EXTRAPOLATED REFINEMENT OF GENERALIZED GAUSS- SEIDEL SCHEME FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

A THESIS SUBMITTED TO THE DEPARTMENT OFMATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS (NUMERICAL ANALYSIS)

By: Amrach Sebro

Advisor: Hailu Muleta(M.Sc.)

Co-advisor: Solomon G/Giorgis (M.Sc.)

JIMMA, ETHIOPIA

June, 2018

Table of content

Content **Pages**

ACKNOWLEDGMENT

The research project outlined in this thesis would not have been possible without inspiration and help provided by people too numerous to mention. I am indebted very thankful to my advisor Mr. Hailu Muleta for his help in designing this research and gave opportunities to do the research by guiding me throughout the work and giving me unreserved advice. My deepest gratitude also goes to my Co-advisor Mr.Solomon G/Giorgis, through his personal example of intellectual curiosity, optimism and integrity; he set tone of free thinking and enormous scientific excitement in my mind.

Abstract

In this thesis, Extrapolated Refinement of Generalized Gauss-Seidel scheme for solving system of linear equations has been developed. In order to accelerate the rate of convergence of the scheme, the one-parameter family of splitting procedure has been introduced in order to attain the largest rate of convergence. To validate the proposed method, two numerical examples were considered. Comparisons were made among Refinement of Generalized Jacobi, Generalized Gauss-Seidel, Refinement of Generalized Gauss-Seidel and Extrapolated Refinement of Generalized Gauss-Seidel schemes with respect to the number of iterations to converge, computational running time and storage capacity. Finally, the results showed that the Extrapolated Refinement of Generalized Gauss-Seidel scheme is more efficient than the other three schemes considered.

CHAPTER ONE

INTRODUCTION

1.1.Background of the Study

Numerical linear algebra is an exciting field of research and much of this research has been triggered by a problem of getting the solution vector(s) $X \in \mathcal{R}^n$ of the equation $AX = B$ where$A \in \mathcal{R}^{n\times n}$and $B \in \mathcal{R}^n$. Many scientific problems lead to the requirement to solve linear systems of equations as part of the computations. From a pure mathematical point of view, this problem can be considered as being solved in the sense that we explicitly know its solution in terms of determinants. The actual computation of the solution(s) may, however, lead to several complications when carried out in infinite precision and when each basic arithmetic operation takes finite time. Even the simple case when $n = m$ and A is non-singular which is a trivial problem from a mathematical point of view and may even turn out to be impossible (Yousef S., 1996).

The major factors to be considered in comparing different numerical methods are the accuracy of the numerical solutions, iteration number and its computational time (Bedet *et al.,* 1985). It is important to note that the comparison of numerical methods is not simple because their performance may depend on the characteristic of the problem at hand. It should also be noted that there are other factors to be considered such as stability, versatility, and proof against run-time error and so on which are being considered in most of the MATLAB built – in routines (Atkinson, 1978).

As it is discussed by (Saeed, 2008) different methods are being used for the solution of system of linear equations that is for all solution. However, one method is not the best at all because any method is to be determined according to its speed and accuracy.

System of linear equations of the form $AX = B$ plays greater role in finance, economics, industry, engineering, physics, chemistry and computer science (Javad, 2012).

The numerical method for the solution of $AX = B$ is classified into two categories by considering the criteria: convergence rate, number of iterations required, memory requirements and accuracy.

These methods are direct methods (Gauss elimination, Gauss Jordan, matrix inversion, etc.) and indirect or iterative methods. The direct methods produce the exact solution after a finite number of steps (disregarding the round-off errors).As it is discussed by (Turner, 1989) the direct methods for solving linear equations have some difficulties. For example, the problem of Gaussian elimination method lies in control of the accumulation of rounding errors. Besides, the solution is obtained directly when the system of equation has some special forms like (diagonal system of equations or upper triangular system of equation).

The iterative method is a technique that starts with an initial guess and attempts to solve the problem or a solution of linear system of equations by finding successive approximations to the solution and is very efficient when applied to large and sparse system of equations that arise in the practical problems (Javad, 2012).

These crucial methods at the first instance of iteration method for solving system of linear equations appeared in the work of Gauss, Jacobi, Seidel and Nekrasovi during the 19^{th} century. At this time, stationary iterated methods such as Successive over relaxation (SOR) and its variants were perfected and widely applied to the solution of large linear systems arising from the discretization of PDEs of elliptic type.

1.2. Statement of the Problem

System of linear equations has a wide application in science and engineering researches, but those applications may face with the problem of solving large and sparse linear system of the form $AX = B$. The iterative methods are also suitable for solving linear equations when the number of equations in a system is very large and are very effective concerning computer storage and time requirements. One of the advantages of using iterative methods is that they require fewer multiplications for large systems and are fast and simple to use when the coefficient matrix is sparse.

Many iterative methods have been developed in an attempt to come up with efficient schemes. It is under this consideration this research is conducted in order to develop efficient iterative method for solving system of linear equations which requires less number of iterations, less storage capacity, less computational time and the convergence of the approximate solution is guaranteed.

