Non-Polynomial Septic Spline Method for Third Order Type Singularly Perturbed Boundary Value Problems

A Thesis Submitted to Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters Science in Mathematics

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Declaration

I hereby declare that this work which is being presented in this thesis entitled "***Non polynomial septic spline method for third order type singularly perturbed boundary value problems***" is an authentic record of my own work. It has not been submitted elsewhere (Universities or Institutions) for the award of any other degree, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Abstract

In this thesis, we present non-polynomial septic spline method for solving third order type singularly perturbed boundary value problems. First, the given system is discretized. Then, the spline coefficients are derived and the consistency relation is obtained by using continuity of second, fourth and fifth derivatives. Further, we reduce the obtained fifteen different systems of equations to a system of equations and develop boundary equation in order to equate system of linear equations. The convergence analysis of the obtained hepta-diagonal scheme is investigated. To validate the applicability of the method, two model examples have been considered for different values of perturbation parameter ε *and different mesh size h. The numerical results are presented in Tables and Figures and compared with some existing numerical method in the literature. Further, the proposed method approximate the exact solution very well when* $\varepsilon << h$*, for which most of the existing methods reported in the literature fail to give good result.*

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Introduction

1.1 Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications. It does not strive for exactness; instead he/she attempts to devise a method, which will give an approximation differing from exactness by less than a specified tolerance. The ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in tractable forms. Such problems may each be formulated, for example, in terms of algebraic or transcendental equation, an ordinary or partial differential equation, or in terms of a set of such equations.

In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving large or small parameters, become more complex, Priyadharshini and Ramanujam (2009). Any differential equation in which the highest order derivative is multiplied by a small positive parameter is called perturbed problem and the parameter is known as the perturbation parameter. Singularly perturbed problems occur in a number of areas of applied mathematics, science and engineering among them fluid mechanics, elasticity and quantum mechanics. This perturbation parameter prevents to obtaining satisfactory numerical solutions, and the treatment of singularly perturbed problems is not trivial because the solution depends on perturbation parameter and mesh size *h*, Doolan et al. (1980). Accordingly, more efficient and simpler numerical methods are required to solve singularly perturbed two-point boundary value problems.

Bawa and Natesan, (2005), Rashidinia et al. (2010), and Fasika et al. (2017) presented one dimensional singularly perturbed reaction-diffusion of the form:

$$-\varepsilon y''(x) + a(x)y(x) = f(x), \quad 0 \le x \le 1$$

with the boundary conditions $y(0) = \alpha, \; y(1) = \beta$, where ε is a small positive parameter such that $0 < \varepsilon << 1$, α and β are constants, and $a(x)$ and $f(x)$ are assumed to be continuously differential functions.

Most of the researchers have studied the numerical solutions of second order singular perturbation problems (Kadalbajoo and Patidar, 2002; Reddy and Chakravarthy, 2004). Recently, Ramadan et al. (2008) used a non-polynomial septic spline function for the numerical solution of sixth-order two point boundary value problems. The authors, Sun and Stynes (1995), and El-Zahar (2013) considered numerical methods for higher order singularly perturbed problems. In recent years, many authors namely, Chen and Huang (2010), and Jalilian et al. (2015) developed different numerical methods for solving such differential equations. It is well known that many methods for solving singularly perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small.

Rashidinia et al. (2010) developed a class of methods based on non-polynomial quintic spline for the numerical solution of singularly perturbed boundary value problems. Quartic non-polynomial spline solution of a third order singularly perturbed boundary value problem is provided by (Akram and Talib, 2014). Christy and Tamilselvan (2014) presents a numerical method for singularly perturbed third order ordinary differential equations of convection diffusion type. In, Christy and Tamilselvan (2017) a numerical method for singularly perturbed third order ordinary differential equations of reaction-diffusion type is described. Yohannis et al. (2018) developed a quintic non-polynomial spline method for third order singularly perturbed boundary value problems. But, still the accuracy and convergence of the numerical methods needs attention. Do to this, numerical treatment of singularly perturbed boundary value problems need improvement. Thus, this study presents an accurate and convergent numerical method for solving third order type singularly perturbed boundary value problems.

1.2. Statement of the Problem

The numerical treatment of singularly perturbed problems yield major computational difficulties and the usual numerical methods fail to produce accurate results for all independent values of x when ε is very small related to the mesh size h ($i.e.$ $\varepsilon \ll h$) for the solution of singularly perturbation two point boundary value problems (Khan and Khandelwal, 2013). That is, there are thin transition layers, where the solution varies rapidly. Howers (1976), Kelevedjiev (2002) and Roos et al. (1996) discussed the existence and uniqueness of singularly perturbed boundary value problems. Lie (2008) constructed a computational method for singularly perturbed two point boundary value problems in the form of series in reproducing Kernel space. Akram (2012) presented a quartic spline solution for third order singularly perturbed boundary value problems and the method is second order of convergence. Akram and Amin (2012) proposed a quintic spline technique to solve fourth order singularly perturbed boundary value problems. Shanthi and Ramanujam (2002) solved singularly perturbed fourth-order ordinary differential equations of convection-diffusion type using asymptotic numerical methods. So, the treatment of singularly perturbed problem presents severe difficulties that have to be addressed to ensure accurate numerical solutions (Doolan et al. 1980). Therefore, it is important to develop more accurate and convergent numerical method for solving third order type singularly perturbed boundary value problems.

As results, this study answers the following research questions:

- ✓ How does the non-polynomial septic spline method be described for solving third order type singularly perturbed boundary value problems?
- ✓ To what extent the method approximate the exact solution?
- ✓ To what extent the method converges?
- ✓ What is the advantage of the present method over the other numerical methods reported in literatures?

1.3. Objectives of the Study

1.3.1. General Objective

The general objective of this study is to develop a non-polynomial septic spline method for solving third order type singularly perturbed boundary value problems.

1.3.2. Specific Objectives

The specific objectives of the study are:

- To describe non polynomial septic spline method for solving third order type singularly perturbed boundary value problems.
- To investigate the accuracy of the proposed method.
- To establish the convergence of the present method.
- To describe the advantage of the present method over the other numerical methods reported in literatures.

1.4. Significance of the Study

The results obtained in this research may:

- ✓ Provide some background information for other researchers who work on this area.
- ✓ Introduce the application of numerical methods in different field of studies.

1.5. Delimitation of the Study

Singularly perturbed problems of the highest order derivative arise in many branches of applied mathematics and engineering and may be solved by different numerical methods. This study delimited to non-polynomial fourth order septic spline method for solving third order type singularly perturbed boundary value problems of the form:

$$-\varepsilon y'''(x)+u(x)y(x)=f(x),\ \ 0\le x\le 1 \tag{1.1}$$

subject to the boundary conditions,

$$y(0)=\phi_1,\quad y(1)=\phi_2,\quad y''(0)=\gamma \tag{1.2}$$

where ϕ_1, ϕ_2 and γ are constants, ε is a perturbation parameter $0<\varepsilon<<1$, $u(x)$ and $f(x)$ are continuous functions.

Chapter Two

Review of Related Literatures

2.1. Singularly Perturbed Problems

Science and technology develops many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving small parameters have become increasingly complex and therefore require the use of asymptotic methods. The term 'singular perturbations' was first used by Friedrichs and Wasow (1946) in a paper presented at a seminar on non-linear vibrations at New York University. Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, civil engineering, optimal control, etc. The classification of singularly perturbed higher order problems depend on how the order of the original equation is affected if one sets $\varepsilon = 0$, where ε is a small positive parameter multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is of convection-diffusion type and of reaction-diffusion type if the order is reduced by two.

It is well known that the solution of singularly perturbed boundary value problems is described by slowly and rapidly varying parts. So there are thin transition layers where the solution can jump suddenly, while away from the layers the solution varies slowly and behaves regularly (Akram and Afia, 2013). Many scholars have studied the analytical and numerical solutions of these problems. Abrahamsson et al. (1974) solved singularly perturbed ordinary differential equations using difference approximations. Numerical treatment of singularly perturbed boundary value problems for higher-order non-linear ordinary differential equations has a great role in fluid dynamics.

The development of numerical methods for solving singularly perturbed problems started with methods aimed at solving ordinary differential equations, an account of which can be found in the first monograph on this subject by Doolan et al. (1980). Ilicasu and Schultz (2004) introduced three finite-difference techniques for second-order singularly perturbed linear boundary value problems using convergent tension spline and on uniform tension spline methods. Valaramathi and Ramanujam (2002) solved singularly perturbed two-point boundary value problems for third-order ordinary differential equations.

2.2. Spline Method

In spline based methods, the differential equation is discretized by using approximate methods based on spline. The end conditions are derived for the definition of spline. The algorithm developed not only approximates the solutions, but their higher order derivatives as well. The theory of spline function and their applications is relatively recent development. The rapid development of spline functions is primarily due to their great usefulness in applications. Splines have many applications in the numerical solution of a variety of problems in applied mathematics and engineering; some of them are, data fitting, function approximation, integro-differential equations, optimal control problems, computer-aided geometric design, wavelets and so on. Programs based on spline functions have found their way in most of computer applications.

Splines are types of curves, originally developed for ship building in the days before computer modeling. Naval architects needed a way to draw a smooth curve through a set of points. The solution was to place metal weights (called knots) at the control points, and bends a thin metal or wooden beam (called a spline) through the weights. The physics of the bending spline meant that the influence of each weight was greatest at the point of contact and decreased smoothly further along the spline. To get more control over a certain region of the spline, the drafts man simply added more weights. This scheme had obvious problems with data exchange. There was a need for mathematical way to describe the shape of the curve. Univariate splines were studied intensely in the 60s, and by the mid-70s they were sufficiently well understood to permit a fairly comprehensive treatment in books form.

The application of splines for the numerical solution of singularly perturbed boundary-value problems has been described in many authors (Rashidinia, 1990). The numerical techniques for a class of singularly perturbed two point singular boundary value problems on a non-uniform mesh using spline in compression are reported by Mohanty and Jha (2005).

2.2.1. Polynomial Spline Functions

Polynomials have long been the functions most widely used to approximate other functions mainly because of their simple mathematical properties. However, it is well-known that polynomials of high degree tend to oscillate strongly and in many cases they are liable to produce very poor approximations. Spline functions can be integrated and differentiated due to being piecewise polynomials and can be easily stored and implemented on digital computers. Cubic polynomials splines are the mathematical equivalent of the draftsman's wooden beam. Through the advent of computers, splines have gained more importance. They were first used as a replacement for polynomials in interpolation and then as a tool to construct smooth and flexible shapes in computer graphics.

Thus, spline functions are adapted to numerical methods to get the solution of the differential equations. Numerical methods with spline functions in getting the approximate solution of the differential equations lead to a matrices which are solvable easily with algorithms having low cost of computation. Siddiqi and Twizell, (1996) presented a second-order method using a polynomial spline for solving an eighth-order boundary value problem. Ramadan et al. (2007) have solved second-order two-point boundary value problems using polynomial and non-polynomial spline functions.

2.2.2. Non-Polynomial Spline Functions

Ordinary and partial differential equations are useful in describing mathematical models for various physical processes. Non-polynomial spline method has turned out to be an effective tool for solving ordinary and partial differential equations. Most of non-polynomial spline functions are consists of a polynomial and trigonometric parts. In many papers various techniques using quadratic, cubic, quartic, quintic, sextic, septic and higher degree non-polynomial splines have been discussed for the numerical solution of linear and nonlinear boundary value problems. Islam (2005) established the numerical solutions of a system of third-order boundary value problems using a non-polynomial spline.

In particular the non-polynomial septic spline function has the form:

$$T_7 = span\{1, x, x^2, x^3, x^4, x^5, sin\, kx\, , cos\, kx\};$$

where k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary.

