Enhancement of Squeezing and Entanglement of a Non-degenerate Three-Level Laser with Squeezed Vacuum Reservoir

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To My Family

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First of all I would like to express my thanks to the only Allah who help me starting from my womb to still and in all my way,Next to Allah I would like to express my deep thanks to my advisor Tamirat Abebe(PhD) in his unforgettable help and advise from the starting to the end of my work.

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Abstract

In this thesis we analyze a nondegenerate three-level cascade laser with cavity mode coupled to a two-mode squeezed vacuum reservoir with the aid of master equation, We obtain the stochastic differential equations associated with the normal ordering,correlations of the noise force and the solutions of the resulting differential equations.Applying the solutions of the resulting differential equations.we calculate quadrature variances, photon entanglement Photon number correlations,normalized second-order correlation functions and fluctuation of intensity difference for the cavity modes.We also determine the mean photon number sum and difference and the photon number variance sum and difference for the two mode cavity light employing the Q function.We study the squeezing properties and entanglement of the two mode cavity light.It turns out that the generated light exhibits a two-mode squeezing and entanglement when initially there are more atoms in the lower level.Moreover,a strong correlation between photon numbers along with a significant fluctuation in the intensity difference is found.

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Introduction

Entanglement is one of the fundamental tools for quantum information processing and communication protocols. The generation and manipulation of entanglement has attracted a great deal of interest with wide applications in quantum teleportation[1], quantum dense coding[2], quantum computation[3],quantum error correction[4],and quantum cryptography[5]. Squeezing is one of the nonclassical features of light that have been extensively studied by several authors[1 8].squeezed light has potential applications in the detection of week signals and in low-noise communications [1,2].
Squeezed light can be generated by various quantum optical processes such as subharmonic generations [15,10 12],resonance florescence [6,7],and three-level laser under certain conditions[1, 3, 4, 9, 16 27].Hence it proves useful to find some convenient means of generating a bright squeezed light.We define a three-level cascade laser as a quantum optical system in which three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate into a cavity coupled to a two mode squeezed vacuum reservoir.
These atoms are removed from the cavity after some time, which is long enough for the atoms to decay spontaneously to levels other than the middle or bottom. using these mechanisms simultaneously [10, 12, 13, 14]. The top, intermediate, and bottom levels of a three-level cascade atom can be conveniently denoted by $|a\rangle$, $|b\rangle$, and$|c\rangle$, respectively.A direct transition between levels $|a\rangle$ and $|c\rangle$ is taken to be dipole forbidden. When a three-level cascade atom decays from the top level to the bottom level via the interme-

diate level, two photons are emitted. If the two photons have the same frequency, the three-level atom is called degenerate otherwise nondegenerate. Nondegenerate three-level cascade laser is a source of a two-mode squeezed light that is characterized by a strong correlation of the modes at two different frequencies.

The emission of light when the atoms makes the transition from the top level to intermediate level is light mode a, and the emission of light when the atoms makes the transition from the intermediate level to the bottom level is light mode b. The squeezing does not exist in each mode, but in the correlated state formed by the two modes.Due to the strong correlation between the modes, the two-mode squeezed light generally violates certain classical inequalities and hence can be applied in preparing Einstein-Podolsky-Rosen (EPR) [28].

The generation and evolution of macroscopic entanglement in non-degenerate three-level laser with driven coherence has been examined [29] using the sufficient entanglement test proposed by Duan et al. Opposed to this work,Tesfa [30] and Alebachew [31] have studied the entanglement properties when the atomic coherence is induced by the superposition of atomic states.It is found that the entanglement obtained in such lasers highly depends on the amount of squeezing in the two-mode light. Moreover, it has been shown that the quadrature squeezing can be enhanced by coupling the cavity mode to a squeezed vacuum reservoir [32].

Since the squeezed vacuum is coupled to the cavity modes via the single port mirror, it makes the experimental realization difficult. However, experimentally realizable scheme has been proposed by Gilles et al.[33] and was shown that the squeezing can be increased effectively.Thus it appears that the combined system we have considered could generate highly squeezed light.

In this thesis,we study the squeezing and entanglement properties of the two mode cavity light generated by a nondegenerate three-level cascade laser with a cavity modes coupled to a two-mode squeezed vacuum reservoir via a single-port mirror. We rigorously derive the master equation for the cavity mode coupled to a two-mode squeezed vacuum reservoir and applying the master equation,we obtain stochastic differential equations, the correlation properties of the noise force for the cavity mode variables associated with the normal ordering and the solutions of stochastic differential equations.

we calculate the quadrature variance and squeezing, the mean and variances of the photon number sum and difference,the photon number correlation, normalized correlation photon number as well as fluctuations of the intensity difference for the cavity modes employing the Q function.The Q function is obtained with the aid of the anti-normally ordered characteristic function defined in the Heisenbeg picture. Moreover, applying the entanglement criterion developed by Duan et al.[34],we investigate the squeezing and entanglement non-degenerate three-level cascade laser coupled to a two-mode squeezed vacuum reservoir.

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Stochastic Differential Equations

In this chapter we seek to study the squeezing and statistical properties of the light produced by a nondegenerate three-level laser whose cavity contains a parametric amplifier and coupled to a two-mode squeezed vacuum reservoir. Three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected into the cavity at a constant rate and removed from the cavity after sometime. We first obtain the master equation and stochastic differential equations for the cavity mode variables associated with the normal ordering. Using the solutions of the resulting differential equations and the correlation properties of the noise forces, we calculate the quadrature variances,.

In addition, we determine the mean and variances of the photon number sum and difference for the cavity modes employing the Q function. The Q function is obtained with the aid of the antinormally ordered characteristic function defined in the Heisenbeg picture.

2.1 Master Equation

In this section we wish to obtain the equation of evolution of the density operator for a nondegenerate three-level laser and coupled to a two-mode squeezed vacuum reservoir. We first drive the equation of evolution of the density operator for the three-level laser applying the linear and the adiabatic approximation schemes [4, 21]. Then after obtaining the properties of the reservoir submode operators, we drive the time

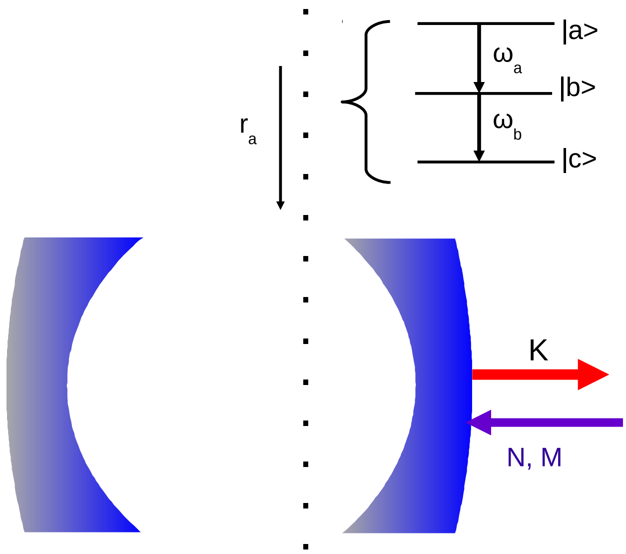

Figure 2.1: Scheme of a non-degenerate three-level cascade laser coupled to a two-mode squeezed vacuum reservoir.

evolution of the reduced density operator for a cavity modes coupled to a two-mode squeezed vacuum reservoir [22]. Finally, with the help of the two resulting equations, we write the master equation for the system under consideration.

We consider a non-degenerate three-level cascade laser coupled to a two-mode squeezed vacuum reservoir. In this quantum optical system, three-level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity at a constant rate r_a and removed after certain time τ.

We denote the top, intermediate, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$ as shown fig.1 we assume the cavity mode to be at resonance with the two transitions$|a\rangle$, $|b\rangle$, and $|c\rangle$and $|b\rangle \rightarrow |c\rangle$, with direct transition between levels$|a\rangle$ and $|c\rangle$ to be dipole forbidden.The interaction of a nondegenerate three-level atom with the cavity modes can be described by the Hamiltonian.

$$\hat{H} \quad = \quad ig(|a\rangle\langle b|\hat{a} - \hat{a}\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^{\dagger}|c\rangle\langle b|) \tag{2.1}$$

where g is the coupling constant, $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes. We consider a three-level atom initially in the state

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle \tag{2.2}$$

The density operator for a single atom can then be written as

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c| \tag{2.3}$$

where

$$\rho_{aa}^{(0)} = |C_a|^2 and \rho_{cc} = |C_c|^2 \tag{2.4}$$

are the probability for the atom to be in the upper and the lower levels at the initial time

$$\rho_{ac}^{(0)} = C_a C_c^* and \rho_{ca}^{(0)} = C_c C_a^* \tag{2.5}$$

represent the atomic coherence at the initial time. We note that

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^2 \tag{2.6}$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for as single atom plus the cavity mode at time t, with the atom injected at time t_j such that $t - \tau \leq t_j \leq t$. Then the density operator for all atoms in the cavity plus the cavity mode at time t can be written as

$$\hat{\rho}_{AR}(t) = r_a \sum \hat{\rho}_{AR}(t, t_j)\Delta t_j, \tag{2.7}$$

in which $r_a\Delta t_j$ represents the number of atoms are injected in to the cavity at timeΔt_j. Now converting the summation into integration in the limit $\Delta t_j \longrightarrow 0$, we have at time t

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt' \tag{2.8}$$

and on differentiating with respect to t, there follows

$$\frac{d\hat{\rho}_{AR}(t)}{dt} = r_a(\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t, t-\tau)) + r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt' \tag{2.9}$$

We observe that $\hat{\rho}_{AR}(t,t)$ is the density operator for the cavity modes plus an atom injected at time t and $\hat{\rho}_{AR}(t,t-\tau)$, represents the density operator for an atom plus the cavity modes at time t with the atom being removed from the cavity at this time. Therefore, these density operators can be decoupled, so that

$$\hat{\rho}_{AR}(t,t) \quad = \quad \hat{\rho}_A(0)\hat{\rho}(t), \tag{2.10}$$

$$\hat{\rho}_{AR}(t,t) \quad = \quad \hat{\rho}_A(t-\tau)\hat{\rho}(t), \tag{2.11}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone. In view of Eqs. (2.10), Eq.(2.11) and Eq. (2.9) can be written as

$$\frac{d\hat{\rho}_{AR}(t)}{dt} \quad = \quad r_a(\hat{\rho}_{AR}(0) - \hat{\rho}_{AR}(t,t-\tau))\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t^{'})dt^{'}, \tag{2.12}$$

In the absence of damping of the cavity mode by a vacuum reservoir, the density operator $\hat{\rho}(t,t^{'})$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t^{'})dt^{'} \quad = \quad -i[\hat{H},\hat{\rho}_{AR}(t,t^{'})], \tag{2.13}$$

so that using Eq.(2.13) and taking into account Eq.(2.10),one can put Eq. (2.12), Eq.in the from

$$\frac{d\hat{\rho}_{AR}(t)}{dt} \quad = \quad r_a(\hat{\rho}_{AR}(0) - \hat{\rho}_{AR}(t,t-\tau))\hat{\rho}(t) - i[\hat{H},\hat{\rho}_{AR}(t,t^{'})]. \tag{2.14}$$

Furthermore,tracing over the atomic variables, we have

$$\frac{d\hat{\rho(t)}}{dt} \quad = \quad -iTr_A[\hat{H},\hat{\rho}_{AR}(t)], \tag{2.15}$$

With the aid of Eqs. (2.15) and (2.1), the equation of evolution of the density operator for the cavity modes can be put in the form

$$\frac{d\hat{\rho}}{dt} \quad = \quad g(\hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} - \hat{a}^{\dagger}\hat{\rho}_{ab} + \hat{\rho}_{ab}\hat{a}^{\dagger} + \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{b} - \hat{b}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{bc}\hat{b}^{\dagger}), \tag{2.16}$$

where

$$\hat{\rho}_{\alpha\beta} \quad = \quad \langle\alpha|\hat{\rho}_{AR}(t)|\beta\rangle, \tag{2.17}$$

with α,β = a, b, c.next we proceed to determine the matrix elements $\hat{\rho}_{\alpha\beta}$.We see from Eq. (2.14) that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\{\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle\hat{\rho}(t)\} - \langle\alpha|[\hat{H},\hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}, \quad (2.18)$$

in which the last term is included to account for the decay of the atoms due to spontaneous emission. Here γ, considered to be the same for all the three levels, is the atomic decay rate. We assume that the atoms are removed after they have decayed to a level other than levels $|b\rangle$ and $|c\rangle$. then we see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \quad (2.19)$$

with $\alpha, \beta = a, b, c$. next we proceed to determine the matrix elements $\hat{\rho}_{\alpha\beta}$ involved in Eq. (2.17). Taking into account Eqs. (2.18), (2.19), (2.3), and (2.1), we can write

$$\begin{aligned}\frac{d\hat{\rho}_{\alpha\beta}}{dt} &= r_a(\rho^{(0)}_{aa}\delta_{\alpha a}\delta_{a\beta} + \rho^{(0)}_{ac}\delta_{\alpha a}\delta_{c\beta} + \rho^{(0)}_{ca}\delta_{\alpha c}\delta_{a\beta} + \rho^{(0)}_{cc}\delta_{\alpha c}\delta_{c\beta})\hat{\rho} \\ &+g(\hat{a}\hat{\rho}_{b\beta}\delta_{\alpha a} - \hat{a}^\dagger\hat{\rho}_{c\beta}\delta_{\alpha b} - \hat{b}^\dagger\hat{\rho}_{b\beta}\delta_{\alpha c} \\ &-\hat{\rho}_{\alpha a}\hat{a}\delta_{b\beta} + \hat{\rho}_{\alpha b}\hat{a}^\dagger\delta_{a\beta} - \hat{\rho}_{\alpha b}\hat{b}\delta_{c\beta} + \hat{\rho}_{\alpha c}\hat{b}^\dagger\delta_{b\beta}) - \gamma\hat{\rho}_{\alpha\beta}, \end{aligned} \quad (2.20)$$

Applying Eq.(2.20), we obtain

$$\frac{d\hat{\rho}_{ab}}{dt} = g(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{b}^\dagger) - \gamma\hat{\rho}_{ab}, \quad (2.21)$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b} - \hat{a}^\dagger\hat{\rho}_{ac}) - \gamma\hat{\rho}_{bc}, \quad (2.22)$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\hat{\rho}^{(0)}_{aa}\hat{\rho} + g(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger) - \gamma\hat{\rho}_{aa}, \quad (2.23)$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a\hat{\rho}^{(0)}_{cc}\hat{\rho} - g(\hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}) - \gamma\hat{\rho}_{cc}, \quad (2.24)$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\hat{\rho}^{(0)}_{ac}\hat{\rho} + g(\hat{a}\hat{\rho}_{bc} + \hat{\rho}_{ab}\hat{b}) - \gamma\hat{\rho}_{ac}, \quad (2.25)$$

$$\frac{d\hat{\rho}_{ac}}{dt} = -g(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^\dagger) - \gamma\hat{\rho}_{bb}. \quad (2.26)$$

Dropping the g terms in Eqs. (2.22),(2.23),(2.24),(2.25) and then applying the adiabatic approximation scheme,we get

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{2.27}$$

$$\hat{\rho}_{cc} = \frac{r_a\rho_{cc}^{(0)}}{\gamma}\hat{\rho}, \tag{2.28}$$

$$\hat{\rho}_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{2.29}$$

$$\hat{\rho}_{bb} = 0. \tag{2.30}$$

Putting the above results into Eqs.(2.20) and (2.21), we have

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}(\rho_{ac}^{(0)}\hat{\rho}\hat{b}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{2.31}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}(\hat{b}\rho_{cc}^{(0)}\hat{\rho} - \hat{a}^\dagger\rho_{ac}^{(0)}\hat{\rho}) - \gamma\hat{\rho}_{bc}, \tag{2.32}$$

so that employing once more the adiabatic approximation scheme, we get

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}(\rho_{ac}^{(0)}\hat{\rho}\hat{b}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}), \tag{2.33}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}(\rho_{cc}^{(0)}\hat{b}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}), \tag{2.34}$$