To this end, the intention of this paper was to answer the following basic research questions:

1. What are the procedures and techniques that can be followed to modify Refinement of Generalized Gauss- Seidel scheme by using Extrapolation technique?
2. To what degree of accuracy does the proposed scheme converge?
3. What is the advantage of the proposed scheme over the existing schemes namely, RGJ, GGS and RGGS?

1.3 Objective of the Study

1.3.1. General Objective

The general objective of this thesis is to modify the Refinement of Generalized Gauss- Seidel scheme for solving system of linear equations by using Extrapolation technique..

1.3.2. Specific Objectives

- To describe the procedures(or techniques) that could be implemented to develop Extrapolated Refinement of Generalized Gauss- Seidel scheme;
- To establish condition for convergence of the modified Refinement of Generalized Gauss- Seidel scheme;
- To compare the efficiency of Extrapolated Refinement of Generalized Gauss- Seidel scheme with RGJ, GGS, and RGGS in terms of number of iterations, computational running time and storage capacity.

1.4. Significance of the Study

The results, conclusions and comments of this paper is crucial because it

- ✓ Can be used as spring board for other researchers who want to conduct research on similar and/or related areas;
- ✓ Enhanced the ability of the researcher in conducting scientific research;
- ✓ Serves as an input to the scientific community.

1.5 Delimitation of the study

This study was delimited to Extrapolated Refinement of Generalized Gauss-Seidel scheme for solving system of linear equations of the form $AX = B$ where, $A = \left(a_{ij}\right)$ is known $n \times n$ non-singular real coefficient matrix, B is n dimensional real column vector and X is the solution vector to be determined.

CHAPTER TWO
LITERATURE REVIEW

The limitation of analytical method in practical application is led mathematicians and other scientists to evolve numerical methods. It is clear that exact method often fail in drawing reasonable inference from a given set of tabulated data or in finding solution for different equation. There are many situations where analytical methods are unable to produce desirable results. Even if analytical solutions are not amenable to direct numerical interpretations (Goyal, 2007).

Consistent linear systems in real life are solved in one of the ways: by direct calculation (using a matrix factorization, for example) or by iterative procedure that generates sequence of vectors that approach the exact solution. When the coefficient matrix is large and sparse, iterative algorithms can be more rapid than direct methods and can require less computer memory. Also an iterative process may be stopped as soon as an approximate solution is sufficiently accurate for practical work. However, iterative methods can also fail or be extremely slow for some problems.

Stationary iterations suffer from serious limitations such as lack of sufficient generality and dependency on convergence parameters that might be difficult to estimate without prior information, for example on the spectrum of the coefficient matrix for many problems of practical interest. Stationary iterations diverge or converge very slowly (Michael, 2002). Much work has been done to overcome this limitation. Adoptive parameter estimation procedures together with acceleration techniques based on several emerging krylov sub-space methods are covered in the monograph by Hegemony and Young (Hegeman and Young, 1981).

Yousef et al., (2000) states iterative methods have traditionally been used for the solution of large linear systems with diagonally dominant sparse matrices. For such systems the method of Gauss Jacobi and Gauss-Seidel could be used with some success, not so much because of the reduction in computational work, but mainly because of the limited amount of memory that is required. Of course, reduction of the computation also a serious concern, and this led Jacobi (1846) to apply plane rotations to the matrix in order to force stronger diagonal dominance

giving up sparsely. Jacobi had to solve much similar system in the context of eigenvalue computations; his linear systems were rather small of order seven.

Iterative methods were also predominantly applied for solving discretized elliptic self-ad joint partial differential equations together with a local parameter for accelerating the iterative process. The first and simplest of this method is Richardson's method (David, 1954).

The major factors to be considered in evaluating (comparing) different numerical methods are the accuracy of the numerical solution and its computational time (Bedet and Hall, 1985).

Performance actually depends on several factors: the computation time taken for one iteration of the algorithm, the time step for one iteration which represents the time discretization required to reach a given accuracy or numerical stability for a given method, the desired accuracy of the method, the numerical stability of the method which also limits the time step for a given method (Volino and Magnenat, 2000).

The approximate methods for solving system of linear equations make it possible to obtain the values of the roots system with the specified accuracy as the limit of the sequence of some vector. The process of constructing such a sequence is known as iteration. Unlike the direct methods, it attempts to calculate an exact solution in a finite number of operations, and starts with an initial approximation and generates successively improved approximations in an infinite sequence whose limit is the exact solution. In practical terms, this has more advantage because the direct solution will be subject to rounding off errors.

As discussed by Turner (1989), the direct methods for solving linear equations have some difficulties. He faced difficulty with gauss-elimination approach because of round off errors and slow convergence for large systems of equations. To get rid of these problems many authors like Kalambi (1998) and Turner (1989) were encouraged to investigate solutions of linear equations by iterative methods.

As it is discussed by Salkuyeh (2007), the Gauss- Seidel method is more efficient if it combined with other methods. It has also been proved that if the given coefficient matrix is strictly diagonally dominant or irreducible diagonally dominant or symmetric positive definite matrix the Gauss–Seidel method converges for any initial approximation.