2.3. Numerical versus Analytical Methods

The analytical and numerical methods to solve singular perturbation problems have been widely used in many fields of fluid dynamics, reaction-diffusion processes, particle physics, and combustion processes. These types of problems are represented by differential equations including ε which is assumed to be a small parameter and solutions of the problems have non-uniform behavior when the parameter $\varepsilon \rightarrow 0$. Analytical solution is exact solution to a problem that can be calculated symbolically by manipulating equations. But for higher order or non-linear differential equations with complex co-efficient, it becomes very difficult to find exact solution. Therefore, we need numerical method for solving these equations. Numerical methods give an approximate solution to any equations. It is important to realize that a numerical solution is always numeric but analytical methods usually give a result in terms of mathematical functions that can be evaluated for specific instances. However, numerical results can be plotted to show some of the behavior of the solution. A variety of numerical methods to solve singularly perturbed boundary value problem for ordinary differential equations are available.

One of the important subjects in applied mathematics is the theory of singular perturbation problem. The mathematical model for this kind of problem usually is in the form of either ordinary differential equations or partial differential equations in which the highest derivative is multiplied by positive small parameter. The purpose of the theory of singularly perturbations is to solve a differential equation with some initial or boundary conditions with small parameter. A spline is a numeric function defined piecewise by polynomials or non-polynomials. The numerical solution of two point boundary value problems using spline methods has been considered by many authors. However, this study focuses on non-polynomial spline method for solving third order singularly perturbed boundary value problems.

Chapter Three

Methodology

This chapter consists; study area and period, study design, source of information and mathematical procedures.

3.1. Study Area and Period

This study will be conducted at Jimma university department of mathematics from September 2017 to June 2018 G.C. Conceptually, the study focus on non-polynomial septic spline method for third order type singularly perturbed boundary value problems.

3.2. Study Design

This study will be employed mixed design (*i.e*. documentary review and experimental design).

3.3. Source of Information

The sources of the information are books, journals and internet, and the experimental result will be obtained by MATLAB version R2013a software.

3.4. Mathematical Procedures

To achieve the stated objectives, the study followed the next steps:

1. Defining the problem.
2. Discretizing the given interval.
3. Replacing the differential equation by spline approximation.
4. Developing the end conditions for the definition of spline.
5. Reducing the obtained schemes into hepta-diagonal system and solved by using Gauss elimination method.
6. Establishing the convergence of the obtained scheme.
7. Writing MATLAB code for the hepta-diagonal system obtained,
8. Validating the scheme by using numerical examples.
9. Compare the obtained result with the result of previous numerical methods.

Chapter Four

Description of the Method, Analysis and Results

4.1. Description of the Method

In order to develop the septic spline approximation for the third-order type boundary value problem in Eqs. (1.1) and (1.2), the interval $[0,1]$ is divided into N equal sub-intervals. For this, we introduce the set of grid points $x_i = x_0 + ih, \quad i = 0,1,2,...,N$, so that,

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \text{ where } h = \frac{x_N - x_0}{N} = \frac{1}{N}.$$

Let $y(x)$ be the exact solution of the Eqs. (1.1) and (1.2) and y_i be an approximation to $y(x_i)$, obtained by the segment $S_\Delta(x)$ of the spline function passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). For each i^{th} segment, the non-polynomial septic spline function $S_\Delta(x)$ in subinterval $[x_i,\ x_{i+1}]$, $i = 0,1,2,\cdots,N-1$ has the form:

$$\begin{aligned} S_\Delta(x) &= a_i \cos(k(x-x_i)) + b_i \sin(k(x-x_i)) + c_i (x-x_i)^5 + d_i (x-x_i)^4 \\ &\quad + e_i (x-x_i)^3 + f_i (x-x_i)^2 + g_i (x-x_i) + r_i \ , \qquad \text{for } i = 0,1,\cdots,N \end{aligned} \tag{4.1}$$

where, $a_i, b_i, c_i, d_i, e_i, f_i, g_i$ and r_i are constants and $k \neq 0$ is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary, and which will be used to raise the accuracy of the method.

To derive expression for the coefficients, we first denote:

$$\left. \begin{aligned} S_i(x_i) &= y_i, & S_i(x_{i+1}) &= y_{i+1}, \\ S_i'(x_i) &= M_i, & S_i'(x_{i+1}) &= M_{i+1}, \\ S_i'''(x_i) &= T_i, & S_i'''(x_{i+1}) &= T_{i+1}, \\ S_i^{(6)}(x_i) &= F_i, & S_i^{(6)}(x_{i+1}) &= F_{i+1}. \end{aligned} \right\} \tag{4.2}$$

Now, by successively differentiating Eq. (4.1) with respect to x, we obtain:

$$\begin{aligned} S_\Delta'(x) &= -ka_i \sin(k(x-x_i)) + kb_i \cos(k(x-x_i)) + 5c_i (x-x_i)^4 + 4d_i (x-x_i)^3 \\ &\quad + 3e_i (x-x_i)^2 + 2 f_i (x-x_i) + g_i \end{aligned} \tag{4.3}$$

$$S_\Delta''(x) = -k^2 a_i \cos(k(x-x_i)) - k^2 b_i \sin(k(x-x_i)) + 20c_i(x-x_i)^3 + 12d_i(x-x_i)^2 + 6e_i(x-x_i) + 2f_i \tag{4.4}$$

$$S_\Delta'''(x) = k^3 a_i \sin(k(x-x_i)) - k^3 b_i \cos(k(x-x_i)) + 60c_i(x-x_i)^2 + 24d_i(x-x_i) + 6e_i \tag{4.5}$$

$$S_\Delta^{(4)}(x) = k^4 a_i \cos(k(x-x_i)) + k^4 b_i \sin(k(x-x_i)) + 120c_i(x-x_i) + 24d_i \tag{4.6}$$

$$S_\Delta^{(5)}(x) = -k^5 a_i \sin(k(x-x_i)) + k^5 b_i \cos(k(x-x_i)) + 120c_i \tag{4.7}$$

$$S_\Delta^{(6)}(x) = -k^6 a_i \cos(k(x-x_i)) - k^5 b_i \sin(k(x-x_i)) \tag{4.8}$$

Evaluating Eqs. (4.3) – (4.8) at x_i and using the relation in Eq. (4.2), and $h = x_{i+1} - x_i$, we have:

$$y_i = a_i + r_i \tag{4.9}$$

$$y_{i+1} = a_i \cos(kh) + b_i \sin(kh) + c_i h^5 + d_i h^4 + e_i h^3 + f_i h^2 + g_i h + r_i \tag{4.10}$$

$$M_i = kb_i + g_i \tag{4.11}$$

$$M_{i+1} = -ka_i \sin(kh) + kb_i \cos(kh) + 5c_i h^4 + 4d_i h^3 + 3e_i h^2 + 2f_i h + g_i \tag{4.12}$$

$$T_i = -k^3 b_i + 6e_i \tag{4.13}$$

$$T_{i+1} = k^3 a_i \sin(kh) - k^3 b_i \cos(kh) + 60c_i h^2 + 24d_i h + 6e_i \tag{4.14}$$

$$S_\Delta^{(5)}(x_i) = k^5 b_i + 120c_i \tag{4.15}$$

$$S_\Delta^{(5)}(x_{i+1}) = -k^5 a_i \sin(kh) + k^5 b_i \cos(kh) + 120c_i \tag{4.16}$$

$$F_i = -k^6 a_i \tag{4.17}$$

$$F_{i+1} = -k^6 a_i \cos(kh) - k^6 b_i \sin(kh) \tag{4.18}$$

From Eqs. (4.17), (4.18), (4.13), (4.11), (4.9) in order and letting $\theta = kh$, we get:

$$a_i = -\frac{h^6 F_i}{\theta^6} \tag{4.19}$$

$$b_i = \frac{h^6 (F_i \cos\theta - F_{i+1})}{\theta^6 \sin\theta} \tag{4.20}$$

$$e_i = \frac{T_i}{6} + \frac{h^3 (F_i \cos\theta - F_{i+1})}{6\theta^3 \sin\theta} \tag{4.21}$$

$$g_i = M_i - kb_i = M_i + \frac{h^5 (F_{i+1} - F_i \cos\theta)}{\theta^5 \sin\theta} \tag{4.22}$$

$$r_i = y_i - a_i = y_i + \frac{h^6 F_i}{\theta^6} \tag{4.23}$$

From Eq. (4.14), we get: $60h^2 c_i = T_{i+1} - k^3 a_i \sin\theta + k^3 b_i \cos\theta - 24 d_i h - 6e_i$ and then, using Eqs. (4.19), (4.20) and (4.21), we obtain:

$$c_i = \frac{T_{i+1}}{60h^2} + \frac{hF_i \sin\theta}{60\theta^3} + \left(\frac{hF_i \cos\theta - hF_{i+1}}{60\theta^3 \sin\theta} \right) \cos\theta - \frac{24 d_i}{60h} - \frac{T_i}{60h^2} + \frac{hF_{i+1} - hF_i \cos\theta}{60\theta^3 \sin\theta} \tag{4.24}$$

Again, using Eqs. (4.19) - (4.24) in order into Eqs. (4.12) and (4.10), we get:

$$\begin{aligned} f_i = &-h^2 d_i - \frac{h^4 F_i \sin\theta}{2\theta^5} + \left(\frac{h^4 F_{i+1} - h^4 F_i \cos\theta}{2\theta^5 \sin\theta} \right) \cos\theta - \frac{h^4 \sin\theta F_i}{24\theta^3} + \frac{h^4 F_{i+1} \cos\theta - h^4 F_i \cos^2\theta}{24\theta^3 \sin\theta} \\ &+ \frac{5h^4 F_{i+1} - 5\theta^3 F_i \cos\theta}{24\theta^3 \sin\theta} + \frac{h^4 F_i \cos\theta - h^4 F_{i+1}}{2\theta^5 \sin\theta} - \frac{hT_{i+1}}{24} - \frac{5hT_i}{24} + \frac{M_{i+1} - M_i}{2h} \end{aligned} \tag{4.25}$$

$$d_i = \frac{-h^2}{48\theta^6}\left\{\frac{3\theta^6 T_{i+1}}{h^3} + \frac{7\theta^6 T_i}{h^3} + 120F_{i+1} - 120F_i + 3\theta^3 \sin\theta F_i + 3\theta^3 \cos\theta \cot\theta F_i \right.$$
$$- 3\theta^3 \cot\theta F_{i+1} - 7\theta^3 \csc\theta F_{i+1} + 7\theta^3 \cot\theta F_i + 60\theta \sin\theta F_i$$
$$- 60\theta \cot\theta F_{i+1} + 60\theta \cot\theta \cot\theta F_i - 60\theta \csc\theta F_{i+1} + 60\theta \cot\theta F_i \tag{4.26}$$
$$\left. - \frac{60\theta^6 M_i + 60\theta^6 M_{i+1}}{h^5} + \frac{120\theta^6 y_{i+1} - 120\theta^6 y_i}{h^6}\right\}$$

Substituting the values of Eq. (4.26) into Eq. (4.25), we obtain:

$$f_i = \frac{h^4}{48\theta^6}\left\{\frac{\theta^6 T_{i+1}}{h^3} - \frac{3\theta^6 T_i}{h^3} + 120F_{i+1} - 120F_i + \theta^3 \sin\theta F_i + 3\theta^3 \cos\theta \cot\theta F_i \right.$$
$$- \theta^3 \cot\theta F_{i+1} - 3\theta^3 \csc\theta F_{i+1} + 3\theta^3 \cot\theta F_i + 36\theta \sin\theta F_i$$
$$- 36\theta \cot\theta F_{i+1} + 36\theta \cot\theta \cot\theta F_i - 84\theta \csc\theta F_{i+1} + 84\theta \cot\theta F_i \tag{4.27}$$
$$\left. - \frac{36\theta^6 M_{i+1}}{h^5} - \frac{84\theta^6 M_i}{h^5} + \frac{120\theta^6 y_{i+1}}{h^6} - \frac{120\theta^6 y_i}{h^6}\right\}$$