Finally on account of Eqs. (2.22) and (2.23), the equation of evolution of the density operator for the cavity modes given by Eq. (2.7)turn out to be

$$\begin{aligned}\frac{d\rho(t)}{dt} = {} & \frac{A\rho_{aa}^{(0)}}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{A\rho_{cc}^{(0)}}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}) \\ & - \frac{A\rho_{ac}^{(0)}}{2}(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger) \\ & - \frac{A\rho_{ca}^{(0)}}{2}(2\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho}). \end{aligned} \tag{2.35}$$

Two-Mode Squeezed Vacuum Reservoir

The density operator for a two-mode squeezed vacuum reservoir can be expressed as

$$\hat{\rho}(r) = \prod_i \hat{S}_i|0_i0_i\rangle\langle 0_i0_i|\hat{S}_i^{\dagger}(r) \tag{2.36}$$

where

$$\hat{S}_i(r) = e^{r(\hat{c}_i^{\dagger}\hat{d}_i - \hat{c}_i\hat{d}_i)} \tag{2.37}$$

is the squeeze operator with the squeeze parameter$r$ taken for convenience to be real and positive, and $\hat{c}_i$and $\hat{d}_i$ represent the annihilation operators for the reservoir sub-modes. With the aid of Eq. (2.37) and the identity operator $\hat{I} = \hat{S}_i(r)\hat{S}_i^{\dagger}(r)$, we can write as

$$\langle \hat{C}_j\hat{d}_k\rangle = \prod_i \langle 0_i0_i|\hat{c}_j(r)\hat{d}_k|0_i0_i\rangle, \tag{2.38}$$

in which

$$\hat{c}_j(r) = \hat{S}_i^{\dagger}(r)\hat{c}_j\hat{S}_i(r) \tag{2.39}$$

and

$$\hat{d}_k(r) = \hat{S}_i^{\dagger}(r)\hat{d}_k(r)\hat{S}_i(r). \tag{2.40}$$

Differentiating Eq.(2.40) and the adjoint of Eq.(2.41) with respect to r, we obtain

$$\frac{d}{dr}\hat{c}_j = \hat{d}_i^{\dagger}(r)\delta_{ij} \tag{2.41}$$

and

$$\frac{d}{dr}\hat{d}_k^{\dagger}(r) = \hat{c}_i(r)\delta_{ik}. \tag{2.42}$$

In order to decouple these equations, we differentiate Eq. (2.42) once more with respect to r. We then get

$$\frac{d^2}{dr^2}\hat{c}_j(r) = \hat{c}_j(r). \tag{2.43}$$

The solution of this equation can be put in the form

$$\hat{c}_j(r) = Ae^r + Be^{-r}. \tag{2.44}$$

Applying the condition $r = 0$, we see that

$$\hat{c}_j|_{r=0} = A + B = \hat{c}_i\delta_{ij}, \tag{2.45}$$

$$\frac{d}{dr}\hat{c}_j|_{r=0} = A - B = \hat{d}^\dagger\delta_{ij}. \tag{2.46}$$

It then follows that

$$A = \frac{1}{2}(\hat{c}_i + d_i^\dagger)\delta_{ij}, \tag{2.47}$$

$$B = \frac{1}{2}(\hat{c}_i - \hat{d}_i^\dagger)\delta_{ij}, \tag{2.48}$$

so that on account of these, Eq. (2.45) takes the form

$$\hat{c}_j(r) = (\hat{c}_i\cosh r + \hat{d}_i^\dagger\sinh r)\delta_{ij}. \tag{2.49}$$

Following a similar procedure, we can easily verify that

$$\hat{d}_k(r) = (\hat{d}_i\cosh r + \hat{c}_i^\dagger\sinh r)\delta_{ik} \tag{2.50}$$

In view of Eqs. (2.49) and (2.50), we have

$$\langle\hat{c}_j\hat{d}_k\rangle = \prod_i\langle 0_i0_i|\hat{c}_i\hat{d}_i\cosh^2 r + (\hat{d}^\dagger\hat{d}_i + \hat{c}_i\hat{c}_i^\dagger)\cosh r\sinh r + \hat{d}_i^\dagger\sinh^2 r|0_i0_i\rangle\delta_{ij}\delta_{ik}. \tag{2.51}$$

Applying the relation $\hat{c}_i\hat{c}_i^\dagger = 1 + \hat{c}_i^\dagger\hat{c}_i$, we find

$$\langle\hat{c}_j\hat{d}_k\rangle = M\delta_{jk} \tag{2.52}$$

where $M = \sinh r\cosh r$. We assume that k is of the order of the central wave number k_0, so that we can replace k by $2k_0 - k$. In view of this, Eq. (2.53) can be rewritten as

$$\langle\hat{c}_j\hat{d}_k\rangle = M\delta_{j,2k_0-k}. \tag{2.53}$$

One can also easily establish in a similar manner that

$$\langle \hat{c}_k \rangle = \langle \hat{d}_k \rangle = \langle \hat{c}_j \hat{c}_k \rangle = \langle \hat{d}_j \hat{d}_k \rangle = \langle \hat{c}_j \hat{d}_k^\dagger \rangle = \langle \hat{c}_j^\dagger \hat{c}_k^\dagger \rangle = \langle \hat{d}_j^\dagger \hat{d}_k^\dagger \rangle = \langle \hat{c}_j^\dagger \hat{d}_k \rangle = 0, \quad (2.54)$$

$$\langle \hat{c}_j^\dagger \hat{d}_k^\dagger \rangle = M\delta_{j,2k_0-k}, \quad (2.55)$$

$$\langle \hat{c}_j^\dagger \hat{c}_k \rangle = \langle \hat{d}_j^\dagger \hat{d}_k \rangle = N\delta_{jk}, \quad (2.56)$$

$$\langle \hat{c}_j \hat{c}_k^\dagger \rangle = \langle \hat{d}_j \hat{d}_k^\dagger \rangle \quad (2.57)$$

in which

$$N = \sinh^2 r \quad (2.58)$$

We now proceed to drive the equation of evolution of the density operator for cavity modes coupled to a two-mode squeezed vacuum reservoir.

In general, the reduced density operator for cavity modes coupled to a reservoir can be expressed in Born approximation as

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = & -i[\langle \hat{H}_{SR}(t) \rangle, \hat{\rho}(0)] - \int dt' TrR(\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t)' R) \\ & + \int dt' TrR(\hat{H}_{SR}(t)\hat{\rho}(t')R\hat{H}_{SR}(t') \\ & + \int dt' TrR(\hat{H}_{SR}(t')\hat{\rho}(t')R\hat{H}_{SR}(t) \\ & - \int dt' TrR(\hat{\rho}(t')R\hat{H}_{SR}(t')\hat{H}_{SR}(t).\end{aligned} \quad (2.59)$$

We seek to obtain the equation of evolution of the reduced density operator for cavity modes coupled to a two-mode squeezed vacuum. The Hamiltonian describing the interaction of the cavity modes with the two-mode squeezed vacuum reservoir can be written as

$$\hat{H}_{SR}(t) = i\sum_k \lambda_k(\hat{a}^\dagger \hat{c}_k e^{i(\omega_a-\omega_k)t} - \hat{a}\hat{c}_k^\dagger e^{-i(\omega_a-\omega_k)t} + \hat{b}^\dagger \hat{d}_k e^{i(\omega_a-\omega_k)t} + \hat{b}\hat{b}_k^\dagger e^{-i(\omega_b-\omega_k)t}), \quad (2.60)$$

where $\hat{c}_k$ and $\hat{d}_k$ are the annihilation operators for reservoir submodes and λ_k is the coupling constant. Using Eqs.(2.60), one can write Eq.(2.61) as

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) &= \int dt'(I_1(\hat{a}^{\dagger 2}\hat{\rho}(t')\hat{a}^{\dagger 2} - 2\hat{a}^{\dagger}\hat{\rho}(t')\hat{a}^{\dagger}) + I_2(\hat{a}^{\dagger}\hat{a}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^{\dagger}\hat{a} - 2\hat{a}\hat{\rho}(t')\hat{a}^{\dagger} \\ &+ I_3(\hat{a}\hat{a}^{\dagger}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}\hat{a}^{\dagger} - 2\hat{a}^{\dagger}\hat{\rho}(t')\hat{a}) + I_4(\hat{a}^2\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^2 - 2\hat{a}\hat{\rho}(t')\hat{a}) \\ &+ I_5(\hat{b}^{\dagger 2}\hat{\rho}(t') + \hat{\rho}(t')\hat{b}^{\dagger 2} - 2\hat{b}^{\dagger}\hat{\rho}(t')\hat{b}^{\dagger}) + I_6(\hat{b}^{\dagger}\hat{b}\hat{\rho}(t') + \hat{\rho}(t')\hat{b}^{\dagger}\hat{b} - 2\hat{a}\hat{\rho}(t')\hat{b}^{\dagger}) \\ &+ I_7(\hat{b}\hat{b}^{\dagger}\hat{\rho}(t') + \hat{\rho}(t')\hat{b}\hat{b}^{\dagger} - 2\hat{b}^{\dagger}\hat{\rho}(t')\hat{b}) + I_8(\hat{b}^2\hat{\rho}(t') + \hat{\rho}(t')\hat{b}^2 - 2\hat{b}\hat{\rho}(t')\hat{b}) \\ &+ 2I_9(\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^{\dagger}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{\rho}(t')\hat{b}^{\dagger}) \\ &+ 2I_{10}(\hat{a}^{\dagger}\hat{b}\hat{\rho}(t' + \hat{\rho}(t')\hat{a}^{\dagger}\hat{b} - \hat{b}\hat{\rho}(t')\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{\rho}(t')\hat{b}) \\ &+ 2I_{11}(\hat{a}\hat{b}^{\dagger}\hat{\rho}(t' + \hat{\rho}(t')\hat{a}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{\rho}(t')\hat{a} - \hat{a}\hat{\rho}(t')\hat{b}^{\dagger}) \\ &+ 2I_{12}(\hat{a}\hat{b}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}\hat{b} - \hat{b}\hat{\rho}(t')\hat{a} - \hat{a}\hat{\rho}(t')\hat{b}), \end{aligned} \tag{2.61}$$

where

$$I_1 = \sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{c}_k\hat{c}_{k'}\rangle e^{i(\omega_a-\omega_k)}t + i(\omega_a - \omega_k')t', \tag{2.62}$$

$$I_2 = \sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{c}_k\hat{c}^{\dagger}_{k'}\rangle e^{i(\omega_a-\omega_k)}t - i(\omega_a - \omega_k')t', \tag{2.63}$$

$$I_3 = -\sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{c}^{\dagger}_k\hat{c}_{k'}\rangle e^{-i(\omega_a-\omega_k)}t + i(\omega_a - \omega_k')t', \tag{2.64}$$

$$I_4 = -\sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{c}^{\dagger}_k\hat{c}^{\dagger}_{k'}\rangle e^{-i(\omega_a-\omega_k)}t + i(\omega_a - \omega_k')t', \tag{2.65}$$

$$I_5 = \sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{d}_k\hat{d}_{k'}\rangle e^{i(\omega_a-\omega_k)}t - i(\omega_a + \omega_k')t', \tag{2.66}$$

$$I_6 = -\sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{c}^{\dagger}_k\hat{c}^{\dagger}_{k'}\rangle e^{i(\omega_a-\omega_k)}t - i(\omega_a - \omega_k')t', \tag{2.67}$$

$$I_7 = -\sum_{kk'} \lambda_k\lambda_{k'}\langle\hat{d}^{\dagger}_k\hat{d}_{k'}\rangle e^{-i(\omega_a-\omega_k)}t + i(\omega_a - \omega_k')t', \tag{2.68}$$

$$I_8 = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{d}_k^\dagger \hat{d}_{k'}^\dagger \rangle e^{-i(\omega_a - \omega_k)} t - i(\omega_a - \omega_k^{'})t^{'}, \quad (2.69)$$

$$I_9 = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k \hat{d}_{k'} \rangle e^{-i(\omega_a - \omega_k)} t + i(\omega_a - \omega_k^{'})t^{'}, \quad (2.70)$$

$$I_{10} = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k \hat{d}_{k'}^\dagger \rangle e^{i(\omega_a - \omega_k)} t - i(\omega_a - \omega_k^{'})t^{'}, \quad (2.71)$$

$$I_{11} = -\sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k^\dagger \hat{d}_{k'} \rangle e^{-i(\omega_a - \omega_k)} t + i(\omega_a - \omega_k^{'})t^{'}, \quad (2.72)$$

$$I_{12} = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k^\dagger \hat{d}_{k'}^\dagger \rangle e^{i(\omega_a - \omega_k)} t + i(\omega_a - \omega_k^{'})t^{'}, \quad (2.73)$$

In view of Eqs. (2.55), We easily see that

$$I_1 = I_4 = I_5 = I_8 = I_{10} = I_{11} = 0. \quad (2.74)$$

On account of Eqs.(2.69),we have

$$I_2 = -(N+1) \sum_k \lambda_k^2 e^{i(\omega_a - \omega_k)}(t - t^{'}) \quad (2.75)$$

Now replacing ω_a by the average value $\omega_0 = \frac{\omega_a - \omega_b}{2}$, and assuming the reservoir submode frequencies to be closely spaced then the summation can be converted into integration. We then write

$$I_2 = -(N+1) \int_0^\infty d\omega g(\omega) \lambda^2(\omega) \lambda(\omega) e^{i(\omega_0 - \omega)}(t - t^{'}), \quad (2.76)$$

where $g(\omega)$ is the density of the reservoir submodes. We expect $\omega$to vary little around ω_0.In view of this, we can replace $g(\omega)$and$\lambda(\omega)$ by $g(\omega_0)$ and $\lambda(\omega_0)$ and extend the lower limit of the integration to $-\infty$, so that

$$I_2 = -(N+1) g(\omega_O) \lambda^2(\omega_0) \int_{-\infty}^\infty d\omega e^{i(\omega_0 - \omega)}(t - t^{'}) \quad (2.77)$$

Moreover, upon setting $\omega^{'} = \omega - \omega_0$, we see that

$$I_2 = -(N+1) g(\omega_O) \lambda^2(\omega_0) \int_{-\infty}^\infty d\omega^{'} e^{i(t - t^{'})\omega^{'}}, \quad (2.78)$$

From which follows

$$I_2 = -k(N+1)\delta(t-t^{'}) \tag{2.79}$$

where

$$k = 2\pi g(\omega_0)\lambda^2(\omega_0) \tag{2.80}$$

is defined to be the cavity damping constant.Following a similar procedure, we can also easily obtain

$$I_3 = I_7 = -kN\delta(t-t^{'}), \tag{2.81}$$

$$I_6 = -k(N+1)\delta(t-t^{'}), \tag{2.82}$$

$$I_9 = I_{12} = kM\delta(t-t^{'}). \tag{2.83}$$

Upon substituting Eqs.(2.75), (2.80),(2.82),(2.83), and (2.84) into Eq.(2.62), we have