On the other hand,(Davod,2007) developed the method called Generalized Gauss- Seidel method and the result shows that the method is more efficient than conventional Gauss-Seidel method and he also introduced Generalized Jacobi method which is more efficient than conventional Jacobi method. Further, Genanew Gofe (2016) developed the method called Refinement of Generalized Gauss-Seidel method and states that the method is more efficient than the other methods like Refinement of Generalized Jacobi (RGJ) and successive over relation (SOR).

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was being conducted in Jimma University under the supervision of Department of Mathematics in 2017/2018 academic year.

3.2. Study Design

The study employed mixed design (documentary review and experimental results).

3.3. Source of Information

Relevant books, published articles, journals, and experimental results obtained from numerical test examples were used as the source of information.

3.4. Mathematical procedures

The following procedures were performed in orderly manner so as to attain the predefined objectives of the paper:

1. Splitting the coefficient matrix A as $A = T_m - E_m - F_m$, where T_m is a banded matrix with band length $2m+1$, $-E_m$ and $-F_m$ are strictly lower and strictly upper triangular parts of $A - T_m$, respectively;
2. modifying the algorithm for the proposed scheme;
3. proving whether the modified scheme is completely consistent with $AX = B$or not;
4. establishing the convergence of the proposed scheme;
5. Validating the proposed scheme by using numerical examples using MATLAB software.

CHAPTER FOUR

PRELIMINARIES AND DESCRIPTION OF THE METHOD

4.1. Preliminaries

Consider a system of linear equations

$$AX = B, \tag{1}$$

where A is a given nonsingular real nxn matrix with non-vanishing diagonal entries, B is a given real column vector and X is a solution-vector to be determined.

According to Salkuyeh (2011), let $A = \left(a_{ij}\right)$ be an $n \times n$ matrix and $T_m = \left(t_{ij}\right)$ be a banded matrix with band length $2m + 1$ defined as

$$t_{ij} = \begin{cases} a_{ij}, & |j-i| \le m \\ 0, & otherwise \end{cases}.$$

We consider the decomposition

$$A = T_m - E_m - F_m, \tag{2}$$

$-E_m$ and $-F_m$ are strictly lower and upper triangular parts of $A - T_m$ respectively and are defined as follows:

$$T_m = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m+1} & & \\ \vdots & & & \ddots & \\ a_{m+1,1} & \ddots & & \vdots & \\ & \ddots & & \ddots & a_{n-m,n} \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{bmatrix}, \quad E_m = \begin{bmatrix} & & \\ -a_{m+2,1} & \ddots & \\ \vdots & \ddots & \\ -a_{n,1} & \cdots & -a_{n-m-1,n} \end{bmatrix},$$

$$F_m = \begin{bmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m,n} \\ & & & \\ & & & \end{bmatrix}.$$

Then the Generalized Gauss-Seidel (GGS), Refinement of Generalized Jacobi (RGJ), and Refinement of Generalized Gauss-Seidel (RGGS) schemes for solving Eq. (1) in line with the decomposition given in Eq. (2) are, respectively, defined by

$$X^{(k+1)}=\left(T_m-E_m\right)^{-1}F_m X^{(k)}+\left(T_m-E_m\right)^{-1}B \tag{3}$$

$$X^{(k+1)}=\left[T_m^{-1}\left(F_m+E_m\right)\right]^2 X^{(k)}+\left[I+T_m^{-1}\left(F_m+E_m\right)\right]T_m^{-1}B, \tag{4}$$

$$X^{(k+1)}=\left[\left(T_m-E_m\right)^{-1}F_m\right]^2 X^{(k)}+\left[I+\left(T_m-E_m\right)^{-1}F_m\right]\left(T_m-E_m\right)^{-1}B;$$

$$X^{(k+1)}=\overline{B}_{RGGS}^{m}X^{(k)}+\left[I+\left(T_m-E_m\right)^{-1}F_m\right]\left(T_m-E_m\right)^{-1}B \tag{5}$$

, where I is an identity matrix of order n.

For the derivation and proof of Eq. (3), (4) and (5), see (Salkuyeh, 2007 and Genanew, 2016).

Note that if $m=0$, then Eq. (3) results in Gauss-Seidel method which is given by

$X^{(k+1)}=\left(T-E\right)^{-1}F X^{(k)}+\left(T-E\right)^{-1}B$, where T is the diagonal, $-E$ and $-F$ are strictly lower and upper parts of the matrix A.

Definition1. (Gananew, 2016).

A banded matrix is a square matrix with zeros after "m" elements above and below the main diagonal. "m" is usually significantly less than n.

Definition2.A square matrix $A=\left(a_{ij}\right)$ is said to be strictly diagonally dominant (SDD) if

$$|a_{ii}|>\sum_{j\neq i}^{n}|a_{ij}|, i=1,2,3,\ldots,n.$$

Definition 3. (Nicoly and Anton, 2013).

