DYNAMICS OF PARTICLES AROUND STELLAR BLACK HOLES

By

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The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a thesis entitled "**Dynamics of particles around stellar black holes**" by **Fikru Degife** in partial fulfillment of the requirements for the degree of **Master of Science in Physics(Astrophysics)**.

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To my lovely children.

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Abstract

The Schwarzschild solution is unique and its metric can be interpreted as the exterior gravitational field of a spherically symmetric mass. Most of the experimental tests of general relativity were based on the Schwarzschild geometry in the region $r > \frac{2GM}{c^2}$. Some are based on the trajectories of massive particles and others on photons trajectories.In this thesis we did study this issue theoretically. In our derivations,we considered the Einstein's field equations and equations of motion for massive particle and photons where we did derive the relevant dynamical equations and observable parameters. Then, we did generate numerical solutions of effective potentials using MATHEMATICA. As a result ,Schwarzschild solution possesses stable circular orbits for $r > \frac{6GM}{c^2}$ and unstable circular orbits for $\frac{3GM}{c^2} < r < \frac{6GM}{c^2}$.

Key words: Stellar black hole,Accretion on to black hole and General Relativity.

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Chapter 1

General Introduction

1.1 Thesis Scheme

In this introductory chapter we discussed about the historical development of black holes(stellar black holes). We give also the relevant literature review. In chapter two,we briefly discuss about general relativity and Einstein's field equation. We begin this chapter by deriving Einstein's field equations and also we derive the Schwarzschild solution to general relativity. In chapter three we review on black holes and their historical developments. In chapter four we discuss our work. Finally, in chapter five we discuss the results with graphs and summarize our work in the last chapter.

1.2 Background

The term black hole was introduced by American physicist J.A. Wheeler because of everything, including the light, that went into that astronomical zone was unable to get out and consequently it appeared black. In the 18th century Laplace and Michell hypothesized for the first time the existence of a celestial body provided with a greatest mass that was able to cause an escape velocity greater than the speed of light for which neither light was able to resist the strongest gravitational force generated by the celestial body. On this account the concept of black hole was abandoned. Some month after the publication of General Relativity by Einstein (1916) the black hole was again contemplated because gravitation in General Relativity was considered as a geometric variation of the space and not a force. In 1919 Eddington on the occasion of a total solar eclipse measured the deflection of light coming from a remote star when light passed near the sun. He deduced that in place of the sun a greatest celestial mass should have produced a so great deflection of light that this once gone into the even horizon was unable to get out any longer. In the same years also Karl Schwarzschild calculated that the black hole should have possessed a greatest mass because the calculus implied a smallest radius of the celestial body $R = \frac{2GM}{c^2}$ and consequently in order to have an acceptable value of radius a very great mass was necessary. Lastly a few published papers have denied the existence of black holes. In the gravitational theory the black hole looks like an astronomical monster that devours all what passes in the proximity of its even horizon, the relativistic theory intends also to propose a more explanation of the black hole[1].Also Einstein's general theory of relativity predicts the existence of these black holes as astrophysical objects so dense that even light cannot escape from them. The boundary around the black hole, where the light cannot escape, is called the event horizon [2].

Black holes have captured the imagination of scientists, as well as the general public. In addition to their intrinsic appeal, black holes potentially impact on a number of fundamental problems in physics and astronomy. They are a possible end point of stellar evolution. They provide a unique laboratory in which to study strong gravity. Knowledge of the mass and spatial distributions of black holes could also provide information about stellar evolution, galaxy formation, and dark matter.

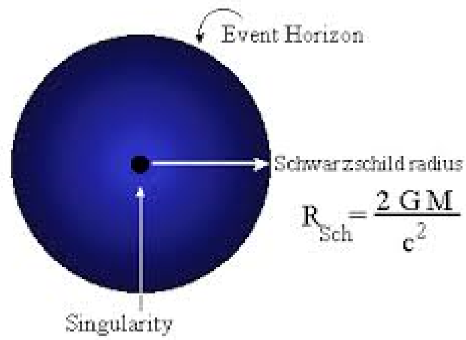

Figure 1.1: Basic structure of black holes

While they are among the most interesting astrophysical objects, black holes, by their very nature black, are difficult to isolate and study. Since the intrinsic Hawking radiation from black holes (of the mass greater than about a solar mass) is quite weak, the search for black holes must concentrate on the interaction of the black hole with the surrounding medium.

As black holes will not themselves be luminous, the key to detecting them is to observe their effect on their surroundings. Black holes will accrete and radiate some fraction of the accreting matter into energy [3].

Observation with in our own galaxy, the milky-way galaxy, reveals a plethora of x-ray radiation from astronomical sources(objects). However in the fittings of the observations to the existing models are much controversial. But to the current understanding they are more linked to neutron stars and black holes.

Motivated by this, we are interested to work on the dynamics of particles around stellar black holes.

1.3 Literature Review

The general theory of relativity is the geometric theory of gravitation published by Albert Einstein in 1916. Most of the General relativity are Einsteins predictions which were subject to interpretation. Mathematicians and Physicists have tried to test and apply the predictions. At presents five different astronomical tests have verified the theory. First of all the orbit of a planet is no longer a closed Keplerian ellipse. The effect is strongest for the innermost planets, whose perihelia should turn little by little.Most of the motion of the perihelion of Mercury is predicted by Newtonian mechanics; only a small excess of 43 arc seconds per century remains unexplained. And it so happens that this is exactly the correction suggested by general relativity [4].

