda_{1\pm}-a_{1\pm})e^{-\lambda_{2\pm}t}-(\lambda_{2\pm}-a_{1\pm})e^{-\lambda_{1\pm}t}, \quad (2.1.97)$$

$$R_\pm(t)=\frac{1}{2(\lambda_{1\pm}-\lambda_{2\pm})}e^{-\lambda_{1\pm}t}-e^{-\lambda_{2\pm}t}, \quad (2.1.98)$$

$$S_\pm(t)=\frac{(\lambda_{1\pm}-a_\pm)(\lambda_{2\pm}-a_\pm)}{2(\varepsilon_1\lambda_{1\pm}-\lambda_{2\pm})}(e^{-\lambda_{2\pm}t}-e^{-\lambda_{2\pm}t}), \quad (2.1.99)$$

$$T_\pm(t) = \frac{1}{2(\lambda_{1\pm} - \lambda_{2\pm})}(\lambda_{1\pm} - a_{1\pm})e^{-\lambda_{2\pm} t} - (\lambda_{2\pm} - a_{1\pm})e^{-\lambda_{1\pm} t}, \qquad (2.1.100)$$

$$\begin{aligned} E_\pm(t) &= \frac{1}{2(\lambda_{1\pm} - \lambda_{2\pm})}\int_0^t ((\lambda_{1\pm} - a_{1\pm})e^{-\lambda_{2\pm}(t-t')} - (\lambda_{2\pm} - a_{1\pm})e^{-\lambda_{1\pm}(t-t')}) \\ &\quad \times (g^*(t') \pm g(t'))dt', \end{aligned} \qquad (2.1.101)$$

$$\begin{aligned} F_\pm(t) &= \frac{(\lambda_{1\pm} - a_\pm)(\lambda_{2\pm} - a_\pm)}{(2\varepsilon_1 \lambda_{1\pm}}\int_0^t (e^{-\lambda_{2\pm}(t-t')} - e^{-\lambda_{2\pm}(t-t')}) \\ &\quad \times (g^*(t') \pm g(t'))dt'. \end{aligned} \qquad (2.1.102)$$

Moreover, using Eqs.(2.1.46) and (2.1.47), we have

$$\alpha_\pm(0) = \alpha'^*(0) \pm \alpha'(0)), \qquad (2.1.103)$$

$$\beta_\pm(0) = \beta'^*(0) \pm \beta'(0)). \qquad (2.1.104)$$

Inserting Eqs.(2.1.103) and (2.1.104) into Eqs. (2.1.95) and (2.1.96), we find

$$\begin{aligned} \alpha'(t) &= (P_+(t) - P_-(t))\alpha'^*(0) + (P_+(t) - P_-(t))\alpha'^(0) \\ &\quad (R_+(t) - R_-(t))\beta'^*(0) + (R_+(t) - R_-(t))\beta'^(0) \\ &\quad +(E_+(t) - E_-(t)) \end{aligned} \qquad (2.1.105)$$

$$\begin{aligned} \beta'(t) &= (S_+(t) - S_-(t))\alpha'^*(0) + (S_+(t) - S_-(t))\alpha'^(0) \\ &\quad (T_+(t) - T_-(t))\beta'^*(0) + (T_+(t) - T_-(t))\beta'^(0) \\ &\quad +(F_+(t) - F_-(t)) \end{aligned} \qquad (2.1.106)$$

Finally, with the aid of Eqs.(2.1.42) and (2.1.43) together with Eqs.(2.1.101) and (2.1.102), we get

$$\alpha(t) = \alpha'(t) + \alpha_0 \qquad (2.1.107)$$

and

$$\beta(t) = \beta'(t) + \beta_0. \qquad (2.1.108)$$

For a cavity mode initially in vacuum state, the above expressions take the form

$$\alpha(t) = E_+(t) - E_-(t) + \alpha_0, \qquad (2.1.109)$$

$$\beta(t) = F_+(t) - F_-(t) + \beta_0. \qquad (2.1.110)$$

2.2 The Q function

Next we wish to find the Q function for the second harmonic light employing the anti-normally ordered chracterstic function.

2.2.1 Q function for the second harmonic light

The Q function for second harmonic light is expressible as

$$Q(\beta^*, \beta, t) = \frac{1}{\pi^2} \int d^2\eta \phi_a(\eta^*, \eta, t) e^{\eta^*\beta - \eta\beta^*}, \tag{2.2.1}$$

where the antinormally ordered characterstic function $\phi_a(\eta^*, \eta, t)$ is defined as

$$\phi_a(\eta^*, \eta, t) = Tr(\hat{\rho} e^{-\eta^*\hat{b}} e^{\eta\hat{b}^\dagger}). \tag{2.2.2}$$

Then applying the identity $e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}$ along with c-number variable associated with normal ordering, we get

$$\phi_a(\eta^*, \eta, t) = e^{-\eta^*\eta} \langle e^{\eta\beta^* + \eta^*\beta} \rangle. \tag{2.2.3}$$

Assuming the fundamental mode to be initially in a vacuum state, we see from Eq.(2.1.106) and (2.1.108), that

$$\langle \beta(t) \rangle = 0. \tag{2.2.4}$$

β is Gaussian variable with zero mean, we can write [5]

$$\langle e^{-\eta^*\beta + \eta\beta^*} \rangle = e^{(\frac{1}{2}\langle -\eta^*\beta + \eta\beta^* \rangle^2)} = e^{\frac{1}{2}(\eta^{*2}\langle\beta^2\rangle + \eta^2\langle\beta^{*2}\rangle) - \eta^*\eta\langle\beta^*\beta\rangle} \tag{2.2.5}$$

It then follows that

$$\phi_a(\eta^*, \eta, t) = e^{-(1+\langle\beta^*\beta\rangle)\eta^*\eta + \frac{1}{2}(\eta^{*2}\langle\beta^2\rangle + \eta^2\langle\beta^{*2}\rangle)}. \tag{2.2.6}$$

Taking into account Eqs.(2.1.109) and (2.1.110) along Eqs.(2.1.101) and (2.1.102) and the correlation properties in Eqs.(2.1.25) and (2.1.26), we obtain

$$\langle \beta^*\beta \rangle = (F_+(t) - F_-(t) + \beta_0)(F_+(t) + F_-(t) + \beta_0),$$

then it follows

$$\langle \beta^*\beta \rangle = F_+^2 - 2F_+F_- + 2\beta_0(F_+ + F_-) + \beta_0^2 + F_-^2.$$

Using Eq.(2.1.19), we obtain

$$\langle \beta^*\beta \rangle = \beta_0^2 + \langle F_+^2 \rangle - \langle F_-^2 \rangle. \tag{2.2.7}$$

Similarly we have

$$\langle\beta^2\rangle = \langle\beta^{*2}\rangle = \beta_0^2 + \langle F_+^2\rangle - \langle F_-^2\rangle. \tag{2.2.8}$$

Then the antinormally ordered characterstic function takes the form

$$\phi_a(\eta^*, \eta, t) = exp[-c\eta^*\eta + \frac{d}{2}(\eta^2 + \eta^{*2})], \tag{2.2.9}$$

where

$$\begin{aligned} c \;=\; & 1 + \beta_0^2 + \frac{\varepsilon_2(\lambda_{1-} - a_-)^2(\lambda_{2-} - a_-)^2}{4\varepsilon_1^2(\lambda_{1-} - \lambda_{2-})^2}[\frac{1}{\lambda_{2-}}(1 - e^{-\lambda_{2-}t}) \\ & + \frac{1}{\lambda_{1-}}(1 - e^{-\lambda_{1-}t}) - \frac{4}{\lambda_{1-} + \lambda_{2-}}(1 - e^{-(\lambda_{1-}+\lambda_{2-})t})] \\ & - \frac{\varepsilon_2(\lambda_{1+} - a_+)^2(\lambda_{2+} - a_+)^2}{4\varepsilon_1^2(\lambda_{1+} - \lambda_{2+})^2}[\frac{1}{\lambda_{2+}}(1 - e^{-\lambda_{2+}t}) \\ & + \frac{1}{\lambda_{1+}}(1 - e^{-\lambda_{1+}t}) - \frac{4}{\lambda_{1+} + \lambda_{2+}}(1 - e^{-(\lambda_{1+}+\lambda_{2+})t})] \end{aligned} \tag{2.2.10}$$

and

$$\begin{aligned} d \;=\; & \beta_0^2 + \frac{\varepsilon_2(\lambda_{1-} - a_-)^2(\lambda_{2-} - a_-)^2}{4\varepsilon_1^2(\lambda_{1-} - \lambda_{2-})^2}[\frac{1}{\lambda_{2-}}(1 - e^{-\lambda_{2-}t}) \\ & + \frac{1}{\lambda_{1-}}(1 - e^{-\lambda_{1-}t}) - \frac{4}{\lambda_{1-} + \lambda_{2-}}(1 - e^{-(\lambda_{1-}+\lambda_{2-})t})] \\ & - \frac{\varepsilon_2(\lambda_{1+} - a_+)^2(\lambda_{2+} - a_+)^2}{4\varepsilon_1^2(\lambda_{1+} - \lambda_{2+})^2}[\frac{1}{\lambda_{2+}}(1 - e^{-\lambda_{2+}t}) \\ & + \frac{1}{\lambda_{1+}}(1 - e^{-\lambda_{1+}t}) - \frac{4}{\lambda_{1+} + \lambda_{2+}}(1 - e^{-(\lambda_{1+}+\lambda_{2+})t})] \end{aligned} \tag{2.2.11}$$

On account of Eqs.(2.1.63), (2.1.64) and (2.1.55) the steady state form of Eqs.(2.2.10) and (2.2.11), can be written as

$$c = 1 + \frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{\frac{\kappa_a}{2} + \varepsilon_2}]^2 + \frac{\varepsilon_2^2}{2}[\frac{1}{(\frac{\kappa_a+\kappa_b}{2} - \varepsilon_2)(\frac{\kappa_a}{2} + \varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2} + \varepsilon_2)(\frac{\kappa_a}{2} + 3\varepsilon_2)}] \tag{2.2.12}$$

and

$$d = \frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{\frac{\kappa_a}{2} + \varepsilon_2}]^2 + \frac{\varepsilon_2^2}{2}[\frac{1}{(\frac{\kappa_a+\kappa_b}{2} - \varepsilon_2)(\frac{\kappa_a}{2} + \varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2} + \varepsilon_2)(\frac{\kappa_a}{2} + 3\varepsilon_2)}]. \tag{2.2.13}$$

Substituting Eq.(2.2.9) into (2.2.1), we find

$$Q(\beta^*, \beta, t) = \frac{1}{\pi^2}\int d^2z exp[-cz^*z - \beta^*z + \beta z^* + \frac{d}{2}(z^2 + z^{*2})]. \tag{2.2.14}$$

Performing the integration, over the variable z the Q function for the second harmonic light takes the form

$$Q(\beta^*, \beta, t) = \frac{1}{\pi}\sqrt{l^2 - m^2} exp[-l\beta^*\beta + \frac{m}{2}(\beta^2 + \beta^{*2})], \tag{2.2.15}$$

where

$$l = \frac{c}{c^2 - d^2}, \tag{2.2.16}$$

$$m = \frac{d}{c^2 - d^2}. \tag{2.2.17}$$

2.3 Photon statistics

We now proceed to calculate the mean and variance of the photon number of the second harmonic light using the Q function.