A matrix A is an $M-$matrix ($MP-$matrix) if $a_{ii}>0$ for $i=1,2,3,...,n$, $a_{ij}\leq 0$ for $i\neq j$, A is nonsingular and $A^{-1}\geq O\left(A^{-1}>O\right)$, where O is the $n\times n$ zero matrix.

Definition 4. (Salkuyeh, 2011).

Let $A\in\Re^{n\times n}$. The splitting $A=M-N$ is called:

a. Weak regular if $M^{-1}\geq O$ and $M^{-1}N\geq O$;
b. Regular if $M^{-1}\geq O$ and $N\geq O$.

Definition5. Let $A=\left(a_{ij}\right)$ and $B=\left(b_{ij}\right)$ be $n\times n$ matrices. Then $A\le B\left(A<B\right)$ if $a_{ij}\le b_{ij}\left(a_{ij}<b_{ij}\right)$ for $1\le i$, $j\le n$.

Theorem 1. (Saad, 1995).

Let $A=(a_{ij})$ and $B=\left(b_{ij}\right)$ be two matrices such that $A\le B$ and $b_{ij}\le 0$ for all $i\ne j$. Then, if A is an M-matrix, so is the matrix B.

Theorem 2. (Saad, 1995).

Let $A=M-N$ be the regular splitting of the matrix A. Then $\rho\left(M^{-1}N\right)<1$ if and only if A is nonsingular and $A^{-1}\ge O$.

4.2. Description of the Method

In this section, we develop Extrapolated Refinement of Generalized Gauss-Seidel scheme.

Suppose $X^{(0)}$is an initial approximation to Eq. (1) and

$$\bar{X}^{(n+1)}=T\bar{X}^{(n)}+D \tag{6}$$

In the sequel, we extrapolate the Refinement of Generalized Gauss-Seidel Iterative method in order to increase its rate of convergence in such a way that the proposed method will be completely consistent with the original system in Eq.(1).

Given any iteration procedure it may be possible to improve its rate of convergence by a simple modifications (which we call it acceleration) are frequently termed "Extrapolation," "Overreaction," or various other names based on the problem to which they are applied (Eugene, I. and Herbert, B. K., 1966).

Let the definite splitting given in Eq. (2) be

$$A=J_o+K_o, \tag{7}$$

where $J_o=T_m-E_m$ and $K_o=-F_m$ are fixed matrices with $\det\left(J_o\right)\ne 0$.

Let the eigenvalues of

$$L_o = J_o^{-1} K_o \quad \text{be} \quad \lambda_i, \quad i = 1, 2, ..., n. \tag{7a}$$

Then we introduce the one-parameter family of splitting

$$J(\omega) = \omega J_o \text{ and } K(\omega) = A - \omega J_o = (1-\omega) J_o + K_o \tag{8}$$

We only require $\omega \neq 0$ in order that $\det(J(\omega)) \neq 0$.

Using of Eq. (8) and the clue given in (Yeyios, 1981), the extrapolation scheme for solving Eq. (1) is defined as:

$$X^{(n+1)} = \left[(1-\omega) I + \omega T\right] X^{(n)} + \omega D, \tag{9}$$

Where $T = \left[\left(T_m - E_m\right)^{-1} F_m\right]^2$, $D = \left[I + \left(T_m - E_m\right)^{-1} F_m\right]\left(T_m - E_m\right)^{-1} B$

The scheme in Eq. (9) is called Extrapolated Refinement of Generalized Gauss-Seidel (ERGGS) and ω is a nonzero constant called the extrapolation parameter.

If $\omega = 1$, then Eq. (9) reduces to RGGS.

Let us now show the consistency of the method.

$$X^{(n+1)} = \left[(1-\omega) I + \omega T\right] X^{(n)} + \omega D$$

However, $A^{-1}B = \left[(1-\omega) I + \omega T\right] A^{-1}B + \omega D$. Thus,

$$\omega(I - T) A^{-1} B = \omega D \tag{10}$$

Similarly,

$$X^{(n+1)} = T(\omega) X^{(n)} + \omega D$$
$$\Rightarrow A^{-1}B = T(\omega) A^{-1}B + \omega D$$

$$\Rightarrow \left(I - T(\omega)\right) A^{-1} B = \omega D \tag{11}$$

The comparison of Eq. (10) and Eq. (11) gives

$$\omega(I - T) A^{-1} B = \left(I - T(\omega)\right) A^{-1} B = \omega D$$

Hence, $\det\left(I - T(\omega)\right) = \det\left(\omega(I - T)\right) = \omega^{n} \det(I - T) \neq 0$

Thus, the ERGGS scheme is completely consistent with Eq. (1).

4.3 Convergence of the Scheme

The extrapolation parameter ω must be chosen so as to have $\rho\left[T(\omega)\right] < \rho(T)$ with $\rho\left[T(\omega)\right] < 1$, where $\rho(T)$ and $\rho\left[T(\omega)\right]$ are the spectral radii of the matrices T and $T(\omega)$, respectively.