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) = & k\int(dt^{'}(N+1)(2\hat{a}\hat{\rho}(t^{'})\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho}(t^{'}) - \hat{\rho}(t^{'})\hat{a}^{\dagger}\hat{a})) \\ & +N(2\hat{a}\hat{\rho}(t^{'})\hat{a}^{\dagger} - \hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho}(t^{'}) - \hat{\rho}(t^{'})\hat{a}\hat{a}^{\dagger}) \\ & +(N+1)(2\hat{b}\hat{\rho}(t^{'})\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho}(t^{'}) - \hat{\rho}(t^{'})\hat{b}^{\dagger}\hat{a}) \\ & +N(2\hat{b}^{\dagger}\hat{\rho}(t^{'})\hat{b} - \hat{b}\hat{b}^{\dagger}\hat{\rho}(t^{'}) - \hat{\rho}(t^{'})\hat{b}\hat{b}^{\dagger}) \\ & -2M(\hat{b}^{\dagger}\hat{\rho}(t^{'})\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{\rho}(t^{'})\hat{b}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}(t^{'}) - \hat{\rho}(t^{'})\hat{a}^{\dagger}\hat{b}^{\dagger}) \\ & -2M(\hat{b}\hat{\rho}(t^{'})\hat{a} + \hat{a}\hat{\rho}(t^{'})\hat{b} - \hat{a}\hat{b}\hat{\rho}(t^{'}) - \hat{\rho}(t^{'})\hat{a}\hat{b})\delta(t-t^{'}), \end{aligned} \tag{2.84}$$

so that carrying out the integration, we get

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) = & \frac{k}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}) \\ & \frac{k}{2}N(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger} + 2\hat{b}^{\dagger}\hat{\rho}\hat{b} - \hat{b}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^{\dagger}) \\ & -kM(\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} + \hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}) \end{aligned} \tag{2.85}$$

This represents the equation of evolution of the reduced density operator for cavity modes coupled to a two-mode squeezed vacuum reservoir. Taking into account Eqs. (2.45)and(2.85), the master equation for the cavity modes of a nondegenerate three-level laser whose cavity mode coupled to a two-mode squeezed vacuum reservoir can be written as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \frac{1}{2}[(A\rho_{aa}^{(0)} + kN)(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}) + (A\rho_{cc}^{(0)}k(N+1))(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{\rho}\hat{b}^{\dagger}\hat{b} - \hat{b}^{\dagger}\hat{b}\hat{\rho})] \\ &-\frac{1}{2}[(A\rho_{ac} + kM)(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}) + (A\rho_{ca}^{(0)} + k(2\hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b})] \\ &+\frac{1}{2}k[(N+1)(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) + N(2\hat{b}^{\dagger}\hat{\rho}\hat{b} - \hat{b}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^{\dagger})] \\ &-\frac{1}{2}kM(2\hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} + 2\hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}) \end{aligned} \tag{2.86}$$

where

$$A = \frac{2g^2 r_a}{\gamma^2} \tag{2.87}$$

is the linear gain coefficient with γ being the spontaneous atomic decay rate assumed to be the same for all the three levels.

2.1.1 Stochastic Differential Equations

Next we wish to obtain stochastic differential equations associated with the normal ordering. The expectation value of an operator $\hat{a}$ evolves in time in the Schrodinger picture according to

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{2.88}$$

Employing Eq.(2.86)and Eq.(2.88we see that

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}\rangle &= \frac{1}{2}(A\rho_{aa}^{(0)}+kN)Tr(2\hat{a}^{\dagger}\hat{\rho}\hat{a}\hat{a}-\hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}) \\
&+\frac{1}{2}(A\rho_{cc}^{(0)}+k(N+1))Tr(2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a}-\hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a}-\hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a}) \\
&-\frac{1}{2}(A\rho_{ac}^{(0)}+kM)Tr(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}\hat{a}-\hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho}\hat{a}-\hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}\hat{a}) \\
&-\frac{1}{2}(A\rho_{ca}^{(0)}+kM)Tr(2\hat{b}\hat{\rho}\hat{a}\hat{a}-\hat{a}\hat{b}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{b}\hat{a}) \\
&+\frac{k}{2}[(N+1)Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a}-\hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a})+NTr(2\hat{b}^{\dagger}\hat{\rho}\hat{b}\hat{a}-\hat{b}\hat{b}^{\dagger}\hat{\rho}\hat{a}-\hat{\rho}\hat{b}\hat{b}^{\dagger}\hat{a})] \\
&-\frac{k}{2}MTr(2\hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\hat{a}-\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}+2\hat{a}\hat{\rho}\hat{b}\hat{a}-\hat{b}\hat{a}\hat{\rho}\hat{a}-\hat{\rho}\hat{b}\hat{a}\hat{a}). \qquad (2.89)
\end{aligned}$$

Applying the cyclic property of the trace operation and the commutation relation

$$[\hat{a},\hat{a}^{\dagger}] = [\hat{b},\hat{b}^{\dagger}]=1, \qquad (2.90)$$

we get

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{1}{2}\mu_a\langle\hat{a}\rangle-\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^{\dagger}\rangle \qquad (2.91)$$

where

$$\mu_a = k-A\rho_{aa}^{(0)}, \qquad (2.92)$$

$$\nu_- = -A\rho_{ac}^{(0)}. \qquad (2.93)$$

Following the same procedure, it can also be easily verified that

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{1}{2}\mu_c\langle\hat{b}\rangle+\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}\rangle, \qquad (2.94)$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\mu_a\langle\hat{a}^2\rangle-A\rho_{ac}^{(0)}\langle\hat{b}^{\dagger}\hat{a}\rangle, \qquad (2.95)$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\mu_c\langle\hat{b}^2\rangle+A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}\hat{b}\rangle, \qquad (2.96)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle = -\mu_a\langle\hat{a}^{\dagger}\hat{a}\rangle-\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle+\frac{1}{2}A\rho_{ac}^{(0)*}\langle\hat{a}\hat{b}\rangle+A\rho_{aa}^{(0)}+kN, \qquad (2.97)$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\mu_c\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^\dagger\hat{a}^\dagger\rangle - \frac{1}{2}A\rho_{ac}^{(0)*} + kN, \tag{2.98}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^\dagger 2\rangle - \frac{1}{2}A\rho_{ac}^{(0)*}\langle\hat{b}^2\rangle, \tag{2.99}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\hat{a}\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^\dagger\hat{a}\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}(A\rho_{ac}^{(0)} + 2kM), \tag{2.100}$$

in which

$$\mu_a = k - A\rho_{aa}^{(0)}, \mu_b = k + A\rho_{cc}^{(0)}, \mu_c = \frac{1}{2}[2k + A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)})] \tag{2.101}$$

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{1}{2}\mu_a\langle\alpha\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*\rangle \tag{2.102}$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{1}{2}\mu_c\langle\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*\rangle \tag{2.103}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle - A\rho_{ac}^{(0)}\langle\beta^*\beta\rangle, \tag{2.104}$$

$$\frac{d}{dt}\langle\beta^2\rangle = -\mu_a\langle\beta^2\rangle - A\rho_{ac}^{(0)}\langle\alpha^*\beta\rangle, \tag{2.105}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*\beta^*\rangle + \frac{1}{2}A\rho_{ac}^{(0)*}\langle\alpha\beta\rangle + A\rho_{aa}^{(0)} + kN, \tag{2.106}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\mu_c\langle\beta^*\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*\alpha^*\rangle + \frac{1}{2}A\rho_{ac}^{(0)*}\langle\alpha\beta\rangle + kN, \tag{2.107}$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha^*\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^{*2}\rangle + \frac{1}{2}A\rho_{ac}^{(0)*}\langle\beta^2\rangle, \tag{2.108}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha\beta\rangle + \frac{1}{2}A\rho_{ac}\langle\alpha^*\alpha\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*\beta\rangle + \frac{1}{2}(A\rho_{ac}^{(0)} + 2kM) \tag{2.109}$$

the basis of Eqs. (2.89) and (2.92), we can write(2.108)

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu_a\alpha(t) - \frac{1}{2}A\rho_{ac}^{(0)}\alpha^*(t) + f_\alpha(t), \tag{2.110}$$

$$\frac{d}{dt}\beta^*(t) = -\frac{1}{2}\mu_c\beta^*(t) - \frac{1}{2}A\rho_{ac}^{(0)}\beta^*(t) + f_\beta^*(t), \tag{2.111}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces. The formal solutions of these equations can be put in the form

$$\alpha(t) = \alpha(0)e^{-\mu_a\frac{t}{2}} + \int_0^t dt' e^{\mu_a(t-t')/2}\Big[-\frac{1}{2}A\rho_{ac}^{(0)}\beta^*(t') + f_\alpha(t')\Big], \tag{2.112}$$

$$\beta^*(t) = \beta^*(0)e^{-\mu_c\frac{t}{2}} + \int_0^t dt' e^{\mu_c(t-t')/2}\Big[\frac{1}{2}A\rho_{ac}\alpha(t') + f_\beta^*(t')\Big], \tag{2.113}$$

2.1.2 Correlations of Noise Forces

We proceed to determine the properties of the noise forces. We note that Eq. (2.92) and the expectation value of Eq. (2.100) as well as Eq. (2.93) and the expectation value of Eq.(2.101) will have the same form provided that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.114}$$

Applying the relation $\frac{d}{dt}\langle\alpha^2\rangle = 2\langle\alpha\frac{d}{dt}\alpha\rangle$ along with Eq.(2.95)

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle - A\rho_{ac}^{(0)}\langle\beta^*\alpha\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle \tag{2.115}$$

Comparison of this equation with (2.94) leads to

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{2.116}$$

On account of Eq.(2.102) along with (2.106) we see that

$$\langle\alpha(0)f_\alpha(t)\rangle e^{-\mu_a t/2} + \int_0^t e^{-\mu_a(t-t')/2}\Big[-\frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*(t' f_\alpha(t))\rangle + \langle f_\alpha(t')f_\alpha(t)\rangle\Big]dt' = 0, \tag{2.117}$$

so that taking into account Eq.(2.104) and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, we have

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0. \tag{2.118}$$

Similarly, we can easily establish that

$$\langle f_\beta(t^{'})f_\beta(t)\rangle = \langle f_\alpha^*(t^{'})f_\beta(t)\rangle = 0. \tag{2.119}$$

Furthermore, using Eq.(2.100) and its complex conjugate, we have

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*\beta^*\rangle \tag{2.120}$$

$$+\langle\alpha^*(t)f_\alpha(t)\rangle + \langle f_\alpha^*(t)f_\alpha^*(t)\alpha(t)\rangle. \tag{2.121}$$

Comparison of this equation with Eq.(2.96) shows that

$$\langle f_\alpha^*(t)f_\alpha^*(t)\alpha(t)\rangle = A\rho_{aa}^{(0)} + kN. \tag{2.122}$$

we note that the property of the dirac delta function

$$\int_0^t f(t^{'})\delta(t-t^{'})dt^{'} = \frac{1}{2}f(t) \tag{2.123}$$

In view of Eq.(2.112) so that Eq. (2.111) can be rewritten as

$$\int_0^t e^{-\mu_a(t-t)}/2\langle f_\alpha^*(t^{'})f_\alpha(t)\rangle dt^{'} = \int_0^t e^{-\mu_a(t-t)}/2(A\rho_{aa}^{(0)} + KN)\delta(t-t^{'})dt^{'} \tag{2.124}$$

It then follows that

$$\langle f_\alpha^*(t^{'})f_\alpha(t)\rangle = (A\rho_{aa}^{(0)} + KN)\delta(t-t^{'}). \tag{2.125}$$

It can also be established in a similar fashion that

$$\langle f_\alpha^*(t^{'})f_\alpha(t)\rangle = KN\delta(t-t^{'}) \tag{2.126}$$

$$\langle f_\alpha^*(t^{'})f_\alpha(t)\rangle = \frac{1}{2}(A\rho_{ac}^{(0)} + 2kM)\delta(t-t^{'}) \tag{2.127}$$

The results described by Eqs. (2.109), (2.113), (2.114), (2.119), (2.123) and (2.125) represent the correlation properties of the noise forces and $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

2.1.3 Solutions of Stochastic Differential Equations

In the previous section we have found two Stochastic Differential Equations for $\alpha(t)$ and $\beta^*(t)$.

In this section we seek to obtain the solution of these coupled Stochastic Differential Equations using matrix method.

On account of Eq.(2.108)and Eq.(2.109)

$$\frac{d}{dt}J(t) = -\frac{1}{2}MJ(t) + F(t), \quad (2.128)$$

$$J(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \quad (2.129)$$

$$M = \begin{pmatrix} \mu_a & A\rho_{ac}^{(0)} \\ -A\rho_{ac}^{(0)} & \mu_c \end{pmatrix} \quad (2.130)$$

$$F(t) = \begin{pmatrix} f_\alpha(t) \\ f_\beta^*(t) \end{pmatrix} \quad (2.131)$$

Introducing a matrix defined by

$$V = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} \quad (2.132)$$

$$V_1 = \begin{pmatrix} \nu_{11} \\ \nu_{21} \end{pmatrix}$$

and

$$V_2 = \begin{pmatrix} \nu_{12} \\ \nu_{22} \end{pmatrix}$$

being the eigenvectors of the matrix B, Eq.(2.121)can be written as

$$\frac{d}{dt}J(t) = -\frac{1}{2}VV^{-1}J(t) + F(t). \quad (2.133)$$

Multiplying both sides from the left by V^{-1} we see that

$$\frac{d}{dt}(V^{-1}J(t)) = -\frac{1}{2}R(V^{-1}J(t)) + V^{-1}F(t), \tag{2.134}$$

where

$$R = = V^{-1}MV \begin{pmatrix} \frac{1}{2}(2\kappa + A\eta + \lambda) & 0 \\ 0 & \frac{1}{2}(2\kappa - A\eta - \lambda) \end{pmatrix} \tag{2.135}$$

in which λ_+ and λ_- are the eigenvalues of the matrix M. We note that Eq. (2.124) has a well defined solution for $\lambda_+ > 0$and $\lambda_- > 0$. The solution of this equation can be written as

$$J(t+\tau) = Ve^{-\frac{1}{2}R(\tau)V^{-1}J(t)} + \int_0^\tau Ve^{-\frac{1}{2}R(\tau-\tau')V^{-1}F(t+\tau')})d\tau'. \tag{2.136}$$

We next proceed to find the eigenvalues and eigenvectors of the matrix M. Applying the eigenvalue equation

$$MV_i = \lambda V_i \tag{2.137}$$

along with Eq (2.119), we find the characteristic equation

$$\lambda^2 - (\mu_a + \mu_c)\lambda + (\mu_a\mu_c - A\rho_{ac}^{(0)*}A\rho_{ac}^{(0)}) = 0. \tag{2.138}$$

$$\lambda_\pm = \frac{1}{2}(2k + A\eta \pm \lambda). \tag{2.139}$$

Taking into account Eqs. (2.91), (2.100), and the relation

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{2.140}$$

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}, \tag{2.141}$$

$$\lambda = \sqrt{A^2 - 4A\rho_{ac}^{(0)*}A\rho_{ac}^{(0)}}. \tag{2.142}$$

With the aid of Eqs.(2.80),(2.90),(2.119), (2.131), and (2.132), we have

$$(A+\lambda)\nu_{11} - 2A\rho_{ac}^{(0)}\nu_{21} = 0, \tag{2.143}$$

and taking into account the normalization condition:

$$\nu_{11}^2 + \nu_{21}^2 = 1, \tag{2.144}$$

we get

$$\begin{aligned} \nu_{11} &= \frac{-2(A\rho_{ac}^{(0)})}{\sqrt{(A+\lambda)^2 + 4(-A\rho_{ac}^{(0)})^2}} \\ \nu_{21} &= \frac{A+\lambda}{\sqrt{(A+\lambda)^2 + 4(-A\rho_{ac}^{(0)})^2}} \end{aligned} \tag{2.145}$$

