EXISTENCE OF TWO POSITIVE SOLUTIONS FOR SECOND-ORDER UNDAMPED THREE POINT BOUNDARY VALUE PROBLEMS IN CONE BANACH SPACE

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Declaration

I, the undersigned declared that, the thesis entitled **"Existence of two positive solutions for second-order undamped three point boundary value problems in cone Banach space "** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged.
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Abstract

This thesis concerned with second-order undamped three point boundary value problem. It also focused on constructing Green's function for corresponding homogeneous equation by using its properties. Under the suitable conditions, we established the existence of two positive solution by applying Avery-Henderson fixed point theorem. We provided an example to demonstrate for the applicability of our main result. This study was mostly dependent on secondary source of data such as journals and books which related to our study area.

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Chapter 1

Introduction

1.1 Background of the study

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. Boundary value problems arise in applications where some physical process involves knowledge of information at the edges. For example, it may be possible to measure the electric potential around the edge of a semi-conductor and then use this information to infer the potential distribution near the middle HELM .(2008).

Some theories such as the Krasnoselskii's fixed point theorem, the Leggett-Williams fixed point theorem, Avery's generalization of the Leggett-Williams fixed point theorem and Avery-Henderson fixed point theorem have given a decisive impetus for the development of the modern theory of differential equations. The advantage of these techniques lies in that they do not demand the knowledge of solution, but have great power in application, in finding positive solutions, multiple positive solutions, and eigenvalue intervals for which there exists one or more positive solutions.

In this vast field of research, we focused on the second-order undamped three point boundary value problem. Most results so far have been obtained mainly by using the fixed-point theorems in cone, such as the Guo-Krasnoselkii's fixed point theorem (Krasnoselskii,M.A.(1964)), the Legget-Williams theorem (1979), Avery and Henderson's theorem(2001), and so on.
In the field of differential equations, a boundary value problem (BVP) is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions .

A second-order ordinary differential equation with three boundary conditions is called second-order three point boundary value problems. Gupta(1992), Studied three point boundary value problems for nonlinear ordinary differential equations. Since,then nonlinear three point boundary value problems have been studied by many authors using the fixed point index theory,Leray-Schauder continuation theorem, coincidence degree theory, and fixed point theorem in cones.The existence of positive solutions of boundary value problems was studied by many researchers. We list down few of them which are related to our particular problem.

Liu in 2004 [Liu2014], established the existence, multiplicity, and nonexistence of positive solutions for

$$u''(t) + \beta^2 u(t) + \lambda q(t) f(t, u(t)) = 0, 0 < t < 1$$
$$u(0) = 0, u(t) = \sigma u(\eta)$$
$$\text{where}, \beta \in (0, \frac{\pi}{2}), \eta \in (0, 1)$$

λ is a positive constant, using by the fixed point index theorem, degree theory, and fixed point theorem in cones.

Neito in 2013 [Nieto 2013], established the existence of a solution for a three-point boundary value problem for a second order differential equation at resonance

$$-u''(t) = f(t, u(t)), 0 \leq t \leq T$$
$$u(0) = 0, \alpha u(\eta) = u(T),$$

where $T > 0, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous function $\alpha \in \mathbb{R}$ *and* $\eta \in (0, T)$.

Motivated by the above mentioned results, in this thesis, we established the existence of two positive solutions for second-order undamped three point boundary value problems of the form

$$-u''(t) + k^2 u(t) = f(t, u(t)), \quad 0 \leq t \leq 1 \tag{1.1}$$

$$u(0) = 0, u(1) = \alpha u(\eta), 0 < \eta < 1, \quad k > 0 \tag{1.2}$$

By applying Avery-Henderson fixed point theorems in cone Banach space. And some examples has been demonstrated for the applicability of our main result.

By a positive solution of (1.1), (1.2), we understand a function $u(t)$ which is positive on $0 \leq t \leq 1$ and satisfies the differential equation (1.1) for $0 \leq t \leq 1$ and three-point boundary conditions (1.2).

The rest of this thesis was organized as follows:-We first present some definitions which are needed throughout in this work and construct Green's function by using its properties for corresponding homogeneous boundary value problems and state fixed point result by applying the Avery-Henderson fixed point theorem in a cone Banach space. Finally, we investigate the existence of at least two positive solutions for second-order undamped three point boundary value problems (1.1),(1.2) and as an application, example included to verify the main result.

1.2 Statement of the problem

In this study we focused on establishing the existence of two positive solutions for second-order undamped three point boundary value problems.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this thesis was to establish the existence of at least two positive solutions for second-order undamped three point boundary value problems by applying Avery-Henderson fixed point theorem (1.1),(1.2).

1.3.2 Specific objectives

The study has the following specific objectives:-

- To construct the Green's function of the corresponding homogeneous equation.
- To formulate the problem in the form of equivalent integral equation.
- To prove the existence of at least two positive solutions by using Avery-Henderson fixed point theorem.
- To verify the main result by providing an illustrative example.

1.4 Significance of the study

The outcome of this thesis may have the following importance:-

1. It may build the research skill and scientific communication skill of the researcher.

2. It may be used to solve some problems in applied sciences.

3. It may provide some background information for other researchers who want to conduct a research on related topics.

1.5 Delimitation of the Study

This study was delimited to show the existence of at least two positive solutions for second-order undamped three point boundary value problems.

Chapter 2

Review of Related literatures

2.1 Over view of Positive solutions

Positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important result the existence of positive solution was proved by Erbe and Wang in 1994.
The existence of positive solutions for three- point boundary value problem has been studied by many researchers. We list down few of them which are related to our particular problem.