2.3.1 The mean photon number of second harmonic light

The mean photon number for the second harmonic light is defined as

$$\bar{n} = \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle. \tag{2.3.1}$$

The mean photon number of the second harmonic light interms of Q function can be written as

$$\bar{n} = \int d^2\beta Q(\beta^*, \beta, t)(\beta\beta^* - 1). \tag{2.3.2}$$

Upon substituting Eq.(2.2.15) into Eq.(2.3.2), we see that

$$\bar{n} = \frac{1}{\pi}\sqrt{l^2 - m^2}\int d^2_\beta exp(-l\beta^*\beta + \frac{m}{2}(\beta^2 + \beta^{*2})(\beta\beta^* - 1), \tag{2.3.3}$$

$$\bar{n} = \sqrt{l^2 - m^2}\frac{d^2}{dxdy}\int \frac{d^2\beta}{\pi} exp(-l\beta^*\beta + x\beta + y\beta^* \\ + \frac{m}{2}(\beta^2 + \beta^{*2}))|_{x=y=0} - 1,$$

Then carrying out the differentiation, it becomes

$$\bar{n} = \frac{l}{l^2 - m^2} - 1. \tag{2.3.4}$$

It can be rewritten as

$$\bar{n} = c - 1 \tag{2.3.5}$$

and this leads to

$$\begin{aligned}\bar{n} &= \beta_0^2 + \frac{\varepsilon_2(\lambda_{1-} - a_-)^2(\lambda_{2-} - a_-)^2}{4\varepsilon_1^2(\lambda_{1-} - \lambda_{2-})^2}[\frac{1}{\lambda_{2-}}(1 - e^{-\lambda_{2-}t}) \\ &+ \frac{1}{\lambda_{1-}}(1 - e^{-\lambda_{1-}t)} - \frac{4}{\lambda_{1-} + \lambda_{2-}}(1 - e^{-\lambda_{1-}+\lambda_{2-})t})] \\ &- \frac{\varepsilon_2(\lambda_{1+} - a_+)^2(\lambda_{2+} - a_+)^2}{4\varepsilon_1^2(\lambda_{1+} - \lambda_{2+})^2}[\frac{1}{\lambda_{2+}}(1 - e^{-\lambda_{2+}t}) \\ &+ \frac{1}{\lambda_{1+}}(1 - e^{-\lambda_{1+}t)} - \frac{4}{\lambda_{1+} + \lambda_{2+}}(1 - e^{-\lambda_{1+}+\lambda_{2+})t})]\end{aligned} \tag{2.3.6}$$

Using Eqs.(2.1.63) and (2.1.64) along with (2.1.55) and (2.1.56) the mean photon number of second harmonic light at steady state, have the form

$$\bar{n}_{ss} = \frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{\frac{\kappa_a}{2} + \varepsilon_2}]^2 + \frac{\varepsilon_2^2}{2}[\frac{1}{(\frac{\kappa_a+\kappa_b}{2} - \varepsilon_2)(\frac{\kappa_a}{2} + \varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2} + \varepsilon_2)(\frac{\kappa_a}{2} + 3\varepsilon_2)}]. \tag{2.3.7}$$

By setting $\varepsilon = 0$, (which means that, when the driving light is switched off from the system) we get

$$\bar{n}_{ss} = \frac{\varepsilon_2^2}{2}[\frac{1}{(\frac{\kappa_a+\kappa_b}{2} - \varepsilon_2)(\frac{\kappa_a}{2} + \varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2} + \varepsilon_2)(\frac{\kappa_a}{2} + 3\varepsilon_2)}], \tag{2.3.8}$$

and for $\varepsilon_2 = 0$

$$\bar{n}_{ss} = 0, \tag{2.3.9}$$

meaning that the system is with out the nonlinear crystal.

2.3.2 The variance of the photon number for second harmonic light

We next proceed to obtain the variance of the photon number for second harmonic light employing the Q function. The variance of the photon number for second harmonic light is defined as

$$(\Delta n)^2 = \langle(\hat{b}^\dagger(t)\hat{b}(t))^2\rangle - \langle(\hat{b}^\dagger(t)\hat{b}(t))\rangle^2 \tag{2.3.10}$$

and again

$$\langle(\hat{b}^\dagger(t)\hat{b}(t))^2\rangle = \langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle - 3\langle(\hat{b}(t))\hat{b}^\dagger(t)\rangle + 1. \tag{2.3.11}$$

We know that

$$\langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle = \int d^2\beta Q(\beta^*, \beta, t)\beta^2\beta^{*2}. \tag{2.3.12}$$

Substituting Eq.(2.2.15) into Eq.(2.3.12), we see that

$$\langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle = \frac{1}{\pi}\sqrt{l^2 - m^2}\int d^2_\beta exp(-l\beta^*\beta + \frac{m}{2}(\beta^2 + \beta^{*2})\beta^2\beta^{*2} \tag{2.3.13}$$

Upon carrying out the integration one can readily find

$$\langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle = \sqrt{l^2 - m^2}\frac{\partial^2}{\partial l^2}\int \frac{d^2\beta}{\pi}(exp\bigg(- l\beta^*\beta + \frac{m}{2}(\beta^2 + \beta^{*2})\bigg))$$

$$\langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle = \sqrt{l^2 - m^2}\frac{\partial^2}{\partial l^2}\frac{1}{\sqrt{l^2 - m^2}}.$$

Upon carrying out the differentiation, we readily find

$$\langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle = \frac{2l^2 + m^2}{(l^2 + m^2)^2}.$$

On account of Eqs.(2.2.16) and (2.2.17) along with (2.2.12) and (2.2.13), we get

$$\langle(\hat{b}^2(t)\hat{b}^{\dagger 2}(t))\rangle = 2c^2 + d^2. \tag{2.3.14}$$

Introducing Eq.(2.3.14) togther with Eq.(2.3.5) into Eq.(2.3.11), we see that

$$\langle(\hat{b}^\dagger(t)\hat{b}(t))^2\rangle = 2c^2 + d^2 - 3c + 1. \tag{2.3.15}$$

Finally, using Eq.(2.3.15) togther with Eq.(2.3.5), the variance of photon number becomes

$$(\Delta n)^2 = 2c^2 + d^2 - 3c + 1 - (c-1)^2 \tag{2.3.16}$$

and this leads to

$$(\Delta n)^2 = c^2 + d^2 - c. \tag{2.3.17}$$

Finally, using Eqs.(2.2.12) and (2.2.13), the variance of photon number have the form

$$
\begin{aligned}
(\Delta n)^2 &= \frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{\frac{\kappa_a}{2}+\varepsilon_2}]^2 + \frac{\varepsilon_2^2}{2}[\frac{1}{(\frac{\kappa_a+\kappa_b}{2}-\varepsilon_2)(\frac{\kappa_a}{2}+\varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+3\varepsilon_2)}] \\
&\quad +2[\frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{\frac{\kappa_a}{2}+\varepsilon_2}]^2 - \frac{\varepsilon_2^2}{2(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+3\varepsilon_2)}]^2 \\
&\quad +2[\frac{\varepsilon_2^2}{2(\frac{\kappa_a+\kappa_b}{2}-\varepsilon_2)(\frac{\kappa_a}{2}+\varepsilon_2)}]^2. \qquad (2.3.18)
\end{aligned}
$$

Then by setting $\varepsilon = 0$, (which means that, when the driving light is switched off from the system) we see

$$
\begin{aligned}
(\Delta n)^2 &= \frac{\varepsilon_2^2}{2}\left[\frac{1}{(\frac{\kappa_a+\kappa_b}{2}-\varepsilon_2)(\frac{\kappa_a}{2}+\varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+3\varepsilon_2)}\right] \\
&\quad -\varepsilon_2^2\left[\frac{1}{(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+3\varepsilon_2)} + [\frac{1}{(\frac{\kappa_a+\kappa_b}{2}-\varepsilon_2)(\frac{\kappa_a}{2}+\varepsilon_2)}\right]^2 \qquad (2.3.19)
\end{aligned}
$$

and by setting $\varepsilon_2 = 0$

$$(\Delta n)^2 = 0, \qquad (2.3.20)$$

meaning that the system is with out the nonlinear crystal.

2.4 Quadrature fluctuation

We next proceed to obtain the quadrature variance and quadrature squeezing for second harmonic light.

2.4.1 Quadrature variance

The the plus and minus quadrature operators of the second harmonic light are defined as

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b} \qquad (2.4.1)$$

and

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}). \qquad (2.4.2)$$

The quadrature variance interms of the quadrature operators can be written as

$$(\Delta b_{\pm})^2 = 1 \pm \langle : (\hat{b}_{\pm}(t), \hat{b}_{\pm}(t)) : \rangle. \tag{2.4.3}$$

Note the :: , $\langle$ and $\rangle$ in Eq.(2.4.3) indicates normal ordering.
This leads to

$$(\Delta b_{\pm})^2 = 1 + 2\langle(\hat{b}^{\dagger}\hat{b})\rangle \pm (\langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b^2}\rangle) \mp (\langle\hat{b}^{\dagger 2}\rangle \pm \langle\hat{b^2}\rangle)^2. \tag{2.4.4}$$

The expectation value of $\hat{b}^2$ using the Q function can be put in the form

$$\langle\hat{b}^{\dagger 2}\rangle = \langle\hat{b}^2\rangle = \int d^2\beta Q(\beta^*, \beta, t)\beta^{*2} \tag{2.4.5}$$

which the use of Eq.(2.2.15) becomes

$$\langle\hat{b}^{\dagger 2}\rangle = \langle\hat{b}^2\rangle = \frac{1}{\pi}\sqrt{l^2 - m^2}\int d^2_{\beta} exp(-l\beta^*\beta + \frac{m}{2}(\beta^2 + \beta^{*2})\beta^2, \tag{2.4.6}$$

and can be written as

$$\langle\hat{b}^{\dagger 2}\rangle = (l^2 - m^2)^{\frac{1}{2}}\frac{\partial^2}{\partial p^2}\int \frac{1}{\pi} d^2\beta exp\Bigg(-l\beta^*\beta + p\beta + \frac{m}{2}(\beta^2 + \beta^{*2})\Bigg)|_{p=0}. \tag{2.4.7}$$

We now proceed to evaluate the integral employing standard integration, we see

$$\langle\hat{b}^{\dagger 2}\rangle = \frac{(l^2 - m^2)^{\frac{1}{2}}}{(l^2 - m^2)^{\frac{1}{2}}}\frac{\partial}{\partial p}(\frac{lp}{l^2 - m^2} exp\Bigg(\frac{lp}{2(l^2 - m^2)}\Bigg))|_{p=0},$$

differentiating this equation we get

$$\langle\hat{b}^{\dagger 2}\rangle = \langle\hat{b}^2\rangle = \frac{m}{(m^2 - l^2)^2}, \tag{2.4.8}$$

on account of Eqs.(2.2.16) and (2.2.17) along with (2.2.11), one can easily find

$$\langle\hat{b}^{\dagger 2}\rangle = \langle\hat{b}^2\rangle = d. \tag{2.4.9}$$

On the other hand, the expectation value of $\hat{b}$ in terms of Q function can be given as

$$\langle\hat{b}^{\dagger}\rangle = \langle\hat{b}\rangle = \frac{1}{\pi}\sqrt{l^2 - m^2}\int d^2_{\beta} exp(-l\beta^*\beta + \frac{m}{2}(\beta^2 + \beta^{*2})\beta. \tag{2.4.10}$$

Performing the integration, we get

$$\langle\hat{b}\rangle = (l^2 - m^2)^{\frac{1}{2}}\frac{\partial}{\partial a}\int \frac{1}{\pi} d^2\beta exp\Bigg(-l\beta^*\beta + a\beta + \frac{m}{2}(\beta^2 + \beta^{*2})\Bigg)|_{a=0}, \tag{2.4.11}$$

it follows

$$\langle\hat{b}\rangle = \frac{(l^2-m^2)^{\frac{1}{2}}}{(l^2-m^2)^{\frac{1}{2}}}(\frac{la}{l^2-m^2}exp\left(\frac{la}{2(l^2-m^2)}\right))|_{a=0}, \tag{2.4.12}$$

so that applying the condition $a = 0$ results

$$\langle\hat{b}\rangle = 0. \tag{2.4.13}$$

Now substituting Eqs. (2.1.102), (2.3.5) and (2.4.9) into Eq.(2.4.4), the quadrature variance of the second harmonic light become

$$(\Delta b_{\pm})^2 = 1 \pm 4\langle F_{\pm}^2\rangle. \tag{2.4.14}$$

$$(\Delta b_{\pm})^2 = 1 \mp (c-1) - d.$$

Finally, the quadrature variance can be rewritten as

$$(\Delta b_{\pm})^2 = 1 \mp \frac{\varepsilon_2(\lambda_{1\pm}-a_{\pm})^2(\lambda_{2\pm}-a_{\pm})^2}{\varepsilon_1^2(\lambda_{1\pm}-\lambda_{2\pm})^2}\left[\frac{1}{\lambda_{2\pm}}(1-e^{-\lambda_{2\pm}t}) + \frac{1}{\lambda_{1\pm}}(1-e^{-\lambda_{1\pm}t)} - \frac{4}{\lambda_{1\pm}+\lambda_{2\pm}}(1-e^{-\lambda_{1\pm}+\lambda_{2\pm})t})\right] \tag{2.4.15}$$

At steady state, it become

$$(\Delta b_{\pm})^2_{ss} = 1 \mp \frac{2\varepsilon_2^2}{(\frac{\kappa_a+\kappa_b}{2} \pm \varepsilon_2)(\frac{\kappa_a}{2} + (2\pm 1)\varepsilon_2)}. \tag{2.4.16}$$

It is easy to see that the second harmonic light is in squeezed state and the squeezing occurs in the plus quadrature.