Similarly we can also easily show that the elements of the eigenvector corresponding to λ_- to be

$$\begin{aligned} \nu_{12} &= \frac{2A\rho_{ac}^{(0)}}{\sqrt{(A-\lambda)^2 + 4(-A\rho_{ac}^{(0)})^2}} \\ \nu_{22} &= \frac{A-\lambda}{\sqrt{(A-\lambda)^2 + 4(-A\rho_{ac}^{(0)})^2}} \end{aligned} \tag{2.146}$$

Now substitution of Eqs. (2.136) and (2.137) into Eq.(2.112) yeids

$$v = \begin{pmatrix} \frac{-2A\rho_{ac}^{(0)}}{\sqrt{(A_+^2+4(-A\rho_{ac}^{(0)})^2}} & \frac{2A\rho_{ac}^{(0)}}{\sqrt{A_-^2+4(-A\rho_{ac}^{(0)})^2}} \\ -\frac{A_+}{\sqrt{A_+^2+4(-A\rho_{ac}^{(0)})^2}} & \frac{A_-}{\sqrt{A_-^2+4(-A\rho_{ac}^{(0)})^2}} \end{pmatrix}, \tag{2.147}$$

in which

$$A_\pm = A \pm A\eta \tag{2.148}$$

And the inverse of the matrix V is found to be

$$V^{-1} = \frac{1}{4A\rho_{ac^{(0)}}\lambda} \begin{pmatrix} A_-\sqrt{A_+^2 + 4(-A\rho_{ac}^{(0)})^2} & -2A\rho_{ac}^{(0)}\sqrt{A_+^2 + 4(-A\rho_{ac}^{(0)})^2} \\ A_+\sqrt{A_-^2 + 4(-A\rho_{ac}^{(0)})^2} & -2A\rho_{ac}^{(0)}\sqrt{A_-^2 + 4(-A\rho_{ac}^{(0)})^2} \end{pmatrix} \tag{2.149}$$

Since Eq.(2.125) describes a diagonal matrix, we observe that

$$e^{-\frac{1}{2}R\tau} = \begin{pmatrix} e^{-\frac{1}{4}(2\kappa+A\eta+\lambda)\tau} & 0 \\ 0 & e^{-\frac{1}{4}(2\kappa-A\eta-\lambda)\tau} \end{pmatrix}, \tag{2.150}$$

$$e^{-\frac{1}{2}R(\tau} - \tau^{'}) = \begin{pmatrix} e^{-\frac{1}{4}(2\kappa+A\eta+\lambda)(\tau-\tau^{'})} & 0 \\ 0 & e^{-\frac{1}{4}(2\kappa-A\eta-\lambda)\tau-\tau^{'}} \end{pmatrix}, \tag{2.151}$$

From which follows

$$Ve^{-\frac{1}{2}R\tau V^{-1}} = \begin{pmatrix} T_1(\tau) & C_1(\tau) \\ C_2(\tau) & T_2(\tau) \end{pmatrix} \tag{2.152}$$

and

$$Ve^{-\frac{1}{2}R(\tau-\tau^{'})V^{-1}} = \begin{pmatrix} T_1(\tau-\tau^{'}) & C_1(\tau-\tau^{'}) \\ C_2(\tau-\tau^{'}) & T_2(t-t^{'}) \end{pmatrix}, \tag{2.153}$$

where

$$T_1(\tau) = \frac{A+A\eta}{2A\eta}e^{-\frac{1}{2}(\kappa)\tau} - \frac{A-A\eta}{2A\eta}e^{-\frac{1}{2}(\kappa+A\eta)\tau}, \tag{2.154}$$

$$T_2(\tau) = \frac{A+A\eta}{2A\eta}e^{-\frac{1}{4}(2\kappa+2A\eta)\tau} - \frac{A-A\eta}{2A\eta}e^{-\frac{1}{2}(\kappa+2A\eta+)\tau}, \tag{2.155}$$

$$C_1(\tau) = \frac{-A\rho_{ac}^{(0)}}{A\eta}e^{-\frac{1}{2}(\kappa)\tau} + \frac{A\rho_{ac}^{(0)}}{A\eta}e^{-\frac{1}{2}(\kappa+2A\eta)\tau}, \tag{2.156}$$

$$C_2(\tau) = \frac{A\rho_{ac}^{(0)*}}{A\eta}e^{-\frac{1}{2}(\kappa)\tau} - \frac{A\rho_{ac}^{(0)*}}{A\eta}e^{-\frac{1}{2}(\kappa+A\eta+)\tau} \tag{2.157}$$

$$T_1(\tau-\tau^{'}) = \frac{A+A\eta}{2\kappa}e^{-\frac{1}{2}(\kappa)(\tau-\tau^{'})} - \frac{A-A\eta}{2A\eta}e^{-\frac{1}{2}(\kappa+A\eta+\lambda)(\tau-\tau^{'})}, \tag{2.158}$$

$$T_2(\tau-\tau^{'}) = \frac{A+A\eta}{2A\eta}e^{-\frac{1}{2}(\kappa+A\eta)(\tau-\tau^{'})} - \frac{A-A\eta}{2A\eta}e^{-\frac{1}{2}(\kappa)(\tau-\tau^{'})}, \tag{2.159}$$

$$C_1(\tau-\tau^{'}) = \frac{-A\rho_{ac}^{(0)}}{A\eta}e^{-\frac{1}{2}(\kappa)(\tau-\tau^{'})} + \frac{A\rho_{ac}^{(0)}}{A\eta}e^{-\frac{1}{2}(\kappa+A\eta)(\tau-\tau^{'})}, \tag{2.160}$$

$$C_2(\tau-\tau^{'}) = \frac{A\rho_{ac}^{(0)*}}{A\eta}e^{-\frac{1}{2}(\kappa)(\tau-\tau^{'})} - \frac{A\rho_{ac}^{(0)*}}{A\eta}e^{-\frac{1}{2}(2\kappa+A\eta)(\tau-\tau^{'})}, \tag{2.161}$$

With the aid of Eqs. (2.121), (2.122), (2.123),(2.128), (2.144), and (2.145), we finally obtain

$$\alpha(t+\tau) = T_1\alpha(t) + C_1(\tau)\beta^*(t) + G_+(t+\tau), \tag{2.162}$$

$$\beta^*(t+\tau) = T_2\beta^*(t) + C_2(\tau)\beta^*(t) + G_-(t+\tau), \tag{2.163}$$

where

$$G_+(t+\tau) = \int_0^\tau [T_1(\tau-\tau^{'})f_\alpha(\tau^{'}+t) + C_1(\tau-\tau^{'})f_\beta^*(\tau^{'}+t)]d\tau^{'}, \tag{2.164}$$

$$G_-(t+\tau) = \int_0^\tau [T_2(\tau-\tau^{'})f_\alpha(\tau^{'}+t) + C_2(\tau-\tau^{'})f_\beta^*(\tau^{'}+t)]d\tau^{'}, \tag{2.165}$$

Furthermore, upon setting $t=0$ and $\tau=t$, the cavity mode variables $\alpha(t)$ and $\beta(t)$ take the form

$$\alpha(t) = T_1(t)\alpha(0) + C_1(t)\beta^*(0) + G_+(t), \tag{2.166}$$

$$\beta^*(t) = T_1(t)\alpha(0) + C_2(t)\alpha(0) + G_-(t). \tag{2.167}$$

3

Quadrature Squeezing

In this chapter we seek to analyze the quadrature squeezing properties of the two-mode cavity light.

3.1 Quadrature Variance

A two-mode cavity light can be described by an operator

$$\hat{c} \;=\; \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{3.1}$$

where $\hat{a}$ and $\hat{b}$ represent the separate modes.
On the other hand, employing the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] \;=\; [\hat{b}, \hat{b}^{\dagger}] = 1 \tag{3.2}$$

$$[\hat{a}^{\dagger}, \hat{a}] \;=\; [\hat{b}^{\dagger}, \hat{b}] = 0 \tag{3.3}$$

we have

$$[\hat{c}, \hat{c}^{\dagger}] \;=\; 1 \tag{3.4}$$

$$[\hat{c}, \hat{c}] \;=\; 0 \tag{3.5}$$

The squeezing of the two-mode cavity light can be studied applying the quadrature operators defined by

$$\hat{c}_{+} \;=\; \hat{c}^{\dagger} + \hat{c}, \tag{3.6}$$

$$\hat{c}_- \;=\; i(\hat{c}^\dagger - \hat{c}) \tag{3.7}$$

where $\hat{c}_+$and $\hat{c}_-$are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{c}^\dagger$and $\hat{c}$ are the creation and annihilation oprators of the two-mode cavity light. Usind Eq.(3.7)and (3.8),one can verify that

$$[\hat{c}_+, \hat{c}_-] \;=\; 2i. \tag{3.8}$$

We now seek to calculate the variance of the quadrature operators (3.4) and (3.6).
To begin with, making use of the commutation relations(3.8)we get

$$\Delta c_\pm^2 \;=\; 1 \pm [\langle \hat{c}^{\dagger 2}\rangle + \langle \hat{c}^\dagger\rangle \pm 2\langle \hat{c}^\dagger \hat{c}\rangle + \langle \hat{c}^\dagger\rangle^2 + \langle \hat{c}\rangle^2 \pm 2\langle \hat{c}\rangle^2 \langle \hat{c}^\dagger\rangle^2]. \tag{3.9}$$

We notice that all operators in Eq. (3.9) are in the normal order. Therefore, the c-number equation corresponding to (3.9) can be expressed as

$$\Delta c_\pm^2 \;=\; 1 \pm \langle \gamma_\pm(t), \gamma_\pm(t)\rangle, \tag{3.10}$$

in which

$$\gamma_\pm(t) \;=\; \frac{1}{\sqrt{2}}(\alpha^*(t) + \beta^*(t) \pm \alpha(t) \pm \beta(t)). \tag{3.11}$$

On account of Eq.(3.11), we see that

$$\begin{aligned}\langle \gamma_\pm(t), \gamma_\pm(t)\rangle \;=\; & \frac{1}{2}(\langle \alpha(t), \alpha(t)\rangle + \langle \beta^*(t), \beta^*(t)\rangle 2\langle \alpha(t), \beta(t) \\ & \langle \pm\langle \alpha^*(t), \alpha(t)\rangle \pm \beta^*(t), \beta^*(t) \pm 2\langle \beta^*(t), \alpha(t)\rangle) + s.c.c.,\end{aligned} \tag{3.12}$$

in which c.c. stands for complex conjugate. Using Eqs. (2.104),(2.108), (2.109),(2.154), (2.155), (2.156), (2.157), and assuming the cavity modes are initially in vacuum states along with the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time, we easily find

$$\langle \alpha(t), \beta(t)\rangle \;=\; \langle \beta(t), \beta(t)\rangle = \beta^*(t), \alpha(t) = 0, \tag{3.13}$$

so that in view of these results, Eq. (3.10) reduces to

$$\langle\gamma_{\pm}(t),\gamma_{\pm}(t)\rangle = \langle\alpha(t),\beta(t)\rangle + \langle\alpha(t)\beta^*(t)\rangle \pm \langle\alpha^*(t),\alpha(t)\rangle \pm \langle\beta^*,\beta(t)\rangle. \quad (3.14)$$

Furthermore, taking into account Eq. (2.156) along with its complex conjugate, we get

$$\langle\alpha^*(t)\alpha(t)\rangle = \langle G_1^*(t)G_1(t)\rangle. \quad (3.15)$$

With the aid of Eqs.(2.154),(2.114),(2.115), and (2.116), we have

$$\begin{aligned}\langle\alpha^*(t),\alpha(t)\rangle &= \int_0^t (|p_1(t-t^{'})|^2 f_{\alpha^*\alpha} + p_1^*(t-t^{'})q_1(t-t^{'})f^*_{\alpha\beta}) \\ &+q_1^*(t-t^{'})p_1(t-t^{'})f_{\alpha\beta} + |q_1(t-t^{'})|^2 kN)dt^{'}. \quad (3.16)\end{aligned}$$

Applying (2.148) and (2.150) in Eq. (3.15) and then carrying out the integration, we get

$$\begin{aligned}\langle\alpha^*(t),\alpha(t)\rangle &= \frac{A+A\eta[(A+A\eta)f\alpha^*\alpha - 2A\rho_{ac}^{(0)}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)*}[A+A\eta f_{\alpha\beta} - 2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2\kappa}(1+e^{-(\kappa)t}) \\ &-\frac{A+A\eta[(A-A\eta)f\alpha^*\alpha - 2A\rho_{ac}^{(0)}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)*}[(A-A\eta)f_{\alpha\beta} - 2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2(2\kappa+A\eta)/2} \\ &\times(1+e^{(\kappa-\frac{A\eta}{2})t}) \\ &-\frac{A-A\eta[(A+A\eta)f\alpha^*\alpha - 2A\rho_{ac}^{(0)}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)*}[(A+A\eta)f_{\alpha\beta} - 2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2(\kappa+A\eta)} \\ &\times(1+e^{-(\kappa+\frac{A\eta}{2})t}) \\ &+\frac{A_+^*[A_-f\alpha^*\alpha - 2A\rho_{ac}^{(0)}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)*}[(A+A\eta)f_{\alpha\beta} - 2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2(\kappa+A\eta)}(1+e^{(\kappa-A\eta)t})\end{aligned} \quad (3.17)$$

Following a similar procedure, we also find

$$\begin{aligned}\langle\beta^*(t)\beta(t)\rangle &= \frac{A+A\eta[A_+kN - 2A\rho_{ac}^{(0)*}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)}[A_+f^*_{\alpha\beta} - 2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2(\kappa+A\eta)}(1-e^{-(\kappa+a\eta)t}) \\ &-\frac{A+A\eta[A_-kN - 2A\rho_{ac}^{(0)*}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)}[A_+f^*_{\alpha\beta} - 2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2(\kappa+A\eta)}(1-e^{-(\kappa+A\eta)t})) \\ &-\frac{A-A\eta[A_+kN - 2A\rho_{ac}^{(0)*}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)}[A_-f^*_{\alpha\beta} - 2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2(\kappa+A\eta/2)}(1-e^{-\frac{1}{2}(2\kappa+A\eta)t}) \\ &+\frac{A+A\eta[A_+kN - 2A\rho_{ac}^{(0)*}f_{\alpha\beta}] - 2A\rho_{ac}^{(0)}[A_+f^*_{\alpha\beta} - 2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2\kappa}(1-e^{-(\kappa)t})\end{aligned} \quad (3.18)$$

and

$$\begin{aligned}\langle\alpha(t)\beta(t)\rangle &= \frac{A+A\eta[A_+f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_+f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa+A\eta/2)}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-)t})\\ &-\frac{A+A\eta[A_-f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_-f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa+A\eta)}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+)t})\\ &-\frac{A-A\eta[A_-f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_+f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa)}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t})\\ &+\frac{A+A\eta[A_-f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_-f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa+A\eta/2)}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t}).\end{aligned} \tag{3.19}$$