Ma in 1999 [Ma 1999], established the existence of positive solutions to the boundary-value problem

$$\begin{aligned} u'' + \alpha(t)f(u) = 0, \quad 0 < t < 1 \\ u(0) = 0, \quad \alpha u(\eta) = u(1), \end{aligned}$$

where $0 < \eta < 1, \quad 0 < \alpha < \frac{1}{\eta}$
by applying the fixed point theorem in cones.
Zima in 2004 [Zima2004], established the existence of positive solution of second-order three point boundary value problem

$$\begin{aligned} x''(t) + f(t, x(t)) = 0, \quad 0 \leq t \leq 1 \\ x(0) = 0, \quad \alpha x(\eta) = x(1) \\ 0 < \eta < 1, \alpha \geq 0 \end{aligned}$$

by establishing a norm-type cone expansion and compression fixed point theorem for completely continuous operator.

Sveikate et al. in 2016[Sveikate 2016], established the existence of positive solutions of the boundary value problem

$$x'' + k^2 x = f(t, x, x')$$

$$x(0) = 0, \quad x(1) = \alpha x(\eta), \quad 0 < \eta < 1, \quad \alpha > 0$$

by using the quasilinearization approach.

2.2 Preliminaries

In this section we recall some known definitions, theorems and basic concepts on Green's function that will be used in the proof of our main results.

Definition 2.2.1. *A differential equation together with its boundary conditions is referred to as boundary value problem.*

Definition 2.2.2. *A differential equation together with three point boundary conditions is referred to as three point boundary value problem.*

Definition 2.2.3. *Let X be a non-empty set. A map $T : X \to X$ is said to be a self-map with domain of $T = D(T) = X$ and range of $T = R(T) \subset X$*

Definition 2.2.4. *Let X be a non-empty set and a map $T : X \to X$ be self-map. A point x in X is called a fixed point of T if $Tx = x$.*

Definition 2.2.5 (Agarwal 2008). *We consider the second-order linear DE.*

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = r(x), x \in J = [\alpha, \beta], \tag{2.1}$$

where the functions $p_0(x), p_1(x), p_2(x)$ and $r(x)$ are continuous in J and boundary conditions of the form

$$\begin{aligned} l_1[y] &= a_0 y(\alpha) + a_1 y'(\alpha) + b_0 y(\beta) + b_1 y'(\beta) = A \\ l_2[y] &= c_0 y(\alpha) + c_1 y'(\alpha) + d_0 y(\beta) + d_1 y'(\beta) = B \end{aligned} \tag{2.2}$$

where $a_i, b_i, c_i, d_i, i = 0, 1$ and A & B are given constants and l is differential operator.
The boundary value problems (2.1), (2.2) are called nonhomogeneous two-point linear boundary value problems, where as the homogeneous DE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0 \tag{2.3}$$

together with the homogeneous boundary conditions

$$l_1[y] = 0, l_2[y] = 0 \tag{2.4}$$

be called a homogeneous two-point linear boundary value problem.

The function called a Green's function $G(x,t)$ for the homogeneous boundary value problems (2.3)-(2.4) and the solution of the non homogeneous boundary value problem (2.1)-(2.2) can be explicitly expressed in terms of $G(x,t)$. Obviously, for the homogeneous problem (2.3)-(2.4) the trivial solution always exists. Green's function for the boundary value problem (2.3)-(2.4) is defined in the square $[\alpha,\beta]\times[\alpha,\beta]$ and possesses the following fundamental properties:

i. $G(x,t)$ is continuous $[\alpha,\beta]\times[\alpha,\beta]$.

ii. $\frac{\partial G(x,t)}{\partial x}$ is continuous in each of the triangles $\alpha \le x \le t \le \beta$ and $\alpha \le t \le x \le \beta$.
Moreover, $\frac{\partial G(t^+,t)}{\partial x} - \frac{\partial G(t^-,t)}{\partial x} = -\frac{1}{p_0(t)}$
where, $\frac{\partial G(t^+,t)}{\partial x} = \lim_{x\to t, x>t} \frac{\partial G(x,t)}{\partial x}$, $\frac{\partial G(t^-,t)}{\partial x} = \lim_{x\to t, x<t} \frac{\partial G(x,t)}{\partial x}$.

iii. for every $t \in [\alpha,\beta], z(x) = G(x,t)$ is a solution of the differential equation(2.3) in each of the intervals $[\alpha,t)$ and $(t,\beta]$.

iv. for every $t \in [\alpha,\beta], z(x) = G(x,t)$ satisfies the boundary conditions (2.4).

These properties completely characterize Green's function $G(x,t)$.

Definition 2.2.6. *Let $-\infty < a < b < \infty$ be collection of real valued functions $A = f_i : f_i : [a,b] \to R$ is said to be*

(i) Uniformly bounded, if there exists a constant $M > 0$ with $|f_i(t)| \le M$, for all $t \in [a,b]$ and for all $f_i \in A$, and

(ii) Equi continuous, if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies $|f_i(t_1) - f_i(t_2)| < \epsilon$, for all $t_1, t_2 \in [a,b]$ and for every $f_i \in A$.

Definition 2.2.7. *A normed linear space is a linear space X in which for each vector x,there corresponds a real number, denoted by $||x||$ called the norm of x and has the following properties:*

i. $\|x\| \ge 0$, for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$,

ii. $||x + y|| \le ||x|| + ||y||$for all $x, y \in X$, *iii.* $||\alpha x|| = |\alpha|||x||$, *for all* $x \in X$ *and* α *be a scalar.*

Definition 2.2.8. *Let* E *be a real Banach space. A nonempty closed convex set* P *is called a cone, if it satisfies the following two conditions:*

(i) $u \in P, \alpha \geq 0$ *implies* $\alpha u \in P$, *and*

(ii) $u \in P$ *and* $-u \in P$ *implies* $u = 0$.