Most of the experimental tests of general relativity are based on the Schwarzschild geometry in the region $r > 2GM/c^2$. Some are based on the trajectories of massive particles and others on photon trajectories.

Information about the geometry produced by compact massive objects or black holes can be obtained from observations of the orbits of particles in the accretion disc that often surrounds them [5]

A photon sphere is a spherical region of space where gravity is strong enough that photons are forced to travel in orbits. The radius of the photon sphere, which is also the lower bound for any stable orbit, is for a Schwarzschild black hole:

$r = \frac{3GM}{c^2} = \frac{3}{2}r_s$

where G is the gravitational constant, M is the black hole mass, and c is the speed of light in vacuum and r_s is the Schwarzschild radius (the radius of the event horizon). This equation entails that photon spheres can only exist in the space surrounding an extremely compact object (a black hole or mainly an "ultracompact" neutron star). General relativity predicts apparent bending of light rays passing through gravitational fields [6]. As photons approach the event horizon of a black hole, those with

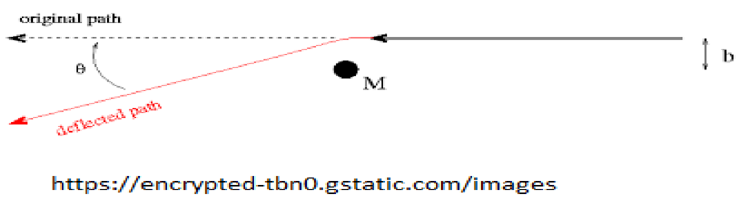

Figure 1.2: Bending of light by massive body, in vacuum

the appropriate energy avoid being pulled into the black hole by traveling in a nearly tangential direction known as an exit cone. A photon on the boundary of this cone does not possess the energy to escape the gravity well of the black hole. Instead, it orbits the black hole. These orbits are rarely stable in the long term.

The photon sphere is located farther from the center of a black hole than the event horizon. For non-rotating black holes, the photon sphere is a sphere of radius $\frac{3}{2}r_s$. There are no stable free fall orbits that exist within or cross the photon sphere. Any free fall orbit that crosses it from the outside spirals into the black hole. Any orbit that crosses it from the inside escapes to infinity. Through the 1930s, the applications of general relativity and quantum mechanics to the studies of the late evolution of stars predicted that stars with different initial masses, after exhausting their thermal nuclear energy sources, may eventually collapse to become exotic compact objects, such as white dwarfs, neutron stars, and black holes. A low-mass star, such as our Sun, will end up as a white dwarf, in which the degeneracy pressure of the electron gas balances the gravity of the object. For a more massive star, the formed compact object can be more massive than around 1.4 solar masses ($M_\odot$), the so-called Chandrasekhar limit, in which the degeneracy pressure of the electron gas cannot resist the gravity, as pointed out by Chandrasekhar. In this case, the compact object has to further contract to become a neutron star, in which most of the free electrons are pushed into protons to form neutrons and the degeneracy pressure of neutrons balances the gravity of the object. Then as Oppenheimer and others noted, if the neutron star is too massive, for example, more than around 3 $M\odot$, the internal pressure in the object also cannot resist the gravity and the object must undergo catastrophic collapse and form a black hole [7].

So the basic process of stellar evolution is gravitational contraction at a rate controlled by luminosity. The key parameter is the initial mass. According to its value, stars evolve through various stages of nuclear burning and finish their lives as white dwarfs, neutron stars or black holes. Any stellar remnant (cold equilibrium configuration) more massive than about $3M_\odot$ can not be supported by degeneracy pressure and is doomed to collapse to a black hole [8]. The formation of stellar BHs is of topical interest for several areas of astrophysics. Stellar BHs are remnants of massive stars, possible seeds for the formation of supermassive BHs, and also sources of the most energetic phenomena in the universe, such as the gravitational waves produced by fusion of BHs [9]. These black holes form when massive stars run out of fuel at the end of their life. When the nuclear fuel is exhausted, stars contract inwards under the influence of their own gravity. Our knowledge about the final stages of this collapse suggests that sufficiently massive stars inevitably leave black hole remnants, i.e. regions of spacetime in which gravity is so strong that neither matter nor light can ever escape [3].The formation of the black holes usually involve catastrophic gravitational collapse. Stellar mass black holes form in the same way as the neutron stars, namely, by core collapse [10].

Compact objects gravitationally capture matter in a process known as accretion. It is the mechanism of matter accumulation of a central mass due to its gravitational force. During this process angular momentum is transferred from matter of the inner parts to matter further out in the disc, which enables matter to move inward finally to fall on to the center [3].

Accretion is the process of growth or enlargement of a gravitating object by infall of material. It is a widespread process in our Universe, relevant to the formation of everything from planets to galaxies. Understanding the physics of accretion is therefore of fundamental importance to many areas of Astrophysics [11].

Accretion disks form whenever a compact object such as a star or a black hole draws matter from its surroundings. More often than not this matter carries angular momentum and therefore cannot fall directly on to the compact object but instead settles in Keplerian motion around it. The accretion could take place via stellar winds as for example on to a black hole or through the Roche-Lobe filling process as in close binary stars. A blob of gas would orbit a compact object with the Keplerian speed. It can spiral inwards only if its energy and angular momentum are removed by some kind of a dissipative process. With this mechanism, the binding energy of its innermost orbit can be extracted. Thus the inflow of the matter could be enhanced by the outflow of the angular momentum [12].