Substituting the values of Eq. (4.26) into Eq. (4.24), we obtain:

$$c_i = \frac{h}{120\theta^6}\left\{\frac{5\theta^6 T_{i+1}}{h^3} + \frac{5\theta^6 T_i}{h^3} + 120F_{i+1} - 120F_i + 5\theta^3 \sin\theta F_i + 5\theta^3 \cos\theta \cot\theta F_i \right.$$
$$- 5\theta^3 \cot\theta F_{i+1} - 5\theta^3 \csc\theta F_{i+1} + 5\theta^3 \cot\theta F_i + 60\theta \sin\theta F_i$$
$$- 60\theta \cot\theta F_{i+1} + 60\theta \cot\theta \cot\theta F_i - 60\theta \csc\theta F_{i+1} + 60\theta \cot\theta F_i \tag{4.28}$$
$$\left. - \frac{60\theta^6 M_i}{h^5} - \frac{60\theta^6 M_{i+1}}{h^5} + \frac{120\theta^6 y_{i+1}}{h^6} - \frac{120\theta^6 y_i}{h^6}\right\}$$

Using the continuity condition of the fifth derivatives, that is $S_{\Delta}^{(5)}(x_i) = S_{\Delta-1}^{(5)}(x_i)$, we have:

$$-k^5 a_i \sin k(x_i - x_i)) + k^5 b_i \cos(k(x_i - x_i)) + 120c_i = -k^5 a_{i-1} \sin k(x_i - x_{i-1}))$$
$$+ k^5 b_{i-1} \cos(k(x_i - x_{i-1})) + 120c_{i-1}$$

$$\Rightarrow k^5 b_i + 120c_i = -a_{i-1}k^5 \sin\theta + b_{i-1}k^5 \cos\theta + 120c_{i-1} \tag{4.29}$$

since, $h = x_i - x_{i-1}$ and $\theta = kh$.

Reducing the indices of Eqs. (4.19), (4.20) and (4.28), substituting into Eq. (4.29), and simplifying, we obtain:

$$h^6\left(\alpha_1 F_{i-1}+\beta_1 F_i+\alpha_1 F_{i+1}\right)=h^3\left(5T_{i+1}-T_{i-1}\right)+h\left(60M_{i+1}-60M_{i-1}\right) \\ -120\left(y_{i+1}+y_{i-1}\right)+240y_i \tag{4.30}$$

where,

$$\alpha_1=\frac{1}{\theta^6}\left(-\theta^5\csc\theta-5\theta^3\cot\theta-5\theta^3\csc\theta-60\theta\cot\theta-60\theta\csc\theta+120\right),$$
$$\beta_1=\frac{1}{\theta^6}\left(10\theta^3\csc\theta+120\csc\theta+(2\theta^4+10\theta^2+120)\theta\cot\theta-240\right)$$

Using the continuity condition of the fourth derivatives, that is $S_{\Delta}^{(4)}(x_i)=S_{\Delta-1}^{(4)}(x_i)$, we have:

$$k^4 a_i+24d_i=a_{i-1}k^4\cos\theta+b_{i-1}k^4\sin\theta+120hc_{i-1}+24d_{i-1} \tag{4.31}$$

Reducing the indices of Eqs. (4.19), (4.20), (4.24) and (4.26), substituting into Eq. (4.31), and simplifying, we obtain:

$$h^6\left(\alpha_2 F_{i+1}-\alpha_2 F_{i-1}\right)=h^3\left(-3T_{i+1}-14T_i-3T_{i-1}\right)+h\left(60M_{i+1}+120M_i+60M_{i-1}\right) \\ +120\left(y_{i+1}+y_{i-1}\right) \tag{4.32}$$

where,

$$\alpha_2=\frac{1}{\theta^6}\left(-3\theta^3\cot\theta-7\theta^3\csc\theta-60\theta\cot\theta-60\theta\csc\theta+120\right)$$

Again, from the continuity condition of the second derivatives, that is $S''_{\Delta}(x_i)=S''_{\Delta-1}(x_i)$, we have:

$$-k^2 a_i+2f_i=-a_{i-1}k^2\cos\theta-b_{i-1}k^2\sin\theta+20h^3c_{i-1}+12h^2d_{i-1}+6e_{i-1}h+2f_{i-1} \tag{4.33}$$

Reducing the indices of Eqs. (4.19), (4.20), (4.24), (4.26), (4.23) and (4.27), substituting into Eq. (4.33), and simplifying, we obtain:

$$h^6\left(\alpha_3 F_{i+1}-\alpha_3 F_{i-1}\right)=h^3\left(-T_{i+1}+6T_i+T_{i-1}\right)+h\left(-36M_{i+1}+168M_i+36M_{i-1}\right) \\ -120\left(y_{i+1}-y_{i-1}\right) \tag{4.34}$$

where,

$$\alpha_3=\frac{1}{\theta^6}\left(-\theta^3\cot\theta+3\theta^3\csc\theta-36\theta\cot\theta-84\theta\csc\theta+120\right)$$

In order to eliminate F_i's and M_i's from Eqs. (4.30), (4.32), and (4.34), let's replace i by $i+2,\ \ i+1,\ \ i-1$ and $i-2$, in Eq. (4.30), and obtain:

$$\begin{aligned}&-60hM_{i+1}+60hM_{i-1}+h^3(5T_{i+1}-5T_{i-1})+h^6(\alpha_1F_{i+1}+\beta_1F_i-\alpha_1F_{i-1})\\&+120y_{i+1}-240y_i+120y_{i-1}=0\end{aligned} \tag{4.35}$$

$$\begin{aligned}&-60hM_{i+2}+60hM_{i}+h^3(5T_{i+2}-5T_{i})+h^6(\alpha_1F_{i+2}+\beta_1F_{i+1}-\alpha_1F_{i})\\&+120y_{i+2}-240y_{i+1}+120y_{i}=0\end{aligned} \tag{4.36}$$

$$\begin{aligned}&-60hM_{i+3}+60hM_{i+1}+h^3(5T_{i+3}-5T_{i+1})+h^6(\alpha_1F_{i+3}+\beta_1F_{i+2}-\alpha_1F_{i+1})\\&+120y_{i+3}-240y_{i+2}+120y_{i+1}=0\end{aligned} \tag{4.37}$$

$$\begin{aligned}&-60hM_{i}+60hM_{i-2}+h^3(5T_{i}-5T_{i-2})+h^6(\alpha_1F_{i}+\beta_1F_{i-1}-\alpha_1F_{i-2})\\&+120y_{i}-240y_{i-1}+120y_{i-2}=0\end{aligned} \tag{4.38}$$

$$\begin{aligned}&-60hM_{i-1}+60hM_{i-3}+h^3(5T_{i-1}-5T_{i-3})+h^6(\alpha_1F_{i-1}+\beta_1F_{i-2}-\alpha_1F_{i-3})\\&+120y_{i-1}-240y_{i-2}+120y_{i-3}=0\end{aligned} \tag{4.39}$$

Replacing i by $i+2,\ \ i+1,\ \ i-1$ and $i-2$, in Eq. (4.32), we obtain:

$$\begin{aligned}&-60hM_{i+1}-120hM_i-60hM_{i-1}+h^3(3T_{i+1}+14T_i+3T_{i-1})+h^6(\alpha_2F_{i+1}-\alpha_2F_{i-1})\\&+120y_{i+1}-120y_{i-1}=0\end{aligned} \tag{4.40}$$

$$\begin{aligned}&-60hM_{i+2}-120hM_{i+1}-60hM_{i}+h^3(3T_{i+2}+14T_{i+1}+3T_{i})+h^6(\alpha_2F_{i+2}-\alpha_2F_{i})\\&+120y_{i+2}-120y_{i}=0\end{aligned} \tag{4.41}$$

$$\begin{aligned}&-60hM_{i+3}-120hM_{i+2}-60hM_{i+1}+h^3(3T_{i+3}+14T_{i+2}+3T_{i+1})+h^6(\alpha_2F_{i+3}-\alpha_2F_{i+1})\\&+120y_{i+3}-120y_{i+1}=0\end{aligned} \tag{4.42}$$

$$\begin{aligned}&-60hM_{i}-120hM_{i-1}-60hM_{i-2}+h^3(3T_{i}+14T_{i-1}+3T_{i-2})+h^6(\alpha_2F_{i}-\alpha_2F_{i-2})\\&+120y_{i}-120y_{i-2}=0\end{aligned} \tag{4.43}$$

$$\begin{aligned}&-60hM_{i-1}-120hM_{i-2}-60hM_{i-3}+h^3(3T_{i-1}+14T_{i-2}+3T_{i-3})+h^6(\alpha_2F_{i-1}-\alpha_2F_{i-3})\\&+120y_{i-1}-120y_{i-3}=0\end{aligned} \tag{4.44}$$

Replacing i by $i+2,\ \ i+1,\ \ i-1$ and $i-2$, in Eq. (4.34), and obtain:

$$\begin{aligned}&-36hM_{i+1}-168hM_i-36hM_{i-1}+h^3(T_{i+1}-6T_i+T_{i-1})+h^6(\alpha_3F_{i+1}-\alpha_3F_{i-1})\\&+120y_{i+1}-120y_{i-1}=0\end{aligned} \tag{4.45}$$

$$\begin{aligned}&-36hM_{i+2}-168hM_{i+1}-36hM_i+h^3(T_{i+2}-6T_{i+1}+T_i)+h^6(\alpha_3F_{i+2}-\alpha_3F_i)\\&+120y_{i+2}-120y_i=0\end{aligned} \tag{4.46}$$

$$\begin{aligned}&-36hM_{i+3}-168hM_{i+2}-36hM_{i+1}+h^3(T_{i+3}-6T_{i+2}+T_{i+1})+h^6(\alpha_3F_{i+3}-\alpha_3F_{i+1})\\&+120y_{i+3}-120y_{i+1}=0\end{aligned} \tag{4.47}$$

$$\begin{aligned}&-36hM_i-168hM_{i-1}-36hM_{i-2}+h^3(T_i-6T_{i-1}+T_{i-2})+h^6(\alpha_3F_i-\alpha_3F_{i-2})\\&+120y_i-120y_{i-2}=0\end{aligned} \tag{4.48}$$

$$\begin{aligned}&-36hM_{i-1}-168hM_{i-2}-36hM_{i-3}+h^3(T_{i-1}-6T_{i-2}+T_{i-3})+h^6(\alpha_3F_{i-1}-\alpha_3F_{i-3})\\&+120y_{i-1}-120y_{i-3}=0\end{aligned} \tag{4.49}$$

Simultaneous solution of Eqs. $(4.35)-(4.39)$, with the help of symbolic toolbox by Matlab 2013a, eliminates $F_i's$ and $M_i's$ terms gives the following important relations in terms of y_i and third order derivative T_i, as:

$$\begin{aligned}&\mu_1(y_{i+3}-y_{i-3})+\mu_2(y_{i+2}-y_{i-2})+\mu_3(y_{i-1}-y_{i+1})+\mu_4y_i\\&\quad=\frac{2}{3}h^3\left(\eta_1(T_{i+3}+T_{i-3})+\eta_2(T_{i+2}+T_{i-2})+\eta_3(T_{i+1}+T_{i-1})+\eta_4T_i\right)\end{aligned} \tag{4.50}$$

where,

$\eta_1=X_1Z_2-Z_1X_2,\quad \eta_2=X_1Z_3-Z_1X_3,\quad \eta_3=X_1Z_4-Z_1X_4,\quad \eta_4=X_1Z_5-Z_1X_5,$
$\mu_1=X_9Z_1-Z_9X_1,\quad \mu_2=X_{10}Z_1-Z_{10}X_1,\quad \mu_3=X_1Z_{11}-Z_1X_{11},\quad \mu_4=X_1Z_{12}-Z_1X_{12},$

$X_i's$ and $Z_i's$ for $i=1(1)12$ are described in Appendix.