Definition 2.2.9. *Let* X *and* Y *be two metric spaces. A map* $T : X \to Y$ *is said to be completely continuous, if it is continuous and maps bounded sets into precompact sets.*

Definition 2.2.10. *A Banach space is a real normed linear space that is a complete metric space in the metric defined by its norm. A complex Banach space is a complex normed linear space that is, as a real normed linear space, a Banach space.*

Definition 2.2.11. *Let* X *and* Y *be Banach Spaces and* $T : X \to Y$. *An operator* T *is said to be completely continuous, if* T *is continuous and for each bounded sequence* $\{x_n\} \subset X$, $\{Tx_n\}$ *has a convergent subsequence.*

Definition 2.2.12. *Let* X *and* Y *be two metric spaces and a map* $T : X \to Y$ *is said to be*

(a) compact, if its range is relatively compact of Y *and*

(b) completely continuous , if it maps each bounded subset of X *into a relatively compact subset in* Y .

Definition 2.2.13. *Let* E *be a real Banach space with cone* P. *A map* $f : P \to [0, \infty)$ *is said to be a nonnegative continuous convex functional on* P, *if* f *is continuous and* $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$, *for all* $x, y \in P$ *and* $\lambda \in [0, 1]$.

Definition 2.2.14. *Let* E *be a real Banach space with cone* P. *A map* $f : P \to [0, \infty)$ *is said to be a nonnegative continuous concave functional on* P *if* f *is continuous and* $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$,*for all* $x, y \in P$ *and* $\lambda \in [0, 1]$.

The function $u(t) \in C^2[0, 1]$ is a positive solution of the boundary value problem,

$$\begin{gathered} -u''(t) + k^2u(t) = f(t, u(t)) \\ u(0) = 0, \quad u(1) = \alpha u(\eta) \quad , 0 < \eta < 1, \quad k > 0 \end{gathered} \tag{2.5}$$

if $u(t)$ is positive on the given interval and satisfies both the differential equation and the boundary conditions.
Let ψ be a nonnegative continuous functional on a cone P of the real Banach space E. Then for a positive real number c, we define the sets $P(\psi, c) = \{y \in P : \psi(u) < c\}$ and $P_a = \{y \in P : \|u\| < c\}$.
In obtaining multiple positive solutions of the boundary value problem (1.1)-(1.2), the following Avery-Henderson functional fixed point theorem will be the fundamental tool.

Theorem 2.2.15. *[Avery and Henderson 2001] Let P be cone in the real Banach space E. Suppose γ and ψ are increasing,nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c and M, $\gamma(u) \leq \theta(u) \leq \psi(u)$ and $||u|| \leq M\gamma(u)$ for all $u \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ such that $\theta(\lambda u) \leq \lambda\theta(u)$, for all $0 \leq \lambda \leq 1$ and $u \in \partial P(\theta, b)$. Further, $T : \overline{P(\gamma, c)} \to P$ be a completely continuous operator such that*

(B1) $\gamma(Tu) > c$ for all $u \in \partial P(\gamma, c)$,

(B1) $\theta(Tu) < b$ for all $u \in \partial P(\theta, b)$,

(B1) $\psi(Tu) > a$ and $P(\psi, a) \neq \emptyset$ for all $u \in \partial P(\psi, a)$.

Then T has at least two fixed points $u_1, u_2 \in \overline{P(\gamma, c)}$ such that $a < \psi(u_1)$ with $\theta(u_1) < b$ and $b < \theta(u_2)$ with $\gamma(u_2) < c$.

2.3 Green's Functions and Bounds

In this section, we construct Green's function for the corresponding homogeneous boundary value problem to (1.1).
Before formulation of Green's function for three point boundary value problems, first we construct Green's function for two point homogeneous boundary value problems,

$$-u'' + k^2 u = 0 \tag{2.6}$$

$$u(0) = 0, u(1) = 0 \tag{2.7}$$

For equation (2.6) two linearly independent solution are $u_1(t) = -\sinh kt + \cosh kt$ and $u_2(t) = \sinh kt + \cosh kt$. Hence, the problem (2.6)-(2.7) has only trivial solution if and only if

$$\Delta = \begin{vmatrix} u_1(0) & u_2(0) \\ u_1(1) & u_2(1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\sinh k + \cosh k & \sinh k + \cosh k \end{vmatrix} = 2\sinh k \neq 0, k > 0$$

To show this $u_1(t)$ and $u_2(t)$ be two linearly independent solution if the differential equation (2.6).
Green's function for second - order two point boundary value problems can be written in the form of

$$H(t,s) = \begin{cases} u_1(t)\lambda_1(s) + u_2(t)\lambda_2(s), & \text{if} \quad 0 \leq t \leq s \leq 1 \\ u_1(t)\mu_1(s) + u_2(t)\mu_2(s), & \text{if} \quad 0 \leq s \leq t \leq 1 \end{cases} \tag{2.8}$$

Where, $\lambda_1(s), \lambda_2(s), \mu_1(s)$, and $\mu_2(s)$ are functions. By applying properties of (i) and (ii) ,we obtain

$$\begin{cases} u_1(t)(\mu_1(s) - \lambda_1(s)) + u_2(t)(\mu_2(s) - \lambda_2(s)) = 0, \\ u_1'(t)(\mu_1(s) - \lambda_1(s)) + u_2'(t)(\mu_2(s) - \lambda_2(s)) = -1. \end{cases} \tag{2.9}$$