The matter rotating in circular Keplerian orbits around the compact object loses angular momentum because of the friction between adjacent layers and spirals inwards.In the process gravitational energy is released, the kinetic energy of the plasma increases and the disk heats up, emitting thermal energy [13].

1.4 Statement of the Problem

General relativity gives predictions for trajectories of particles different from classical physics. Most of the experimental tests of general relativity are based on the Schwarzschild geometry in the region $r > \frac{2GM}{c^2}$. Some are based on the trajectories of massive particles and others on photon trajectories.Most of the classical tests are in the weak field limit,but the more recent observations have begun on strong field regime. Considering strong field regime the thesis must answers the following questions.

Research Question

- How do stellar black holes accrete matter from surrounding astronomical companions?
- What are the relevant parameters responsible in the dynamism of accreting stellar black holes?
- What are the trajectories of massive and massless particles around stellar black holes?

1.5 Objectives

1.5.1 General objective

The general objective of this thesis will be to study the dynamics of particles around stellar black holes with Schwarzschild metric.

1.5.2 Specific objective

The specific objectives of the study will be the following

- To derive dynamical equations from the General theory of Relativity.
- To derive dynamical observable parameters like angular momentum, and energy of the particles which are responsible in the dynamism of accreting stellar black holes.
- To study the trajectories of massive and massless particles around stellar black holes.

1.6 Methodology

The general method is to derive dynamical equations of accreting stellar black holes from general relativity where appropriate boundary conditions be set. The analytically derived equations are used to generate numerical values computationally with MATHEMATICA. Then, the results will be discussed and summarized.

The steps are:

- Provide preliminary boundary conditions to derive the relevant set of dynamical equations from Einstein's field equations of General Relativity.
- Study and examine the effects of the relevant parameters like angular momentum and energy derived from the equations.
- Numerically generate some theoretical data from the formalism using computation.
- Summary and conclusion

Chapter 2

Introduction to General Relativity and Einstein's Field Equations

General theory of relativity is the geometric theory of gravitation published by Albert Einstein in 1916. It is the current description of gravitation in modern physics. Since 1916 Einsteins general theory of relativity, over a hundred years, has remained unaltered and is fundamental to astrophysics and cosmology. General relativity and Einsteins field equations are considered by many as the perfect example of physical law and general relativity continues to be tested. General Relativity generalizes special relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or spacetime. In particular, the curvature of spacetime(x, y, z, ct),is directly related to the four-momentum (mass-energy and linear momentum) of whatever matter and radiation are present, $(p = \frac{E}{c}, p_x, p_y, p_z)$ where E is mass-energy, c is velocity of light and(p_x, p_y, p_z) are space linear momentum. The relation is specified by the Einstein field equations

The fundamental physical postulate of General Relativity is that the presence of matter causes curvature in the spacetime in which it exists. This curvature is taken to be the gravitational field produced by the matter. Einstein's field equation gives the

Mathematical description of how the matter and curvature are related. Moreover, once this curvature is given, General Relativity describes how other objects(such as planets and light beams)move in this gravitational field via the geodesic equation [14]. Among the most important predictions of the general theory of relativity one is the bending of light around a massive object. The orbit described by a photon in the photon sphere is actually an unstable orbit , and a small perturbation in the orbit can lead either to the photon escaping the black hole or diving towards the event horizon [15].

Einsteins equivalence principle was the fundamental basis for the successful development of the general theory of relativity and so testing Einstein's Equivalence Principle and the strong equivalence principle to the highest order validates General Relativity and stipulates that gravity must necessarily be curved spacetime

2.1 Einstein's Field Equation

Having formulated the relativistic, geometric version of the effects of gravity, the question of gravity's source remains. In Newtonian gravity, the source is mass. In special relativity, mass turns out to be part of a more general quantity called the energy-momentum tensor, which includes both energy and momentum densities as well as stress (that is, pressure and shear). Using the equivalence principle, this tensor is readily generalized to curved space-time. Drawing further upon the analogy with geometric Newtonian gravity, it is natural to assume that the field equation for gravity relates this tensor and the Ricci tensor, which describes a particular class of tidal effects: the change in volume for a small cloud of test particles that are initially at rest, and then fall freely. In special relativity, conservation of energy-momentum corresponds to the statement that the energy-momentum tensor is divergence-free

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu - pg_{\mu\nu} \tag{2.1.1}$$

where $T_{\mu\nu}$ is the energy momentum tensor, ρ is the mass density. This formula is readily generalized to curved spacetime by replacing partial derivatives with their curved-manifold counterparts, covariant derivatives studied in differential geometry. With this additional condition, the covariant divergence of the energy-momentum tensor, and hence of whatever is on the other side of the equation, is zero the simplest set of equations are what are called Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.1.2}$$

From equation above the left-hand side is the Einstein tensor, a specific divergence-free combination of the Ricci tensor and the metric. In particular,

$$R_{\mu\nu} = Rg_{\mu\nu} \tag{2.1.3}$$

is the curvature scalar. The Ricci tensor itself is related to the more general Riemann curvature tensor as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \tag{2.1.4}$$

On the right-hand side of (2.1.2), $T_{\mu\nu}$ is the energy-momentum tensor. All tensors are written in abstract index notation. Matching the theory's prediction to observational results for planetary orbits (equivalently, assuring that the weak-gravity, low-speed limit is Newtonian mechanics), the proportionality constant can be fixed as $K = \frac{8\pi G}{c^4}$, with G the gravitational constant and c the speed of light. When there is no matter present, so that the energy-momentum tensor vanishes, the result are the vacuum

Einstein equations,

$$R_{\mu\nu} = 0 \tag{2.1.5}$$

In the Newtonian theory, gravitational field is to be described in terms of a scalar potential function Φ and the equations that determine Φ are

$a.\Phi = 0$ and when there is no gravitation;

$b.\nabla^2\Phi = 0$ (Laplace equation) in empty space (no matter present and no physical fields except a gravitational field); and finally

$c.\nabla^2\Phi = 4\pi G\rho$ (Poisson's equation) in regions of space where matter is present and the material density is ρ. These equations are supplemented by the standard equations of motion.