2.4.2 Quadrature squeezing

The quadrature squeezing for second harmonic light can be defined as[5]

$$S_{+} = 1 - (\Delta b_{+})^2. \tag{2.4.17}$$

Then applying Eq.(2.4.16) into Eq.(2.4.17), we have

$$S_{+} = \frac{2\varepsilon_2^2}{(\frac{\kappa_a+\kappa_b}{2} + \varepsilon_2)(\frac{\kappa_a}{2} + 3\varepsilon_2)}. \tag{2.4.18}$$

For the value of $\kappa_a = \kappa_b = \kappa$, the graph of S_{+} vs ε_2 looks like as shown below and we can easly see that, the maximum degree of squeezing for second harmonic light is about 66.7% [16]

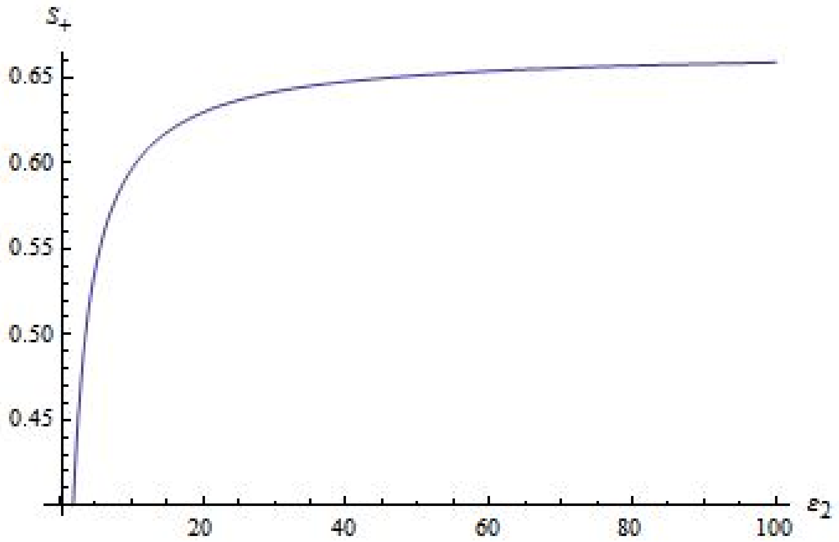

Figure 2.2: a plot of S_+ Vs ε_2

Chapter 3

Twin One-mode Subharmonic Generator

Sub-harmonic generator is one of the most interesting and well characterized optical devices in quantum optics. In this device, a pump photon interacts with a nonlinear crystal in side a cavity and is down-converted into two highly correlated photons. If these photons have the same frequency, the device is called a one-mode subharmonic generator, otherwise it is called a two-mode subharmonic generator [17].

3.1 Master equation

We first obtain the master equation, for the twin light beams produced by subharmonic generator. The process of subharmonic generator leading to the creation of twin light modes with the same frequencies can be described by the Hamiltonian [17]

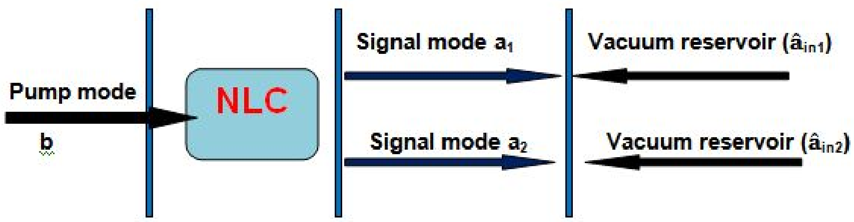

Figure 3.1: Schematic Diagram for twin one-mode subharmonic light

$$\hat{H} = i\mu(\hat{b}^\dagger - \hat{b}) + i\lambda(\hat{b}^\dagger \hat{a}_1 \hat{a}_2 - \hat{b}\hat{a}_1^\dagger \hat{a}_2^\dagger), \tag{3.1.1}$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the light beams, $\hat{b}$ is the annihilation operator for the pump mode, λ is the coupling constant and μ is proportional to the amplitude of the coherent deriving pump mode. We may refer to a Hamiltonian of the form described by Eq.(3.1.1) as first order Hamiltonian [18]. We note that the master equation for a cavity mode coupled to any reservoir can be written as [17]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i[\hat{H}_s, \hat{\rho}] - i[\langle \hat{H}_{SR}\rangle_R, \hat{\rho}(t-h)] - h[\langle \hat{H}_{SR}\rangle_R, \hat{H}_s, \hat{\rho}] \\ &\quad -hTr_R[\hat{H}_{SR}, [\hat{H}_{SR}, \hat{\rho}\hat{R}]], \end{aligned} \tag{3.1.2}$$

where $\hat{H}_{SR}$ is the Hamiltonian describing the interaction of the system with reservior with density operator $\hat{R}$, Tr_R is trace over reservoir variables and $\hat{H}_s$ is given by Eq. (3.1.1). The Hamiltonian describing the interaction of twin light beams coupled to any reservoir is expressiable as [17]

$$\hat{H}_{SR} = i\lambda'(\hat{a}_1^\dagger \hat{a}_{1in} - \hat{a}_{1in}^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_{2in} - \hat{a}_{2in}^\dagger \hat{a}_2), \tag{3.1.3}$$

in whichλ' is coupling constant.
In fact, the expectation value of Eq.(3.1.3), can be written as

$$\langle \hat{H}_{SR}\rangle = i\lambda' Tr_R[\hat{R}(\hat{a}_1^\dagger \hat{a}_{1in} - \hat{a}_{1in}^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_{2in} - \hat{a}_{2in}^\dagger \hat{a}_2)]. \tag{3.1.4}$$

Now applying the density operator of the two mode vacuum reservoir $\hat{R} = |0_1, 0_2\rangle\langle 0_2, 0_1|$ and the trace operator we obtain

$$\begin{aligned}\langle \hat{H}_{SR}\rangle_R &= i\lambda'[(\hat{a}_1^\dagger \langle 0_2, 0_1|\hat{a}_{1in}|0_1, 0_2\rangle - \langle 0_2, 0_1|\hat{a}_{1in}^\dagger|0_1, 0_2\rangle \hat{a}_1 \\ &\quad + \langle 0_2, 0_1|\hat{a}_2^\dagger \hat{a}_{2in}|0_1, 0_2\rangle - \langle 0_2, 0_1|\hat{a}_{2in}^\dagger|0_1, 0_2\rangle \hat{a}_2)]\end{aligned} \tag{3.1.5}$$

Thus with the aid of the eigenvalue equation for the number state

$$\hat{a}_1|n_1, n_2\rangle = \sqrt{n_1}|n_1 - 1, n_2\rangle \tag{3.1.6}$$

and

$$\hat{a}_2|n_1, n_2\rangle = \sqrt{n_2}|n_1, n_2 - 1\rangle, \tag{3.1.7}$$

one can see that

$$\langle \hat{H}_{SR}\rangle_R = 0. \tag{3.1.8}$$

In view of this result, Eq.(3.1.2) becomes

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_s,\hat{\rho}] - hTr_R(\hat{H}_{SR}^2\hat{\rho}\hat{R} - \hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) + hTr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR} - \hat{\rho}\hat{R}\hat{H}_{SR}^2) \tag{3.1.9}$$

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_s,\hat{\rho}] - h\langle\hat{H}_{SR}^2\rangle_R\hat{\rho} - h\hat{\rho}\langle\hat{H}_{SR}^2\rangle_R + 2h\langle\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}\rangle_R \tag{3.1.10}$$

Moreover, in view of Eq.(3.1.3), we can readly obtain

$$\langle\hat{H}_{SR}^2\rangle = \lambda'^2\Bigg[\hat{a}_1^\dagger\hat{a}_1\langle\hat{a}_{1in}^\dagger\hat{a}_{1in}\rangle + \hat{a}_1^\dagger\hat{a}_1 + \langle\hat{a}_{1in}^\dagger\hat{a}_{1in}\rangle\hat{a}_1\hat{a}_1^\dagger + \hat{a}_2^\dagger\hat{a}_2\langle\hat{a}_{2in}^\dagger\hat{a}_{2in}\rangle + \hat{a}_2^\dagger\hat{a}_2 + \langle\hat{a}_{2in}^\dagger\hat{a}_{2in}\rangle\hat{a}_2\hat{a}_2^\dagger\Bigg], \tag{3.1.11}$$

where

$$[\hat{a}_{1in},\hat{a}_{1in}^\dagger] = [\hat{a}_{2in},\hat{a}_{2in}^\dagger] = 1, \tag{3.1.12}$$

has been used.
Now considering the mean photon number of a vacuum reservior to be zero, one can easily obtain

$$\langle\hat{H}_{SR}^2\rangle = \lambda'^2(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2). \tag{3.1.13}$$

Moreover, in view of Eq.(3.1.3), one can easily prove that

$$\begin{aligned} Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = -\lambda' Tr_R\Bigg[&\hat{a}_1^\dagger\hat{\rho}\hat{a}_1^\dagger\hat{a}_{1in}\hat{R}\hat{a}_{1in} \\ &-\hat{a}_1^\dagger\hat{\rho}\hat{a}_1\hat{a}_{1in}\hat{R}\hat{a}_{1in}^\dagger + \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho}\hat{a}_{1in}\hat{R}\hat{a}_{2in} \\ &-\hat{a}_1^\dagger\hat{\rho}\hat{a}_2\hat{a}_{1in}\hat{R}\hat{a}_{2in}^\dagger - \hat{a}_1\hat{\rho}\hat{a}_1^\dagger\hat{a}_{1in}\hat{R}\hat{a}_{1in} \\ &+\hat{a}_{1in}^\dagger\hat{R}\hat{a}_{1in}^\dagger\hat{a}_1\hat{\rho}\hat{a}_1 - \hat{a}_{1in}^\dagger\hat{R}\hat{a}_{2in}\hat{a}_1\hat{\rho}\hat{a}_2^\dagger \\ &+\hat{a}_{1in}^\dagger\hat{R}\hat{a}_{2in}\hat{a}_1\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{\rho}\hat{a}_1^\dagger\hat{a}_{2in}\hat{R}\hat{a}_{1in} \\ &-\hat{a}_2^\dagger\hat{\rho}\hat{a}_1\hat{a}_{2in}\hat{R}\hat{a}_{1in}^\dagger + \hat{a}_2^\dagger\hat{\rho}\hat{a}_2^\dagger\hat{a}_{2in}\hat{R}\hat{a}_{2in} \\ &-\hat{a}_2^\dagger\hat{\rho}\hat{a}_2\hat{a}_{2in}\hat{R}\hat{a}_{2in}^\dagger - \hat{a}_2\hat{\rho}\hat{a}_1^\dagger\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{1in} \\ &+\hat{a}_2\hat{\rho}\hat{a}_1\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{1in}^\dagger - \hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{2in} \\ &+\hat{a}_2\hat{\rho}\hat{a}_2\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{2in}^\dagger\Bigg]. \end{aligned} \tag{3.1.14}$$

Then it follows

$$Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = \lambda'^2 Tr_R[-\hat{a}_1^\dagger\hat{\rho}\hat{a}_1\hat{a}_{1in}\hat{R}\hat{a}_{1in}^\dagger - \hat{a}_1\hat{\rho}\hat{a}_1^\dagger\hat{a}_{1in}\hat{R}\hat{a}_{1in} \\ -\hat{a}_2^\dagger\hat{\rho}\hat{a}_2\hat{a}_{2in}\hat{R}\hat{a}_{2in}^\dagger - \hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{2in}]. \quad (3.1.15)$$

Now the mean photon number of the vacuum reservior to be zero, one can easily check that

$$Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = \lambda'^2(\hat{a}_1\hat{\rho}\hat{a}_1^\dagger + \hat{a}_2\hat{\rho}\hat{a}_2^\dagger). \quad (3.1.16)$$

In view of Eq.(3.1.1), it is easy to get

$$[\hat{H}_S, \hat{\rho}] = i[\mu(\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{b} - \hat{b}\hat{\rho}) \\ +\lambda(\hat{b}^\dagger\hat{a}_1\hat{a}_2\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}_1\hat{a}_2 + \hat{\rho}\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho}). \quad (3.1.17)$$

Finally, applying Eqs.(3.1.9), (3.1.13), (3.1.16) and (3.1.17), into (3.1.10), one can easily finds

$$\frac{d}{dt}\hat{\rho} = \mu(\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{b} + \hat{b}\hat{\rho}) + \lambda(\hat{b}^\dagger\hat{a}_1\hat{a}_2\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}_1\hat{a}_2 + \hat{\rho}\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho}) \\ +\frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1) \\ +\frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2), \quad (3.1.18)$$

which is the master equation for twin light beams and $\kappa_1 = 2h\lambda'^2$ is the cavity damping constant for light mode $\hat{a}_1$ and $\kappa_2 = 2h\lambda'^2$ is the cavity damping constant for light mode $\hat{a}_2$.