Let $v_1(s) = \mu_1(s) - \lambda_1(s)$ and $v_2(s) = \mu_2(s) - \lambda_2(s)$ ————————-(*)
Then

$$\begin{cases} u_1(s)V_1(s) + u_2(s)v_2(s) = 0, \\ u_1'(s)V_1(s) + u_2'(s)v_2(s) = -1. \end{cases}$$

From this we get

$$v_1(s) = \frac{1}{2k(-\sinh ks + \cosh ks)} \text{and } v_2(s) = \frac{-1}{2k(\sinh ks + \cosh ks)}$$

Frome $(*)$, we have $\mu_1(s) = v_1(s) + \lambda_1(s)$ and $\mu_2(s) = v_2(s) + \lambda_2(s)$
By using the boundary condition of (2.8) , we obtain

$$\begin{cases} u_1(0)\lambda_1(s) + u_2(0)\lambda_2(s) = 0, \\ u_1(1)(V_1(s) + \lambda_1(s)) + u_2(1)(v_2(s) + \lambda_2(s)) = 0. \end{cases}$$

$$\begin{cases} \lambda_1(s) + \lambda_2(s) = 0, \\ (-\sinh k + \cosh k)(\lambda_1(s) + \frac{1}{2k(-\sinh ks + \cosh ks)}) \\ +(\sinh k + \cosh k)(\lambda_2(s) + \frac{1}{2k(\sinh ks + \cosh ks)}) = \quad 0. \end{cases}$$

By applying Cramer's rule ,we find the value of $\lambda_1(s)$ and $\lambda_2(s)$.
For $\lambda_1(s)$,

$$\lambda_1(s) = \begin{vmatrix} 0 & 1 \\ \frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{k} & \sinh k + \cosh k \end{vmatrix} = \frac{\cosh k.\sinh ks - \sinh k.\cosh ks}{2k \sinh k}$$

For $\lambda_2(s)$,

$$\lambda_2(s) = \begin{vmatrix} 1 & 0 \\ -\sinh k + \cosh k & \frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{k} \end{vmatrix} = \frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{2k \sinh k}$$

Hence ,

$$u_1(t)\mu_1(s) + u_2(t)\mu_2(s) = u_1(t)(\lambda_1(s) + v_1(s)) + u_2(t)(\lambda_2(s) + v_2(s))$$
$$= (-\sinh kt + \cosh kt)(\frac{\cosh k.\sinh ks - \sinh k.\cosh ks}{2k\sinh k} + \frac{1}{2k(-\sinh ks + \cosh ks)})$$
$$+(\sinh kt + \cosh kt)(\frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{2k\sinh k} - \frac{1}{2k(\sinh ks + \cosh ks)})$$
$$= \frac{\sinh kt \sinh k(1-t)}{k\sinh k}, 0 \le t \le s \le 1$$

Therefore,

$$H(t,s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k\sinh k}, & 0 \le t \le s \le 1 \\ \frac{\sinh ks \sinh k(1-t)}{k\sinh k}, & 0 \le s \le t \le 1 \end{cases} \tag{2.10}$$

And the solution of (2.7) is given by

$$w(t) = \int_0^1 H(t,s)f(s)ds \tag{2.11}$$

and

$$w(0) = 0, \quad w(1) = 0, \quad w(\eta) = \int_0^1 H(\eta,s)f(s)ds \tag{2.12}$$

Lemma 2.3.1. *The Green's function $H(t,s)$ has the following properties:*

(i) $H(t,s) \le H(s,s)$, *for all* $t,s \in [0,1]$*;*

(ii) $H(t,s) \ge NH(s,s)$, *for all* $t \in [\delta, 1-\delta], s \in [0,1]$, $N = \frac{\sinh k\delta}{\sinh k}$

Proof. i. $H(t,s)$ is positive for all $t,s \in [0,1]$.

For $0 \le s \le t \le 1$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh ks.\sinh k(1-t)}{\sinh ks.\sinh k(1-s)} = \frac{\sinh k(1-t)}{\sinh k(1-s)} \le 1.$$

$\Longrightarrow H(t,s) \le H(s,s), t,s \in [0,1]$

For $0 \le t \le s \le 1$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh kt.\sinh k(1-s)}{\sinh ks.\sinh k(1-s)} = \frac{\sinh kt}{\sinh ks} \le 1.$$

$\Longrightarrow H(t,s) \le H(s,s), t,s \in [0,1]$

Therefore, $H(t,s) \le H(s,s)$, for all $t,s \in [0,1]$.

ii. If $s \leq t$ for $t \in [\delta, 1-\delta], s \in [0,1]$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh ks \sinh k(1-t)}{\sinh ks \sinh k(1-s)} \geq \frac{\sinh k\delta}{\sinh k}.$$
$$\Longrightarrow H(t,s) \geq NH(s,s)$$

If $t \leq s$ for $t \in [\delta, 1-\delta], s \in [0,1]$, we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\sinh kt \sinh k(1-s)}{\sinh ks \sinh k(1-s)} \geq \frac{\sinh k\delta}{\sinh k}.$$
$$\Longrightarrow H(t,s) \geq NH(s,s)$$