2.2 Vacuum Field Equations

This follows the method presented by [14]

We begin with the realization that we would like to find an equation which is supersedes the poisson equation for the Newtonian potential.

$$\nabla^2\Phi = 4\pi G\rho \tag{2.2.1}$$

where $\nabla^2 = \nabla^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ is the mass density.The explicit form of $\Phi = \frac{-GM}{r}$ is one solution of the above equation for the case of a point like mass distribution. The tensor generalization of the mass density is the energy momentum tensor $T_{\mu\nu}$. The gravitational potential, mean while, should get replaced by the metric tensor. It is thus reasonable to guess that the new equation will have

$T_{\mu\nu}$ set proportional to some tensor which is second order in derivatives of the metric. In fact using the Newtonian limit, for the metric

$g_{00} = -(1 + 2\Phi)$ and $T_{00} = \rho$,

We are looking for an equation that predicts

$$\nabla^2 h_{00} = -8\pi G T_{00} \tag{2.2.2}$$

with $h_{00} = 2\Phi$ We do though need to generalize it to a completely tensorial equation. The left hand side of Eq. (2.2.2) does not obviously generalize to a tensor.It might be to act the D'Alembertian $\Box = \nabla^\mu \nabla_\mu$ on the metric $g_{\mu\nu}$; but this is automatically zero by metric compatibility $(\equiv g_{\mu\nu}; \lambda = 0)$.

Fortunately, there is an obvious quantity which is constructed from second derivatives and first derivatives of the metric; the Riemann tensor$R^\rho_{\mu\rho\nu}$. We can contract this to form the Ricci tensor $R_{\mu\nu}$; which does further more, it is symmetric. It is there fore reasonable to guess that the gravitational field equations are,

$$R_{\mu\nu} = K T_{\mu\nu} \tag{2.2.3}$$

for some constant K.

According to the principle of equivalence, the statement of energy-momentum conservation in curved spacetime should be

$$\nabla^\mu T_{\mu\nu} = 0 \tag{2.2.4}$$

which would then imply

$$\nabla^\mu R_{\mu\nu} = 0 \tag{2.2.5}$$

from Bianchi identity we have;

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R \tag{2.2.6}$$

But our proposed field equation implies that, $R = Kg^{\mu\nu}T_{\mu\nu} = KT$, so taking these together we have

$$\nabla_{\mu} T = 0 \tag{2.2.7}$$

The covariant derivative of a scalar is just the partial derivative. By now we already know of asymmetric tensor constructed from the Ricci tensor which is automatically conserved:
from the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.2.8}$$

which always obeys $\nabla^{\mu} G_{\mu\nu} = 0$. we are there fore led to propose

$$G_{\mu\nu} = KT_{\mu\nu} \tag{2.2.9}$$

2.3 Schwarzschild Solution to Einstein's General Theory of Relativity

In 1915, Karl Schwarzschild derived an exact solution to Einsteins General Relativity field equations, published in 1916. The Schwarzschild solution is used in most tests of General Relativity. He also derived the Schwarzschild radius, R_s, which is the radius of a sufficiently massive object where all particles, including photons, which will inevitably fall into a massive central object.

Starting from Newton's gravitational theory,the spherically symmetric solution of Einstein's vacuum field equation , the schwarzschild solution is a unique and its metric can be interpreted as the exterior gravitational field of spherically symmetric mass [16]

The Schwarzschild metric

Assumptions

1. The schwarzschild metric assumes that the system is spherically symmetric;it uses spherical coordinates along the metric to achieve this symmetry(it can be seen with the r^2 and $r^2sin^2\theta$ terms of the metric).

2. The solution assumes vacuum conditions($T_{\mu\nu} = 0$).So the solutions only to have to solve $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$

3. The solution assumes that the system is static and time invariant.

2.3.1 The Christoffel symbols

This follows the method presented in [13].

Using the spherical parametrization of the metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \qquad (2.3.1)$$

where $g^{\mu_\mu} = g^{\mu\mu}g_{\mu\mu} = 4$

Generalizing this with functions on each of the infinitesimals

$$ds^2 = Bdt^2 - Adr^2 - Wr^2d\theta^2 - Xr^2sin^2\theta d\phi^2 \qquad (2.3.2)$$

However, since it was assumed that the equations are spherically symmetric, W=X=1. Again since the solution is for a static field, the functions have no dependence on time, and since the only mass is located inside a point mass, the stress energy tensor($T_{\mu\nu}$) will vanish.

$$ds^2 = B(r)dt^2 - A(r)dr^2 - Wr^2d\theta^2 - Xr^2sin^2\theta d\phi^2 \tag{2.3.3}$$

However, the problem is reduced in complexity to solutions of $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ because $T_{\mu\nu} = 0$ from the assumptions.