Here, we find the theoretical optimum for ω, say ω_{opt}, such that

$$\rho\left[T\left(\omega_{Opt}\right)\right] = \min_{\omega} \rho\left[T(\omega)\right] \leq \rho(T) \text{ and } \rho\left[T\left(\omega_{Opt}\right)\right] < 1.$$

If the eigenvalues of $L_o = J_o^{-1} K_o$ and $L(\omega) = J^{-1}(\omega) K(\omega)$ are denoted by λ_i, and $\mu_i(\omega), \quad i = 1, 2, ..., n,$ respectively, then

$$\mu_i(\omega) = \omega(\lambda_i - 1) + 1, \quad i = 1,2, \ldots, n. \tag{12}$$

Moreover, using Eq. (7a) and Eq. (8), we obtain

$$L(\omega) = J^{-1}(\omega) K(\omega) = \frac{1}{\omega} J_o^{-1}\left((1-\omega) J_0 + K_o\right)$$

$$= \frac{(1-\omega)}{\omega} I + \frac{1}{\omega} L_0.$$

Therefore, if $\boldsymbol{u}$ is any eigenvector of L_o corresponding to λ, then $L_o \boldsymbol{u} = \lambda \boldsymbol{u}$ That is,

$$L(\omega)u = \frac{1-\omega}{\omega}u + \frac{\lambda}{\omega}u = \frac{(1-\omega)+\lambda}{\omega}u.$$

This shows that $\boldsymbol{u}$ must also be an eigenvector of $L(\omega)$ corresponding to $\omega(\lambda-1)+1$. On the other hand, if $L(\omega)t = \mu t$,we have

$$\mu t = L(\omega)t = \frac{1-\omega}{\omega}t + \frac{1}{\omega}L_o t.$$

That is, $\left[\mu\omega-(1-\omega)\right]t = L_0\, t$. Hence, every eigenvector of $L(\omega)$ is an eigenvector of L_o and Eq. (12) is established for $\omega \neq 0$.

Most importantly, if a value of ω can be determined such that

$\rho\left[L(\omega)\right] = \max_i \left|\mu_i(\omega)\right| < 1$, then

the scheme converges. Besides, since the rate of convergence is

$$R = -\log\rho\left[L(\omega)\right],$$

the convergence is best for the value $\omega = \omega_{Opt}$ such that

$$\rho\left[L\left(\omega_{Opt}\right)\right] = \min_{\omega}\rho\left[L(\omega)\right]$$

To determine convergent schemes of the form of the family of splitting in Eq. (8), we should study the relation in Eq. (7a).

Lemma 1: Let T has real Eigenvalues λ_j such that

$$\lambda_1 \leq \lambda_j \leq \lambda_n < 1. \tag{13}$$

Then, the scheme in Eq. (8) will converge for any ω such that

$$0 < \omega < \frac{2}{1-\lambda_1}. \tag{14}$$

Besides, such a scheme attains the maximum rate of convergence at

$$\omega = \omega_{opt} = \frac{2}{2-(\lambda_1+\lambda_n)}. \tag{15}$$

for which

$$\rho\left[T\left(\omega_{Opt}\right)\right] = \min_{\omega}\rho\left[T(\omega)\right] = \min_{\omega}\left(\max_{i=1}^{n}\left|\mu_i(\omega)\right|\right) = \frac{\lambda_n-\lambda_1}{2-(\lambda_1+\lambda_n)} < 1. \tag{16}$$

***Proof*:** Obviously, the scheme of the form Eq. (8) converges if $\left|\mu_i(\omega)\right| \prec$,..., n.

For the sake of simplicity, let us introduce

$$x = \omega; \quad m_i = \lambda_i - 1, \quad \text{i= 1, 2,…, n.} \tag{17}$$

Then Eq. (12) can be re-written as

$$\mu_i = m_i x + 1, i = 1,2, \dots, n. \tag{18}$$

All $m_i < 0$ by Eq.(13) and Eq. (17).The equations in Eq. (18) represent n straight lines with negative slopes. Again, by Eq. (17),

we have

$$m_1 \le m_2 \le \dots \le m_n < 0, \tag{19}$$

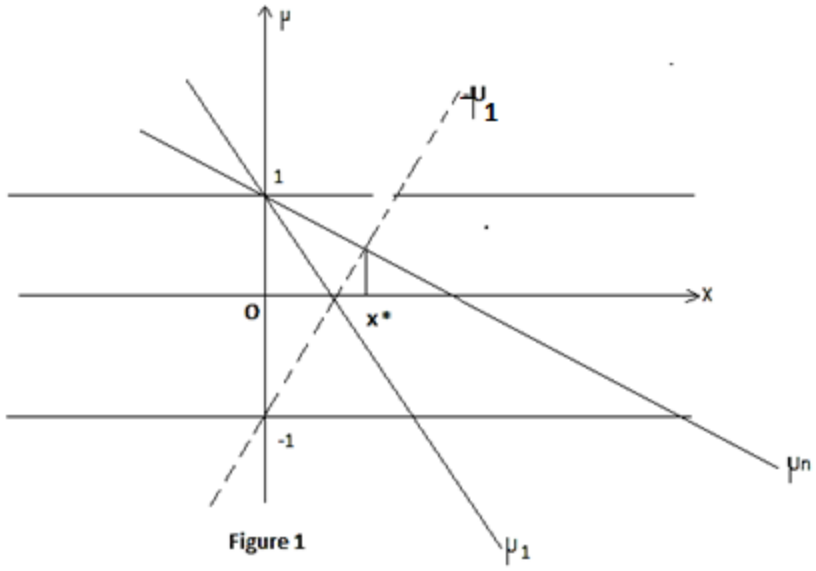

Figure 1

Based on x, whether it is positive or negative, we have the following relations.