Thus, the Lemma follows. □

The three-point boundary value problems (1.1),(1.2) can be obtained by replacing $u(1) = 0$ for $u(1) = \alpha u(\eta)$ in (2.7), thus we suppose the solution of the three point boundary value problems (1.1),(1.2) can be expressed by

$$u(t) = w(t) + A_1 \sinh kt + A_2 \sinh k(1-t) \tag{2.13}$$

Where A_1 and A_2 are constants that will be determined . From (2.12), we know that

$$\begin{cases} u(0) = w(0) + A_1 \sinh k(0) + A_2 \sinh k(1-0) \\ u(1) = w(1) + A_1 \sinh k(1) + A_2 \sinh k(1-1) \\ u(\eta) = w(\eta) + A_1 \sinh k(\eta) + A_2 \sinh k(1-\eta) \end{cases}$$

$$\Rightarrow \begin{cases} 0 = 0 + A_1.0 + A_2 \sinh k(1) \\ u(1) = 0 + A_1 \sinh k(1) + A_2 \sinh k(0) \\ u(\eta) = w(\eta) + A_1 \sinh k(\eta) + A_2 \sinh k(1-\eta) \end{cases}$$

$$\Rightarrow A_2 = 0.$$

$u(1) = \alpha u(\eta)$,we have $A_1 \sinh k = \alpha(w(\eta) + A_1 \sinh k(\eta))$

$$A_1 = \frac{\alpha w(\eta)}{\sinh k - \alpha \sinh k(\eta)}, \qquad \frac{\sinh k}{\sinh k\eta} > \alpha. \tag{2.14}$$

Therefore,

$$u(t) = w(t) + A_1 \sinh kt + A_2 \sinh k(1-t)$$
$$G(t,s) = H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} H(\eta, s)$$

$$\text{Where, } H(t,s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k \sinh k}, & 0 \le t \le s \le 1 \\ \frac{\sinh ks \sinh k(1-t)}{k \sinh k}, & 0 \le s \le t \le 1 \end{cases}$$

$$H(\eta,s) = \begin{cases} \frac{\sinh k\eta \sinh k(1-s)}{k \sinh k}, & 0 \le \eta \le s \le 1 \\ \frac{\sinh ks \sinh k(1-\eta)}{k \sinh k}, & 0 \le s \le \eta \le 1 \end{cases}$$

$$G(t,s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k \sinh k} \\ \frac{\sinh ks \sinh k(1-t)}{k \sinh k} \end{cases} + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} \begin{cases} \frac{\sinh k\eta \sinh k(1-s)}{k \sinh k} \\ \frac{\sinh s \sinh k(1-\eta)}{k \sinh k} \end{cases} \tag{2.15}$$

$$G(t,s) = \frac{1}{k(\sinh k - \alpha \sinh k(\eta))} \begin{cases} [\sinh k(1-s) + \alpha \sinh k(s-\eta)](\sinh kt), & t \le s \le \eta, \\ [\sinh k(1-t) + \alpha \sinh k(t-\eta)](\sinh ks), & s \le t, s \le \eta \\ \sinh kt \sinh k(1-s), & t \le s, \eta \le s, \\ \sinh ks . \sinh k(1-t) + \alpha \sinh k\eta \sinh k(t-s), & \eta \le s \le t \le 1. \end{cases} \tag{2.16}$$

Lemma 2.3.2. *The Green's function $G(t,s)$ satisfies the following inequalities*

i. $G(t,s) \ge 0, \forall t, s \in [0,1]$;

ii. $G(t,s) \le DH(s,s), \forall t, s \in [0,1]$,

iii. $G(t,s) \ge \mathcal{M}H(s,s), \forall t, \in [\delta, 1-\delta], s \in [0,1]$

$$\textit{where, } D = 1 + \frac{\alpha . \sinh k}{\sinh k - \alpha \sinh k\eta}$$

$$\mathcal{M} = \frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha . \sinh k\delta}{\sinh k - \alpha \sinh k\eta}].$$

Proof. (i) It is obvious that $G(t,s)$ is nonnegative. Since $H(t,s) \ge 0$ and $\sinh k - \alpha \sinh k\eta \ > 0$.

(ii) consider the following case
Case (i) if $t \le s, \eta \le s$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta,s) \\ &\le H(s,s) + \frac{\alpha \sinh ks}{\sinh k - \sinh k\eta} H(s,s) \\ &\le H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\le D_1 H(s,s) \end{aligned} \tag{2.17}$$

Case(ii) If $t \leq s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_2 H(s,s). \end{aligned} \tag{2.18}$$

Case(iii) if $s \leq t, s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_3 H(s,s) \end{aligned} \tag{2.19}$$

Case(iv) if $\eta \leq s \leq t \leq 1$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_4 H(s,s) \end{aligned} \tag{2.20}$$

Therefore, $G(t,s) \leq DH(s,s)$,where $D = D_1 = D_2 = D_3 = D_4$

(iii) To prove (iii) we consider the following cases:
Case(i) If $s \leq t, s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\ &\geq NH(s,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(s,s) \geq NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq M_1 H(s,s). \end{aligned} \tag{2.21}$$

Case(ii) If $t \le s, \eta \le s$

$$\begin{aligned}
G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
&\ge NH(s,s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s,s) \\
&\ge NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\
&\ge H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\
&\ge M_2 H(s,s).
\end{aligned} \tag{2.22}$$

Case(iii) If $t \le s \le \eta$

$$\begin{aligned}
G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
&\ge NH(s,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} NH(s,s) \\
&\ge NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\
&\ge H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\
&\ge M_3 H(s,s).
\end{aligned} \tag{2.23}$$

Case(iv) If $\eta \le s \le t \le 1$

$$\begin{aligned}
G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\
&\ge NH(s,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} NH(s,s) \\
&\ge NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\
&\ge H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\
&\ge M_4 H(s,s).
\end{aligned} \tag{2.24}$$

Therefore, $G(t,s) \ge \mathcal{M} H(s,s)$, where $\mathcal{M} = M_1 = M_2 = M_3 = M_4$

□

Chapter 3

Research Design and Methodology

This chapter contains:- study area and period, study design, source of information and mathematical procedures.