The affine connection can be given as:

$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda})$ and our non-vanishing metric components are,

$g_{00} = B$, $g_{11} = -A$, $g_{22} = -r^2$, $g_{33} = -r^2sin^2\theta$

its inverse $g^{\mu\nu}$:

$g^{00} = \frac{1}{B}$, $g^{11} = \frac{-1}{A}$, $g^{22} = \frac{-1}{r^2}$, $g^{33} = \frac{-1}{r^2sin^2\theta}$, Now to find the Ricci scalar and tensor, the Riemann curvature tensor must be calculated.

It is given by:

$$R^{\beta}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma},_{\rho} - \Gamma^{\beta}_{\nu\rho},_{\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma} \tag{2.3.4}$$

To simplify the calculation of the christoffel symbols:

1. Any derivatives with respect to t are zero, as the solution is static and does not depend on time.

2. $g_{\mu\nu}$ and $g^{\mu\nu}$ both equal to zero when $\mu \neq \nu$(the metric is symmetric)

3. $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$ the Christoffel symbols are symmetric in their lower indices).

Indices μ and ν run from 0-3, while i and $j$run from 1-3 in the spatial dimension).

The Christoffel symbols are:

$\Gamma^{1}_{00} = \frac{1}{2}g^{11}(g_{10,0} + g_{10,0} - g_{00,1}) = 0\frac{-1}{2}g^{11}\partial_1 g_{00} = \frac{1}{2}\frac{1}{A}\partial_r B$

$\Gamma^{1}_{11} = \frac{1}{2}g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{1}{2}\frac{1}{A}\partial_r A$

$\Gamma^1_{22} = \frac{1}{2}g^{11}(g_{12,2} + g_{12,2} - g_{22,1}) = \frac{-1}{2}g^{11}\partial_1 g_{22} = \frac{-r}{A}$

$\Gamma^1_{33} = \frac{1}{2}g^{11}(g_{13,3} + g_{13,3} - g_{33,1}) = \frac{-1}{2}g^{11}\partial_1 g_{33} = \frac{-r}{A}sin^2\theta$

$\Gamma^2_{12} = \frac{1}{2}g^{22}(g_{21,2} + g_{22,1} - g_{12,2}) = \frac{1}{2}g^{22}\partial_1 g_{22} = \frac{1}{r}$

$\Gamma^3_{13} = \frac{1}{2}g^{33}(g_{31,3} + g_{33,1} - g_{13,3}) = \frac{1}{2}g^{33}\partial_1 g_{33} = \frac{1}{r}$

$\Gamma^3_{23} = \frac{1}{2}g^{33}(g_{32,3} + g_{33,2} - g_{23,3}) = \frac{1}{2}g^{33}\partial_2 g_{33} = \frac{cos\theta}{sin\theta}$ Any thing not written down explicitly is mean to be zero.Or it is symmetry.

Now all the non- vanishing terms:

$\Gamma^0_{01} = \Gamma^0_{10} = \frac{B'}{2B}$

$\Gamma^1_{00} = \frac{B'}{2A}$

$\Gamma^1_{11} = \frac{A'}{2A}$

$\Gamma^1_{22} = \frac{-r}{A}$

$\Gamma^1_{33} = \frac{-r}{A}sin^2\theta$

$\Gamma^2_{12} = \Gamma^0_{21} = \frac{1}{r}$

$\Gamma^2_{33} = -cos\theta sin\theta$

$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}$

$\Gamma^3_{23} = \Gamma^3_{32} = \frac{cos\theta}{sin\theta}$

2.3.2 The Ricci Tensor

The Ricci tensor is a contruction of the Riemann curvature tensor;

$R_{\mu\nu} = R^{\beta}_{\mu\nu\beta} = \Gamma^{\beta}_{\mu\beta,\nu} - \Gamma^{\beta}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}$

Using the above relation, we obtain:

$R_{00} = -\Gamma^1_{00,1} + \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{11} - \Gamma^1_{00}\Gamma^2_{12} - \Gamma^1_{00}\Gamma^3_{13}$

$= \frac{-B"}{2A} + \frac{B'}{2B}\frac{B'}{2B} - \frac{B'}{2A}\frac{A'}{2A} - \frac{B'}{2A}\frac{1}{r} - \frac{B'}{2A}\frac{1}{r}$

$$R_{00} = \frac{-B''}{2A} + \frac{B'A'}{4A^2} + \frac{(B')^2}{4BA} - \frac{1}{r}\frac{B'}{A} \tag{2.3.5}$$

$R_{11} = \Gamma^0_{10,1} + \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{12,1} + \Gamma^2_{12} + \Gamma^2_{21} - \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13}$

$= \frac{B"}{2B} - \frac{(B')^2}{2B^2} + \frac{(B')^2}{4B^2} - \frac{B'}{2B}\frac{A'}{2A} - \frac{A'}{Ar}$

$$R_{11} = \frac{B''}{2A} - \frac{(B')^2}{4B^2} - \frac{B'}{2B}\frac{A'}{2A} - \frac{A'}{Ar} \tag{2.3.6}$$

$R_{22} = -\Gamma^1_{22}\Gamma^0_{10} - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} + \Gamma^3_{23,2} + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13}$

$= \frac{r}{A}\frac{B'}{2B} + (\frac{1}{A} - \frac{rA'}{A^2}) - \frac{1}{A} + \frac{rA'}{2A^2} + 2\theta cot\theta + cot^2\theta + \frac{1}{A}$