$$x>0:\ \mu_1=m_1x+1\le\mu_i\le m_nx+1=\mu_n;$$

$$x<0:\ \mu_n=m_nx+1\le\mu_i\le m_1x+1=\mu_1,\quad i=1,2,...,n. \tag{20}$$

Hence, all the lines are bounded by μ_1, and μ_n .

Eq. (19) clearly shows that all $\mu_i<1$ if and only if $x>0$ and all $\mu_i>-1$ if and only if $\mu_1>-1$; that is, $x<\dfrac{-2}{m_1}$.

Hence, $\left|\mu_i\right|<1$ if and only if $0<x<\dfrac{-2}{m1}$.

Substituting the values of x and m_1 from Eq. (17) in to the inequality $0<x<\dfrac{-2}{m1}$, we get Eq. (14).

From Eq. (20), we have

$$\mu=\max_{i=1}^{n}\left|\mu_i\right|=\max\left(\left|m_1x+1\right|,\left|m_nx+1\right|\right);x>0.$$

From Figure 1, for $x > 0$, we clearly see that the equation of $-\mu_1$ is given by $-\mu_1 = -m_1 x - 1$ and that of μ_n is given by $\mu_n = m_n x + 1$. However, at $x = x^*$, $-m_1 x^* - 1 = m_n x^* + 1$ which gives

$$x^* = \frac{-2}{m_1 + m_n}.$$

$$x^* = \omega_{Opt} = \frac{-2}{m_1 + m_n} = \frac{-2}{(\lambda_1 + \lambda_n) - 2} = \frac{2}{2 - (\lambda_1 + \lambda_n)}$$

Moreover,

$$\rho\left[T(\omega_{Opt})\right] = \min_{x>0} \mu = |m_1 x^* + 1| = |m_n x^* + 1| = \frac{m_1 - m_n}{m_1 + m_n} = \frac{\lambda_1 - \lambda_n}{(\lambda_1 + \lambda_n) - 2} = \frac{\lambda_n - \lambda_1}{2 - (\lambda_1 + \lambda_n)} < 1.$$

This completes the proof of Lemma 1.

Lemma 2: Let T has real Eigenvalues λ_j such that

$$1 < \lambda_1 \le \lambda_j \le \lambda_n. \tag{21}$$

Then the scheme in Eq. (11) will converge for any ω such that

$$\frac{2}{1 - \lambda_n} < \omega < 0. \tag{22}$$

Further, such schemes attain the maximum rate of convergence at

$$\omega = \omega_{Opt} = \frac{2}{2 - (\lambda_1 + \lambda_n)}, \tag{23}$$

for which $$\rho\left[T(\omega_{Opt})\right] = \frac{\lambda_n - \lambda_1}{(\lambda_1 + \lambda_n) - 2} < 1. \tag{24}$$

Proof:A similar proof procedures can be used as that of Lemma 1.

To study the extrapolation scheme in Eq. (9) to find the ranges for ω in which convergence is guaranteed in the general case (when the eigenvalues of $T(\omega)$ are complex) one can follow similar procedures discussed in (Yeyios, 1981).

4.4. Condition on the Convergence of the Scheme

In this section, we study the convergence of the scheme for special classes of matrices, namely strictly diagonally dominant and M-matrices (MP-matrices).

Theorem3. Let A be a *SDD* matrix. Then for any natural number $m \le n$ the ERGGS scheme is convergent for any initial guess $X^{(0)}$.

***Proof*:** Let $M = \left(M_{ij}\right)$ and $N = \left(N_{ij}\right)$ be an $n \times n$ matrices with M being SDD. Then (see Jin X Q, *et al.*, 2005)

$$\rho\left(M^{-1}N\right) \le \rho = \max_i \rho_i, \qquad \rho_i = \frac{\sum_{j=i}^{n} \left|N_{ij}\right|}{\left|M_{ii}\right| - \sum_{j=1, j\neq i}^{n} \left|M_{ij}\right|}$$

Now, let us choose $M_m = T_m - E_m$ and $N_m = F_m$ in the ERGGS scheme. Since A is SDD, it can be easily shown that $\rho_i < 1$. As a result, $\rho\left(M^{-1}N\right) \le \rho < 1$. Since $M^{-1}N$ is squared in ERGGS, $\rho\left(M^{-1}N\right) < \rho\left(\left[M^{-1}N\right]^2\right) < 1$.

Thus, ERGGS converges for arbitrary initial vectors.