Now, evaluating Eq. (1.1) at the nodal points x_i, and using the relation in Eq. (4.2), we get:

$$-\varepsilon T_i+u_iy_i=f_i \tag{4.51}$$

where, $T_i=y^{(3)}(x_i)$, $y_i=y(x_i)$, $u_i=u(x_i)$ and $f_i=f(x_i)$, for $i=0,1,2,...,N$.

Substituting the values of Eq. (4.51) in to Eq. (4.50) and simplifying, we get:

$$
\begin{aligned}
&(3\varepsilon\mu_1 - 2\eta_1 u_{i+3} h^3) y_{i+3} + (3\varepsilon\mu_2 - 2\eta_2 u_{i+2} h^3) y_{i+2} + (-3\varepsilon\mu_3 - 2\eta_3 u_{i+1} h^3) y_{i+1} \\
&+(3\varepsilon\mu_4 - 2\eta_4 u_i h^3) y_i + (3\varepsilon\mu_3 - 2\eta_3 u_{i-1} h^3) y_{i-1} + (-3\varepsilon\mu_2 - 2\eta_2 u_{i-2} h^3) y_{i-2} \\
&+ (-3\varepsilon\mu_1 - 2\eta_1 u_{i-3} h^3) y_{i-3} = -2h^3 \{\eta_1 (f_{i+3} + f_{i-3}) \\
&+\eta_2 (f_{i+2} + f_{i-2}) + \eta_3 (f_{i+1} + f_{i-1}) + \eta_4 f_i , \qquad \text{for } i = 3(1)N-3.
\end{aligned}
\tag{4.52}
$$

when $k \to 0$, that is $\theta \to 0$, since $\theta = kh$, then

$$
(\alpha_1, \alpha_2, \alpha_3, \beta_1) \to \left(\frac{-25}{168}, \frac{-11}{168}, \frac{-13}{840}, \frac{-59}{84} \right) \text{ and}
$$

$$
(\mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4) \to \left(1, 8, 19, 0, \frac{1}{140}, \frac{120}{140}, \frac{1191}{140}, \frac{2416}{140} \right),
$$

and the relation in Eq. (4.50) reduces into septic polynomial spline (Akram and Siddiqi, 2005). The relation in Eq. (4.52) gives $N-5$ equations in $N-1$ unknowns y_j, $j = 1(1)N-1$. We require four more equations, two at each end of the nodal points.

4.2. Development of the Boundary Equations

For the discretization of the boundary conditions, we define:

$$
\begin{aligned}
&\text{i.} && \sum_{j=0}^{4} e_j^* y_j + f_j^* h^2 y_0'' + h^3 \sum_{j=0}^{5} g_j^* y_k^{(3)} + t_1 = 0, && \text{for } i = 1 \\
&\text{ii.} && \sum_{j=1}^{5} h_j^* y_j + f_2^* h^2 y_0'' + h^3 \sum_{j=1}^{6} m_j^* y_k^{(3)} + t_2 = 0, && \text{for } i = 2 \\
&\text{iii.} && \sum_{j=N-5}^{N} c_j^* y_j + h^3 \sum_{j=N-6}^{N} d_j^* y_k^{(3)} + t_{N-2} = 0, && \text{for } i = N-2 \\
&\text{iv.} && \sum_{j=N-4}^{N} a_j^* y_j + h^3 \sum_{j=N-5}^{N} b_j^* y_k^{(3)} + t_{N-1} = 0, && \text{for } i = N-1
\end{aligned}
\tag{4.53}
$$

where $e_j^*, g_j^*, f_1^*, f_2^*, h_j^*, m_j^*, c_j^*, d_j^*, a_j^*$ and b_j^* are arbitrary parameters to be determined by employing Taylor series expansion.

From Eq. (4.53), we have:

$$
\begin{aligned}
&e_0^* y_0 + e_1^* y_1 + e_2^* y_2 + e_3^* y_3 + e_4^* y_4 + f_1^* h^2 y_0'' + \\
&h^3 \left(g_0^* y_0''' + g_1^* y_1''' + g_2^* y_2''' + g_3^* y_3''' + g_4^* y_4''' + g_5^* y_5''' \right) = 0, \ \text{for } i = 1
\end{aligned}
\tag{4.54}
$$

Expanding each terms of Eq. (4.54) about x_0, we obtain:

$$e_1^* y_1 = e_1^* \left(y_0 + hy_0' + \frac{h^2 y_0''}{2} + \frac{h^3 y_0^{(3)}}{6} + \frac{h^4 y_0^{(4)}}{24} + \frac{h^5 y_0^{(5)}}{120} + \frac{h^6 y_0^{(6)}}{720} + \frac{h^7 y_0^{(7)}}{5040} + o(h^8) \right)$$

$$e_2^* y_2 = e_2^* \left\{ y_0 + 2hy_0' + \frac{4h^2 y_0''}{2} + \frac{8h^3 y_0^{(3)}}{6} + \frac{16h^4 y_0^{(4)}}{24} + \frac{32h^5 y_0^{(5)}}{120} + \frac{64h^6 y_0^{(6)}}{720} \right.$$
$$\left. + \frac{128h^7 y_0^{(7)}}{5040} + o(h^8) \right\}$$

$$e_3^* y_3 = e_3^* \left\{ y_0 + 3hy_0' + \frac{9h^2 y_0''}{2} + \frac{27h^3 y_0^{(3)}}{6} + \frac{81h^4 y_0^{(4)}}{24} + \frac{243h^5 y_0^{(5)}}{120} \right.$$
$$\left. + \frac{729h^6 y_0^{(6)}}{720} + \frac{2187h^7 y_0^{(7)}}{5040} + o(h^8) \right\}$$

$$e_4^* y_4 = e_4^* \left\{ y_0 + 4hy_0' + \frac{16h^2 y_0''}{2} + \frac{64h^3 y_0^{(3)}}{6} + \frac{256h^4 y_0^{(4)}}{24} + \frac{1024h^5 y_0^{(5)}}{120} + \frac{4096h^6 y_0^{(6)}}{720} \right.$$
$$\left. + \frac{16384h^7 y_0^{(7)}}{5040} + o(h^8) \right\}$$

$$g_1^* y_1^{(3)} = g_1^* \left(y_0^{(3)} + hy_0^{(4)} + \frac{h^2 y_0^{(5)}}{2} + \frac{h^3 y_0^{(6)}}{6} + \frac{h^4 y_0^{(7)}}{24} + o(h^8) \right)$$

$$g_2^* y_2^{(3)} = g_2^* \left(y_0^{(3)} + 2hy_0^{(4)} + \frac{4h^2 y_0^{(5)}}{2} + \frac{8h^3 y_0^{(6)}}{6} + \frac{16h^4 y_0^{(7)}}{24} + o(h^8) \right)$$

$$g_3^* y_3^{(3)} = g_3^* \left(y_0^{(3)} + 3hy_0^{(4)} + \frac{9h^2 y_0^{(5)}}{2} + \frac{27h^3 y_0^{(6)}}{6} + \frac{81h^4 y_0^{(7)}}{24} + o(h^8) \right)$$

$$g_4^* y_4^{(3)} = g_4^* \left(y_0^{(3)} + 4hy_0^{(4)} + \frac{16h^2 y_0^{(5)}}{2} + \frac{64h^3 y_0^{(6)}}{6} + \frac{256h^4 y_0^{(7)}}{24} + o(h^8) \right)$$

$$g_5^* y_5^{(3)} = g_5^* \left(y_0^{(3)} + 5hy_0^{(4)} + \frac{25h^2 y_0^{(5)}}{2} + \frac{125h^3 y_0^{(6)}}{6} + \frac{625h^4 y_0^{(7)}}{24} + o(h^8) \right)$$

Substituting these values in Eq. (4.54) and collecting coefficients of the same order, we obtain:

$$(e_0^*+e_1^*+e_2^*+e_3^*+e_4^*)y_0+(e_1^*+2e_2^*+3e_3^*+4e_4^*)hy_0'+\left(\frac{e_1^*+4e_2^*+9e_3^*+16e_4^*}{2}+f_1^*\right)h^2y_0''$$
$$+\left(\frac{e_1^*+8e_2^*+27e_3^*+64e_4^*}{6}+g_0^*+g_1^*+g_2^*+g_3^*+g_4^*+g_5^*\right)h^3y_0^{(3)}$$
$$+\left(\frac{e_1^*+16e_2^*+81e_3^*+256e_4^*}{24}+g_1^*+2g_2^*+3g_3^*+4g_4^*+5g_5^*\right)h^4y_0^{(4)}$$
$$+\left(\frac{e_1^*+32e_2^*+243e_3^*+1024e_4^*}{120}+\frac{g_1^*+4g_2^*+9g_3^*+16g_4^*+25g_5^*}{2}\right)h^5y_0^{(5)}$$
$$+\left(\frac{e_1^*+64e_2^*+729e_3^*+4096e_4^*}{720}+\frac{g_1^*+8g_2^*+27g_3^*+64g_4^*+125g_5^*}{6}\right)h^6y_0^{(6)}$$
$$+\left(\frac{e_1^*+128e_2^*+2187e_3^*+16384e_4^*}{5040}+\frac{g_1^*+16g_2^*+81g_3^*+256g_4^*+625g_5^*}{24}\right)h^7y_0^{(7)}$$
$$+\left(\frac{e_1^*+256e_2^*+6561e_3^*+65536e_4^*}{40320}+\frac{g_1^*+32g_2^*+243g_3^*+1024g_4^*+3125g_5^*}{120}\right)h^8y_0^{(8)}+o(h^9)$$

Equating each coefficients of orders with zero, we obtain the parameters:

$$\left(e_0^*,e_1^*,e_2^*,e_3^*,e_4^*,\ f_1^*,\ g_0^*,\ g_1^*,\ g_2^*,\ g_3^*,\ g_4^*,\ g_5^*\right)$$
$$=\left(\frac{-22}{3},\frac{344}{33},\frac{20}{11},\frac{-184}{33},\frac{2}{3},\frac{120}{11},\frac{124}{33},\frac{332}{33},0,0,0,0\right),\ \text{for } i=1$$

By similar fashion, we obtain the values of the parameters at $i=2,\ i=N-2$ and $N-1$.

$$\left(h_1^*,h_2^*,h_3^*,h_4^*,h_5^*,\ f_2^*,m_1^*,m_2^*,m_3^*,m_4^*,m_5^*,m_6^*\right)$$
$$=\left(\frac{811}{124},\frac{-2377}{124},\frac{2445}{124},\frac{-1003}{124},1,\ \frac{-45}{31},\frac{-949}{248},\frac{1243}{248},0,0,0,0\right),\ \text{for } i=2$$

$$\left(c_{N-5}^*,c_{N-4}^*,c_{N-3}^*,c_{N-2}^*,c_{N-1}^*,c_N^*,d_{N-6}^*,d_{N-5}^*,d_{N-4}^*,d_{N-3}^*,d_{N-2}^*,d_{N-1}^*,d_N^*\right)$$
$$=\left(\frac{-1}{11},\frac{8}{11},\frac{-29}{11},\frac{49}{11},\frac{-38}{11},1,0,0,0,0,0,\frac{-15}{11},\frac{1}{22}\right),\quad \text{for } i=N-2$$

and

$$\left(a_{N-4}^*,a_{N-3}^*,a_{N-2}^*,a_{N-1}^*,a_N^*,b_{N-5}^*,b_{N-4}^*,b_{N-3}^*,b_{N-2}^*,b_{N-1}^*,b_N^*\right)$$
$$=\left(0,-1,\ 3,\ -3,\ 1,0,0,0,\frac{-1}{2},\frac{-1}{2},0\right),\quad \text{for } i=N-1$$