3.2 Operator dynamics

We consider here the case in which the twin light modes and the pump mode are in a cavity coupled to a vacuum reservior via a single-port mirror. Using the relation

$$\frac{d}{dt}\langle\hat{a}_1\rangle = Tr(\frac{d}{dt}\hat{\rho}\hat{a}_1), \quad (3.2.1)$$

one can easily check that

$$\frac{d}{dt}\langle\hat{a}_1\rangle = -\frac{\kappa_1}{2}\langle\hat{a}_1\rangle - \lambda\langle\hat{b}\hat{a}_2^\dagger\rangle, \quad (3.2.2)$$

$$\frac{d}{dt}\langle\hat{a}_2\rangle = -\frac{\kappa_2}{2}\langle\hat{a}_2\rangle - \lambda\langle\hat{b}\hat{a}_1^\dagger\rangle, \quad (3.2.3)$$

$$\frac{d}{dt}\langle\hat{a}_1^\dagger\hat{a}_1\rangle = -\kappa_1\langle\hat{a}_1^\dagger\hat{a}_1\rangle - \lambda\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \lambda\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{3.2.4}$$

$$\frac{d}{dt}\langle\hat{a}_2^\dagger\hat{a}_2\rangle = -\kappa_2\langle\hat{a}_2^\dagger\hat{a}_2\rangle - \lambda\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \lambda\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle \tag{3.2.5}$$

and

$$\frac{d}{dt}\langle\hat{a}_1\hat{a}_2\rangle = -\frac{1}{2}(\kappa_1+\kappa_2)\langle\hat{a}_1\hat{a}_2\rangle - \lambda\langle\hat{b}^\dagger\hat{a}_1\hat{a}_1\rangle - \lambda\langle\hat{b}\hat{a}_2^\dagger\hat{a}_2^\dagger\rangle - \lambda\langle\hat{b}\rangle. \tag{3.2.6}$$

On taking $\kappa_1 = \kappa_2 = \kappa$, the steady state solution of Eqs.(3.2.2), (3.2.3), (3.2.4), (3.2.5) and (3.2.6), is found to be

$$\langle\hat{a}_1\rangle = -\frac{2\lambda}{\kappa}\langle\hat{b}\hat{a}_2^\dagger\rangle, \tag{3.2.7}$$

$$\langle\hat{a}_2\rangle = -\frac{2\lambda}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\rangle, \tag{3.2.8}$$

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = -\frac{\lambda}{\kappa}\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \frac{\lambda}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{3.2.9}$$

$$\langle\hat{a}_2^\dagger\hat{a}_2\rangle = -\frac{\lambda}{\kappa}\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \frac{\lambda}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{3.2.10}$$

and

$$\langle\hat{a}_1\hat{a}_2\rangle = -\frac{\lambda}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_1\rangle - \frac{\lambda}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_2\rangle - \frac{\lambda}{\kappa}\langle\hat{b}\rangle. \tag{3.2.11}$$

Moreover, upon dropping the noise operator and in the absence of subharmonic generation ($\lambda = 0$), one can easily write the quantum Langevin equation for the operator $\hat{b}$ as

$$\frac{d}{dt}\hat{b} = -\frac{1}{2}\kappa\hat{b} + \mu, \tag{3.2.12}$$

where κ is the cavity damping constant. The steady-state solution for Eq.(3.2.12) is

$$\hat{b} = \frac{2\mu}{k}. \tag{3.2.13}$$

Now upon substituting Eq.(3.2.13) into Eqs.(3.2.7), (3.2.8), (3.2.9), (3.2.10) and (3.2.11), one can easily arrive at

$$\langle\hat{a}_1\rangle = -\frac{2\Gamma}{\kappa}\langle\hat{a}_2^\dagger\rangle, \tag{3.2.14}$$

$$\langle \hat{a}_2 \rangle = -\frac{2\Gamma}{\kappa}\langle \hat{a}_1^\dagger \rangle, \tag{3.2.15}$$

$$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = -\frac{\Gamma}{\kappa}\langle \hat{a}_1 \hat{a}_2 \rangle - \frac{\Gamma}{\kappa}\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle, \tag{3.2.16}$$

$$\langle \hat{a}_2^\dagger \hat{a}_2 \rangle = -\frac{\Gamma}{\kappa}\langle \hat{a}_1 \hat{a}_2 \rangle - \frac{\Gamma}{\kappa}\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \tag{3.2.17}$$

and

$$\langle \hat{a}_1 \hat{a}_2 \rangle = -\frac{\Gamma}{\kappa}\langle \hat{a}_1^\dagger \hat{a}_1 \rangle - \frac{\Gamma}{\kappa}\langle \hat{a}_2^\dagger \hat{a}_2 \rangle - \frac{\Gamma}{\kappa}, \tag{3.2.18}$$

where

$$\Gamma = \frac{2\mu\lambda}{k}. \tag{3.2.19}$$

In view of Eq.(3.2.14) and Eq.(3.2.15), we see that

$$\langle \hat{a}_1 \rangle = \langle \hat{a}_2 \rangle = 0. \tag{3.2.20}$$

Moreover, using Eqs.(3.2.16), (3.2.17) and (3.2.18), one can easily find

$$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = \frac{2\Gamma^2}{(\kappa^2 - 4\Gamma^2)}. \tag{3.2.21}$$

$$\langle \hat{a}_2^\dagger \hat{a}_2 \rangle = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \tag{3.2.22}$$

and

$$\langle \hat{a}_1 \hat{a}_2 \rangle = -\frac{\kappa\Gamma}{(\kappa^2 - 4\Gamma^2)}. \tag{3.2.23}$$

It can also be readly verified that

$$\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = \langle \hat{a}_1^\dagger \hat{a}_2 \rangle = \langle \hat{a}_2^\dagger \hat{a}_1 \rangle = 0. \tag{3.2.24}$$

Now we take

$$\hat{a} = \hat{a}_1 + \hat{a}_2, \tag{3.2.25}$$

to be the annihilation operator for the superposition of light mode $\hat{a}_1$ and $\hat{a}_2$ produced by the subharmonic generator coupled to a vacuum reservoir, one can then easily check that

$$[\hat{a}, \hat{a}^\dagger] = 2, \tag{3.2.26}$$

where

$$[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1 \tag{3.2.27}$$

and the remaining constitutes to be zero.

We realize that the superposition of the two light beams, with the same frequencies, constitutes a two-mode light. We wish to call the superposed light beams with the same frequency the signal-signal light beams [17]. Now the result described by Eqs. (3.2.20) and (3.2.24) are valid for signal-signal light beams.

3.3 The Q function

The Q function that describes the twin light beams is expressiable as

$$Q(\alpha_1, \alpha_2, t) = \frac{1}{\pi^4}\int d^2z d^2\eta \phi_a(z, \eta, t) exp[z^*\alpha_1 - z\alpha_1^* + \eta^*\alpha_2 - \eta\alpha_2^*], \tag{3.3.1}$$

where the anti-normally ordered characteristic function $\phi_a(z, \eta, t)$ is defined in the Heisenberg picture by

$$\phi_a(z, \eta, t) = Tr(\hat{\rho}(0)e^{-z^*\hat{a}_1(t)}e^{z\hat{a}_1^\dagger(t)}e^{-\eta^*\hat{a}_2(t)}e^{\eta\hat{a}_2^\dagger(t)}). \tag{3.3.2}$$

Applying the identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \tag{3.3.3}$$

Eq.(3.3.2) can written as

$$\phi_a(z, \eta, t) = e^{-\frac{1}{2}(z^*z+\eta^*\eta)}Tr\left(\hat{\rho}(0)e^{-z^*\hat{a}_1}e^{z\hat{a}_1^\dagger}e^{-\eta^*\hat{a}_2}e^{\eta\hat{a}_2^\dagger}\right).$$

Then this function can be written in terms of c-number variables associated with the normal order as

$$\phi_a(z, \eta, t) = e^{-\frac{1}{2}(z^*z+\eta^*\eta)}\left(exp[z\alpha_1^* - z\alpha_1 + \eta\alpha_2^* - \eta^*\alpha_2]\right).$$

and taking into account the fact that $\hat{a}_1$ and $\hat{a}_2$ are the Gaussian variables with zero mean, Eq.(3.3.2) can be put in the form

$$\phi_a(z, \eta, t) = exp[-p(z^*z + \eta^*\eta) - q(z^*\eta^* + z\eta)], \tag{3.3.4}$$

where

$$p = 1 - \frac{\Gamma}{2(\kappa + 2\Gamma)}(1 - e^{-(\kappa+2\Gamma)t}) + \frac{\Gamma}{2(\kappa - 2\Gamma)}(1 - e^{-(\kappa-2\Gamma)t}), \tag{3.3.5}$$

$$q = \frac{\Gamma}{2(\kappa + 2\Gamma)}(1 - e^{-(\kappa+2\Gamma)t}) + \frac{\Gamma}{2(\kappa - 2\Gamma)}(1 - e^{-(\kappa-2\Gamma)t}). \tag{3.3.6}$$

Now introdusing Eq.(3.3.4) into Eq.(3.3.1) and carrying out the integration, using the standared integration

$$\int \frac{d_z^2}{\pi} exp[-az^*z + bz + cz^* + Az^2 + Bz^{*2}] = [\frac{1}{a^2 - 4AB}]^{\frac{1}{2}}[\frac{abb + Ac^2 + Bb^2}{a^2 - 4AB}], a > 0,$$

we get the Q function as

$$Q(\alpha_1, \alpha_2, t) = \frac{1}{\pi^2}(u^2 - v^2)exp[-u(\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2) - v(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)], \tag{3.3.7}$$

in which

$$u = \frac{p}{(p^2 - q^2)} \tag{3.3.8}$$

and

$$v = \frac{q}{(p^2 - q^2)} \tag{3.3.9}$$

3.4 Photon statistics

It would be helpful to classify the photon statistics of light modes based on the relation between the variance and the mean of the photon number. Hence the photon statistics of a light mode for which $(\Delta n)^2 = \bar{n}$ is referred to as Poissonian and the photon statistics of a light mode for which $(\Delta n)^2 > \bar{n}$ is called super- Poissonian. Otherwise the photon statistics is said to be sub-Poissonian.[17]

3.4.1 The mean photon number

We define the mean photon number of twin one-mode subharmonic light by

$$\bar{n} = \langle \hat{a}^{\dagger} \hat{a} \rangle, \tag{3.4.1}$$

where

$$\hat{a}^{\dagger} = \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger} \tag{3.4.2}$$

and

$$\hat{a} = \hat{a}_1 + \hat{a}_2. \tag{3.4.3}$$

Then using Eq.(3.4.2) and (3.4.3), Eq.(3.4.1) can be rewritten as

$$\bar{n} = \langle (\hat{a}_1^{\dagger} + \hat{a}_2^{\dagger})(\hat{a}_1 + \hat{a}_2) \rangle \tag{3.4.4}$$

which on account of Eq.(3.2.24), becomes

$$\bar{n} = \langle \hat{a}_1^{\dagger} \hat{a}_1 \rangle + \langle \hat{a}_2^{\dagger} \hat{a}_2 \rangle, \tag{3.4.5}$$

so that in view of Eqs.(3.2.21) and (3.2.22), the mean photon number can put as

$$\bar{n} = \frac{2\Gamma^2}{\kappa^2 - 4\Gamma^2} + \frac{2\Gamma^2}{\kappa^2 - 4\Gamma^2}. \tag{3.4.6}$$

This result shows that the mean photon number of the signal-signal mode is the sum of the mean photon number of the signal light beams.