Theorem4. Let A be an $M-$ matrix. Then for a given natural number $m \le n$, the ERGGS method converges for any initial guess $X^{(0)}$.

***Proof*:** Let $M_m = T_m - E_m$ and $N_m = F_m$ in the ERGGS. Clearly, $A \le M_m$. As a result of Theorem 1, we assert that the matrix M_m is an $M-$matrix. Besides, $N_m \ge O$. Thus, according to Definition 4b, $A = M_m - N_m$ is a regular splitting of the matrix A. From Theorem 2 and the fact that $A^{-1} \ge O$, we conclude that $\rho\left(M^{-1}N\right) < \rho\left(\left[\left(M^{-1}N\right)\right]^2\right) < 1$.

Thus, $\rho\left(\overline{B}_{ERGGS}^{m}\right) < 1$.

Hence, the ERGGS scheme converges faster than the RGGS.

4.5. Numerical Experiments

Though the comparison of different numerical methods is not simple because their performance may rely on the characteristic of the problems, the major parameters to be taken into account are the accuracy of the numerical solution, storage capacity and its computational time (Bedet,R. A. 1975).

The numerical examples presented in this section were computed with some MATLAB codes on a personal computer Intel® Core™ i3-3120CPU@2.50GHZwith4.00GB memory(RAM) and 32 bits operating system(window 7 home premium). The stopping criteria used is $\left|X_i^{(k+1)} - X_i^{(k)}\right| \leq 5\times 10^{-6}$, where $X_i^{(k+1)}$ and $X_i^{(k)}$ are the approximate solutions at the $(k+1)th$, and kth iteration, respectively.

Example 1. Solve the system of linear equations, which was considered by (*Jain, et al*. 1994) by GGS, RGJ, RGGS and ERGGS.

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ -2 & 0 & 4 & -1 \\ 0 & -2 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1/4 \\ -1/4 \\ -1/4 \\ -1/4 \end{pmatrix}$$

Example 2. Solve the system of linear equations, which was considered by (*Jain, et al.* 1994), by GGS, RGJ, RGGS and ERGGS.

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -3 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -5/3 \\ 2/3 \\ 3 \\ -4/3 \\ -1/3 \\ 5/3 \end{pmatrix}$$

Obviously, the coefficient matrix of the system in example 1 is diagonally dominant and that of in Example 2 is an M-matrix. Thus, the solution produced by the aforementioned methods converges to the exact solution for any initial guess.

Table 1. Numerical solution of **Example 1** by GGS, RGJ, RGGS and ERGGS when $m=1$

Numerical Method	Spectral Radius	Number of Iterations	CPU Running Time	Maximum Absolute Error
RGJ	0.299119	11	0.004745	5.06479×10^{-7}
GGS	0.299119	11	0.004745	5.06479×10^{-7}
RGGS	0.089472	6	0.002771	1.95871×10^{-7}
ERGGS	0.046831	4	0.002512	1.19108×10^{-7}

Table 2. Numerical solution of **Example 1** by GGS, RGJ, RGGS and ERGGS when $m=2$

Numerical Method	Spectral Radius	Number of Iterations	CPU Running Time	Maximum Absolute Error
RGJ	0.00000000	2	0.002035	0.000000
GGS	0.00000000	2	0.002035	0.000000
RGGS	0.00000000	2	0.001941	0.000000
ERGGS	0.00000000	2	0.001829	0.000000

Table 3. Numerical solution of **Example 2** by GGS, RGJ, RGGS and ERGGS when $m=1$

Numerical Method	Spectral Radius	Number of Iterations	CPU Running Time	Maximum Absolute Error
RGJ	0.25000000	10	0.004529	9.53674×10^{-7}
GGS	0.25000000	10	0.004529	7.62939×10^{-7}
RGGS	0.06250000	6	0.003550	4.76837×10^{-8}
ERGGS	0.03225800	4	0.003175	4.65397×10^{-8}

Table 4. Numerical solution of **Example 2** by GGS, RGJ, RGGS and ERGGS when $m=3$

Numerical Method	Spectral Radius	Number of Iterations	CPU Running Time	Maximum Absolute Error
RGJ	0.000000	2	0.002399	0.000000
GGS	0.000000	2	0.002399	0.000000
RGGS	0.000000	2	0.002130	0.000000
ERGGS	0.000000	2	0.002050	0.000000