Hence, by rearranging the coefficients of the end conditions and using Eq. (1.1), we obtain:

$$\begin{aligned}&(\varepsilon e_1^* + g_1^* u_1 h^3) y_1 + (\varepsilon e_2^* + g_2^* u_2 h^3) y_2 + (\varepsilon e_3^* + g_3^* u_3 h^3) y_3 + (\varepsilon e_4^* + g_4^* u_4 h^3) y_4 \\ &= h^3 \left(g_0^* f_0 + g_1^* f_1 + g_2^* f_2 + g_3^* f_3 + g_4^* f_4 + g_5^* f_5 \right) - (\varepsilon e_0^* + g_0^* u_0 h^3)\phi_1 - \varepsilon f_1^* \gamma, \quad \text{for } i = 1\end{aligned} \tag{4.55}$$

$$\begin{aligned}&(\varepsilon h_1^* + m_1^* u_1 h^3) y_1 + (\varepsilon h_2^* + m_2^* u_2 h^3) y_2 + (\varepsilon h_3^* + m_3^* u_3 h^3) y_3 + (\varepsilon h_4^* + m_4^* u_4 h^3) y_4 + (\varepsilon h_5^* \\ &+ m_5^* u_5 h^3) y_5 + m_6^* u_6 h^3 y_6 = h^3 \left\{ m_1^* f_1 + m_2^* f_2 + m_3^* f_3 + m_4^* f_4 + m_5^* f_5 + m_6^* f_6 \right\} - \varepsilon f_2^* h^2 \gamma, \text{ for } i = 2\end{aligned} \tag{4.56}$$

$$\begin{aligned}&(d_{N-6}^* u_{N-6} h^3) y_{N-6} + (\varepsilon c_{N-5}^* + d_{N-5}^* u_{N-5} h^3) y_{N-5} + (\varepsilon c_{N-4}^* + d_{N-4}^* u_{N-4} h^3) y_{N-4} \\ &+ (\varepsilon c_{N-3}^* + d_{N-3}^* u_{N-3} h^3) y_{N-3} + (\varepsilon c_{N-2}^* + d_{N-2}^* u_{N-2} h^3) y_{N-2} + (\varepsilon c_{N-1}^* + d_{N-1}^* u_{N-1} h^3) y_{N-1} \\ &= h^3 \left\{ d_{N-6}^* f_{N-6} + d_{N-5}^* f_{N-5} + d_{N-4}^* f_{N-4} + d_{N-3}^* f_{N-3} + d_{N-2}^* f_{N-2} + d_{N-1}^* f_{N-1} + d_N^* f_N \right\} \\ &- (\varepsilon c_N^* + d_N^* u_N h^3) y_N, \qquad \text{for } i = N-2\end{aligned} \tag{4.57}$$

$$\begin{aligned}&b_{N-5}^* u_{N-5} h^3 y_{N-5} + (\varepsilon a_{N-4}^* + b_{N-4}^* u_{N-4} h^3) y_{N-4} (\varepsilon a_{N-3}^* + b_{N-3}^* u_{N-3} h^3) y_{N-3} \\ &+ (\varepsilon a_{N-2}^* + b_{N-2}^* u_{N-2} h^3) y_{N-2} + \left\{ \varepsilon a_{N-1}^* + b_{N-1}^* u_{N-1} h^3 \right\} y_{N-1} = h^3 \left\{ b_{N-5}^* f_{N-5} + b_{N-4}^* f_{N-4} \right. \\ &\left. + b_{N-3}^* f_{N-3} + b_{N-2}^* f_{N-2} + b_{N-1}^* f_{N-1} + b_N^* f_N \right\} - (\varepsilon a_N^* + b_N^* u_N h^3)\phi_2, \quad \text{for } i = N-1\end{aligned} \tag{4.58}$$

By expanding Eq. (4.50) in Taylor's series about x_0, we obtain the following local truncation error t_i as:

$$t_i = w_0^* y_i + w_1^* h y_i + w_3^* h^3 y_i^{(3)} + w_5^* h^5 y_i^{(5)} + w_7^* h^7 y_i^{(7)} + o(h^8) \tag{4.59}$$

where,

$$\begin{aligned}w_0^* &= 3\mu_4 \\ w_1^* &= 18\mu_1 + 12\mu_2 - 6\mu_3 \\ w_3^* &= 162\mu_1 + 48\mu_2 - 6\mu_3 - 24\eta_1 - 24\eta_2 - 24\eta_3 - 12\eta_4 \\ w_5^* &= 1458\mu_1 + 192\mu_2 - 6\mu_3 - 2160\eta_1 - 960\eta_2 - 240\eta_3 \\ w_7^* &= 13122\mu_1 + 768\mu_2 - 6\mu_3 - 68040\eta_1 - 13440\eta_2 - 840\eta_3\end{aligned} \tag{4.60}$$

and $\mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3$ and η_4 are arbitrary parameter's.

By using Eq. (4.54) and eliminating the coefficients of the powers of h for different choices of the parameters we obtain class of methods. In order to obtain the boundary equations of fourth order method the coefficients of h^0, h, h^3 and h^5 equal to zero. (*i.e.*, $w_0^* = 0 = w_1^* = w_3^* = w_5^*$).

So, for $(\mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4) = \left(1, 8, 19, 0, \frac{1}{120}, \frac{1}{3}, \frac{632}{89}, \frac{604}{35}\right)$ the truncation error in Eq. (4.59) is reduced to:

$$t_i = \frac{2365}{994} \varepsilon h^7 y^7 + O(h^8).$$

Hence, Eqs. (4.52) and (4.55) – (4.58) gives hepta-diagonal system for $i = 1, 2, \cdots, N-1$ and can easily be solved using Gauss-Elimination method.

4.3. Convergence Analysis

We investigate the convergence analysis for the developed method. The non-polynomial spline solution of Eq. (1.1) with the boundary conditions of Eq. (1.2) is based on the linear system of

Eqs. (4.52) and (4.55) – (4.58). For this, let $Y = y_i$, $\bar{Y} = y(x_i)$, $T = t_i$, and $E = e_i = \bar{Y} - Y$, for $i = 1, 2, ..., N-1$ be column vectors, where $Y, \bar{Y}, T, E$ are approximate solution, exact solution, local truncation error and discretization error respectively. These equations can be written in the following matrix-vector form:

$$\left(A + h^3 B\right) Y + h^3 DF = C \tag{4.61}$$

$$A = \begin{bmatrix}
\varepsilon e_1^* & \varepsilon e_2^* & \varepsilon e_3^* & \varepsilon e_4^* & & & & & & \\
\varepsilon h_1^* & \varepsilon h_2^* & \varepsilon h_3^* & \varepsilon h_4^* & \varepsilon h_5^* & & & & & \\
-3\varepsilon\mu_2 & 3\varepsilon\mu_3 & 3\varepsilon\mu_4 & -3\varepsilon\mu_3 & 3\varepsilon\mu_2 & 3\varepsilon\mu_1 & & & & \\
-3\varepsilon\mu_1 & -3\varepsilon\mu_2 & 3\varepsilon\mu_3 & 3\varepsilon\mu_4 & -3\varepsilon\mu_3 & 3\varepsilon\mu_2 & 3\varepsilon\mu_1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & -3\varepsilon\mu_1 & -3\varepsilon\mu_2 & 3\varepsilon\mu_3 & 3\varepsilon\mu_4 & -3\varepsilon\mu_3 & 3\varepsilon\mu_2 & \\
 & & & & \varepsilon c_{N-5}^* & \varepsilon c_{N-4}^* & \varepsilon c_{N-3}^* & \varepsilon c_{N-2}^* & \varepsilon c_{N-1}^* & \\
 & & & & & \varepsilon a_{N-4}^* & \varepsilon a_{N-3}^* & \varepsilon a_{N-2}^* & \varepsilon a_{N-1}^* &
\end{bmatrix}$$

$$
B=\begin{bmatrix}
g_1^*u_1 & g_2^*u_2 & g_3^*u_3 & g_4^*u_4 & & & & & & & \\
m_1^*u_1 & m_2^*u_2 & m_3^*u_3 & m_4^*u_4 & m_5^*u_5 & & & & & & \\
-2\eta_2u_1 & -2\eta_3u_2 & -2\eta_4u_3 & -2\eta_3u_4 & -2\eta_2u_5 & -2\eta_1u_6 & & & & & \\
-2\eta_1u_1 & -2\eta_2u_2 & -2\eta_3u_3 & -2\eta_4u_4 & -2\eta_3u_5 & -2\eta_2u_6 & -2\eta_1u_7 & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & -2\eta_1u_{N-6} & -2\eta_2u_{N-5} & -2\eta_3u_{N-4} & -2\eta_4u_{N-3} & -2\eta_3u_{N-2} & -2\eta_2u_{N-1} & & \\
 & & & & d_{N-5}^*u_{N-5} & d_{N-4}^*u_{N-4} & d_{N-3}^*u_{N-3} & d_{N-2}^*u_{N-2} & d_{N-1}^*u_{N-1} & & \\
 & & & & & b_{N-4}^*u_{N-4} & b_{N-3}^*u_{N-3} & b_{N-2}^*u_{N-2} & b_{N-1}^*u_{N-1} & &
\end{bmatrix}
$$

$$
D=\begin{bmatrix}
g_1^* & g_2^* & g_3^* & g_4^* & & & & & \\
m_1^* & m_2^* & m_3^* & m_4^* & m_5^* & & & & \\
2\eta_2 & 2\eta_3 & 2\eta_4 & 2\eta_3 & 2\eta_2 & 2\eta_1 & & & \\
2\eta_1 & 2\eta_2 & 2\eta_3 & 2\eta_4 & 2\eta_3 & 2\eta_2 & 2\eta_1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & 2\eta_1 & 2\eta_2 & 2\eta_3 & 2\eta_4 & 2\eta_3 & 2\eta_2 \\
 & & & & d_{N-5}^* & d_{N-4}^* & d_{N-3}^* & d_{N-2}^* & d_{N-1}^* \\
 & & & & & b_{N-4}^* & b_{N-3}^* & b_{N-2}^* & b_{N-1}^*
\end{bmatrix}
$$

$$C=[c_1,\, c_2,\, ...,\, c_{N-1}]^T$$

where,

$$c_1=-\varepsilon f_1^*\gamma-(\varepsilon e_0^*+g_0^*u_0h^3)\phi_1+g_0^*f_0h^3$$

$$c_2=-\varepsilon\gamma f_2^*h^2$$

$$c_3=\ -2h^3\eta_1 f_0+(3\varepsilon\mu_1+2\eta_1u_0h^3)\phi_1$$

$$c_i=0,\qquad \text{for}\quad i=4(1)N-4$$

$$c_{N-3}=-(3\varepsilon\mu_1-2\eta_1u_Nh^3)y_N-2h^3\eta_1 f_N$$

$$c_{N-2}=-\left(\varepsilon c_N^*+d_N^*u_Nh^3\right)\phi_2+h^3d_N^*f_N$$

$$c_{N-1}=-\left(\varepsilon a_N^*+b_N^*u_Nh^3\right)\phi_2+h^3b_N^*f_N$$

$$Y=[y_1,y_2,...y_{N-1}]^T\quad \text{and}\quad F=[f_1,\, f_2,...,\, f_{N-1}]^T$$

Now considering the above system with the exact solution $\bar{Y} = [y(x_1), y(x_2), ..., y(x_{N-1})]^T$, we have:

$$\left(A + h^3 B\right)\bar{Y} + h^3 DF = T(h) + C \qquad (4.62)$$

where, $T(h) = [t_1(h),\, t_2(h),\, ...,\, t_{N-1}(h)]^T$ defined as:

$$\begin{aligned}
t_1 &= \varepsilon h^7 \left(\frac{71}{330}\right) y^7(\xi_1)\,, && x_0 \le \xi_1 \le x_1, && \text{for} \quad i = 1 \\
t_2 &= \varepsilon h^7 \left(\frac{306}{667}\right) y^7(\xi_2)\,, && x_1 \le \xi_2 \le x_2, && \text{for} \quad i = 2 \\
t_i &= \varepsilon h^7 \left(\frac{2365}{994}\right) y^7(\xi_i)\,, && x_{i-1} \le \xi_i \le x_{i+1}, && \text{for} \quad i = 3(1)N-3 \\
t_{N-2} &= \varepsilon h^7 \left(\frac{-277}{2640}\right) y^7(\xi_{N-2})\,, && x_i \le \xi_{N-2} \le x_{i+1}, && \text{for} \quad i = N-2 \\
t_{N-1} &= \varepsilon h^7 \left(\frac{-57}{80}\right) y^7(\xi_{N-1})\,, && x_i \le \xi_{N-1} \le x_{i+1}, && \text{for} \quad i = N-1
\end{aligned}$$

Subtracting Eq. (4.61) from Eq. (4.62), we obtain the error equation,

$$(A + Bh^3)(\bar{Y} - Y) = T(h) \quad \Rightarrow A_0 E = T(h) \qquad (4.63)$$

where, $A_0 = A + h^3 B$ and $E = \bar{Y} - Y = (e_1, e_2, ..., e_{N-1})^T$.