At stades state Eqs.(3.17),(3.18),(3.19)takes the form

$$\begin{aligned}\langle\alpha^*(t),\alpha(t)\rangle_{ss} &= \frac{A+A\eta[(A+A\eta)f\alpha^*\alpha-2A\rho_{ac}^{(0)}f_{\alpha\beta}]-2A\rho_{ac}^{(0)*}[A+A\eta f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2\kappa}\\ &-\frac{A+A\eta[(A-A\eta)f\alpha^*\alpha-2A\rho_{ac}^{(0)}f_{\alpha\beta}]-2A\rho_{ac}^{(0)*}[(A-A\eta)f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2(\kappa+A\eta/2)}\\ &-\frac{A-A\eta[(A+A\eta)f\alpha^*\alpha-2A\rho_{ac}^{(0)}f_{\alpha\beta}]-2A\rho_{ac}^{(0)*}[(A+A\eta)f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2(\kappa+A\eta)}\\ &+\frac{A_+^*[A_-f\alpha^*\alpha-2A\rho_{ac}^{(0)}f_{\alpha\beta}]-2A\rho_{ac}^{(0)*}[(A+A\eta)f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]}{4A^2\eta^2(\kappa+A\eta)}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\langle\beta^*(t)\beta(t)\rangle_{ss} &= \frac{A+A\eta[A_+kN-2A\rho_{ac}^{(0)*}f_{\alpha\beta}]-2A\rho_{ac}^{(0)}[A_+f^*_{\alpha\beta}-2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2(\kappa+A\eta)}\\ &-\frac{A+A\eta[A_-kN-2A\rho_{ac}^{(0)*}f_{\alpha\beta}]-2A\rho_{ac}^{(0)}[A_+f^*_{\alpha\beta}-2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2(\kappa+A\eta)}\\ &-\frac{A-A\eta[A_+kN-2A\rho_{ac}^{(0)*}f_{\alpha\beta}]-2A\rho_{ac}^{(0)}[A_-f^*_{\alpha\beta}-2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2(\kappa+A\eta/2)}\\ &+\frac{A+A\eta[A_+kN-2A\rho_{ac}^{(0)*}f_{\alpha\beta}]-2A\rho_{ac}^{(0)}[A_+f^*_{\alpha\beta}-2A\rho_{ac}^{(0)*}f\alpha^*\alpha]}{4A^2\eta^2\kappa},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\langle\alpha(t)\beta(t)\rangle_{ss} &= \frac{A+A\eta[A_+f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_+f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa+A\eta/2)}\\ &-\frac{A+A\eta[A_-f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_-f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa+A\eta)}\\ &-\frac{A-A\eta[A_-f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_+f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa)}\\ &+\frac{A+A\eta[A_-f_{\alpha\beta}-2A\rho_{ac}^{(0)}kN]-2A\rho_{ac}^{(0)}[A_-f\alpha^*\alpha-2A\rho_{ac}^{(0)}f^*_{\alpha\beta}]}{4A^2\eta^2(\kappa+A\eta/2)}.\end{aligned} \tag{3.22}$$

where ss stands for steady state

To simplify our task it is more convenient to introduce parameter η as follows

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{3.23}$$

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{3.24}$$

$$\rho_{ac}^{(0)} = \frac{\sqrt{1-\eta^2}}{2}. \tag{3.25}$$

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}, \tag{3.26}$$

We notice that the steady state solutions Eqs. (2.91)(2.95) are valid only for non-negative values of η, that is, $0 \leq \eta \leq 1$.

The atomic coherence introduced by the superposition of atomic states is given by Eq. (2.98).

We clearly see that the atomic coherence is maximum when there is equal probabilities of finding for an atom to be in the upper or lower level and it is zero when all atoms are initially in lower level.

It worth mentioning that the degree of squeezing of the two-mode cavity light produced by the three-level cascade laser highly depends on the atomic coherence.

We note that the c-number equations corresponding to Eqs.(2.79)-(2.90) which are in the normal order,now substitution of Eqs. (3.18), (3.19), (3.20), and the complex conjugate of Eq. (3.21) into Eq. (5.13) leads to

$$\begin{aligned}\langle\gamma_\pm(t),\gamma_\pm(t)\rangle &= \pm\frac{1}{4A^2\eta^2}\bigg[\frac{(A+A\eta\pm 2A\rho_{ac}^{(0)*})[(A_+^*\pm 2A\rho_{ac}^{(0)})f_\alpha^*,\alpha\mp(A_-^*+2A\rho_{ac}^{(0)*})f_{\alpha,\beta}]}{2\kappa}\\ &+\frac{(A-A\eta+2A\rho_{ac}^{(0)})(A_-^*+2A\rho_{ac}^{(0)*})kN\mp(A_+^*\pm 2A\rho_{ac}^{(0)})f_{\alpha,\beta}^*}{2\kappa}\bigg]\\ &\pm\frac{1}{4A^2\eta^2}\bigg[\frac{(A_-\pm 2A\rho_{ac}^{(0)*})[(A_-^*\pm 2A\rho_{ac}^{(0)})f_{\alpha^*,\alpha}\mp(A_+^*+2A\rho_{ac}^{(0)*})f_{\alpha,\beta}]}{2\kappa+2A\eta}\\ &+\frac{(A+A\eta\mp 2\nu_-)[(A_+^*+2A\rho_{ac}^{(0)*})kN\mp(A_-^*\pm 2A\rho_{ac}^{(0)})f_{\alpha,\beta}^*]}{2\kappa+2A\eta}\bigg]\\ &\mp\frac{1}{2A^2\eta^2}\bigg[\frac{(A_-\pm 2A\rho_{ac}^{(0)*})[(A_+^*\pm 2A\rho_{ac}^{(0)})f_{\alpha^*,\alpha}\mp(A_-^*+2A\rho_{ac}^{(0)*})f_{\alpha,\beta}]}{2\kappa+A\eta}\\ &+\frac{(A+A\eta+2A\rho_{ac}^{(0)})[(A_-^*+2A\rho_{ac}^{(0)*})kN\mp(A_+^*\pm 2A\rho_{ac}^{(0)})f_{\alpha,\beta}^*]}{2\kappa+A\eta}\bigg]\\ &+c.c.\end{aligned}\tag{3.27}$$

On account of Eqs.(3.19),(3.22),(3.23),(3.28) to gether with Eq.(3.4),the quadrature variances of the two-mode cavity light turn out to be

$$\begin{aligned}\Delta c_\pm^2 &= 1+\frac{1}{2A^2\eta^2}\bigg[\frac{|A+A\eta\pm 2A\rho_{ac}^{(0)*}|^2}{2\kappa}+\frac{|A-A\eta\pm 2A\rho_{ac}^{(0)*}|^2}{2\kappa+2A\eta}\\ &-\frac{(A+A\eta\pm 2A\rho_{ac}^{(0)})(A-A\eta\pm 2A\rho_{ac}^{(0)*})}{2\kappa+A\eta}-\frac{(A+A\eta\pm 2A\rho_{ac}^{(0)*})(A_A\eta\pm 2A\rho_{ac}^{(0)})}{2\kappa+A\eta}\bigg]f\alpha^*\alpha\\ &+\frac{1}{2A^2\eta^2}\bigg[\frac{|A-A\eta+2A\rho_{ac}^{(0)}*|^2}{2\kappa}+\frac{|A+A\eta+2A\rho_{ac}^{(0)}|^2}{2\kappa+2A\eta}\\ &-\frac{(A+A\eta+2A\rho_{ac}^{(0)})(A-A\eta+2A\rho_{ac}^{(0)*})}{2\kappa+A\eta}-\frac{(A+A\eta+2A\rho_{ac}^{(0)*})(A_-+2A\rho_{ac}^{(0)})}{2\kappa+A\eta}\bigg]kN\\ &\mp\frac{1}{2A^2\eta^2}\bigg[\frac{(A+A\eta\pm 2A\rho_{ac}^{(0)*})(A-A\eta+2A\rho_{ac}^{(0)*})(A-A\eta\pm 2A\rho_{ac}^{(0)*})}{2\kappa+2A\eta}\\ &-\frac{(A-A\eta+2A\rho_{ac}^{(0)*})(A-A\eta\pm 2A\rho_{ac}^{(0)*})}{2\kappa}-\frac{(A+A\eta\mp 2A\rho_{ac}^{(0)*})(A+a\eta\pm 2A\rho_{ac}^{(0)*})}{2\kappa+A\eta}\bigg]f_{\alpha\beta}\\ &\mp\frac{1}{2A^2\eta^2}\bigg[\frac{(A+A\eta\pm 2A\rho_{ac}^{(0)*})-(A-A\eta+2A\rho_{ac}^{(0)})}{2\kappa}+\frac{(A_++A\rho_{ac}^{(0)})(A-A\eta\pm 2A\rho_{ac}^{(0)})}{2\kappa+2A\eta}\\ &-\frac{(A+A\eta+2A\rho_{ac}^{(0)})(A+a\eta\pm 2A\rho_{ac}^{(0)})}{2\kappa+A\eta}-\frac{(A_-^*\pm 2A\rho_{ac}^{(0)})(A-A\eta+2A\rho_{ac}^{(0)})}{2\kappa+A\eta}\bigg]f_{\alpha\beta}^*.\end{aligned}\tag{3.28}$$

$$\begin{aligned}\Delta c_{\pm}^2 &= 1+\frac{1}{C}\left[\frac{|E\pm B|^2}{2\kappa}+\frac{|F\pm B|^2}{2\kappa+2A\eta}\right.\\ &\left.-\frac{(E\pm B)(F\pm B}{2\kappa+A\eta}-\frac{(E\pm B)(F\pm B)}{2\kappa+A\eta}\right]f\alpha^*\alpha\\ &+\frac{1}{C}\left[\frac{|F+B|^2}{2\kappa}+\frac{|E+B|^2}{2\kappa+2A\eta}\right.\\ &\left.-\frac{(E+B)(F+F}{2\kappa+A\eta}-\frac{(E+B)(F+B}{2\kappa+A\eta}\right]kN\\ &\mp\frac{1}{C}\left[\frac{(E\pm B)(F+B)(F+B)}{2\kappa+2A\eta}\right.\\ &\left.-\frac{(F+B)(F+B}{2\kappa}-\frac{(F+B)(E\pm)B}{2\kappa+A\eta}\right]f_{\alpha\beta}\\ &\mp\frac{1}{C}\left[\frac{(E\pm B)-(F+B)}{2\kappa}+\frac{(E+B)(F\pm B)}{2\kappa+2A\eta}\right.\\ &\left.-\frac{(E+B)(E\pm B)}{2\kappa+A\eta}-\frac{(F\pm B)(F+B)}{2\kappa+A\eta}\right]f_{\alpha\beta}^*. \end{aligned} \tag{3.29}$$

in which

$$C = 2A^2\eta^2 \tag{3.30}$$

$$E = A+A\eta, \tag{3.31}$$

$$F = A-A\eta, \tag{3.32}$$

$$B = 2A\rho_{ac}^{(0)*}=2A\rho_{ac}^{(0)} \tag{3.33}$$

$$f_{\alpha^*\alpha} = (A(1-\eta)+2\kappa N)/2=A\rho_{aa}^{(0)}+kN, \tag{3.34}$$

$$f_{\alpha\beta} = f_{\alpha\beta}^*=(A\sqrt{1-\eta^2}+4\kappa M)/4 \tag{3.35}$$

$$f_{\alpha\beta}=2A\rho_{ac}^{(0)}+kM)/2. \tag{3.36}$$

$$\lambda_- = 2\kappa and \lambda_+=2\kappa+A\eta \tag{3.37}$$

Now experecing Eq.(3.20),(3.21)and (3.22) interims of η as

$$\begin{aligned}\langle\alpha^*(t),\alpha(t)\rangle_{ss} &= \frac{\kappa A(1-\eta)(4\kappa+3A\eta+A)}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{[2\kappa(2\kappa+2A\eta+A)+A^2(1+\eta)]2\kappa N}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{[-A\sqrt{1-\eta^2}](2\kappa+A\eta+A)2\kappa M}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}. \qquad (3.38)\end{aligned}$$

$$\begin{aligned}\langle\beta^*(t)\beta(t)\rangle_{ss} &= \frac{k(A\sqrt{1-\eta^2})+(A\sqrt{1-\eta^2})[2k+A\eta-A]2kM+[A^2(1-\eta)]2kN}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{[A^2\sqrt{1-\eta^2}]2kN}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)} \qquad (3.39)\\ & \qquad (3.40)\end{aligned}$$

$$\langle\alpha(t)\beta(t)\rangle_{ss} = \frac{k(A\sqrt{1-\eta^2})^2[2k+A\eta+A]+[(2k+A\eta)^2-A^2]2kM}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)} \qquad (3.41)$$

$$\langle\alpha^2(t)\rangle=\langle\beta^2(t)\rangle=\langle\alpha^\dagger(t)\beta(t)\rangle=0. \qquad (3.42)$$

Hence on account of Eqs.(3.29)-(3.42),the quadrature variances of the two-mode cavity light at a steady state turn out to be

$$\begin{aligned}\Delta c_\pm^2 &= 1+\frac{2\kappa A(1-\eta)(2\kappa+2A\eta+A)-4\kappa A^2\eta^2N}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &\pm\frac{2\kappa(A\sqrt{1-\eta^2})(2\kappa+A\eta+A)}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{4\kappa[(2\kappa+A\eta)(2\kappa+A\eta)(N\pm M)+A^2(\sqrt{1-\eta^2})(N\mp M)]}{4[\kappa(\kappa+A\kappa)](2\kappa+A\kappa)}. \qquad (3.43)\end{aligned}$$

We clearly shows that in Fig.3.1, we plot the variance of the minus quadrature versus η for different values of the linear gain coefficient(A). We note from this figure that the two-mode cavity light exhibits a two-mode squeezing.We also see that the degree of squeezing increase with the linear gain coefficient which is in a complete agreement with previous studies [9,13,16,20].

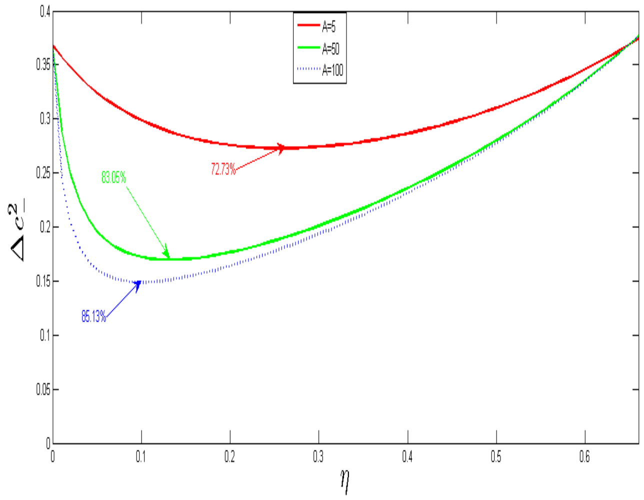

Figure 3.1: Plot of the quadrature variance of the two-mode cavity light at steady state[Eq. 3.43]versus η for different values of the linear gain coefficient when $r = 0.5$,and $\kappa = 0.8$.

The plots in Fig. 3.1 represent the quadrature variance for different values of linear gain coefficients,$r = 0.5$ and $\kappa = 0.8$. As depicted in the Fig.3.1 we note that the quadrature variance attained its small values for larger gain coefficients and slightly small values of η.

This implies that the quadrature squeezing increases with linear gain coefficients for η between 0and 0.85. The minimum value of the quadrature variance is found to be $\Delta c_{\pm}^2 = 0.1487$ and occurs at $\eta = 0.1$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$,. This result indicates that the maximum squeezing for the above value is found to be 85.13% below the coherent-state level.

Table 3.1: : Maximum squeezing occurs for $\eta = 0.1$, $\kappa = 0.8$,$r = 0.5$ and different values of linear gain coefficient (A).

η	Maximum squeezing	Maximum squeezing occurs for
0.26	72.73%	$A = 5$
0.13	83.05%	$A = 50$
0.1	85.13%	$A = 100$

Table 3.2: Maximum squeezing occurs for $A = 100$, $\kappa = 0.8$, $\eta = 0.09$ and different values of squeezed parameter (r).