3.1 Study period and site

The study was conducted from September 2018 G.C to February 2020 G.C in Jimma University under the department of Mathematics.

3.2 Study Design

In order to achieve the objective of the study we employed analytical method of design.

3.3 Source of Information

The relevant sources of for this study were different mathematics books, published articles, journals and related studies from internet services.

3.4 Mathematical Procedures

This study we followed the following procedures:-

1. Defining second-order undamped three point boundary value problems.
2. Constructing Green's function by following its properties for the corresponding homogeneous equation.
3. Formulating equivalent integral equation for the boundary value problems (1.1)-(1.2).
4. Determining existence of fixed point of the integral equation by applying Avery-Henderson fixed point theorem.
5. Verifying main result by providing an illustrative example.

Chapter 4

Result And Discussion

4.1 Main Result

In this section, we discuss the existence of at least two positive solutions for second order undamped three point boundary value problems (1.1)-(1.2) by applying Avery-Henderson fixed point theorem [Theorem 2.2.15].
Obviously, $u(t) \in C^2([0,1], \mathbb{R}^+)$ is solution of (1.1)-(1.2) if and only if $u(t)$ is a solution of the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds. \tag{4.1}$$

Let $E = C([0,1])$ be a real Banach space with the norm $||u|| = \max_{0 \le t \le 1} |u(t)|$ and define a cone P,

$$P = \{u \in E; u(t) \ge 0 \text{ and } \min_{\delta \le t \le 1-\delta} u(t) \ge \omega ||u||\}$$

where $\omega = \frac{\mathcal{M}}{D}$, then P is anon empty closed subset of E.

Define an operator $T : P \to E$ as

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s)) ds \tag{4.2}$$

We assume the following conditions hold through out this research:

H1. $f \in C([0,1] \times [0,\infty), [0,\infty)), k \in (0,\infty)$,

H2. $0 \le G(t,s) < \infty, \forall t, s \in [0,1]$,

H3. $\alpha < \frac{\sinh k}{\sinh k\eta}$.

For the notational convenience, we denote R and S by

$$R = \frac{1}{\int_0^1 H(s,s)ds} > 0, \qquad S = \frac{1}{\int_0^1 \omega H(s,s)ds} > 0. \tag{4.3}$$

By applying fixed point theorem on T and establishing suitable conditions on f we determined the existence of at least two fixed points in a cone P.

Lemma 4.1.1. *Let H_1, $H_2$and H_3 hold the operator $T : P \longrightarrow P$ is completely continuous.*

Proof. First we prove the following

1. The operator T is self map on P. Now (1.1)-(1.2) has a solution $u = u(t)$ if and only if u is the fixed point of the operator T defined in equation 4.2.

   Now $G(t,s)$ is the Green's function for the boundary value problem, by Lemma 2.3.1 and Lemma 2.3.2 we have $G(t,s) \leq DH(s,s)$, for all $t, s \in [0,1]$ and $Tu \in E$, for each $u \in P$. We have

   $$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 DH(s,s)f(s,u(s))ds \tag{4.4}$$

   and we have $Tu(t) \leq \int_0^1 DH(s,s)f(s,u(s))ds$ which implies that

   $$||Tu|| \leq \int_0^1 DH(s,s)f(s,u(s))ds$$

   then,

   $$\begin{aligned}\min_{t\in[\delta,1-\delta]}(Tu)(t) &= \min_{t\in[\delta,1-\delta]} \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \mathcal{M}\int_0^1 H(s,s)f(s,u(s))ds = \frac{\mathcal{M}}{D}\int_0^1 DH(s,s)f(s,u(s))ds \\ &\geq \frac{\mathcal{M}}{D}||T(u)|| \\ &\geq \omega||T(u)||.\end{aligned}$$

   $TP \subset P$. Therefore, T is a self map on P.

2. The operator T is uniformly bounded on P. Let $u \in P$, in view of the positivity and continuity of $G(t,s), H(t,s)$ and f, we have $T : P \longrightarrow P$ is continuous .

Let $\{u_k\}$ be a bounded sequence in P, say $||u_k|| \leq M$ for all k .Since f is continuous, there exist $N > 0$ such that $|f(t, u(t))| \leq N$ for all $u \in [0, \infty)$ with $0 \leq u \leq M$ then, for each $t \in [0, 1]$ and for each k,

$$\begin{aligned} |Tu_k(t)| &= |\int_0^1 G(t,s)f(s,u_k)ds| \\ &\leq \int_0^1 G(t,s)(N)ds \\ &\leq N\int_0^1 G(t,s)ds < +\infty. \end{aligned}$$

That is for each $t \in [0, 1]$, u_k is abounded sequence of real numbers. By choosing successive subsequences, for each t, there exists a subsequence $\{u_k\}$ which converges uniformly for $t \in [0, 1]$. Hence, T is uniformly bounded.

3. The operator T is equicontinuous on P. To prove T is equicontinuous. Let $u \in P$ and $\epsilon > 0$ be given. By the continuity of $G(t, s)$, for $t \in [0, 1]$, there exist a $\delta > 0$ such that $|G(t_2, s) - G(t_1, s)| < \frac{\epsilon}{N}$ whenever $|t_1 - t_2| < \delta$, for t_1 ,$t_2 \in [0, 1]$.