To determine the error bounds the row sums $s_1, s_2, ..., s_{N-1}$ of the matrix A_0 are calculated as follows:

$$\begin{aligned}
s_1 &= \sum_{j=1}^{n-1} a_{1j} = \varepsilon(e_1^* + e_2^* + e_3^* + e_4^*) + (g_1^* u_1 + g_2^* u_2 + g_3^* u_3 + g_4^* u_4)h^3, && \text{for } i = 1 \\
s_2 &= \sum_{j=1}^{n-1} a_{2j} = \varepsilon(h_1^* + h_2^* + h_3^* + h_4^* + h_5^*) + (m_1^* u_1 + m_2^* u_2 + m_3^* u_3 + m_4^* u_4 + m_5^* u_5)h^3, && \text{for } i = 2 \\
s_3 &= \sum_{j=1}^{n-1} a_{ij} = 3\varepsilon\mu_1 - 2\left\{\eta_4 u_3 + \eta_3(u_2 + u_4) + \eta_2(u_1 + u_5) + \eta_1 u_6\right\} h^3\,, && \text{for } i = 3 \\
s_i &= \sum_{j=1}^{n-1} a_{ij} = 2h^3\left\{-\eta_1(u_{i-3} + u_{i+3}) - \eta_2(u_{i-2} + u_{i+2}) - \eta_3(u_{i-1} + u_{i+1}) - \eta_4 u_i\right\}, && \text{for } i = 4(1)N-4 \\
s_{N-3} &= \sum_{j=1}^{N-1} a_{ij} = -3\varepsilon\mu_1 - 2\left\{\eta_4 u_{N-3} + \eta_3(u_{N-4} + u_{N-2}) + \eta_2(u_{N-5} + u_{N-1}) + \eta_1 u_{N-6}\right\} h^3, && \text{for } i = N-3
\end{aligned}$$

$$s_{N-2}=\sum_{j=1}^{N-1} a_{n-2\,j}=\varepsilon(c^*_{N-5}+c^*_{N-4}+c^*_{N-3}+c^*_{N-2}+c^*_{N-1})$$
$$+(d^*_{N-5}u_{N-5}+d^*_{N-4}u_{N-4}+d^*_{N-3}u_{N-3}+d^*_{N-2}u_{N-2}+d^*_{N-1}u_{N-1})h^3, \quad \text{for} \quad i=N-2$$
$$s_{N-1}=\sum_{j=1}^{N-1} a_{n-1\,j}=\varepsilon(a^*_{N-4}+a^*_{N-3}+a^*_{N-2}+a^*_{N-1})$$
$$+(b^*_{N-4}u_{N-4}+b^*_{N-3}u_{N-3}+b^*_{N-2}u_{N-2}+b^*_{N-1}u_{N-1})h^3, \text{ for } i=N-1$$

Since $0<\varepsilon<<1$, we choose h sufficiently small so that the matrix A_0 is irreducible and monotone, according to Mohanty and Jha, (2005). Then, it follows that A_0^{-1} exists and its elements are non-negative.

Hence, from Eq. (4.63), we have:

$$E=A_0^{-1}T \quad \Rightarrow \|E\|\le\|A_0^{-1}\|.\|T(h)\| \tag{4.64}$$

Let $\bar{a}_{i,j}$ is the $(i,j)^{th}$ element of the matrix A_0^{-1}, we define:

$$\|\bar{a}_{i,j}\|=\max\sum_{j=1}^{N-1}\bar{a}_{i,j} \quad \text{and} \quad \|T\|=\max_{1\le i\le N-1}|t_i| \tag{4.65}$$

Also, from the theory of matrices, we have $\sum_{j=1}^{N-1}\bar{a}_{i,j}s_j=1, \quad i=1,2,\ldots,N-1$

That is,

$$\bar{a}_{1,1}(a_{11}+a_{12}+a_{13},\ldots,a_{1,N-1})+\bar{a}_{1,2}(a_{21}+a_{22}+,\ldots,+a_{2,N-1})+,\ldots,+\bar{a}_{1,N-1}(a_{N-1,1}+a_{N-1,2}+,\ldots,+a_{N-1,N-1})$$

$$\Rightarrow \bar{a}_{1,1}(s_1)+\bar{a}_{1,2}(s_2)+,\ldots,+\bar{a}_{1,N-1}(s_{N-1})=1. \tag{4.66}$$

Defining $s_{k*}=\min_{1\le i\le N-1}s_i>0$, then from Eq. (4.66), we have $s_{k*}\left(\bar{a}_{1,1}+\bar{a}_{1,2}+,\ldots,+\bar{a}_{1,N-1}\right)\le 1$.

It follows that: $$\sum_{j=1}^{N-1}\bar{a}_{i,j}\le\frac{1}{s_{k*}}=\frac{1}{h^3M_{k*}}, \tag{4.67}$$

where, $M_{k*}=2\left|\eta_1(u_{k-3}+u_{k+3})+\eta_2(u_{k-2}+u_{k+2})+\eta_3(u_{k-1}+u_{k+1})+\eta_4u_k\right|$.

And also Eq. (4.64) can be written as:

$$e_j=\sum_{j=1}^{N-1}\bar{a}_{i,j}T_j(h) \qquad i=1,2,\ldots,N-1$$

$$\Rightarrow \| e_i \| \le \left\| \sum_{i=1}^{N-1} \bar{a}_{i,j} \right\| \| T(h) \|.$$

From Eqs. (4.65) and (4.67), we get:

$$\| e_i \| \le \max_{1 \le i \le N-1} \| \frac{1}{h^3 M_{k^*}} \| \| T_i \|$$

$$\Rightarrow \| e_i \| \le \frac{N^*}{h^3 M_{k^*}} \left(\frac{\varepsilon h^7 2365}{994} \right) < \left(\frac{2365 N^*}{994 M_{k^*}} \right) h^4 = \psi h^4, \text{ since } 0 < \varepsilon << 1$$

where, $N^* = \max_{x_{i-1} \le \xi_i \le x_{i+1}} \| y^7(\xi_i) \|$ and $\psi = \dfrac{2365 N^*}{994 M_{k^*}}$ which is independent of *h*. It follows that $\| E \| = O(h^4)$ and hence the present method is of fourth order convergence.

4.4. Numerical Examples and Results

In order to test the validity of the proposed method and to demonstrate their convergence computationally, we have taken two model examples of singularly perturbed boundary value problems with exact solutions. The maximum absolute errors at the nodal points, $\max_{1 \le i \le N-1} | y(x_i) - y_i |$ are tabulated in Tables (4.1) - (4.4). The model examples have been solved by taking different values of mesh size, and perturbation parameters. Computed solutions are compared with the exact solutions at nodal points and compared with the methods in (Mustafa and Ejaz, 2017; Akram and Talib, 2014 and Akram, 2012).

Remark: All numerical results of Examples 4.1 and 4.2 are obtained for different values of $\mu_1 = 1,\ \mu_2 = 8,\ \mu_3 = 19,\ \mu_4 = 0,\quad \eta_1 = \dfrac{1}{120},\ \eta_2 = \dfrac{1}{3},\ \eta_3 = \dfrac{632}{89}$ and $\eta_4 = \dfrac{604}{35}$. Because, these values satisfies Eq. (4.60) and they are near to the values of polynomial septic spline but gives an accurate solution.

Example 4.1: Consider the third order singularly perturbed boundary value problem:

$$-\varepsilon y'''(x) + y(x) = 6\varepsilon x^3 (1-x)^5 - 6\varepsilon^2 \left\{ 6(1-x)^5 - 90x(1-x)^4 + 180x^2(1-x)^3 - 60x^3(1-x)^2 \right\}$$

subject to, $y(0) = 0,\quad y(1) = 0,\quad y''(0) = 0.$

The analytical solution of this problem is $y(x) = 6\varepsilon x^3 (1-x)^5.$

Table 4.1: Maximum absolute errors for Example 4.1 with different values of h and ε.

$\varepsilon\downarrow$	$N=10$	$N=20$	$N=40$
Present Method			
$1/16$	2.8930e-04	5.3006e-06	2.6033e-08
$1/32$	1.0962e-04	1.9394e-06	1.3221e-08
$1/64$	3.8007e-05	6.8026e-07	6.2298e-09
Mustafa and Ejaz, (2017)			
$1/16$	6.2854e-03	-	-
$1/32$	1.9707e-03	-	-
$1/64$	3.9065e-04	-	-
Akram, (2012)			
$1/16$	1.3e-02	1.1e-03	7.8e-05
$1/32$	3.2e-03	2.7e-04	1.8e-05
$1/64$	3.4e-04	2.2e-05	1.1e-06

Table 4.2: Maximum absolute errors for Example 4.1 when $\varepsilon \ll h$.

$\varepsilon\downarrow$	$N=10$	$N=50$	$N=100$	$N=150$	$N=200$
Present Method					
10^{-1}	5.3363e-04	5.9813e-08	9.1841e-09	2.1617e-09	7.3808e-10
10^{-2}	1.8773e-05	2.2337e-09	2.5972e-10	6.0042e-11	2.0403e-11
10^{-3}	1.5441e-06	5.5630e-11	3.8697e-12	8.5417e-13	2.8902e-13
10^{-4}	1.8248e-08	3.2291e-12	5.6471e-14	1.0585e-14	3.3903e-15
10^{-5}	1.1853e-10	3.3754e-13	3.6503e-15	2.7729e-16	4.4615e-17
10^{-6}	1.1362e-12	4.5181e-15	2.9874e-16	1.8873e-17	3.6840e-18
10^{-7}	1.1314e-14	2.5989e-17	2.5748e-18	2.4307e-18	2.7110e-19
10^{-8}	1.1309e-16	2.4638e-19	1.6575e-20	3.7102e-21	1.4808e-21
10^{-9}	1.1309e-18	2.4508e-21	1.5883e-22	3.2062e-23	1.0424e-23
10^{-10}	1.1309e-20	2.4495e-23	1.5815e-24	3.1605e-25	1.0074e-25
10^{-11}	1.1309e-22	2.4490e-25	1.5805e-26	3.1550e-27	1.0035e-27
10^{-12}	1.1307e-24	2.4469e-27	1.5764e-28	3.1729e-29	3.6037e-29

Mustafa and Ejaz, (2017)					
10^{-1}	1.6190e-02	7.3371e-04	6.4463e-04	6.3671e-04	6.3496e-04
10^{-2}	5.4777e-04	3.5302e-05	3.2708e-05	3.3005e-05	3.3331e-05
10^{-3}	4.3814e-05	2.4150e-06	1.3966e-06	1.1544e-06	1.2348e-06
10^{-4}	7.5623e-06	2.4329e-07	1.1223e-07	7.6323e-08	6.1521e-08

Example 4.2: Consider the third order singularly perturbed boundary value problem:

$$-\varepsilon y'''(x)+y(x)=81\varepsilon^2\cos(3x)+3\varepsilon\sin(3x),\quad x\in[0,1],$$

subject to, $y(0)=0,\ y(1)=3\varepsilon\sin(3),\ y''(0)=0.$

The analytical solution of this problem is $y(x)=3\varepsilon\sin(3x).$

Table 4.3: Maximum absolute errors for Example 4.2 with different values of h and ε.