Then the mean photon number of the twin one-mode subharmonic light beams can have the form

$$\bar{n} = \frac{4\Gamma^2}{\kappa^2 - 4\Gamma^2}. \tag{3.4.7}$$

Thus the result in Eq.(3.4.7) represents the mean photon number of the signal-signal light mode.

In addition, we note that the equation of evolution of the mean photon number for the pump mode can be written as

$$\frac{d}{dt}\langle \hat{b}^{\dagger} \hat{b} \rangle = -i\langle [\hat{b}^{\dagger} \hat{b}, \hat{H}] \rangle + \frac{\kappa}{2} Tr[(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}\hat{b})\hat{b}^{\dagger}\hat{b}]. \tag{3.4.8}$$

Then using Eq.(3.1.1) and the fact that

$$-i\langle [\hat{b}^{\dagger}\hat{b}, \hat{H}] \rangle = \mu \langle \hat{b} \rangle + \mu \langle \hat{b}^{\dagger} \rangle + \lambda \langle \hat{b}^{\dagger} \hat{a}_1 \hat{a}_2 \rangle + \lambda \langle \hat{b} \hat{a}_1^{\dagger} \hat{a}_2^{\dagger} \rangle \tag{3.4.9}$$

and

$$\frac{\kappa}{2}Tr[(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}\hat{b})\hat{b}^{\dagger}\hat{b}] = -\kappa\langle\hat{b}^{\dagger}\hat{b}\rangle \tag{3.4.10}$$

we readly obtain

$$\frac{d}{dt}\langle\hat{b}^{\dagger}\hat{b}\rangle = -\kappa\langle\hat{b}^{\dagger}\hat{b}\rangle + \mu\langle\hat{b}\rangle + \mu\langle\hat{b}^{\dagger}\rangle + \lambda\langle\hat{b}^{\dagger}\hat{a}_1\hat{a}_2\rangle + \lambda\langle\hat{b}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\rangle. \tag{3.4.11}$$

Now the steady state solution of Eq.(3.4.11) is

$$\langle\hat{b}^{\dagger}\hat{b}\rangle = \frac{\mu}{\kappa}\hat{b} + \frac{\mu}{\kappa}\hat{b}^{\dagger} + \frac{\lambda}{\kappa}\langle\hat{b}^{\dagger}\hat{a}_1\hat{a}_2\rangle + \frac{\lambda}{\kappa}\langle\hat{b}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\rangle, \tag{3.4.12}$$

so that in view of Eqs.(3.2.19) and (3.2.23), then the mean photon number of the pump mode takes the form

$$\langle\hat{b}^{\dagger}\hat{b}\rangle = \frac{4\mu^2}{\kappa^2} - \frac{2\Gamma^2}{\kappa^2 - 4\Gamma^2}. \tag{3.4.13}$$

We observe that the first term represents the mean photon number of the pump mode in the absence of the harmonic generation and the second term represents the mean photon number of light mode $\hat{a}_1$ or light mode $\hat{a}_2$.

3.4.2 The variance of photon number

Now we proceeds to obtain the variance of the photon number of twin one-mode subharmonic light beams. On account of Eq.(3.4.1), the variance of the photon number of twin one-mode subharmonic light beams is defined as

$$(\Delta n)^2 = \langle(\hat{a}^{\dagger}\hat{a})^2\rangle - \langle\hat{a}^{\dagger}\hat{a}\rangle^2, \tag{3.4.14}$$

or

$$(\Delta n)^2 = \langle(\hat{a}^{\dagger}\hat{a})^2\rangle - \bar{n}^2. \tag{3.4.15}$$

Then one can put Eq.(3.4.15) in the form

$$(\Delta n)^2 = \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle + 2\bar{n} - \bar{n}^2. \tag{3.4.16}$$

Now applying the fact that $\hat{a}$ is the a Gaussian variable with zero mean, we get

$$(\Delta n)^2 = 2\bar{n} + \bar{n}^2 + \langle\hat{a}^{\dagger 2}\rangle\langle\hat{a}^2\rangle \tag{3.4.17}$$

and on taking into account of Eq.(3.2.25) along with Eq.(3.2.24), we arrive

$$(\Delta n)^2 = 2\bar{n} - \bar{n}^2 + 4\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \langle \hat{a}_1 \hat{a}_2 \rangle. \tag{3.4.18}$$

Hence in view of Eq.(3.4.7) and Eq.(3.2.23) the variance of the photon number for twin one-mode subharmonic light beams takes the form

$$(\Delta n)^2 = \frac{8\Gamma^2}{\kappa^2 - 4\Gamma^2} + \frac{16\Gamma^4}{(\kappa^2 - 4\Gamma^2)^2} + \frac{4\kappa^2\Gamma^2}{(\kappa^2 - 4\Gamma^2)^2}. \tag{3.4.19}$$

Eq.(3.4.19) can be rewritten as

$$\begin{aligned}(\Delta n)^2 \;=\; & \left[\frac{4\Gamma^2}{\kappa^2 - 4\Gamma^2} + \frac{8\Gamma^4}{(\kappa^2 - 4\Gamma^2)^2} + \frac{2\kappa^2\Gamma^2}{(\kappa^2 - 4\Gamma^2)^2}\right] \\ & + \left[\frac{4\Gamma^2}{\kappa^2 - 4\Gamma^2} + \frac{8\Gamma^4}{(\kappa^2 - 4\Gamma^2)^2} + \frac{2\kappa^2\Gamma^2}{(\kappa^2 - 4\Gamma^2)^2}\right]. \end{aligned} \tag{3.4.20}$$

As we see from Eq.(3.4.20), the variance of the photon number of signal-signal mode is the sum of the variance of the individual signal beam.

3.5 Quadrature fluctuation

We wish here to study the squeezing poroperties of twin one-mode subharmonic light beams.

3.5.1 Quadrature variance

We now proceed to calculate the quadrature variance of twin one-mode subharmonic light beams. To this end, we note that the variance of the plus and minus quadrature operators for a twin one-mode subharmonic light is given by

$$(\Delta a_\pm)^2 = \langle \hat{a}_\pm, \hat{a}_\pm \rangle, \tag{3.5.1}$$

where

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{3.5.2}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{3.5.3}$$

and $\hat{a}$ is the annihilation operator for the twin one-mode subharmonic light beams. Now on account of Eq.(3.2.20) along with Eqs.(3.2.25), (3.5.2) and (3.5.3), we see that

$$\langle \hat{a}_{\pm} \rangle = 0. \tag{3.5.4}$$

Hence in view of this, Eq.(3.5.1) can be written as

$$(\Delta a_{\pm})^2 = \langle \hat{a}_{\pm}^2 \rangle. \tag{3.5.5}$$

We note that

$$(\Delta a_{\pm})^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle \pm \langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle \tag{3.5.6}$$

and applying Eq.(3.2.26), the quadrature variance can be put in the form

$$(\Delta a_{\pm})^2 = 2 + 2\langle \hat{a}^\dagger \hat{a} \rangle \pm \langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle. \tag{3.5.7}$$

With the aid of Eq.(3.2.25), one can rewrite Eq.(3.5.7) as

$$\begin{aligned}(\Delta a_{\pm})^2 \;&=\; 2 + 2\langle \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \rangle \pm \langle \hat{a}_1^2 \rangle + \langle \hat{a}_1^{\dagger 2} \rangle \\ &\quad + \langle \hat{a}_2^2 \rangle + \langle \hat{a}_2^{\dagger 2} \rangle + 2\langle \hat{a}_1 \hat{a}_2 \rangle + 2\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle. \end{aligned} \tag{3.5.8}$$

Therefore, using Eqs.(3.2.21), (3.2.22), (3.2.23) and (3.2.24), the quadrature variance take the form

$$(\Delta a_{+})^2 = 2 - \frac{4\Gamma}{\kappa + 2\Gamma}, \tag{3.5.9}$$

and

$$(\Delta a_{-})^2 = 2 + \frac{4\Gamma}{\kappa - 2\Gamma}. \tag{3.5.10}$$

We imimediately note that the twin one-mode subharmonic cavity light beam is in a squeezed state and the squeezing occures in the plus quadrature. In addition, we see that for $\kappa = 2\Gamma$ the variance of the minus quadrature diverges. We identify $\kappa = 2\Gamma$ as the threshold condition. Upon setting $\Gamma = 0$, in Eqs.(3.5.9) and (3.5.10), we find

$$(\Delta a_{\pm})^2 = 2. \tag{3.5.11}$$

Thus we see that for $\Gamma = 0$ the cavity light is in a twin one-mode vacuum state in which the uncertainites in the two quadrature are equal and satisfy the minimum uncertainity relation.

3.5.2 Quadrature squeezing

Next we determine the quadrature squeezing for twin one-mode subharmonic light beams. We therefore define the quadrature squeezing of the twin one-mode subharmonic cavity light beams by

$$S = \frac{2 - (\Delta a_+)^2}{2}. \tag{3.5.12}$$

Since on account of Eq.(3.5.9), the squeezing occures at the plus quadrature, by substituting this equation into Eq.(3.5.12), we see that

$$S_+ = \frac{2 - (2 - \frac{4\Gamma}{\kappa+2\Gamma})}{2}. \tag{3.5.13}$$

Then Eq.(3.5.13) can be rewriten as

$$S_+ = \frac{2\Gamma}{\kappa + 2\Gamma}. \tag{3.5.14}$$

Finally, we note that at steady state and at threshold there is 50% quadrature squeezing below the vacuum state level.

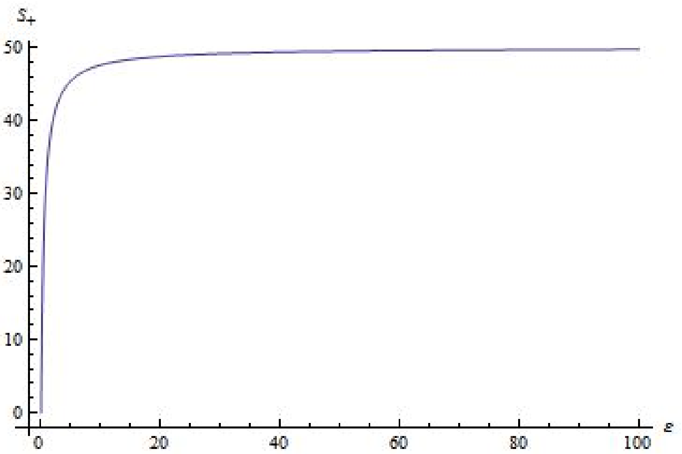

Figure 3.2: a plot of S_+ vs ε for $\kappa = 2$

Chapter 4

Superposition of Second Harmonic and One-mode Subharmonic Light Beams

In chapter two and three, we discussed the statistical and squeezing properties of the light produced by second harmonic light and twin one-mode subharmonic light beams. Here we seek to study the statistical and squeezing properties of the light produced by superposition of second harmonic light with twin one-mode subharmonic light beams. To this end, first we determine the density operator for this light mode, by assuming that the central frequency of second harmonic light and subharmonic light beams are the same. With the aid of the resulting density operator, we calculate the mean and variance of photon number and also the quadrature fluctuations.