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &= |\int_0^1 (G(t_1,s) - G(t_2,s))f(s,u(s))ds| \\ &\leq \int_0^1 |G(t_2,s) - G(t_1,s)|Nds, \\ &\leq N\int_0^1 |G(t_1,s) - G(t_2,s)|ds, \\ &< \epsilon. \end{aligned}$$

Therefore, by a standard application of the Arzela-Ascoli theorem [Royden, 2010] and the result from 1,2 and 3, T is completely continuous.

□

From the above arguments, we know that the existence of at least two positive solutions of (1.1), (1.2) can be equivalent to the existence of at least two fixed points of the operator of T.

Define the nonnegative, increasing, continuous functionals γ, θ, and ψ on the cone P by:- $\gamma(u) = \min_{\delta \leq t \leq 1-\delta} u(t)$
$\theta(u) = max_{\delta \leq t \leq 1-\delta} u(t), \psi(u) = max_{0 \leq t \leq 1} u(t)$
We observe that for any $u \in P$,

$$\gamma(u) \leq \theta(u) \leq \psi(u), \tag{4.5}$$

And

$$||u|| = \frac{\gamma(u)}{\omega} \leq \frac{\theta(u)}{\omega} \leq \frac{\psi(u)}{\omega}. \tag{4.6}$$

Theorem 4.1.2. *Under the conditions* $(H1), (H2)$ *and* $(H3)$ *there exist real numbers* $0 < a < b < c$ *such that* $f(t, u(t))$ *satisfies the following:*

(A1) $f(t, u(t)) > \frac{cS}{D}, t \in (\delta, 1-\delta)$ *and* $u \in (c, \frac{c}{\omega})$,

(A2) $f(t, u(t)) < \frac{bR}{D}, t \in [0, 1]$ *and* $u \in [0, \frac{b}{\omega}]$,

(A3) $f(t, u(t)) > \frac{aS}{D}, t \in (\delta, 1-\delta)$ *and* $u \in [\frac{a}{,} \frac{a}{\omega}]$.

Then the three point BVP (1.1)-(1.2) *has at least two positive solutions* u_1 *and* u_2 *such that* $a < \psi(u_1)$ *with* $\psi(u_1) < b$ *and* $b < \theta(u_2)$ *with* $\gamma(u_2) < c$ *.*

Proof. Consider the operator T: $P \to E$, where T as defined (4.2). We seek two fixed points $u_1, u_2 \in P$ of T. For each $u \in P$, from (4.5), (4.6)

$$\gamma(u) \leq \theta(u) \leq \psi(u)$$

and

$$||u|| \leq \frac{\gamma(u)}{\omega}.$$

Also, for any $u \in P$, $\quad Tu \in P$ by first part of Lemma 4.1.1. Also, for any $0 \leq \lambda \leq 1$ and $u \in P$, $\theta(\lambda u) = \lambda\theta(u) = \max_{\delta \leq t \leq 1-\delta}(\lambda u)(t) = \lambda \max_{\delta \leq t \leq 1-\delta} u(t) = \lambda\theta(u)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of Theorem 2.2.15, are satisfied. Firstly, we shall verify that condition(B1) of Theorem 2.2.15 is satisfied. Since

$$u \in \partial P(\gamma, c),$$

from (4.6) we have that
$c = min_{\delta \leq t \leq 1-\delta} u(t) \leq ||u|| \leq \frac{c}{\omega}$. Then

$$\begin{aligned} \gamma(Tu) &= \min_{\delta \leq t \leq 1-\delta} \int_0^1 G(t, s) f(s, u(s)) ds \\ &\geq \int_0^1 \mathcal{M} H(s, s) \frac{cS}{D} ds \\ &> cS \int_0^1 \omega H(s, s) ds = c, \end{aligned} \tag{4.7}$$

using condition (A1) of Theorem 4.1.2.

Now we shall show that condition (B2) of Theorem 2.2.15 is satisfied. Since $u \in \partial P(\theta, b)$, from (4.6) we have that $0 \le u(t) \le ||u|| \le \frac{b}{\omega}$, for $t \in [0,1]$. Thus

$$\begin{aligned}\theta(Tu) &= \max_{\delta \le t \le 1-\delta} \int_0^1 G(t,s) f(s,u(s))ds \\ &\le \int_0^1 DH(s,s) f(s,u(s))ds \\ &\le \int_0^1 H(s,s) D \frac{bR}{D} ds = bR \int_0^1 H(s,s)ds = b,\end{aligned} \tag{4.8}$$

by the condition(A2) of Theorem 4.1.2. Finally, using condition (A3) of 4.1.2, to settle criteria (B3) of Theorem 2.2.15. Since $0 \in P$ and $a > 0, P \in (\psi, a) \neq \emptyset$. Since,$u \in \partial P(\psi, a)$, $a = max_{0 \le t \le 1} u(t) \le ||u|| \le \frac{a}{\omega}$, for $t \in \ (\delta, 1-\delta)$. Therefore,

$$\begin{aligned}\psi(Tu) &= \max_{0 \le t \le 1} \int_0^1 G(t,s) f(s,u(s))ds \\ &\ge \int_0^1 \mathcal{M} H(s,s) f(s,u(s))ds \\ &> \int_0^1 H(s,s) \mathcal{M} \frac{aS}{D} ds = aS \int_0^1 \omega H(s,s)ds = a.\end{aligned} \tag{4.9}$$