4.6. Discussion of Results and Conclusion

In this thesis, the Extrapolated Refinement of Generalized Gauss-Seidel scheme for solving system of linear equations has been presented. Two numerical test examples were considered and studied by making use of MATLAB software codes Version R2013a. The results obtained by RGS, GGS, RGGS and ERGGS were presented in Tables 1 through 4 for comparison. The analysis of the results indicated that the ERGGS elapses shorter time (0.002512 and 0.003175 seconds for Example 1 and Example 2, respectively, when m=1 and 0.001829 and 0.002050 seconds for Example 1 when m=2 and Example 2 when m=3, respectively) to converge to the exact solution as compared to the other methods used for comparison. In terms of the number of iterations required to converge to the exact solution with the predefined error of tolerance, the ERGGS iterative method took 4 iterations for example 1 and example 2 as compared to RGJ (took 11 and 10 iterations for example 1 and example 2, respectively), GGS (also took 11 and 10 iterations for example 1 and example 2, respectively), and the RGGS (took 6 iterations for example 1 and example 2, respectively) when $m = 1$. Ibrahim B.K., (2008) states that numerical methods that register small number of iterations requires less computer storage to store its data; consequently, the ERGGS requires less computer storage compared to GGS, RGJ, and RGGS. Salkuyeh D.K.,(2007) witnesses that large m (for instance, as in the splitting in Eq. (2) of this paper) results in smaller spectral radius of the iteration matrix of the iterative methods. Vatti V. B.and Genanew Gofe, (2011) further indicates that any reasonable modification of the iterative method that will reduce the spectral radius increase the rate of convergence of the method. On top of this, all the iterative methods, GGS, RGJ, RGGS and ERGGS took only 2 iterations to converge to the exact solution for example 1(when m=2) and example 2(when m= 3) and approximate the exact solution exactly up to 15 places of decimals. The rates of convergence, R, of ERGGS for example 1and example 2 are 3.0612 and 3.4398, respectively. This implies the rate of convergence of the ERGGS scheme is 3.

In conclusion, in all the parameters (computational running time, number of iterations and storage capacity) used for comparison, in this particular paper, the ERGGS iterative method is more efficient than the other methods considered for comparison.

4.7. Open Problem

It is recommended that the rate of convergence of the method that has been developed in this paper can be further enhanced by using preconditioned Conjugate Gradient method so that higher order systems can also be accommodated.

References

Atkinson. K.E. (1978). **An Introduction to Numerical Analysis 2nd edition**. John Wiley & Sons Inc. 325-355.

Bedet R. A., Enright, W. H. and Hull, T. E (1975).STIFF DETEST: A Program for Comparing Numerical Methods for Stiff Ordinary Differential Equations. Computing Surveys 17(1), 25-48

Eugene I. and Herbert B. K., (1966). Analysis of Numerical Methods, Dover Publications, INC., New York

Genanew Gofe (2016). Refined Iterative Method for Solving System of Linear Equations.*American Journal of Computational and Applied Mathematics.*6(3): 144-147

Goyal.M. (2007). Computer Based Numerical and Statistical Technique, Infinity. Science Press Llc, New Dell.

Hageman D. M and Young D. M., (1981).**Applied Iterative Methods**, New York.

Ibrahim, B. K., (2008). A Comparison of Three Iterative Methods for the Solution of Linear Equations, *J. Appl. Sci. Environ.* 12(4), 53-55.

Ibrahim, B. K., (1998). Solution of Simultaneous Equations by Iterative Methods Vol.3

Jain M. K., Iyenger,S. R. and Kanchi, M. B. (1994). Computational Methods for Partial differential equations. New Delhi : Wiley Eastern Ltd. : New Age International Ltd.

Javed Iqbal, Muhammed Islam Noor and Eisa Al-Said (2012). Residual Iterative Method for solving absolute value Equations.

Michale.B.,(2002).Preconditioning Technique for Large Linear Systems.*Journal of Computational Physics* 182: 418-477.

Nicolay K. and Anton I. (2013). A Refinement of some Overrelaxation Algorithms for Solving a System of Linear Equations. Serdica J. Computing 7(3), 245-256.

SAAD, Y., (1995). Iterative Methods for Sparse Linear Systems, PWS press, New York.

Saeed N.A. and Bhatti, A. (2008).**Numerical Analysis** Shahryar.

Salkuyeh D. K. (2011). Generalized AOR method for solving System of Linear Equations. Australian Journal of Basic and Applied Sciences, 5(3):351-358.

Salkuyeh D.K., (2007).Generalized Jacobi and Gauss-Seidel Methods for Solving Linear System of Equations,*Numer.Math.J. Chinese Uni. (Englisher.), 16(2), 164-170*

Turner P., (1989). **Guide to Numerical Analysis.** Macmillan Education Ltd. Hong Kong

VATTI V. B.and GENANEW, G.(2011).Refinement of generalized Jacobi (RGJ) method for solving system of linear equations, Int. J. Contemp. Math. Sci., **6**(3), 109-116.

Volino P. and Magnenat T.N. (2000).Comparing Efficiency of Integration Methods for Cloth Simulation VSMM.5,109-118.

Yeyios A., (1981). On accelerated procedure of extrapolation. Internat. J. Math. And math. Sci. 4(4), 753-762.

Young D.M., (1971). **Iterative Solution of Large Linear System**, Academic press, New York and London

Young D.M., (1954). Richardson's Method for Solving Linear System with Positive Definite Matrices *J. Math's Phys* 32, 243-255.

Yousef D. and Henk A. V., (2000). Iterative Solution of Linear System In The 20th Century.Journal of Computational and applied mathematics-special issue on numerical analysis. 123, 1-33.

Yousef S., (1996). **Iterative Method for Sparse Linear System Paws,** New York.