$= \frac{rB'}{2BA} + \frac{1}{A} - \frac{rA'}{2A^2} + (-1 - cot^2\theta) + cot^2\theta$

$$R_{22} = \frac{rB'}{2BA} + \frac{1}{A} - \frac{rA'}{2A^2} - 1 \tag{2.3.7}$$

$R_{33} = -\Gamma^1_{33}\Gamma^0_{10} - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12}$

$= \frac{rB'}{2BA}sin^2\theta + (\frac{sin^2\theta}{A} - \frac{rsin^2\theta A'}{A^2}) - \frac{sin^2\theta}{A} + \frac{rA'}{2A^2}sin^2\theta + 2\theta(cos\theta sin\theta - cos\theta sin\theta cot\theta + \frac{sin^2\theta}{A})$

$= \frac{rB'}{2BA}sin^2\theta + \frac{sin^2\theta}{A} - \frac{rA'}{2A^2}sin^2\theta + (-sin^2\theta + cos^2\theta) - cos^2\theta$

$= \frac{rB'}{2BA}sin^2\theta + \frac{sin^2\theta}{A} - \frac{rA'}{2A^2}sin^2\theta - sin^2\theta$

$$R_{33} = (\frac{rB'}{2BA} + \frac{1}{A} - \frac{rA'}{2A^2} - 1)sin^2\theta = sin^2\theta R_{22} \tag{2.3.8}$$

2.3.3 The Ricci scalar

The Ricci scalar is obtained from the Ricci tensor, which is as follows

$$R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \tag{2.3.9}$$

$= \frac{1}{B}R_{00} - \frac{1}{A}R_{11} - \frac{1}{r^2}R_{22} - (\frac{1}{r^2 sin^2\theta}sin^2\theta R_{22}$

$= \frac{1}{B}R_{00} - \frac{1}{A}R_{11} - \frac{2}{r^2}R_{22}$

$$R = -\frac{B''}{BA} + \frac{B'}{2}\frac{A'}{BA^2} + \frac{(B')^2}{2B^2A} - \frac{2B'}{rBA} + \frac{2A'}{rA^2} + \frac{2}{r^2}(1 - \frac{1}{A}) \qquad (2.3.10)$$

2.3.4 Einstein's equation for Schwarzschild spacetime

Since the schwarzschild solution concerns with exterior spacetime of a spherically symmetric body,Einstein's equation takes a simpler form.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \qquad (2.3.11)$$

This leads us to four equations that must be satisfied.

$R_{00} - \frac{1}{2}g_{00}R = 0$

$R_{00} - \frac{1}{2}AR = 0$

$\frac{-B''}{2A} + \frac{B'A'}{4A^2} + \frac{(B')^2}{4BA} - \frac{1}{r}\frac{B'}{A} + \frac{B''}{2A} - \frac{B'}{4}\frac{A'}{A^2} - \frac{(B')^2}{4BA} + \frac{1}{r}\frac{B'}{A} + \frac{1}{r}\frac{BA'}{A^2} + \frac{B}{r^2}(1 - \frac{1}{A}) = 0$

$$\frac{1}{r}\frac{A'}{A^2} + \frac{1}{r^2}(1 - \frac{1}{A}) = 0 \qquad (2.3.12)$$

$R_{11} - \frac{1}{2}g_{11}R = 0$

$R_{11} + \frac{1}{2}AR = 0$

$\frac{B''}{2B} - \frac{B'^2}{4B^2} - \frac{B'A'}{4BA} - \frac{A'}{Ar} - \frac{B''}{2B} + \frac{B'}{4}\frac{A'}{BA} + \frac{(B')^2}{4B^2} - \frac{1}{r}\frac{B'}{B} + \frac{1}{r}\frac{A'}{A} + \frac{A}{r^2}(1 - \frac{1}{A}) = 0$

$\frac{-1}{r}\frac{B'}{B} + \frac{A}{r^2}(1 - \frac{1}{A}) = 0$

$$\frac{-B'}{rBA} + \frac{1}{r^2}(1 - \frac{1}{A}) = 0 \qquad (2.3.13)$$

$R_{22} - \frac{1}{2}g_{22}R = 0$

$R_{22} + \frac{1}{2}r^2R = 0$

$\frac{-B'}{B} + \frac{A'}{A} - \frac{rB'}{B} + \frac{rB'}{2}\frac{A'}{BA} + \frac{r(B')^2}{2B^2} = 0$

The last equation is not independent of the previous.

$R_{33} - \frac{1}{2}g_{33}R = 0$

$sin^2\theta R_{22} + \frac{1}{2}r^2 sin^2\theta R$

$R_{22} + \frac{1}{2}r^2 R = 0$

Solving and substituting in to the metric

The first equation $R_{11} - \frac{1}{2}g_{11}R = 0$ is a function of A.Thus we can solve this differential equation to find A and then use that in the other equations to find B.Now re expressing the equation we get,

$\frac{A'}{A} + \frac{1}{r}(A-1) = 0$

$\frac{A'}{A(A-1)} + \frac{1}{r} = 0$

$$\frac{-dr}{r} = \frac{dA}{A(A-1)} \tag{2.3.14}$$

Integrating this and using the formula, $\int \frac{dx}{ax+bx^2} = \frac{-1}{a}\ln\frac{a+bx}{x}$,we get