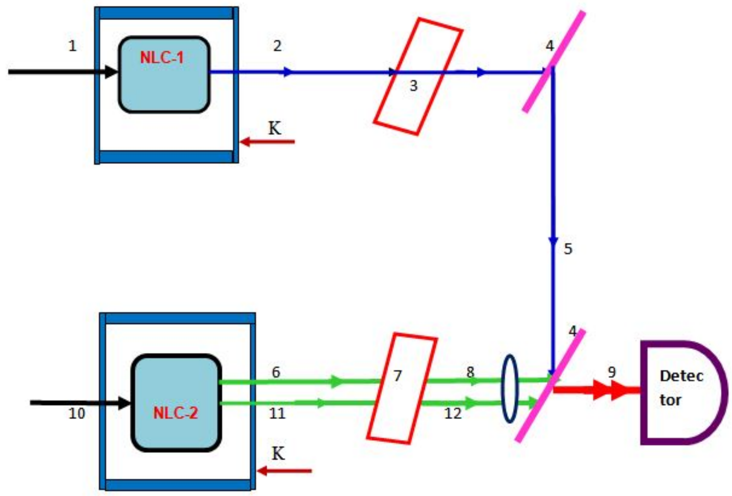

Figure 4.1: Schematic Diagram for Superposed Light Beams

Where

1. Pump mode-1
2. SH-output light ($\hat{b}^{\dagger}_{out}$)
3. Horizontally Polarizing beam splitter-1
4. Mirror
5. Refracted SH-output light ($\hat{b}^{\dagger}_{out}$)
6. Sub-harmonic output light beam ($\hat{a}_1$)
7. Horizontally Polarizing beam splitter-2
8. Horizontally polarized subharmonic light($\hat{a}_1$)
9. Superposed SH-light and signal light beams ($\hat{c}^{\dagger}_{out}$)
10. Pump mode-2
11. Sub-harmonic output light beam ($\hat{a}_2$)
12. Horizontally polarized subharmonic light($\hat{a}_2$)

4.1 The density operator

Here we wish to determine the superposed density operator for second harmonic light and one-mode subharmonic light beams. Suppose $\hat{\rho}'(\hat{b}^{\dagger}, \hat{b})$ be the density operator for a certain light beam. By expanding this density operator in normal ordering we have

$$\hat{\rho}' = \sum_{k,l} C_{k,l}\hat{b}^{\dagger k}\hat{b}^{l}. \tag{4.1.1}$$

Recalling the completeness relation for one mode coherent light is gven by

$$I = \frac{1}{\pi}\int d^2\gamma|\gamma\rangle\langle\gamma|, \tag{4.1.2}$$

we can work Eq.(4.1.1) inthe form

$$\hat{\rho}' = \frac{1}{\pi}\int d^2\gamma\sum_{k,l} C_{k,l}\gamma^{*k}|\gamma\rangle\langle\gamma|\hat{b}^{l}. \tag{4.1.3}$$

Employing the relation

$$|\gamma\rangle\langle\gamma|\hat{b}^{l} = (\gamma + \frac{\partial}{\partial\gamma^*})^{l}|\gamma\rangle\langle\gamma|, \tag{4.1.4}$$

we can express Eq.(4.1.3), as

$$\hat{\rho}' = \int d^2\gamma\frac{1}{\pi}\sum_{k,l} C_{k,l}\gamma^{*k}(\gamma + \frac{\partial}{\partial\gamma^*})^{l}|\gamma\rangle\langle\gamma|, \tag{4.1.5}$$

we then note that

$$Q(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}) = \frac{1}{\pi} \sum_{k,l} C_{k,l} \gamma^{*k} (\gamma + \frac{\partial}{\partial \gamma^*})^l. \tag{4.1.6}$$

Introducing Eq.(4.1.6) into Eq.(4.1.5), we see that

$$\hat{\rho}' = \int d^2\gamma Q(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}) |\gamma\rangle\langle\gamma|. \tag{4.1.7}$$

In terms of displacement operator, the state vectors $|\gamma\rangle$ and $\langle\gamma|$ can be put in the form of

$$|\gamma\rangle = \hat{D}(\gamma)|0\rangle \tag{4.1.8}$$

and

$$\langle\gamma| = \langle 0|\hat{D}(-\gamma). \tag{4.1.9}$$

By introducing Eqs.(4.1.8) and (4.1.9) into Eq.(4.1.7), we get

$$\hat{\rho}' = \int d^2\gamma Q(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}) \hat{D}(\gamma)|0\rangle\langle 0|\hat{D}(-\gamma). \tag{4.1.10}$$

Now we realize that the density operator for the superposition of second harmonic light beam with one-mode subharmonic light beams is expressed as

$$\hat{\rho}(\hat{c}^\dagger, \hat{c}, t) = \int d^2\alpha Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}) \hat{D}(\alpha)\hat{\rho}'\hat{D}(-\alpha), \tag{4.1.11}$$

in which

$$\alpha = \alpha_1 + \alpha_2. \tag{4.1.12}$$

Then substituting Eq.(4.1.10) into Eq.(4.1.11), we see that

$$\begin{aligned} \hat{\rho}(\hat{c}^\dagger, \hat{c}, t) &= \int d^2\alpha Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}) \hat{D}(\alpha) \int d^2\gamma Q(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}) \hat{D}(\gamma)|0\rangle\langle 0| \\ &\times \hat{D}(-\gamma)\hat{D}(-\alpha). \end{aligned} \tag{4.1.13}$$

On account of Eqs.(4.1.8), (4.1.9) and (4.1.10), Eq.(4.1.13) can be rewritten as

$$\begin{aligned} \hat{\rho}(\hat{c}^\dagger, \hat{c}, t) &= \int d^2\alpha d^2\gamma Q(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}) Q(\gamma^*, \gamma + \frac{\partial}{\partial \gamma^*}) \hat{D}(\alpha) \\ &|\gamma\rangle\langle\gamma|\hat{D}(-\alpha). \end{aligned} \tag{4.1.14}$$

we recall that

$$\hat{D}(\alpha)|\gamma\rangle = e^{-\alpha^*\gamma+\alpha\gamma^*}|\alpha+\gamma\rangle \tag{4.1.15}$$

and

$$\langle\gamma|\hat{D}(-\alpha) = e^{-\alpha^*\gamma+\alpha\gamma^*}|\gamma+\alpha\rangle. \tag{4.1.16}$$

Finally, using Eqs.(4.1.15), (4.1.16) and (4.1.14), the density operator for the superposed light beams can be written as [18]

$$\hat{\rho}(\hat{c}^\dagger,\hat{c},t) = \int d^2\alpha d^2\gamma Q(\alpha^*,\alpha+\frac{\partial}{\partial\alpha^*})Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma^*})|\alpha+\gamma\rangle\langle\gamma+\alpha|. \tag{4.1.17}$$

Furthermore, the expectation value of an operator $\hat{A}(\hat{c^\dagger},\hat{c},t)$ can be expressed in the form [18]

$$\langle\hat{A}(\hat{c^\dagger},\hat{c},t)\rangle = Tr(\rho(t)\hat{A}(0)). \tag{4.1.18}$$

Introducing Eq.(4.1.17) into Eq.(4.1.18), we find

$$\begin{aligned}\langle\hat{A}(\hat{c^\dagger},\hat{c},t)\rangle \ &= \int d^2\alpha d^2\gamma Q(\alpha^*,\alpha+\frac{\partial}{\partial\alpha^*},t)Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma^*},t)\\ &\quad A_n(\gamma^*,\alpha),\end{aligned} \tag{4.1.19}$$

in which $\hat{A}_n(\gamma^*,\alpha) = \langle\alpha+\gamma|\hat{A}(\hat{c}^\dagger,\hat{c})|\gamma+\alpha\rangle$ is c-number function corresponding to $\hat{A}$ in the normal order.

$$Q(\gamma^*,\gamma,t) = \frac{1}{\pi}\sum_{k,l} C_{k,l}\gamma^{*k}(\gamma+\frac{\partial}{\partial\gamma^*})^l \tag{4.1.20}$$

and

$$Q(\alpha^*,\alpha,t) = \frac{1}{\pi}\sum_{n,m} C_{n,m}\alpha^{*n}(\alpha+\frac{\partial}{\partial\alpha^*})^m, \tag{4.1.21}$$

are the Q function associated with the three light beams [18].

4.2 Photon statistics

The stastistical properties of light beam is described in terms of the mean photon number and the variance of photon number. Here, we calculate the mean photon number and the variance of photon number of the light produced by superposed second harmonic light beam and twin one-mode subharmonic light beams.

4.2.1 The photon mean number

The mean photon number of the superposed light beams can be expressed in terms of density operator as

$$\bar{n} = Tr(\hat{\rho}(t)\hat{c}^{\dagger}(0)\hat{c}(0)), \tag{4.2.1}$$

where $\hat{c}$ repersents the annihilation operator for the superposed light beams and is defined as

$$\hat{c} = \hat{b} + \hat{a}. \tag{4.2.2}$$

Employing Eq.(4.2.2), we see that

$$\bar{n} = Tr[\hat{\rho}(t)(\hat{b}^{\dagger}\hat{b} + \hat{b}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{b} + \hat{a}^{\dagger}\hat{a})]. \tag{4.2.3}$$

Thus introducing Eq.(4.1.17) into Eq.(4.2.3), we have

$$\begin{aligned} \bar{n} &= \int d^2\alpha d^2\gamma Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t) \\ &\quad \times Tr[|\alpha+\gamma\rangle\langle\gamma+\alpha|(\hat{b}^{\dagger}\hat{b} + \hat{b}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{b} + \hat{a}^{\dagger}\hat{a}]. \end{aligned} \tag{4.2.4}$$

Then applying the cyclic property of a trace, we get

$$\begin{aligned} \bar{n} &= \int d^2\alpha d^2\gamma Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t) \\ &\quad \times[\gamma^*\gamma + \alpha^*\alpha + \gamma^*\alpha + \alpha^*\gamma]. \end{aligned} \tag{4.2.5}$$

It follows that

$$\begin{aligned} \bar{n} &= \int d^2\alpha Q_1(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha^*\alpha + \int d^2\gamma Q_2(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t)\gamma^*\gamma \\ &\quad + \left(\int d^2\alpha Q_1(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha\right)\left(d^2\gamma Q_2(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t)\gamma^*\right) \\ &\quad + \left(\int d^2\alpha Q_1(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}, t)\alpha^*\right)\left(d^2\gamma Q_2(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*}, t)\gamma\right). \end{aligned} \tag{4.2.6}$$

With the aid of Eq.(4.1.19), one can easly put Eq.(4.2.6) in the form

$$\bar{n} = \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle \hat{b}^\dagger(t)\rangle\langle \hat{a}(t)\rangle + \langle \hat{b}(t)\rangle\langle \hat{a}^\dagger(t)\rangle. \tag{4.2.7}$$

In view of the fact that $\hat{a}$ is Gaussian operator with zero mean, we see that

$$\langle \hat{b}^\dagger(t)\rangle\langle \hat{a}(t)\rangle = 0 \tag{4.2.8}$$

and

$$\langle \hat{b}(t)\rangle\langle \hat{a}^\dagger(t)\rangle = 0. \tag{4.2.9}$$

On account of Eq.(4.2.7) along with Eqs.(4.2.8) and (4.2.9), we obtain

$$\bar{n} = \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle. \tag{4.2.10}$$

With the aid of Eqs.(2.3.7) and (3.4.7), the mean photon number of the superposed light beams takes the form

$$\begin{aligned}\bar{n}_{ss} = & \Bigg[\frac{\varepsilon_2^2}{2}\Big[\frac{1}{(\frac{\kappa_a+\kappa_b}{2}-\varepsilon_2)(\frac{\kappa_a}{2}+\varepsilon_2)} - \frac{1}{(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+3\varepsilon_2)}\Big] \\ & + \frac{\varepsilon_2}{\kappa_b}\Big[\frac{\varepsilon}{\frac{\kappa_a}{2}+\varepsilon_2}\Big]^2\Bigg] + \Big[\frac{4\Gamma^2}{\kappa^2-4\Gamma^2}\Big].\end{aligned} \tag{4.2.11}$$

This result shows that the mean photon number of the superposed light beam is the sum of the mean photon number of the individual light beams. By setting $\varepsilon_2 = 0$, we find that the mean photon number of the one-mode subharmonic light beams and if $\Gamma = 0$, we obtain the mean photon number of the second harmonic light beam.