η	Maximum squeezing	Maximum squeezing occurs for
0.18	65.3%	$r = 0$
0.12	79.2%	$r = 0.3$
0.09	87.4%	$r = 0.6$

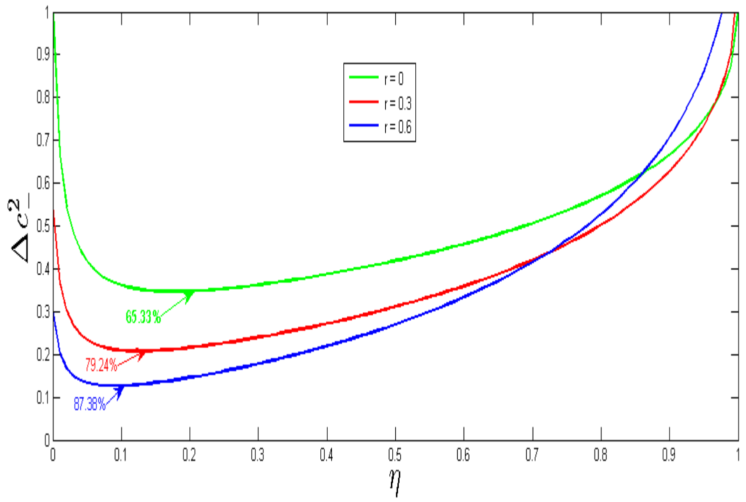

Figure 3.2: Plot of the quadrature variance of the two-mode cavity light at steady state[3.43]versus η for different values of the squeeze parameter r when $A = 100$,and $\kappa = 0.8$.

4

Photon Statistics

In this section we study the statistical properties of the cavity modes produced by a nondegenerate three-level laser coupled to a two-mode squeezed vacuum reservoir. We first obtain, using the anti-normally ordered characteristic function defined in the Heisenberg picture, the Q function for the cavity modes then applying the resulting Q function, we calculate the mean photon number ,the mean photon variances and the photon number correlations.

4.1 Photon Statistics of the Cavity Modes

The Q function for a two-mode cavity light can be expressed as

$$Q(\alpha,\beta,t) \;=\; \frac{1}{\pi^2}\int \frac{d^2z}{\pi}\frac{d^2\omega}{\pi}\Phi_A(z,\omega,t)e^{z^*\alpha - z\alpha^* + \omega^*\beta - \omega\beta^*}, \tag{4.1}$$

where

$$\Phi_A(z,\omega,t) = Tr(\rho(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\omega^*\hat{b}(t)}e^{\omega}\hat{b}^\dagger(t)) \tag{4.2}$$

is the antinormally ordered characteristic function defined in the Heisenberg picture. Employing the Baker-Hausdorff identity, we can rewrite Eq. (4.2) in the normal order as

$$\Phi_A(z,\omega,t) \;=\; e^{-z^*z-\omega^*\omega}Tr(\rho(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\omega^*\hat{b}(t)}e^{\omega}\hat{b}^\dagger(t)) \tag{4.3}$$

so that the corresponding c-number equation is

$$\Phi_A(z,\omega,t) \;=\; e^{-z^*z-\omega^*\omega}\langle e^{z\alpha^*(t)-z^*\alpha(t)+\omega\beta^*(t)-\omega^*\beta(t)}\rangle. \tag{4.4}$$

Now taking into account Eqs. (2.165) and (2.166) along with their complex conjugates, Eq. (4.4) can be put in the form

$$\Phi_A(z,\omega,t) \quad = \quad e^{-z^*z-\omega^*\omega}\langle e^{z\alpha^*(t)-z^*\alpha(t)+\omega\beta^*(t)-\omega^*\beta(t)}\rangle. \tag{4.5}$$

where

$$\alpha^{'}(t) \quad = \quad p_1(t)\alpha(0)+q_1(t)\beta^*(0)+G_1(t), \tag{4.6}$$

$$\beta^{'}(t) \quad = \quad p_2(t)\beta(0)+q_2(t)\alpha^*(0)+G_2^*(t). \tag{4.7}$$

With the aid of Eqs. (2.133), (2.151), (2.152), (2.153), (2.154), (2.157), and (2.158), it can be easily established that

$$\frac{d}{dt}\langle\alpha^{'}(t)\rangle \quad = \quad \frac{1}{2}\mu_a\langle\alpha^{'}(t)\rangle+\frac{1}{2}\nu_-\langle\beta^{'}(t)\rangle, label8c \tag{4.8}$$

$$\frac{d}{dt}\langle\beta^{'}(t)\rangle \quad = \quad \frac{1}{2}\mu_a\langle\beta^{'}(t)\rangle+\frac{1}{2}\nu_+\langle\beta^{'}(t)\rangle, \tag{4.9}$$

We see that Eqs. (2.91) and (4.9) are linear differential equations for $\alpha^{'}(t)$ and $\beta^{'}(t)$. On the other hand, taking into account Eqs. (4.14), (4.15) and (2.147), and the assumption that the cavity modes are initially in a vacuum state, we have

$$\langle\alpha^{'}(t)\rangle \quad = \quad \langle\beta^{'}(t)\rangle = 0. \tag{4.10}$$

Thus we observe that $\alpha^{'}(t)$ and $\beta^{'}(t)$ are Gaussian variables with a vanishing mean. In view of this, Eq. (4.5) can be expressed as [32]

$$\begin{aligned}\Phi_A(z,\omega,t) \quad = \quad & e^{-z^*z-\omega^*\omega} \\ & \times exp\left[\left\langle\frac{1}{2}(z\alpha^{'}(t)-z^*\alpha^{'}(t)+\omega\beta^{'*}(t)-\omega^*\beta^{'}(t))^2\right\rangle\right]\end{aligned} \tag{4.11}$$

or

$$\begin{aligned}\Phi_A(z,\omega,t) &= exp\Big[-z^*z(1+\langle\alpha^{'*}(t)\alpha^(t)\rangle)+\frac{1}{2}(z^2\langle\alpha^{'*2}(t)\rangle+z^{*2}\langle\alpha^{'2}\rangle)\\&+z^*(\omega\langle\alpha^{'}(t)\beta^(t)\rangle-\omega\langle\alpha^{'}(t)\beta^{'*}\rangle)\\&+z(\omega\langle\alpha^{'*}(t)\beta^{'*}(t)\rangle-\omega^*\langle\alpha^{'*}\beta^{'}(t)\rangle\\&-\omega^*\omega(1+\langle\alpha^{'*}(t)\beta^{'*}(t)\rangle)+\frac{1}{2}(\omega^2\langle\beta^{'*2}(t)\rangle)\\&+\omega^{*2}\langle\beta^{'2}\rangle\Big]\end{aligned} \tag{4.12}$$

Now on account of Eq. (4.6), we have

$$\langle\alpha^{'2}(t)\rangle = \langle(p_1\alpha(0)+q_1(t)\beta^*(0))^2\rangle+2\langle(p_1\alpha(0)+q_1(t)\beta^*(0))G_1(t)\rangle$$
$$+\langle G_1(t)G_1(t)\rangle. \tag{4.13}$$

With the aid of Eqs. (2.164), (2.117), and (2.118) along with the assumption that initially the cavity modes are in vacuum state and the fact that a noise force at a given instant does not affect the cavity mode variables at earlier time, we obtain

$$\langle\alpha^{'2}\rangle = 0. \tag{4.14}$$

Similarly, we easily get

$$\langle\beta^{'*2}(t)\rangle = \langle\beta^{'*}(t)\alpha^{'}(t)\rangle = 0, \tag{4.15}$$

$$\langle\beta^{'}(t)\beta^{'}(t)\rangle = \langle G_2^*G_1(t)\rangle, \tag{4.16}$$

$$\langle\alpha^{'*}(t)\alpha^{'}(t)\rangle = \langle G_1(t)G_1(t)\rangle, \tag{4.17}$$

$$\langle\beta^{'*}(t)\beta^{'}(t)\rangle = \langle G_1^*(t)G_2(t)\rangle \tag{4.18}$$

Hence on account of Eqs. (4.14), (4.15), (4.16), (4.17), and (4.18),the characteristic function can be put in the form

$$\Phi_A(z,\omega,t) = e^{-a_\alpha z^*z+z^*(\omega^*b)+z(\omega b^*)}e^{-a\beta\omega^*\omega}, \tag{4.19}$$

where

$$a_\alpha = 1 + \langle G_1^*(t)G_1(t)\rangle \tag{4.20}$$

$$a_\beta = 1 + \langle G_2^*(t)G_2(t)\rangle, \tag{4.21}$$

$$b = \langle G_2^*(t)G_2(t)\rangle \tag{4.22}$$

With the aid of Eqs. (2.165), (2.166), (2.125),(2.126), (2.127), and (4.18), we can write (4.20), (4.21), and (4.22) as

$$\begin{aligned} a_\alpha &= 1 + \frac{\kappa A(1-\eta)(4\kappa + 3A\eta + A)}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \\ &+ \frac{[2\kappa(2\kappa + 2A\eta + A) + A^2(1+\eta)]2\kappa N}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \\ &+ \frac{[-A\sqrt{1-\eta}^2](2\kappa + A\eta + A)2\kappa M}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \end{aligned} \tag{4.23}$$

$$\begin{aligned} a_\beta &= 1 + \frac{\kappa(A\sqrt{1-\eta^2})^2 + A\sqrt{1-\eta^2}[2\kappa + A\eta - A]2\kappa N}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \\ &+ \frac{[2\kappa(2\kappa + 2A\eta - A) + A^2(1-\eta)]2\kappa M}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} b &= \frac{\kappa(A + \sqrt{1-\eta^2})(2\kappa + A\eta + A) + [(2\kappa + A\eta)^2 - A^2]2\kappa M}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \\ & \frac{[(A^2\sqrt{1-\eta^2})]2\kappa N}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \end{aligned} \tag{4.25}$$

Now using Eq.(4.19) in Eq (4.1), we have

$$\begin{aligned} Q(\alpha,\beta,t) &= \frac{1}{\pi_2}\int \frac{d^2z}{\pi}\frac{d^2\omega}{\pi} exp\Big\{-a_\alpha z^* z + z^*(\alpha + \omega^* b) - z(\alpha - \omega b^*)\Big\} \\ &\times exp\Big\{-a_\beta \omega^*\omega + \omega^*\beta - \omega\beta^*\Big\}, \end{aligned} \tag{4.26}$$

Employing the relation

$$\int \frac{d^2\alpha}{\pi} e^{-}\alpha^*\alpha + b\alpha + c\alpha^* + B\alpha^2 + c\alpha^{*2} = \frac{1}{\sqrt{a^2 - 4BC}} exp\Big\{\frac{abc + BC^2cb^2}{a^2 + 4BC}\Big\}, a > 0 \tag{4.27}$$

so that carrying out the integration with the help of Eq. (4.27), the Q function is found to be

$$Q(\alpha,\beta,t) = \frac{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon}{\pi^2} \times exp\Big[-\upsilon_\beta\alpha^*\alpha + \alpha(p^* + \upsilon\beta) + \alpha^*(p + \upsilon\beta^*) - \upsilon_\alpha\beta^*\beta + \beta q^*\Big], \quad (4.28)$$

where

$$\upsilon_\alpha = \frac{a_\alpha}{a_\alpha a_\beta - b^* b}, \quad (4.29)$$

$$\upsilon_\beta = \frac{a_\alpha}{a_\alpha a_\beta - b^* b}, \quad (4.30)$$

$$\upsilon = \frac{b}{a_\alpha a_\beta - b^* b} \quad (4.31)$$

$$p = \upsilon_\beta - \upsilon, \quad (4.32)$$

$$q = \upsilon_\alpha - \upsilon. \quad (4.33)$$

4.1.1 Mean Photon Number Sum and Difference

We next proceed to calculate the mean and variances of the photon number sum and difference of mode a and mode b applying the Q function. We define the operators representing the photon number sum and difference of mode a and mode b by

$$\hat{n}_\pm = \hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b}. \quad (4.34)$$

Then the mean of the photon number sum and difference can be written in terms of the Q function as

$$\bar{n}_\pm = \int d^2\alpha d^2\beta Q(\alpha,\beta,t)(\alpha^*\alpha \pm \beta^* - 1 \mp 1). \quad (4.35)$$

On account of Eq.(4.28), we see that

$$\bar{n}_{\pm} = \frac{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon}{\pi^2} e^{-p^*-q} \int d^2\alpha d^2\beta(\alpha^*\alpha \pm \beta^*\beta - 1 \mp 1) \times e^{-\upsilon_\beta|\alpha|^2+\alpha^*(p+\upsilon_\beta)+\alpha(p^*+\upsilon^*\beta)-\upsilon_\alpha|\beta|^2+\beta^*q+\beta q^*}. \quad (4.36)$$

This equation can be rewritten as

$$\bar{n}_{\pm} = \frac{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon}{\pi^2} e^{-p^*-q}\left(\frac{\partial^2}{\partial p^*\partial p} \pm \frac{\partial^2}{\partial q^*\partial q} - 1 \mp 1\right) \times \int d^2\alpha d^2\beta e^{-\upsilon_\beta|\alpha|^2+\alpha^*(p+\upsilon_\beta^*)+\alpha(p^*+\upsilon\beta^*)+\alpha(p^*+\upsilon^*\beta)-\upsilon_\alpha|\beta|^2+q+\beta^* q\beta q^*}. \quad (4.37)$$

Upon carrying out the integration with the help of Eq.(4.27), we obtain

$$\bar{n}_{\pm} = e^{p^*-q}\left(\frac{\partial^2}{\partial p^*\partial p} \pm \frac{\partial^2}{\partial q^*\partial q} - 1 \mp\right) exp\left\{\frac{\upsilon_\alpha p^*p + \upsilon_\beta q^*q + \upsilon^*pq + \upsilon p^*q^*}{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon}\right\}, \quad (4.38)$$

from which follows

$$\bar{n}_{\pm} = \bar{n}_a \pm \bar{n}_b, \quad (4.39)$$

where

$$\bar{n}_a = \frac{\upsilon_\alpha}{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon} - 1 \quad (4.40)$$

$$\bar{n}_b = \frac{\upsilon_\beta}{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon} - 1 \quad (4.41)$$

are the mean photon numbers of mode a and mode b. With the aid of Eqs.(4.27),(4.28), and (4.24) we can write

$$\bar{n}_a = \frac{\kappa A(1-\eta)(4\kappa + 3A\eta + A)}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} + \frac{[2\kappa(2\kappa + 2A\eta + A) + A^2(1+\eta)]2\kappa N}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} + \frac{[-A\sqrt{1-\eta^2}](2\kappa + A\eta + A)2\kappa M}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)}. \quad (4.42)$$

and

$$\bar{n}_b = \frac{\kappa(A\sqrt{1-\eta^2}) + (A\sqrt{1-\eta^2})[2\kappa + A\eta - A]2\kappa M}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} + \frac{[2\kappa(2\kappa + 2A\eta - A) + A^2(1-\eta)]2\kappa N}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)}. \quad (4.43)$$

On account of Eqs. (4.41) and (4.42), the mean of the photon number sum and difference can be written in the form

$$\begin{aligned}\bar{n}_{\pm} &= \frac{2\kappa A(1-\eta)(2\kappa+A\eta)+(1\pm1)\kappa A^2[1-\eta^2]}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{[(1\pm1)[2\kappa(2\kappa+2A\eta)+A^2]+(1\mp1)A[2\kappa+A\eta]]2\kappa N}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{[(1\pm1)[-A^2\sqrt{1-\eta^2}]+(1\mp1)A[(2\kappa-A\eta)\sqrt{1-\eta^2}]]2\kappa M}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)} \qquad (4.44)\end{aligned}$$

This represents the mean of the photon number sum and difference of the cavity modes for a nondegenerate three-level laser and coupled to a two-mode squeezed vacuum reservoir.

$$\begin{aligned}\bar{n}_{+} &= \frac{2\kappa A(1-\eta)(2\kappa+A\eta)+2(kA^2[1-\eta^2])}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}\\ &+\frac{[(4k^2+4kA\eta)+A^2(4\kappa+A\eta)]+(2-A^2\sqrt{1-\eta^2})2kM}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)}. \qquad (4.45)\end{aligned}$$

$$\bar{n}_{-} = \frac{2\kappa A(1-\eta)(2\kappa+A\eta)+[2A(2\kappa+A\eta)]2\kappa N+[2A(2\kappa+A\eta)]2\kappa M}{4[\kappa(\kappa+A\eta)](2\kappa+A\eta)} \qquad (4.46)$$

Fig.4.1 shows that a relatively better squeezing can be achieved for large values of the linear gain coefficient A. And Fig.4.2 indicates that the degree of squeezing increases with the squeeze parameter r.