$-\ln r + C = \ln\frac{1}{r} + C$

$= \ln\frac{(A-1)}{A}$

$\frac{C}{r} = \frac{A-1}{A}$

$B - 1 = \frac{C}{r}A = A(1-\frac{C}{r}) = 1$

$$A = \frac{1}{1-\frac{C}{r}} \tag{2.3.15}$$

To find A we insert this solution in to equation(4.1.13) and we obtain,

$\frac{B'}{B}(1-\frac{C}{r}) - \frac{1}{r}(1-(1-\frac{C}{r})) = 0$

$\frac{B'}{B} = \frac{C}{r^2(1-\frac{C}{r}} = \frac{C}{r^2-Cr}$

$\frac{dB}{B} = \frac{Cdr}{r^2 - Cr}$

Inserting both sides with the same formula as above;

$\int \frac{dB}{B} = \ln B = C \int \frac{dr}{r^2 - Cr} = \frac{C}{C} \ln \frac{r-C}{r}$

$\ln A = \ln(1 - \frac{C}{r})$ and exponentiating gives,

$$B = 1 - \frac{C}{r} \tag{2.3.16}$$

Our final result is the schwarzschild metric.

$$ds^2 = (1 - \frac{2GM}{c^2 r})c^2 dt^2 - \frac{dr^2}{(1 - \frac{2GM}{c^2 r})} - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{2.3.17}$$

where $C = \frac{2GM}{c^2}$

This is true for any spherically symmetric vacuum solution to Einstein's equations. Finally these equations describes the spacetime manifold around a point mass. This equation shows that there are singular points when $r = 0$ and when $r = \frac{2GM}{c^2}$;

The first is the center of the black hole-no particle can be exactly the same place as another.The second singular point represents the event horizon of a black hole, and the schwarzschild radius- the radius at which a ball of mass M collapses in to a black hole.

Chapter 3

Black Holes

3.1 Formation of black holes

A star is held up by a mixture of gas and radiation pressure which is the relative contributions depending on its mass. The energy to provide this pressure support is derived from the fusion of light nuclei into heavier ones predominantly hydrogen into helium, which releases about 26MeV for each atom of He that is formed. When all the nuclear fuel is used up, the star begins to cool and collapse under its own gravity. For most stars the collapse ends in a high density stellar remnant known as a white dwarf. In fact, we expect that in around 5 billion years the Sun will collapse to form a white dwarf with a radius of about 5000 km and a spectacularly high mean density of about $10^9 kgm^{-3}$. Astronomers have known about white dwarfs since as long ago as 1915 (the earliest example being the companion to the bright star Sirius, known as Sirius B), but nobody at the time knew how to explain them. The physical mechanism providing the internal pressure to support such a dense object was a mystery. The answer had to await the development of quantum mechanics and the formulation of Fermi$-$Dirac statistics. Fowler realized in 1926 that white dwarfs were held up by electron degeneracy pressure. The electrons in a white dwarf behave like the free electrons in a metal, but the electron states are widely spaced in energy because of the small size of the star in its white-dwarf form. Because of the Pauli exclusion principle, the electrons completely fill these states up to a high characteristic Fermi energy. It is these high electron energies that save the star from collapse. In 1930, Chandrasekhar realized that the more massive a white dwarf, the denser it must be and so the stronger the gravitational field. For white dwarfs over a critical mass of about $1.4 M_{\odot}$ (now called the Chandrasekhar limit), gravity would overwhelm the degeneracy pressure and no stable solution would be possible. Thus, the gravitational collapse of the object must continue. At first it was thought that the white dwarf must collapse to a point. After the discovery of the neutron, however, it was realized that at some stage in the collapse the extremely high densities occurring would cause the electrons to interact with the protons via inverse -decay to form neutrons (and neutrinos, which simply escape). A new stable configuration a neutron star was therefore possible in which the pressure support is provided by degenerate neutrons. A neutron star of one solar mass 260 Schwarzschild black holes would have a radius of only 30km, with a density of around $10^{16} kgm^{-3}$. Since the matter in a neutron star is at nuclear density, the gravitational forces inside the star are extremely strong. In fact, it is the first point in the evolution of a stellar object at which general relativistic effects are expected to be important. Given the extreme densities inside a neutron star, there remain uncertainties in the equation of state of matter. Nevertheless, it is believed that as for white dwarfs, there exists a maximum mass above which no stable neutron star configuration is possible. This maximum mass is believed to be about $3M\odot$ [5]. Thus, we believe that stars more massive than this limit should collapse to form black holes. Moreover, if the collapse is spherically symmetric then it must produce a Schwarzschild black hole. Some theorists were very sceptical about the formation of black holes. The Schwarzschild solution in particular is very special. It is exactly spherically symmetric by construction [16].

The basic idea of a black hole is simply an object whose gravity is so strong that light can not escape from it. It is black because it does not reflect light, nor does its surface emit any light. These black holes can be organized in groups according to their mass. Super massive black holes with mass of order $10^9 M_\odot$ are thought to reside in the nuclei of galaxies. In addition to their role in the dynamics of galaxies and galaxy formation, they are believed to be the central engines of energetic phenomenon associated with active galactic nuclei. Evidence for intermediate mass black holes of around 10^2 to $10^5 M_\odot$ has been found.

Lower mass black holes formed at the endpoint of stellar evolution are known as remnant black holes. They are expected to have masses from about 3 to $100 M_\odot$. These remnant black holes are referred to as stellar black holes [17]. A black hole is a region of space from which nothing can escape. It is the result of the deformation of spacetime caused by a very compact mass a lot of mass in a small volume (actually zero) volume. Around the black hole there is an undetectable surface called the event horizon, which marks the point of no return. Once inside nothing can escape. A black hole is called "black" because it absorbs all the light that hits it and reflecting nothing, just like a perfect blackbody in thermodynamics [18].