On the other hand, well known input-output relation can be written as [18]

$$\hat{c}_{out}(t) = \sqrt{\kappa}\hat{c}(t) - \hat{c}_{in}(t), \tag{4.2.12}$$

where $\hat{c}_{in}$ is input resevoir mode operator.
Since the cavity mode is coupled to vacuum resevior, Eq.(4.2.12) can be written

$$\hat{c}_{out}(t) = \sqrt{\kappa}\hat{c}(t). \tag{4.2.13}$$

In view of Eq.(4.2.13), the mean output photon number of the superposed light beam turns to

$$\bar{n}_{out} = \kappa\bar{n}, \tag{4.2.14}$$

where $0 < \kappa < 1$.
This result indicates that the mean photon number of the superposed output light is κ times the cavity light beams.

4.2.2 The variance of photon number

The variance of the photon number can be defined as

$$(\Delta n)^2 = \langle(\hat{c}^\dagger(t)\hat{c}(t))^2\rangle - \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle^2. \tag{4.2.15}$$

Using the commutation relation

$$[\hat{c}, \hat{c}^\dagger] = 3. \tag{4.2.16}$$

for the superposed light beams Eq.(4.2.15) can be rewritten in the form

$$(\Delta n)^2 = \langle(\hat{c}^{\dagger 2}(t)\hat{c}(t))^2\rangle + 3\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle - \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle^2. \tag{4.2.17}$$

With the aid of Eqs.(4.2.2) and (4.2.16), we can write Eq.(4.2.17) as

$$\begin{aligned}(\Delta n)^2 \;=\; & \langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle + \langle\hat{b}^2(t)\hat{a}^{\dagger 2}(t)\rangle \\ & +4\langle\hat{b}^\dagger(t)\hat{b}(t)\hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle\hat{b}^{\dagger 2}(t)\hat{a}^2(t)\rangle \\ & +\langle\hat{b}^{\dagger 2}(t)\hat{b}^2(t)\rangle + 3\bar{n} - \bar{n}^2\end{aligned} \tag{4.2.18}$$

Next we seek to calculate the expectation value of the operators described by Eq.(4.2.18), so we see that

$$\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = Tr[\hat{\rho}(\hat{a}^{\dagger 2}\hat{a}^2)]. \tag{4.2.19}$$

Introducing Eq.(4.1.17) into Eq.(4.2.19) and using Eq.(3.2.25), we find

$$\begin{aligned}\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle \;=\; & \Bigg[\int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t)Q(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t) \\ & Tr[|\alpha_1 + \alpha_2\rangle\langle\alpha_2 + \alpha_1|(\hat{a}_1^{\dagger 2}\hat{a}_1^2 + \hat{a}_1^{\dagger 2}\hat{a}_2^2 + \hat{a}_2^{\dagger 2}\hat{a}_1^2 + \hat{a}_2^{\dagger 2}\hat{a}_2^2) \\ & +4\hat{a}_1^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_2]\Bigg].\end{aligned} \tag{4.2.20}$$

Applying the cyclic property of a trace to Eq.(4.2.20), we get

$$\begin{aligned}\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle \;=\; & \Bigg[\int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t)Q(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t) \\ & [\langle\alpha_2 + \alpha_1|\hat{a}_1^{\dagger 2}\hat{a}_1^2|\alpha_1 + \alpha_2\rangle + \langle\alpha_2 + \alpha_1|\hat{a}_1^{\dagger 2}\hat{a}_2^2|\alpha_1 + \alpha_2\rangle \\ & +\langle\alpha_2 + \alpha_1|\hat{a}_2^{\dagger 2}\hat{a}_1^2|\alpha_1 + \alpha_2\rangle + \langle\alpha_2 + \alpha_1|\hat{a}_2^{\dagger 2}\hat{a}_2^2|\alpha_1 + \alpha_2\rangle]\Bigg].\end{aligned} \tag{4.2.21}$$

One can also easily write Eq.(4.2.21) as

$$\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \Bigg[\int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t)Q(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t) \times[\alpha_1^{*2}\alpha_1^2 + \alpha_1^{*2}\alpha_2^2 + \alpha_2^{*2}\alpha_1^2 + \alpha_2^{*2}\alpha_2^2]\Bigg]. \quad (4.2.22)$$

Then it follows that

$$\begin{aligned}\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \Bigg[&\int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t)\alpha_1^{*2}\alpha_1^2 + \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t)\alpha_2^{*2}\alpha_2^2 \\ &+ \int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t)Q(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t)\alpha_1^{*2}\alpha_2^2 \\ &+ \int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \frac{\partial}{\partial\alpha_1^*}, t)Q(\alpha_2^*, \alpha_2 + \frac{\partial}{\partial\alpha_2^*}, t)\alpha_2^{*2}\alpha_1^2]\Bigg]. \quad (4.2.23)\end{aligned}$$

In view of Eq.(4.1.19), one can put Eq.(4.2.23) inthe form

$$\langle\hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle = \Bigg[\langle\hat{a}_1^{\dagger 2}\hat{a}_1^2\rangle + \langle\hat{a}_1^{\dagger 2}\hat{a}_2^2\rangle + \langle\hat{a}_2^{\dagger 2}\hat{a}_1^2\rangle + \langle\hat{a}_2^{\dagger 2}\hat{a}_2^2\rangle\Bigg]. \quad (4.2.24)$$

Using similar procedure we get

$$\begin{aligned}\langle\hat{b}^{\dagger 2}\hat{b}^2\rangle &= \langle\hat{b}^2\hat{b}^{\dagger 2}\rangle - 4\langle\hat{b}^\dagger\hat{b}\rangle - 2 = 2c^2 - d^2 - 4c - 2 \\ \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle &= \langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_1^{\dagger 2}\hat{a}_2^2\rangle + \langle\hat{a}_2^{\dagger 2}\hat{a}_1^2\rangle + \langle\hat{a}_2^{\dagger 2}\hat{a}_2^2\rangle \\ \langle\hat{b}^2\hat{a}^{\dagger 2}\rangle &= \langle\hat{b}^2\hat{a}_1^{\dagger 2}\rangle + \langle\hat{b}^2\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + \langle\hat{b}^2\hat{a}_2^\dagger\hat{a}_1^\dagger\rangle + \langle\hat{b}^2\hat{a}_2^{\dagger 2}\rangle \\ 4\langle\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{a}\rangle &= 4\Bigg[\langle\hat{b}^\dagger\hat{b}\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{b}^\dagger\hat{b}\hat{a}_1^\dagger\hat{a}_2\rangle + \langle\hat{b}^\dagger\hat{b}\hat{a}_1\hat{a}_2^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\hat{a}_1\hat{a}_2\rangle\Bigg] \\ \langle\hat{b}^{\dagger 2}\hat{a}^2\rangle &= \langle\hat{b}^{\dagger 2}\hat{a}_1\rangle + \langle\hat{b}^{\dagger 2}\hat{a}_2\rangle \\ \langle\hat{a}_1^{\dagger 2}\hat{a}_1^2\rangle &= \langle\hat{a}_1^\dagger\hat{a}_1\rangle^2 - \langle\hat{a}_1^\dagger\hat{a}_1\rangle \\ \langle\hat{a}_2^{\dagger 2}\hat{a}_2^2\rangle &= \langle\hat{a}_2^\dagger\hat{a}_2\rangle^2 - \langle\hat{a}_2^\dagger\hat{a}_2\rangle, \quad (4.2.25)\end{aligned}$$

and on the besis of Eqs.(3.2.20), (3.2.24), (3.2.25) and (3.2.26) along with Eq.(4.2.2), we obtain

$$\begin{aligned}\langle\hat{c}^{\dagger 2}(t)\hat{c}^2(t)\rangle = &\langle\hat{b}^{\dagger 2}(t)\hat{b}^2(t)\rangle - 4\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle - 2 + 2\langle\hat{b}^{\dagger 2}(t)\rangle\langle\hat{a}_1(t)\rangle\langle\hat{a}_2(t)\rangle \\ &+\langle\hat{a}_1(t)\hat{a}_1(t)\rangle^2 - \langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + \langle\hat{a}_2(t)\hat{a}_2(t)\rangle^2 - \langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle \\ &+4\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle + 4\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle \\ &+\langle\hat{a}_1^\dagger(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^\dagger(t)\hat{a}_2(t)\rangle \quad (4.2.26)\end{aligned}$$

On account of Eq.(4.2.17), Eq.(4.2.25), can be rewritten in the form

$$\begin{aligned}(\Delta n)^2 &= \langle\hat{b}^{\dagger 2}(t)\hat{b}^2(t)\rangle - 4\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle - 2 + 2\langle\hat{b}^{\dagger 2}(t)\rangle\langle\hat{a}_1(t)\hat{a}_2(t)\rangle \\ &+\langle\hat{a}_1(t)\hat{a}_1(t)\rangle^2 - \langle\hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle + \langle\hat{a}_2(t)\hat{a}_2(t)\rangle^2 - \langle\hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle \\ &+4\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle\langle\hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle + 4\langle\hat{b}^{\dagger}(t)\hat{b}(t)\rangle\langle\hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle \\ &+\langle\hat{a}_1^{\dagger}(t)\hat{a}_1(t)\rangle\langle\hat{a}_2^{\dagger}(t)\hat{a}_2(t)\rangle + 3\bar{n} - \bar{n}^2. \end{aligned} \quad (4.2.27)$$

Then making the use of Eqs.(2.2.12), (2.2.13), (2.3.7), (2.4.9), (3.2.21 - 3.2.24) and (4.2.11), the variance of the photon number of the superposed light beams is expressable as

$$\begin{aligned}(\Delta n)^2 &= \frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{y}]^2 - \frac{\varepsilon_2^3}{\kappa_b}[\frac{\varepsilon}{y}]^2(\frac{1}{sh}) - 2[\frac{\varepsilon_2}{2}(\frac{1}{sh})]^2 - 2\frac{\varepsilon_2^3}{\kappa_b}[\frac{\varepsilon}{y}]^2(\frac{1}{xy} - \frac{1}{sh})[\frac{\kappa\Gamma}{\kappa^2 - 4\Gamma^2}] \\ &+[\frac{2\kappa\Gamma}{\kappa^2 - 4\Gamma^2}]^2 + 2[\frac{\varepsilon_2}{2}(\frac{1}{xy})]^2\frac{2\kappa\Gamma}{\kappa^2 - 4\Gamma^2} + 2\varepsilon_2^4(\frac{1}{xy} - \frac{1}{sh})^2[\frac{\kappa\Gamma}{\kappa^2 - 4\Gamma^2}]^2 \\ &+\frac{8\varepsilon_2}{\kappa_b}(\frac{\varepsilon}{y})^2\frac{2\Gamma^2}{\kappa^2 - 4\Gamma^2} + 4\varepsilon_2^2[\frac{1}{xy} - \frac{1}{sh}]\frac{2\Gamma^2}{\kappa^2 + 4\Gamma^2} + 4(\frac{2\Gamma^2}{\kappa^2 + 4\Gamma^2}) \\ &+4(\frac{2\Gamma^2}{\kappa^2 + 4\Gamma^2})^2 - \frac{\varepsilon_2^2}{2}(\frac{1}{xy} - \frac{1}{sh}) + \frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{y}]^2 - \frac{2\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{y}]^2(\frac{2\Gamma^2}{\kappa^2 + 4\Gamma^2}) \\ &+[\frac{\varepsilon_2}{\kappa_b}[\frac{\varepsilon}{y}]^2 - \frac{\varepsilon_2^2}{2}(\frac{1}{xy} - \frac{1}{sh})](\frac{2\Gamma^2}{\kappa^2 + 4\Gamma^2}), \end{aligned} \quad (4.2.28)$$

where

$$x = \frac{\kappa_a + \kappa_b}{2} - \varepsilon_2, y = \frac{\kappa_a}{2} + \varepsilon_2 \quad (4.2.29)$$

and

$$s = \frac{\kappa_a + \kappa_b}{2} + \varepsilon_2, h = \frac{\kappa_a}{2} + 3\varepsilon_2. \quad (4.2.30)$$

This result show that unlike the mean photon number, the variance of the photon number of the superposed light beams is not the sum of the variance of the photon number of the separate light beams. However, by setting $\varepsilon = \varepsilon_2 = 0$, we easily get the variance of the photon number of one-mode subharmonic light beams. While by setting $\Gamma = 0$, we obtain the variance of the photon number of the second harmonic light beam.