4.1.2 The Photon Number Variance Sum and Difference

Next we proceed to calculate the variances of the photon number sum and difference of mode $a$and mode b. The variances of the photon number sum and difference defined by

$$\Delta n_{\pm}^2 = \langle(\hat{a}\hat{a}\pm\hat{b}^{\dagger}\hat{b})^2\rangle-\langle\hat{a}^{\dagger}\hat{a}\pm\hat{b}^{\dagger}\hat{b}\rangle^2 \qquad (4.47)$$

can be expressed as

$$\Delta n_{\pm}^2 = \Delta n_a^2+\Delta n_b^2\pm 2n_{ab}, \qquad (4.48)$$

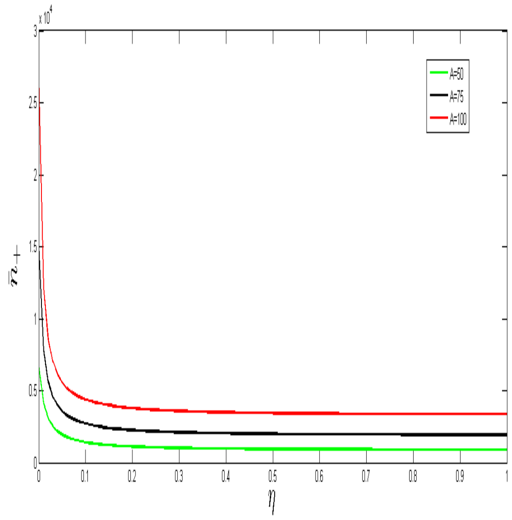

Figure 4.1: Plot of the mean photon number sum of the two-mode cavity light at steady state [Eq.4.45]versus η for different values of the linear gain coefficient when $r = 0.5$,and $\kappa = 0.8$.

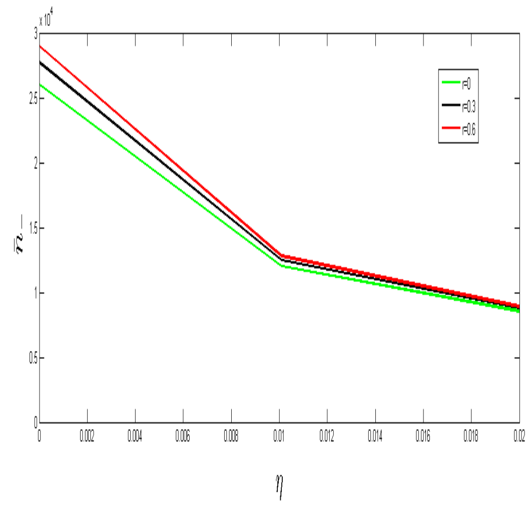

Figure 4.2: Plot of the mean photon number difference of the two-mode cavity light at steady state[Eq.4.46]versus η for different values of the squeezed parameter r when $A = 100$,and $\kappa = 0.8$.

in which

$$\Delta n_a^2 \quad = \quad \langle(\hat{a}^\dagger\hat{a})\rangle - \bar{n}_a^2 \tag{4.49}$$

is the photon number variance of mode$a$,

$$\Delta n_b^2 \quad = \quad \langle(\hat{b}^\dagger\hat{b})\rangle - \bar{n}_b^2 \tag{4.50}$$

is the photon number variance of mode b, and

$$n_{ab} \quad = \quad \langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle - \bar{n}_a\bar{b} \tag{4.51}$$

with $\bar{n}_a = \langle\hat{a}^\dagger\hat{a}\rangle$ and $\bar{n}_b = \langle\hat{b}^\dagger\hat{b}\rangle$. Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$,we can write

$$\Delta n_a^2 \quad = \quad \langle\hat{a}^2\hat{a}^{\dagger 2}\rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{4.52}$$

The first term on the right side of Eq.(4.52) can be expressed in terms of the Q function as

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad = \quad \int d\alpha^2 d\beta^2 Q(\alpha,\beta,t)\alpha^*\alpha^2. \tag{4.53}$$

On account of Eq.(4.1)and ,we have

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad = \quad \frac{\upsilon_\alpha\upsilon_\beta - \upsilon^*}{\pi^2} \int d^2\alpha d^2\beta\alpha^2 e^{-\upsilon_\beta|\alpha|^2+\alpha^*\alpha^*(p+\upsilon\beta^*)+\alpha(p^*+\upsilon\beta)-\upsilon_\alpha|\beta|^2+\beta^* q+\beta q^*}. \tag{4.54}$$

or

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad = \quad \frac{\upsilon_\alpha\upsilon_\beta - \upsilon^*}{\pi^2} \times\frac{\partial^4}{\partial p^2\partial p^{*2}} \int d^2\alpha d^2\beta\alpha^2 e^{-\upsilon_\beta|\alpha|^2+\alpha^*\alpha^*(p+\upsilon\beta^*)+\alpha(p^*+\upsilon\beta)-\upsilon_\alpha|\beta|^2+\beta^* q+\beta q^*}. \tag{4.55}$$

Hence carrying out the integration, we get

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle \quad = \quad \frac{\partial^4}{\partial p^2\partial p^{*2}} exp\left\{\frac{\upsilon_a p^* p + \upsilon_\beta q^* q + \upsilon^* pq + \upsilon p^* q^*}{\upsilon_\alpha\upsilon_\beta - \upsilon^*\upsilon}\right\}. \tag{4.56}$$

Then performing the differentiation, we find

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{2\upsilon_\alpha^2}{\upsilon_\alpha \upsilon_\beta - \upsilon^* \upsilon} + \frac{4\upsilon_\alpha}{\upsilon_\alpha \upsilon_\beta - \upsilon^* \upsilon}\left|\frac{\upsilon_\alpha p + \upsilon q^*}{\upsilon_\alpha \upsilon_\beta - \upsilon^* \upsilon}\right|^2 + \left|\frac{\upsilon_\alpha p + \upsilon q^*}{\upsilon_\alpha \upsilon_\beta - \upsilon^* \upsilon}\right|^2 \quad (4.57)$$

With the aid of Eqs.(4.30), (4.31), and (4.38), we can write as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2(\bar{n}_a + 1)^2 \quad (4.58)$$

Therefore,substitution of [Eq.(4.54)] into [Eq.(4.48)] yields

$$\Delta n_a^2 = \bar{n}_a^2 + \bar{n}_a. \quad (4.59)$$

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \bar{n}_b^2 + \bar{n}_b \quad (4.60)$$

and

$$n_{ab} = \left|\frac{\kappa(A + \sqrt{1-\eta^2})(2\kappa + A\eta + A) + [(2\kappa + A\eta)^2 - A^2]2\kappa M}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)} \right. \\ \left. \frac{[(A^2\sqrt{1-\eta^2})]2\kappa N}{4[\kappa(\kappa + A\eta)](2\kappa + A\eta)}\right|^2 \quad (4.61)$$

Hence combination of Eqs. (4.41), (4.42), (4.43), and (4.49) results in

$$\Delta n_+^2 = \bar{n}_a^2 + \bar{n}_a + \bar{n}_b^2 + \bar{n}_b + 2|b|^2 \quad (4.62)$$

$$\Delta n_-^2 = \bar{n}_a^2 + \bar{n}_a + \bar{n}_b^2 + \bar{n}_b - 2|b|^2 \quad (4.63)$$

Fig.4.3.shows that the variance of the photon number sum increase with the amount of the linear gain coefficient increases.

Fig.(4.4) shows that the variance of the photon number sum in the absence of squeezed vacuum reservoir and in the presence of squeezed vacuum reservoir.

Fig.(4.5) shows that the variance of the photon number difference increase with the linear gain coefficient A increase.

Fig.(4.6) shows that the variance of the photon number difference in the absence of squeezed vacuum reservoir and in the presence of squeezed vacuum reservoir.

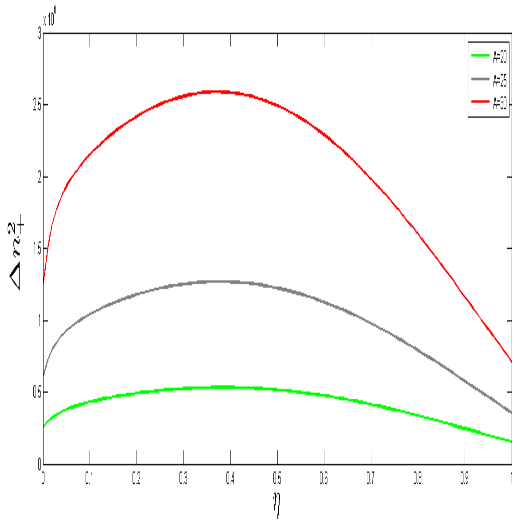

Figure 4.3: Plots of the variance of the photon number sum [Eq.4.62] versus η for $\kappa = 0.8$,$r = 0.5$and for different values of the linear gain coefficient.

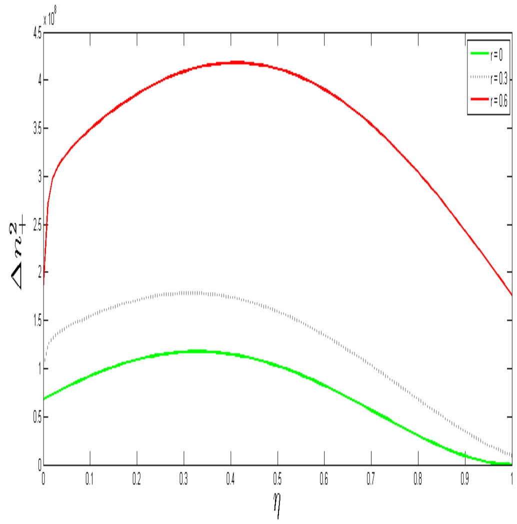

Figure 4.4: Plots of the variance of the photon number sum [Eq. 4.62] versus η for $A = 100$, $\kappa = 0.8$,and $r = 0.5$and for different values of the squeezed parameter r.

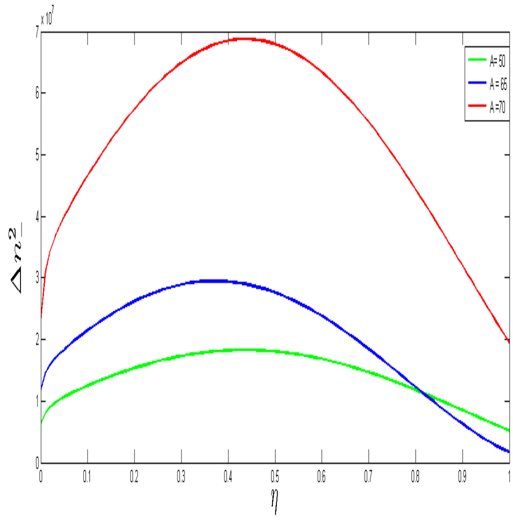

Figure 4.5: Plots of the variance of the photon number different [Eq.4.63] versus η for $\kappa = 0.8$,$r = 0.5$and for different values of the linear gain coefficient.

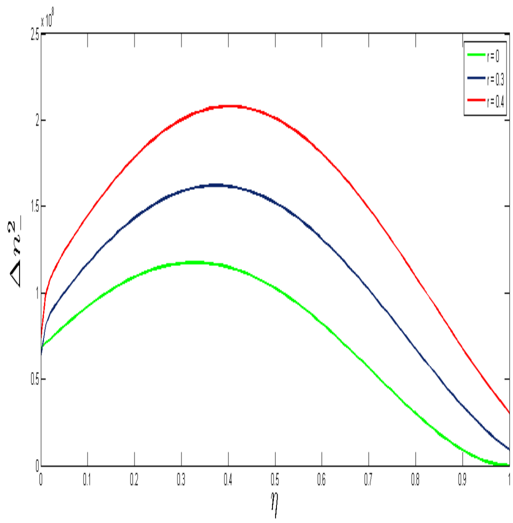

Figure 4.6: Plots of the variance of the photon number difference[Eq. 4.63] versus η for $A = 100$,$\kappa = 0.8$,and for different values of the squeezed parameter r

5

Photon Entanglement

In this chapter, we wish to study entanglement properties of the two modes cavity light. It is known that a state of a system ρ of two modes a and b is said to be entangled or not separable if it is not possible to express in the form

$$\rho = \sum_j \hat{\rho}_j^{(1)} \otimes \hat{\rho}_j^{(2)} \tag{5.1}$$

where $\rho_i^{(a)}$ $a$and $\rho_i^{(b)}$ a $b$are assumed to be the normalized density operators of modes a and b, respectively, with $Pi \geq 0$ and $\sum_i P_i = 1$. A maximally entangled continuous variable state can be expressed as a co-eigenstate of a pair of Einstein- Podolsky-Rosen EPR-type operators (34) such as $\hat{x}_a - \hat{x}_b$and $\hat{p}_a + \hat{p}_b$. Thus the sum of the variances of these operators is reduced to zero for the maximally entangled continuous variable state .According to Duan et al. [29] criterion, a quantum state of a system is said to be entangled if the sum of the variances of the two EPR-like operators of the two modes

$$\hat{u} = \hat{x}_a - \hat{x}_b \tag{5.2}$$

$$\hat{\nu} = \hat{p}_a + \hat{p}_b label3d \tag{5.3}$$

where

$$\hat{x}_a = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \tag{5.4}$$

$$\hat{x}_b = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger), \tag{5.5}$$

$$\hat{p}_a = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}), \tag{5.6}$$

$$\hat{p}_b = \frac{i}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}), \tag{5.7}$$

are quadrature operators for mode a and mode b,satisfy

$$\Delta\nu^2 + \Delta u^2 < 2 \tag{5.8}$$

Furthermore,the variance of these quadrature operators can be put,in terms of the c number variables associated with the normal ordering,in the form

$$\Delta\nu^2 = \Delta u^2 = 1 + \langle\alpha^*(t)\alpha(t)\rangle_{ss} + \langle\beta^*(t)\beta(t)\rangle_{ss} - 2(\langle\alpha(t)\beta(t)\rangle_{ss}. \tag{5.9}$$

Thus in view of Eq. (5.8) together with (3.37),the sum of the variances of $\hat{u}$ and $\hat{\nu}$ can be expressed as

$$\Delta\nu^2 + \Delta u^2 = 2\Delta c_-^2 \tag{5.10}$$

Now making use of Eqs.(3.38)together with (5.10),we see that

$$\begin{aligned}\Delta\nu^2 + \Delta u^2 = 2\Bigg[&\frac{A^2(1-\eta^2)(2N+1) - 2A^2(M+N)(\sqrt{(}1-\eta^2))}{2(2k+A\eta)(k+A\eta)} \\ &+\frac{2(N-M)(2k+A\eta)^2 - 2MA^2}{2(2k+A\eta)(k+A\eta)} \\ &+\frac{(2k+A(1+\eta))(2k+A\eta-A\sqrt{1-\eta^2})}{2(2k+A\eta)(k+A\eta)}\Bigg]\end{aligned} \tag{5.11}$$

We immediately note that this particular entanglement measure is directly related to the two-mode squeezing. This direct relationship shows that, whenever there is a two-mode squeezing in the system, there will be entanglement in the system as well.This is attributed to the fact that the coherent fields do not introduce an additional atomic coherence to the system, and the same is true in the case of squeezing.Using criterion (5.8) that a significant entanglement between the states of the light generated in the cavity of anondegenerate three-level cascade laser can be manifested due to the strong correlation between the light emitted, when the atoms decay from the upper energy level to the lower via the intermediate energy level. Based on criterion (5.8), we clearly see from Fig.5.1 that the two states of the generated light are strongly entangled in the steady state.