$\varepsilon\downarrow$	$N=10$	$N=20$	$N=40$
Present Method			
$1/16$	9.4405e-06	5.4886e-07	2.5658e-08
$1/32$	3.1645e-06	1.9215e-07	9.1282e-09
$1/64$	9.9920e-07	6.1969e-08	2.9364e-09
Akram and Talib, (2014)			
$1/16$	1.02e-02	1.40e-03	1.73e-04
$1/32$	3.80e-03	4.84e-04	6.15e-05
$1/64$	1.40e-03	1.00e-04	2.00e-05
Akram, (2012)			
$1/16$	2.5e-03	1.9e-04	1.4e-05
$1/32$	6.8e-04	5.7e-05	5.0e-06
$1/64$	1.2e-04	1.3e-05	1.6e-06

Table 4.4: Maximum absolute errors for Example 4.2 when $\varepsilon \ll h$.

$\varepsilon\downarrow$	$N=10$	$N=50$	$N=100$	$N=150$	$N=200$
Present Method					
10^{-3}	4.5809e-08	8.2990e-12	4.1392e-13	7.3211e-14	2.6225e-14
10^{-4}	9.3573e-10	4.6893e-14	5.1063e-15	9.5009e-16	3.4922e-16
10^{-5}	6.1195e-12	1.5171e-14	9.3190e-17	6.0495e-18	7.5145e-18

10^{-6}	5.8655e-14	2.0745e-16	1.3348e-17	8.2310e-19	1.5064e-19
10^{-7}	5.8409e-16	1.1931e-18	1.1543e-19	1.0764e-19	1.1465e-20
10^{-8}	5.8386e-18	1.1336e-20	7.5708e-22	1.7702e-22	7.8479e-23
10^{-9}	5.8398e-20	1.1575e-22	8.7888e-24	3.3087e-24	3.3087e-24
10^{-10}	5.8564e-22	1.5574e-24	4.1359e-25	3.6189e-25	1.6529e-25
10^{-11}	6.0132e-24	4.8872e-26	2.9081e-26	3.8774e-26	3.8774e-26
10^{-12}	7.2701e-26	3.2817e-27	3.2312e-27	2.8273e-27	2.8273e-27
Mustafa and Ejaz, (2017)					
10^{-3}	2.5276e-03	1.7508e-04	7.0100e-05	3.6747e-05	2.0355e-05
10^{-4}	1.9938e-03	2.5243e-05	1.1440e-05	7.3469e-06	5.3722e-06
10^{-5}	…	2.0415e-05	1.3333e-06	8.3294e-07	6.0941e-07

The following Figures (4.1) and (4.2) shows the comparison numerical solution and exact solution and Figures (4.3) - (4.5) shows the absolute errors for different values of mesh size h and perturbation parameter ε.

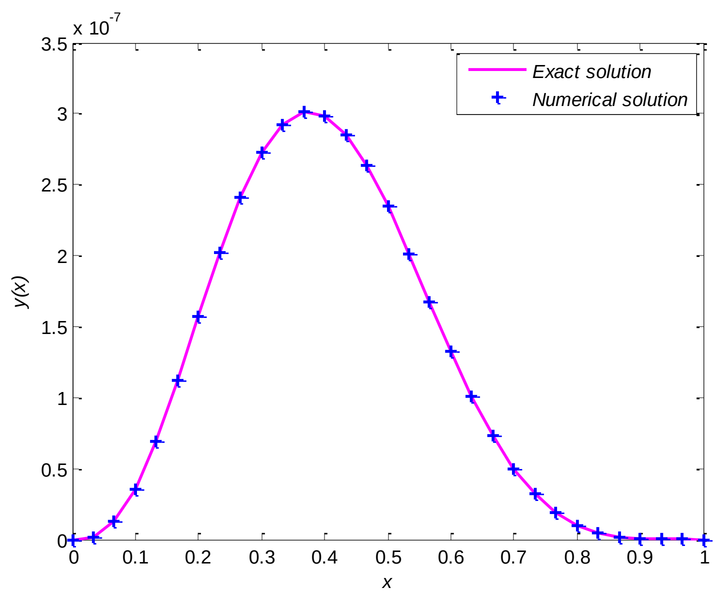

Figure 4.1: Numerical solution versus exact solution of Example 4.1 for $N = 30$ and $\varepsilon = 10^{-5}$.

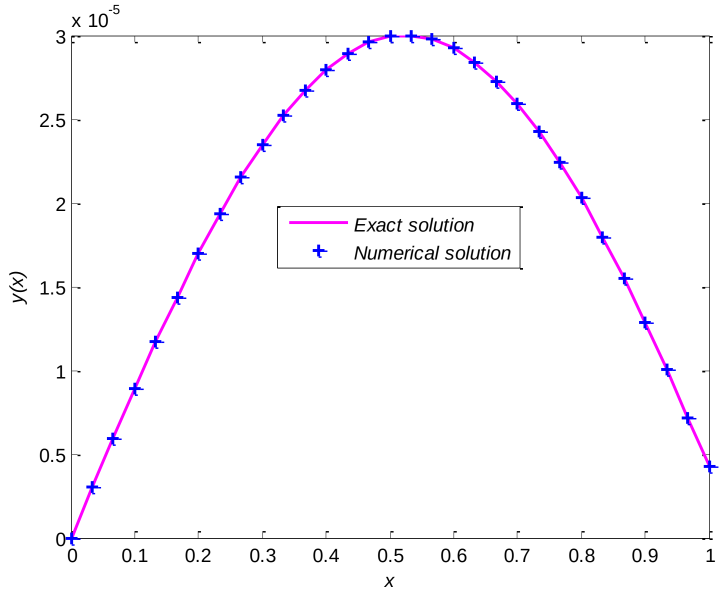

Figure 4.2: Numerical solution versus exact solution of Example 4.2 for $N = 30$ and $\varepsilon = 10^{-5}$.

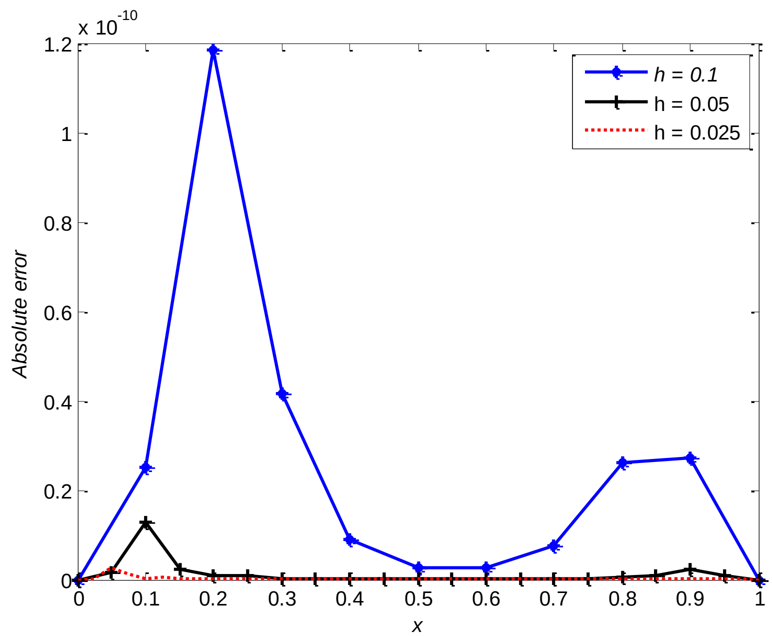

Figure 4.3: Absolute errors of Example 4.1 for different values of h and $\varepsilon = 10^{-5}$.

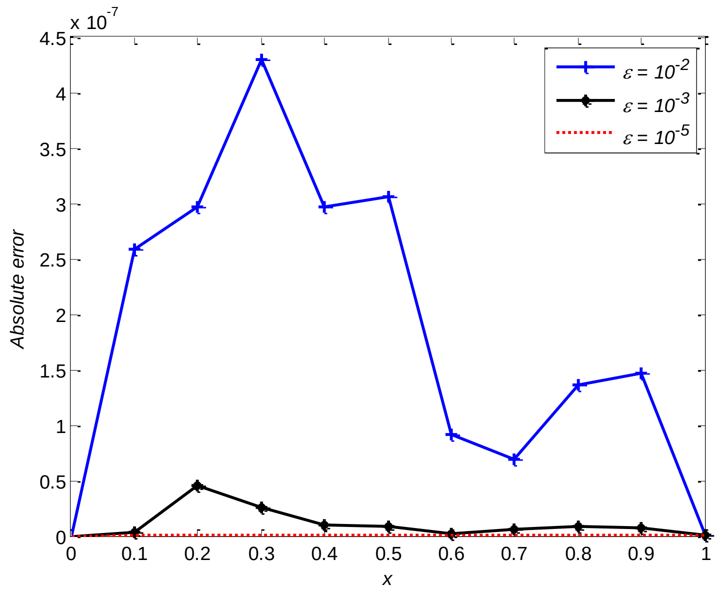

Figure 4.4: Absolute errors of Example 4.2 for different values of ε and $h = 10^{-1}$.

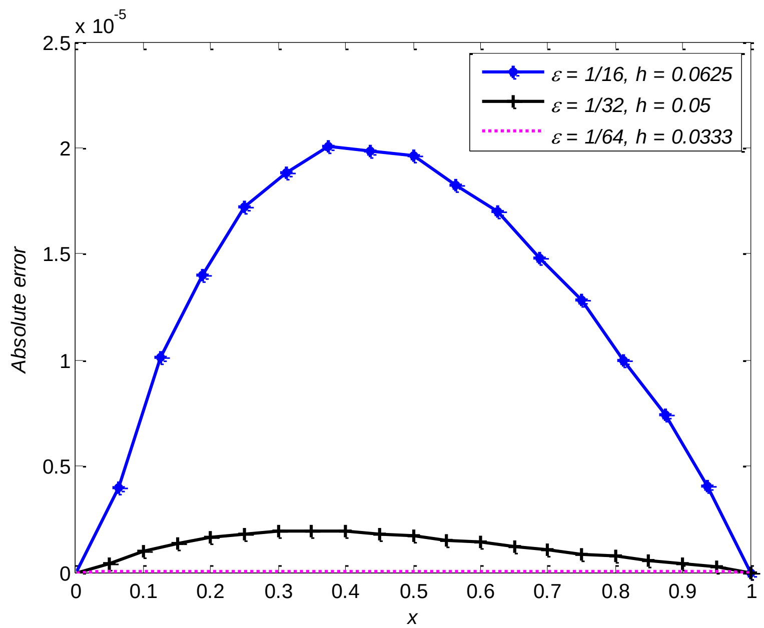

Figure 4.5: Absolute errors of Example 4.1 for different values of ε and h.

4.5. Discussion

In this thesis, the non-polynomial septic spline method has been presented for solving third order type singularly perturbed boundary value problems. First, the given system is discretized. Then, the spline coefficients are derived and the consistency relation is obtained by using continuity of second, fourth and fifth derivatives. Further, we reduce the obtained fifteen different systems of equations to a system of equations and develop boundary equation in order to equate the equal system of linear equations. The convergence analysis of the obtained hepta-diagonal scheme is investigated.