On the other hand, the variance of the photon number of the superposed output light beams can be defined as

$$(\Delta n)^2_{out} = \langle(\bar{n}_{out})^2\rangle - \langle\bar{n}_{out}\rangle^2. \quad (4.2.31)$$

Employing Eq.(4.2.12), (4.2.13) and (4.2.30), we find

$$(\Delta n)^2_{out} = \kappa^2(\Delta n)^2. \quad (4.2.32)$$

We see that the variance of the the photon number of the superposed output light beams is κ^2 times that of the cavity light beams.

4.3 Quadrature fluctuation

Here we study the squeezing property of the superposed light beams.

4.3.1 Quadrature variance

Here we determine the quadrature variance for the superposed light beams. We define the quadrature variance for the superposed light beams as

$$(\Delta c_{\pm})^2 = \langle \hat{c}^2_{\pm}(t)\rangle - \langle \hat{c}_{\pm}(t)\rangle^2, \tag{4.3.1}$$

where

$$\hat{c}_{+}(t) = \hat{c}^{\dagger}(t) + \hat{c}(t) \tag{4.3.2}$$

and

$$\hat{c}_{-}(t) = i(\hat{c}^{\dagger}(t) - \hat{c}(t)), \tag{4.3.3}$$

are the plus and the minus quadratures for the superposed light beams.

Using Eqs.(4.3.2) and (4.3.3) along with the commutation relation given by Eq.(4.2.16), Eq.(4.3.1) can be put in the form

$$(\Delta c_{\pm})^2 = 3 + 2[\langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle \pm \langle \hat{c}^2(t)\rangle], \tag{4.3.4}$$

in which

$$\langle \hat{c}^{\dagger 2}\rangle = \langle \hat{c}^2\rangle, \tag{4.3.5}$$

has been used.

On account of Eq.(4.1.18), Eq.(4.3.4) leads to

$$\begin{aligned}(\Delta c_{\pm})^2 &= 3 + 2[\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle \pm \langle \hat{a}^2(t)\rangle \pm \langle \hat{b}^2(t)\rangle \\ &\quad \pm\langle \hat{a}(t)\hat{b}(t)\rangle \pm \langle \hat{b}(t)\hat{a}(t)\rangle].\end{aligned} \tag{4.3.6}$$

Next we calculate the expectation value of $\hat{a}\hat{b}$, then we write

$$\langle \hat{a}\hat{b}\rangle = Tr(\hat{\rho}(\hat{a}\hat{b})). \tag{4.3.7}$$

Using Eq.(4.1.17), we see that

$$\langle \hat{a}\hat{b}\rangle = \int d^2\alpha d^2\gamma Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*})Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*})Tr(|\gamma + \alpha\rangle\langle\alpha + \gamma|\hat{a}\hat{b}). \tag{4.3.8}$$

Then it follows

$$\langle\hat{a}\hat{b}\rangle = \int d^2\alpha d^2\gamma Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*})Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*})\langle\alpha+\gamma|\hat{a}\hat{b}|\gamma+\alpha\rangle. \quad (4.3.9)$$

Employing Eq.(4.1.12), we find that

$$\langle\hat{a}\hat{b}\rangle = \int d^2\alpha d^2\gamma Q(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*})\gamma Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*})\alpha \quad (4.3.10)$$

On account of Eqs.(4.1.18), Eq.(4.3.10) can be put in the form

$$\langle\hat{b}\rangle\langle\hat{a}\rangle = 0. \quad (4.3.11)$$

In view of Eq.(4.3.11), Eq.(4.3.4) yields

$$(\Delta c_{\pm})^2 = [1 + 2\langle\hat{b}^\dagger\hat{b}\rangle \pm 2\langle\hat{b}^2\rangle] + [2 + 2\langle\hat{a}^\dagger\hat{a}\rangle \pm 2\langle\hat{a}^2\rangle]. \quad (4.3.12)$$

Now Eq.(4.3.12) can be rewritten as

$$(\Delta c_{\pm})^2 = (\Delta b_{\pm})^2 + (\Delta a_{\pm})^2. \quad (4.3.13)$$

This show that the quadrature variance of the superposed light beams is the sum of the separate light beams.

On account of Eqs.(2.4.16), (3.5.9) and (3.5.10), we easily find the quadrature variance of the superposed light beams as

$$(\Delta c_{\pm})^2 = 3 \mp \left[\frac{2\varepsilon_2^2}{(\frac{\kappa_a+\kappa_b}{2} \pm \varepsilon_2)(\frac{\kappa_a}{2} + (2\pm 1)\varepsilon_2)} + \frac{4\Gamma}{\kappa + 2\Gamma}\right]. \quad (4.3.14)$$

Moreover, the squeezing for both second harmonic light beam and one-mode subharmonic light beams occurs in the plus quadrature.

4.3.2 Quadrature squeezing

The quadrature squeezing of the superposition of second harmonic light beam and one-mode subharmonic light beams can be defined as

$$S_+ = \frac{3 - (\Delta c_{\pm})^2}{3}. \quad (4.3.15)$$

It then follows

$$S_+ = \frac{[1 - (\Delta\hat{b}_+)^2(t)] + [2 - (\Delta\hat{a}_+)^2(t)]}{3}. \quad (4.3.16)$$

where $\Delta\hat{a}_+$ is the quadrature variance of one-mode subharmonic light beams and $\Delta\hat{b}_+$ is second harmonic light beam.

Introducing Eq.(2.4.16) and Eq.(3.5.9) into Eq.(4.3.16), we obtain

$$S_+ = \frac{1}{3}[1-(1-\frac{2\varepsilon_2^2}{(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+(2+1)\varepsilon_2)})+2-(2-\frac{4\Gamma}{\kappa+2\Gamma})]. \quad (4.3.17)$$

Finally, on account of Eq.(4.3.17), quadrature variance of the superposed light beams is

$$S_+ = \frac{1}{3}[\frac{2\varepsilon_2^2}{(\frac{\kappa_a+\kappa_b}{2}+\varepsilon_2)(\frac{\kappa_a}{2}+(2+1)\varepsilon_2)}+\frac{4\Gamma}{\kappa+2\Gamma}]. \quad (4.3.18)$$

This result show that the degree of squeezing for the superposed squeezed light beams is the average of the separate light beams that is 55.57% at steady state and at threshold.

We have seen that the degree of squeezing of subharmonic generation enhanced by superposition of the second harmonic generation.

Onthe other hand, the quadrature squeezing of the superposed output light beams can be defined as

$$S_{+out} = \frac{3\kappa-(\Delta c_\pm)_{out}}{3\kappa}, \quad (4.3.19)$$

where 3κ to be the quadrature squeezing of the coherent light beams.

By inspection of Eq.(4.3.13) and in view of Eq.(4.2.31), the quadirature variance of the superposed output light beams to be

$$(\Delta c_\pm)^2_{out} = (\Delta a_\pm)^2_{out} + (\Delta b_\pm)^2_{out}, \quad (4.3.20)$$

in which

$$(\Delta a_\pm)^2_{out} = \kappa(\Delta a_\pm)^2_{out}, \quad (4.3.21)$$

and

$$(\Delta b_\pm)^2_{out} = \kappa(\Delta b_\pm)^2_{out}. \quad (4.3.22)$$

Introducing Eq.(4.3.20) into Eq.(4.3.19), we have

$$S_{+out} = \frac{3\kappa-[\kappa(\Delta a_\pm)^2_{out}+\kappa(\Delta b_\pm)^2_{out]}}{3\kappa}, \quad (4.3.23)$$

$$(S_+)_{out} = S_+ \quad (4.3.24)$$

This result shows that the quadirature squeezing of the superposed light beams is the same as that of the cavity light.

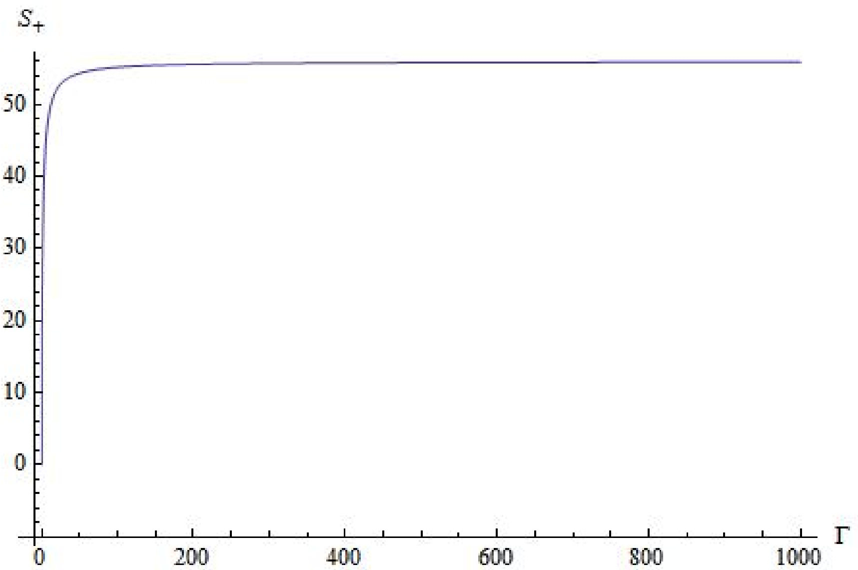

Figure 4.2: a plot of S_+ Vs Γ for $\kappa_a = \kappa_b = \kappa = 1$ and $\varepsilon_2 = \Gamma$.

Chapter 5

Conclusion

In this thesis, we studied the squeezing and statistical properties of light produced by superposed second harmonic light beam and twin one-mode subharmonic light beams. Using the solution of the c-number Langevin equations we calculated the antinormally orderd characteristic function and then the Q function of the separate light beams.

First we have seen the second harmonic light beam, employing the linearizion scheme approximation, to get the Q function. Then using the Q function, we analyzed the mean photon number, the variance of the photon number and the quadrature fluctuations. We have seen that the squeezing occurs in the plus quadrature and maximum degree squeezing is 66.67%.

Next we have considered the one-mode subharmonic generator by writing the Hamiltonian, the master equation and the operator dynamics. Then we get the Q function of the one-mode subharmonic light beams. Using the Q function we calculated the mean photon number, the variance of the photon number and the quadrature fluctuations of the one-mode subharmonic light beams. We observed that the squeezing occurs in the plus quadrature and its maximum squeezing is 50% at steady state and at threshold.

Finally, we analyzed the statistical and squeezing properties of the superposition of second harmonic light beam and twin one-mode subharmonic light beams. Employing the superposed density operator, we obtained the mean photon number, which is the sum of the mean photon number the separate light beams and the output mean photon number is κ times that of the cavity light beams. Unlike the mean photon number, the variance of the photon number is not the sum of the individual light beams and the variance of the photon number of the superposed output light beams is κ^2 times that of the cavity light beams.

Moreover, the quadrature variance of the superposed light beams is simply the sum of the quadrature variance of the individual light beams. Like that of second harmonic light beams and one-mode subharmonic light beams, the squeezing of the superposed light beams occurs in the plus quadrature. Furthermore, we see that the degree of squeezing for superposed squeezed light beams is the average of the separate light beams and 55.57% at steady state and at threshold. In addition we find that the quadrature squeezing of the superposed output light beams is the same as that of the cavity light. In the same way we see that the variance of the photon number is greater than the mean photon number therefore, the photon statistics of the one-mode subharmonic light beams is super-Poisson.

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