Thus, all the conditions of Theorem 2.2.15 are satisfied. Therefore, the boundary value problem (1.1),(1.2) has at least two positive solutions $(u_1), (u_2)$ in cone P. This completes the proof of the theorem. □

4.2 Example

Example 4.2.1. *Consider the boundary value problem:*

$$-u''(t) + \frac{1}{9} u(t) = f(t, u(t)), \qquad 0 \le t \le 1 \tag{4.10}$$

$$u(0) = 0, u(1) = 2u(\frac{1}{5}) \tag{4.11}$$

where $k = \frac{1}{3}, \quad \alpha = 2, \quad \eta = \frac{1}{5}$

By the help of Equation (2.15) *and* (2.16) *the Green's function for the* *corresponding homogeneous BVP of* (4.10)-(4.11) *is*

$$G(t,s)=\frac{1}{\frac{1}{3}(\sinh\frac{1}{3}-2\sinh\frac{1}{15})}\begin{cases}[\sinh\frac{1}{3}(1-s)+2\sinh\frac{1}{3}(s-\frac{1}{5})](\sinh\frac{1}{3}t), & t\le s\le\frac{1}{5},\\ [\sinh\frac{1}{3}(1-t)+2\sinh\frac{1}{3}(t-\frac{1}{5})](\sinh\frac{1}{3}s), & s\le t, s\le\frac{1}{5}\\ \sinh\frac{1}{3}t\sinh\frac{1}{3}(1-s), & t\le s,\frac{1}{5}\le s,\\ \sinh\frac{1}{3}s\sinh\frac{1}{3}(1-t)+2\sinh\frac{1}{15}\sinh\frac{1}{3}(t-s), & \frac{1}{5}\le s\le t\le 1.\end{cases}$$

Clearly, the Green's function G(t,s) is positive. There is a continuous function $f(t,u(t))$ *which satisfies all the conditions of Theorem(4.1.2).Therefore, by theorem (4.1.2) the given BVP (4.10)-(4.11)has two solutions.*

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

Based on the obtained result the following conclusion can be derived:- In this study, we defined second-order undamped three point boundary value problems and used the properties of Green's function to construct it for corresponding homogeneous equation.
We established the existence of two positive solutions for second-order undamped three point boundary value problems by applying Avery-Henderson fixed point theorem.
Finally, It was established that, there exists at least two positive solutions for second-order undamped three point boundary value problems.

5.2 Future Scope

This study focused on existence of two positive solutions for second-order undamped three point boundary value problems. Any interested researcher may conduct the research on:-

- Existence of two positive solutions for n^{th}-order three point boundary value problems.

- Recently there are a number of published research papers related to this area of study. So, the researchers recommends the upcoming Post Graduate students of the department and any other interested researchers to do their research work in area of study.

References

Agarwal, R. P. and ORegan, D. (2009). Ordinary and partial differential equations with special functions, Fourier series and boundary value problems. Springer Science Business Media, New York.

Avery,R.I., Hendrson,J (2001).Two positive fixed points of nonlinear operators on ordered Banach spaces, Comm. Appl. Nonlinear Anal., 8 (2001), 27-36.

Gupta,C.P (1992). "Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation",Journal of Mathematical Analysis and Applications,vol.168.no.2,pp.540-551.

Helm, (2008). Two-point boundary value problems:Work book 33, Numerical boundary value problem.

Krasnoselskii, M.A. (1964). Positive Solutions of Operators Equations, P. Noordhof Groningen, the Netherlands.

Lian, W. C, Wong, F. H, and Yeh, C, C, (1996). On the existence of positive solutions of nonlinear second order differential equations, proceedings of the American mathematical society 124, 4.

Liu, Feng and Fang, F. (2014). Multiplicity of Positive Solutions for a singular Second-Order Three-Point Boundary Value Problem with a Parameter, Journal of Applied Mathematics.

Ma,R.(1998). Positive solutions of a nonlinear three-point boundary-value problem Electronic Journal of Deferential Equations, l.

Ma,R.(1999). Positive solutions of a nonlinear three-point boundary value problem, Electron. J Differential Equations, 1-8.

Ma,R.(2000). Multiplicity of positive solutions for second-order three-point boundary value problems, Comput. Math. Appl. 40 , 193-204.

Ma,R.(2001). Positive solutions for second-order three-point boundary value problems, Appl.Math. Lett.14 , 1-5.

Nieto, J. J. (2013). Existence of a solution for a three-point boundary value problem for a second-order differential equation at resonance. Nieto Boundary Value Problems (2013:130).

Sveikatea, Z.(2016).On Three-Point Boundary Value Problem, Mathematical Modeling and Analysis Publisher: Taylor Francis and VGTU, 21(2), 270-281.

YanleiZhang.(2017). positive solutions for boundary value problems second order ordinary differential equations Trent university Peterborough, Ontario, Canada.

Yongxiang. (2016). Existence of positive solution for the cantilever beam equations with fully nonlinear terms, Nonlinear Analysis: Real World Applications 27, 221-237.

Yongfang, W,. Zhanbing, B., Sujing, S. (2019). On positive solutions for some second order three-point boundary value problems with convection term Wei et al. Journal of Inequalities and Applications. 2019:72.

Zengqin Zhao. (2008).Solutions and Greens functions for some linear second-order three-point boundary value problems: Computers and Mathematics with Applications 56, 104-113.

Zima, M. (2004). Fixed point theorem of LeggettWilliams type and its application J. Math. Anal. Appl. 299, 254-260.

Zhang and Xuan(2016) . Multiple positive solutions for a second-order boundary value problem with integral boundary conditions Boundary Value Problems 2016:60.