The entanglement disappears,when there is no atomic coherence, and it would be stronger for certain values of the atomic coherence for every value of the linear gain coefficient. It can easily be seen that the degree of entanglement increases with the rate, at which the atoms are injected into the cavity, A is less than 2 for all values of except for $\eta = 1$, hence the entanglement criterion is satisfied.This indicates that the state of the system is entangled at steady state provided that there is injected atomic coherence. Moreover, the degree of entanglement increases with the linear gain coefficient. On the other hand comparison of Fig 5.1 and 5.2 shows that there is a strong entanglement of the two modes in the cavity when there is a substantial degree of squeezing. This strong entanglement is observed for relatively small values of η , that is, when slightly more atoms are in the lower level at the initial time.It is also easy to see that the entanglement disappears when the squeezing vanishes. This is due to the fact that the squeezing and entanglement are directly related as given in Eq.(5.7).

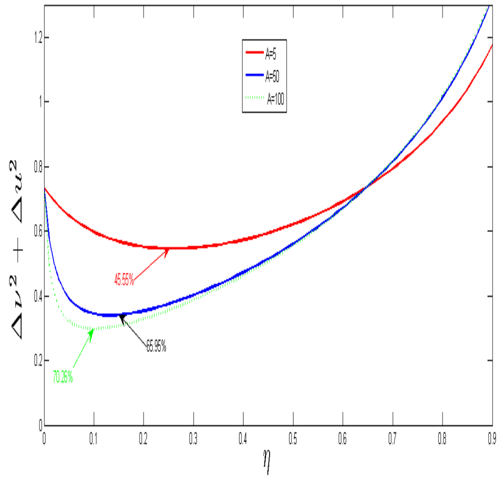

Figure 5.1: Plots of $\Delta\nu^2 + \Delta u^2$ of the two-mode cavity light at steady state[Eq.(5.11)] vs η for $\kappa = 0.8$, $r = 0.5$, and for different values of the linear gain coefficient.

Table 5.1: Maximum Entanglement occurs for $r = 0.5$, $\kappa = 0.8$ and different values of A.

η	Maximum Entanglement	Maximum Entanglement occurs for
0.25	45.5%	$A = 5$
0.15	67%	$A = 50$
0.1	70.3%	$A = 100$

From Fig.5.1 we note that a relatively better Entanglement can be achieved for large values of the linear gain coefficient A.

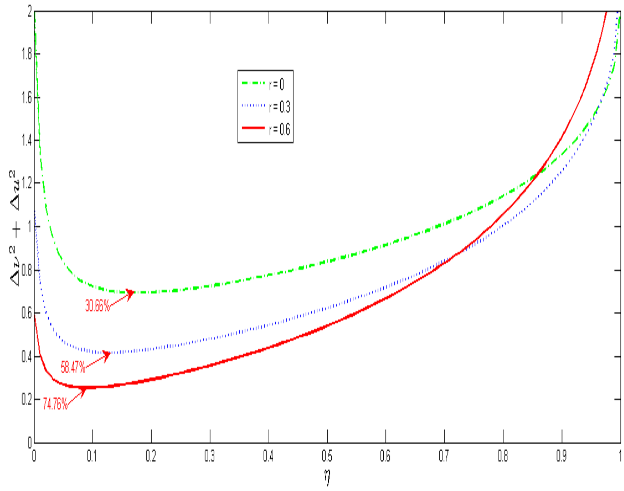

Figure 5.2: Plots of the entanglement of the two-mode cavity light at steady state [Eq. (5.11)]vs η for $A = 100$,$\kappa = 0.8$ and for different values of the squeeze parameter$r$.

Table 5.2: Maximum Entanglement occurs for $A = 100$,$\kappa = 0.8$,and different values of r.

η	Maximum Entanglement	Maximum Entanglement occurs for
0.15	30.7%	$r = 0$
0.12	58.5%	$r = 0.3$
0.09	74.8%	$r = 0.6$

Fig.5.2 indicates that the degree of squeezing increases with the squeeze parameter r.

5.1 Photon Number Correlations

The photon number correlation for two mode light can be defined as

$$g(n_a, n_b) = \frac{\langle \hat{n}_a \hat{n}_b \rangle}{\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle} \tag{5.12}$$

$$\langle \alpha^*(t)\beta^*(t) \rangle = \langle \alpha(t)\beta(t) \rangle \tag{5.13}$$

and

$$\langle \hat{n_a} \hat{n_b} \rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\hat{b}^\dagger(t)\hat{b}(t) \rangle. \tag{5.14}$$

We realize that the operators in Eq.(4.38)-(4.41) are in the normal order. Therefore,Eq.(5.14),The correlation of the photon number at steady state can be expressed in terms of the c number variables associated with the normal ordering as

$$g_{(n_a,n_b)} = 1 + \frac{\langle \alpha(t)\beta(t) \rangle^2}{\langle \alpha^\dagger(t)\alpha(t) \rangle \langle \beta^*(t)\beta(t) \rangle}. \tag{5.15}$$

We realize that the operators in Eq.(4.38)-(4.41) are in the normal order. Therefore, Eq.(5.15),The correlation of the photon number at steady state can be expressed in terms of the c number variables associated with the normal ordering as

$$
\begin{aligned}
g_{(n_a,n_b)} &= 1+\frac{k(A\sqrt{1-\eta^2})^2[2k+A\eta+A]}{(k(A\sqrt{1-\eta^2})+(A\sqrt{1-\eta^2})[2k+A_{-}]2kM)} \\
&+\frac{k(A\sqrt{1-\eta^2})^2[2k+A\eta+A]}{[A^2(1-\eta)]2kN[A\sqrt{1-\eta^2}](2k+A\eta+A)2kM} \\
&+\frac{(2k+A_{-}^2)2kM}{(k(A\sqrt{1-\eta^2})+(A\sqrt{1-\eta^2})[2k+A\eta-A]2kM+[A^2(1-\eta)]2kN)} \\
&+\frac{(2k+A_{-}^2)2kM}{[-A\sqrt{1-\eta^2}](2k+A\eta+A)2kM} \\
&\times\frac{k(A\sqrt{1-\eta^2})^2 2k+A\eta+A}{k(A\sqrt{1-\eta^2})+(A\sqrt{1-\eta^2})[2k+A\eta-A]2kM} \\
&+\frac{k(A\sqrt{1-\eta^2})^2 2k+A\eta+A}{[A^2(1-\eta)]2kN+[A^2\sqrt{1-\eta^2}]2kN} \\
&\times\frac{[2k+A\eta+A^2]2kM}{k(A\sqrt{1-\eta^2})+(A\sqrt{1-\eta^2})[2k+A\eta-A]2kM+[A^2(1-\eta)]2kN+[A^2\sqrt{1-\eta^2}]2kN}
\end{aligned}
\quad (5.16)
$$

This represent the photon number correlation for a two mode cavity light for a nondegenerate threelevel laser coupled to a two-mode squeezed vacuum reservoir at steady state.

We see from Fig.5.3 shows that the correlation of the photon number decreases with increasing injected atomic coherence.Moreover, as shown in Fig.5.3 shows that the squeezing is maximum at vicinity of $\eta = 0.15$ for $A = 10$, where the correlation of the photon number is a little above 2. We also found that for η very close to 1 the correlation of the photon number would be significantly large, since the mean photon number of the light in mode b is very close to zero when initially almost all atoms are populated in the lower level.

Furthermore, Fig.5.1 clearly shows that the correlation of the photon number increse. However, we have found out that the degree of squeezing increases with the linear gain coefficient. We hence infer from these results that the correlation between the photon numbers tend to be minimum in regions where the squeezing is maximum and the presence of squeezed vacuum reservoir

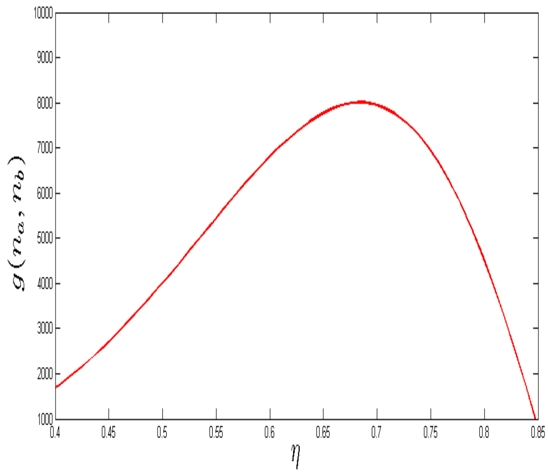

Figure 5.3: Plots of photon number correlation of the two-mode cavity light at steady state versus [Eq. (5.17)] η for $A = 100$,$\kappa = 0.8$ and for different values of the squeeze parameter r.

5.1.1 Normalized Second-order Correlation Functions

In this section we analyze the second-order correlation function for the separate mode as well as for the superposition of the two modes. Moreover, we calculate the linear correlation coefficient between the cavity modes. The normalized second-order correlation function for the two-mode cavity light can be expressed as

$$g^2_{(n_a,n_b)}(0) = \frac{\langle\alpha^*(t)\alpha(t)\rangle\langle\beta^*(t)\beta(t)\rangle}{\langle\alpha^*(t)\alpha(t)\rangle\langle\beta^*(t)\rangle} \tag{5.17}$$

Since Eqs.(2.90)and (2.93) are linear differential equations, we see that $\hat{a}$ and $\hat{b}$ are Gaussian variables. Moreover, on account of Eq.(2.101),$\hat{a}$ and $\hat{b}$are Gaussian variables with vanishing mean. One can then express Eq.(5.17)in the form

$$g^2_{(n_a,n_b)}(0) = 1 + \frac{\langle\alpha(t)\beta(t)\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta^*(t)\rangle\langle\alpha(t)\beta(t)\rangle}{\langle\alpha^*(t)\alpha(t)\rangle\langle\beta^*(t)\beta(t)\rangle}. \tag{5.18}$$

This represent the normalized second-order correlation functions for a two mode cavity light for a nondegenerate threelevel laser coupled to a two-mode squeezed vacuum reservoir at steady state.

5.1.2 Fluctuations of Intensity Difference

In this section we wish to calculate the variance of the intensity difference and compare with that of a coherent light. The variance of the intensity difference defined by.

$$\hat{I}_D = \hat{n}_a - \hat{n}_b \tag{5.19}$$

then

$$\hat{I}_D = \hat{a}^\dagger(t)\hat{a}(t) - \hat{b}^\dagger(t)\hat{b}(t) \tag{5.20}$$

The fluctuations of intensity difference can be

$$\begin{aligned} \Delta I_D^2 = {} & \langle\alpha^{*2}(t)\rangle\langle\alpha^2(t)\rangle + 2\langle\alpha^*(t)\alpha(t)\rangle^2 + \langle\beta^{*2}(t)\beta^2(t)\rangle + 2\langle\beta^*(t)\beta(t)\rangle^2 \\ & -2\langle\alpha(t)\beta^*(t)\rangle^2 - 2\langle\alpha^*(t)\alpha(t)\rangle \times \langle\beta^*(t)\beta(t)\rangle \\ & -2\langle\alpha^*(t)\beta(t)\rangle^2 + \langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle, \end{aligned} \tag{5.21}$$

We note that $\hat{a}$ and $\hat{b}$ are gaussian variables with vanishing mean.For such types of variables, we can express Eq.(5.21)Since the cavity mode operator and are a Gaussian variables.On account of Eq.(5.21)the steady state variance of the intensity difference becomes.

$$\Delta I_D^2 = \langle\alpha^*(t)\alpha(t)\rangle[1 + \langle\alpha^*(t)\alpha(t)\rangle] + \langle\beta^*(t)\beta(t)\rangle \times [1 + \langle\beta^*(t)\beta(t)\rangle] - 2\langle\alpha(t)\beta(t)\rangle^2 \tag{5.22}$$

In Fig. 5.4,we observe that even though the nature of the fluctuations are different,

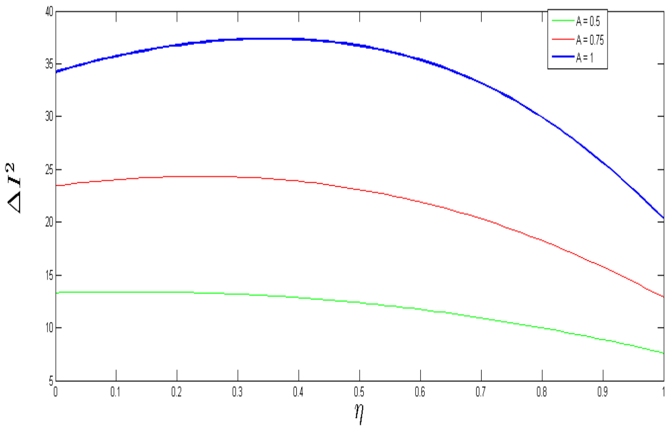

Figure 5.4: Plots of the variance of the intensity difference of the two-mode cavity light at steady state [Eq.5.22] versus η for $\kappa = 0.8$,$r = 0.5$ and for different values of the linear gain coefficient.

they show qualitatively similar behavior. That is, when the intensity difference fluctuations is less than one, the two-mode light exhibits both squeezing and entanglement. However, when the intensity difference fluctuations increases to the coherent level, the squeezing and entanglement decreases and even disappears for values of η close to one.

6

conclusion

In this thesis we have studied the squeezing and entanglement properties of the cavity modes produced by a nondegenerate three-level laser whose cavity mode coupled to a two-mode squeezed vacuum reservoir.

First We have derived the master equation in the linear and adiabatic approximation.Then we have obtained stochastic differential equations associated with the normal ordering and noise force correlations.

Applying the solutions of the resulting differential equations, we have calculated the quadrature variances,photon entanglement,photon number correlations,normalized second order functions and intensity difference of the two mode cavity light at steady state.

We have found for a linear gain coefficient of 100 for a cavity damping constant (κ) of 0.8 and squeezed parameter r of 0.5 the maximum squeezing at a steady state and at threshold 85%. We have also seen that the squeezed parameter of $r = 0.6$ for a cavity damping constant of 0.8, for linear gain coefficient of 100 and for η of 0.13 the maximum entanglement at a steady state 87.4%.

It is found that the squeezed parametric and the squeezed vacuum reservoir increase the degree of squeezing as well as entanglement.We have also seen that the degree of squeezing increases with the linear gain coefficient for small values of η and almost perfect squeezing can be obtained for large values of the linear gain coefficient and squeezed parameter.We have determined employing the Q function the mean photon number sum and difference and the variance of the photon number sum and difference for the cavity modes.

The mean photon number increases considerably due to the squeezed vacuum reservoir, and squeezed parameter.

Our analysis showed that the quadrature squeezing is enhanced due to the injected atomic coherence for certain initial conditions.

Almost perfect squeezing can be achieved for relatively large squeeze parameter and linear gain coefficient. We have established that the squeezing and entanglement in the two-mode cavity light is directly related. As a result, an increase in the degree of squeezing directly implies an increase in the degree of entanglement and vise versa.

In addition,our calculation of the photon number correlation shows that when correlation between the states of the emitted light is stronger, the correlation between the photon number tends to be smaller. Contrary to this fact the variance of the intensity difference is found to be relatively larger in a region where the squeezing and entanglement are significant.

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