To validate the applicability of the proposed method two model examples have been considered for different values perturbation parameter and different mesh sizes. The numerical results are presented in Tables and figures. The result obtained by the present method has been compared with the numerical results developed by Mustafa and Ejaz, (2017), Akram and Talib, (2014) and Akram, (2012), and it is observed that the present method gives better result. Further, the proposed method approximate the exact solution very well when $\varepsilon << h$, for which most of the existing methods fail to give good result. Moreover, the maximum absolute error decreases rapidly as N increases.

Chapter Five

Conclusion and Future Work

5.1. Conclusion

The septic non-polynomial spline method is developed for the approximate solution of a third order type singularly perturbed boundary value problems. The convergence analysis is investigated and shows that the present method is of fourth order convergent. Two examples are considered for numerical illustration of the method. As a result, from Tables (4.1) – (4.4) and Figures (4.3)- (4.5), one can see that the maximum absolute error decreases as a mesh size h and also perturbation parameter decreases, which in turn shows the convergence of the computed solution. Furthermore, the result of the present method is compared with current findings and shows that it is more accurate than some existing numerical methods reported in the literature. Figures (4.1) and (4.2), shows that the present method approximates the exact solution very well.

Moreover, the study has been analyzed by taking large number of mesh size and sufficiently small parameter ε, good accuracy result is obtained. So, this study developed a better method for solving singularly perturbed boundary value problems for most numerical schemes fail to give good result at small mesh size h and for sufficiently small perturbation parameter $\varepsilon << h$.

Generally, the present method is convergent and more accurate for solving third order type singularly perturbed boundary value problems.

5.2. Scope for Future Work

In this thesis, the numerical method based on septic non-polynomial scheme has presented for solving third order type singularly perturbed boundary value problems. Hence, the scheme proposed in this thesis can also be extended to sixth and higher order non-polynomial septic spline method for third order type singularly perturbed boundary value problems.

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Appendix (*Relation for reducing fifteen equations to one equation*)

$$X_i = R_2 S_{i+1} - S_2 R_{i+1}\,, \quad i = 1(1)15\,, \ i \neq 2$$
$$Z_i = S_2 T_{i+1} - T_2 S_{i+1}\,, \quad i = 2(1)15, \ i \neq 1$$

$$S_i = \begin{cases} J_1 G_{i+1} - G_1 J_{i+1}\,, & i = 1,2 \\ J_1 G_{i+1} - G_1 J_i\,, & i = 4(1)7 \\ J_1 G_{i-1} - G_1 J_{i-2}\,, & i = 12,13 \\ -G_1 J_{i-2}\,, & i = 14,15,16 \\ -G_1 J_i\,, & i = 8,9 \\ J_1 G_{i-1}\,, & i = 3,10,11 \end{cases}$$

$$H_i = \begin{cases} A_1 D_{i+1} - D_1 A_{i+1}\,, & i = 1(1)4 \\ A_1 D_{i+1} - D_1 A_i\,, & i = 6(1)10 \\ A_1 D_i\,, & i = 12,13 \\ A_1 D_i - D_1 A_{i-2}\,, & i = 14(1)17 \\ -D_1 A_i\,, & i = 11 \\ -D_1 A_i\,, & i = 18 \\ A_1 D_{i+1}\,, & i = 5 \end{cases}$$

$$I_i = B_4 C_i - C_4 B_i\,, \quad i = 1(1)14$$

$$J_i = \begin{cases} C_4 F_i - F_4 C_i\,, & i = 1(1)3 \\ C_4 F_{i+1} - F_4 C_i\,, & i = 5(1)14 \\ C_4 F_{i+1}\,, & i = 4 \end{cases}$$

$$\begin{array}{ll} G_1 = E_2 M_1 - M_2 E_1 & E_1 = U_2 W_1 - W_2 U_1 \\ G_2 = E_3 M_1 - M_3 E_1 & E_2 = U_3 W_1 - W_3 U_1 \\ G_3 = -M_4 E_1 & E_3 = U_4 W_1 - W_4 U_1 \\ G_4 = E_4 M_1 & E_4 = 110 W_1 - 42 U_1 \\ G_5 = E_5 M_1 - 6 E_1 & E_5 = -1650 W_1 + 126 U_1 \\ G_6 = E_6 M_1 + 18 E_1 & E_6 = -990 W_1 - 126 U_1 \\ G_7 = E_7 M_1 - 18 E_1 & E_7 = -110 W_1 + 42 U_1 \\ G_8 = 6 E_1 & E_8 = 2640 W_1, \quad E_9 = -7920 W_1 \\ G_i = M_1 E_{i-1}\,, \ i = 9(1)12 & E_{10} = 7920 W_1, \ E_{11} = -2640 W_1 \end{array}$$

$$R_i = \begin{cases} G_1 I_{i+1} - G_{i+1} C_1\,, & i = 1,2 \\ -G_i I_i\,, & i = 3 \\ G_1 I_{i-1} - G_{i+1} C_1\,, & i = 5,6,7 \\ G_1 I_{i-3} - G_{i-1} C_1\,, & i = 12,13 \\ G_1 I_{i-1}\,, & i = 8,9 \\ -G_{i+1} I_1\,, & i = 4 \\ -G_{i-1} I_1\,, & i = 10,11 \end{cases}$$

$$\begin{array}{ll} O_i = H_i F_4 - F_i H_4\,, & i = 1(1)3 \\ O_i = H_{i+1} F_4 - F_i H_4\,, & i = 5(1)10 \\ O_i = H_{i+1} F_4 - F_i H_{i-2}\,, & i = 13(1)17 \\ O_i = H_{i+1} F_4\,, & i = 4,11,12 \end{array}$$

$$T_i = \begin{cases} J_1 O_{i+1} - O_1 J_{i+1}\,, & i = 1,2 \\ J_1 O_{i+1} - O_1 J_i\,, & i = 4(1)9 \\ J_1 O_{i+1} - O_1 J_{i-2}\,, & i = 12(1)16 \\ J_1 O_{i+1}\,, & i = 10,\ 11 \\ -O_1 J_{i+1}\,, & i = 3 \end{cases}$$

$$D_i = W_1K_{i+1} - K_1W_{i+1} \quad i = 2(1)3$$
$$D_4 = K_5W_1, \quad D_5 = K_6W_1$$
$$D_6 = 68W_1 - 42K_1$$
$$D_7 = -1536W_1 + 126K_1$$
$$D_8 = -732W_1 - 126K_1$$
$$D_9 = -68W_1 + 42K_1$$
$$D_{10} = -384W_1, \; D_{11} = 12W_1,$$
$$D_{12} = 2640W_1, \; D_{13} = -8640W_1,$$
$$D_{14} = 10800W_1,$$
$$D_{15} = -6960W_1$$
$$D_{16} = 2880W_1,$$
$$D_{17} = -720W_1$$

$$K_1 = -89\alpha_2 + 56\alpha_1 + 55\alpha_3$$
$$K_2 = 40\alpha_3 + 56\beta_1$$
$$K_3 = 89\alpha_2 + 56\alpha_1 - 55\alpha_3$$
$$K_4 = -12\alpha_1 - 21\alpha_2 - 25\alpha_3$$
$$K_5 = -12\beta_1$$
$$K_6 = 21\alpha_2 - 12\alpha_1 - 15\alpha_3$$
$$N_1 = -3M_1$$
$$N_2 = -9\beta_1 - 9\alpha_1 - 15\alpha_3$$
$$N_3 = -9\alpha_2 - 9\alpha_1 + 40\alpha_3 - 9\beta_1$$
$$N_4 = -89\alpha_2 - 65\alpha_1 + 70\alpha_3$$
$$N_5 = -40\alpha_3 - 56\beta_1$$
$$N_6 = 89\alpha_2 - 56\alpha_1 - 55\alpha_3$$

$$F_i = A_1M_{i+1} - M_1A_{i+1} \quad i = 1(1)3$$
$$F_4 = -A_5M_1$$
$$F_5 = 6A_1 - A_6M_1$$
$$F_6 = -18A_1 - A_7M_1$$
$$F_7 = 18A_1 - A_8M_1$$
$$F_8 = -6A_1 - A_9M_1$$
$$F_i = -M_1A_{i-1} \quad i = 9(1)15$$

$$A_1 = N_1P_5,$$
$$A_i = P_5N_i - N_6P_i, \; i = 2(1)5$$
$$A_6 = -18P_5,$$
$$A_7 = 66P_5 + 8N_6$$
$$A_8 = 202P_5 - 20N_6$$
$$A_9 = -702P_5 + 132N_6$$
$$A_{10} = -1536P_5 + 124N_6$$
$$A_{11} = 68P_5 - 4N_6, \; A_{12} = -720P_5,$$
$$A_{13} = 4800P_5 - 240N_6$$
$$A_{14} = -10080P_5 + 720N_6$$
$$A_{15} = 8640P_5 - 720N_6$$
$$A_{16} = -2640P_5 + 240N_6$$

$$B_i = Q_iP_5 - P_iQ_5, \; i = 1(1)4$$
$$B_5 = -4P_5 + 8Q_5$$
$$B_6 = 128P_5 - 20Q_5$$
$$B_7 = 132Q_5$$
$$B_8 = -128P_5 + 124Q_5$$
$$B_9 = 4P_5 - 4Q_5$$
$$B_{10} = -240P_5$$
$$B_{11} = 960P_5 - 240Q_5$$
$$B_{12} = -1440P_5 + 720Q_5$$
$$B_{13} = 960P_5 - 720Q_5$$
$$B_{14} = -240P_5 + 240Q_5$$

$$C_1 = -Q_1L_4$$
$$C_i = L_{i-1}Q_5 - Q_iL_4, \; i = 2(1)4$$
$$C_5 = 4L_4$$
$$C_6 = -110Q_5 - 128L_4$$
$$C_7 = -990Q_5$$
$$C_8 = -1650Q_5 + 128L_4$$
$$C_9 = 110Q_5 - 4L_4$$
$$C_{10} = 240L_4$$
$$C_{11} = 2640Q_5 - 960L_4$$
$$C_{12} = -7920Q_5 + 1440L_4$$
$$C_{13} = 7920Q_5 - 960L_4$$
$$C_{14} = -2640Q_5 + 240L_4$$

$$M_1 = -3\alpha_2 + 3\alpha_1$$
$$M_2 = 3\alpha_2 + 3\alpha_1 + 3\beta_1$$
$$M_3 = M_2$$
$$M_4 = M_1$$
$$W_1 = 7M_1$$
$$W_2 = 7M_2$$
$$W_3 = W_2$$
$$W_4 = W_1$$

$$U_1 = -110\alpha_2 + 77\alpha_1 + 55\alpha_3$$
$$U_2 = 33\alpha_1 + 55\alpha_3 + 77\beta_1$$
$$U_3 = 110\alpha_2 + 77\alpha_1 - 55\alpha_3 + 33\beta_1$$
$$U_4 = 33\alpha_1 - 55\alpha_3$$
$$L_1 = -U_4$$
$$L_2 = -U_3$$
$$L_3 = -U_2$$
$$L_4 = -U_1$$

$$P_1 = 4\alpha_2 - 4\alpha_1$$
$$P_2 = -7\alpha_2 - 4\alpha_1 + 5\alpha_3 - 4\beta_1$$
$$P_3 = -11\alpha_2 - 8\alpha_1 + 5\alpha_3 - 4\beta_1$$
$$P_4 = 7\alpha_2 - 4\alpha_1 - 5\alpha_3 - 4\beta_1$$
$$P_5 = 7\alpha_2 - 4\alpha_1 - 5\alpha_3$$
$$Q_1 = P_5$$
$$Q_2 = -4\beta_1$$
$$Q_3 = -14\alpha_2 - 8\alpha_1 + 10\alpha_3$$
$$Q_4 = Q_2$$
$$Q5 = P_5$$