Chapter 4

Particle Dynamics Around Stellar Black hole

4.1 Particle motion around stellar black holes

From the Schwarzschild line element given as,

$$ds^2 = c^2(1 - \frac{2GM}{c^2 r})dt^2 - (1 - \frac{2GM}{c^2 r})^{-1}dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{4.1.1}$$

This follows the method presented in [5]

The connection coefficients $\Gamma^{\mu}_{\nu\rho}$ for this metric could be written the geodesic equations for the Schwarzschild geometry in the form:

$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$ where λ is some affine parameter along the geodesic $x^\mu(\lambda)$.

Let consider the "Lagrangian"

$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$

where

$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, Using equation (4.1.1), L is given by

$$L = c^2(1 - \frac{2GM}{c^2r})\dot{t}^2 - (1 - \frac{2GM}{c^2r})^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2) \tag{4.1.2}$$

where L is the Lagrangian

The geodesic equation are then obtained by substituting this form for L in to the Euler-Lagrange equations.

$\frac{d}{d\lambda}(\frac{\partial L}{\partial \dot{x^\mu}}) - \frac{\partial L}{\partial x^\mu} = 0$ Performing this calculation, we find the four resulting geodesic equations for $(\mu = 0, 1, 2, 3)$ are given by:

$$(1 - \frac{2GM}{c^2r})\dot{t} = k \tag{4.1.3}$$

$$(1 - \frac{2GM}{c^2r})^{-1}\ddot{r} + \frac{GM}{r^2}\dot{t}^2 - (1 - \frac{2GM}{c^2r})^{-2} - r(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2) = 0 \tag{4.1.4}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - sin\theta cos\theta\dot{\phi}^2 = 0 \tag{4.1.5}$$

$$r^2 sin^2\theta\dot{\phi} = h \tag{4.1.6}$$

In equation (4.1.3) and (4.1.6) respectively, the quantities k and h are constants. These two equations are derived immediately since L is not an explicit function of t or ϕ. B/c of the spherical symmetry of the Schwarzschild metric we can therefore, without lose of generality, give attention to particles moving in the equatorial plane, given by $\theta = \frac{\pi}{2}$.

We see that $\theta = \frac{\pi}{2}$ satisfies the third geodesic equation (4.1.5). Then our set of geodesic equations reduces to

$$(1 - \frac{2GM}{c^2 r})\dot{t} = k \tag{4.1.7}$$

$$(1 - \frac{2GM}{c^2 r})^{-1}\ddot{r} + \frac{GM}{r^2}\dot{t}^2 - (1 - \frac{2GM}{c^2 r})^{-2}\frac{GM}{c^2 r^2}\dot{r}^2 - r\dot{\phi}^2 = 0 \tag{4.1.8}$$

$$r^2\dot{\phi} = h \tag{4.1.9}$$

These equations are valid for both null and non null affinity parameterized geodesics. For a non-null geodesic the first integral is simply

$$g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = x^2 \tag{4.1.10}$$

where x is some constant

For a null geodesic it is as follows,

$$g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = 0 \tag{4.1.11}$$

4.1.1 Massive particles

The trajectory of a massive particle is a timelike geodesic.Considering motion in the equatorial plane,we replace the geodesic equation (4.1.8) by equation (4.1.10),where $g_{\mu\nu}$ is taken from Eq.(4.1.1) with $\theta = \frac{\pi}{2}$. Moreover, since we are considering a timelike geodesic we can choose our affine parameter λ to be the proper time τ along the path.

Thus we find that the worldline $x^\mu(\tau)$ of a massive particle moving in the equatorial plane of the Schwarzschild geometry must satisfy the equations,

$$(1-\frac{2GM}{c^2r})\dot{t}=k \tag{4.1.12}$$

$$c^2(1-\frac{2GM}{c^2r})\dot{t}^2-(1-\frac{2GM}{c^2r})^{-1}\dot{r}^2-r^2\dot{\phi}^2=x^2 \tag{4.1.13}$$

$$r^2\dot{\phi}=h \tag{4.1.14}$$

Substituting Eq.(4.1.12) and Eq.(4.1.14) in to Eq.(4.1.13), we obtain the combined "Energy" equation for the r- coordinate.

$$\dot{r^2}+\frac{h^2}{c^2}(1-\frac{2GM}{c^2r})-\frac{2GM}{r}=x^2(k^2-1) \tag{4.1.15}$$

We use this "Energy" equation to discuss radial free fall and the stability of orbits. Note that the right hand side of Eq.(4.1.15) is a constant of the motion. The constant of proportionality is fixed by requiring that, for a particle at rest at $r=\infty$, we have $E=m_0c^2$. Letting $r\to\infty$ and $\dot{r}=0$,in Eq.(4.1.15) we get $k^2=1$.

Hence we must have $k=\frac{E}{m_0c^2}$,

where E- is the total energy of the particle in its orbit.

A second useful equation which help us to determine the shape of a particle orbit(i.e r as a function of ϕ) We found by using $h=r^2\dot{\phi}$ to express $\dot{r}$ in the energy equation (4.1.15) as,

$\frac{dr}{d\tau}=\frac{dr}{d\phi}\frac{d\phi}{d\tau}=\frac{h}{r^2}\frac{dr}{d\phi}$ Thus we obtain

$(\frac{h}{r^2}\frac{dr}{d\phi})^2+\frac{h^2}{r^2}=x^2(k^2-1)+\frac{2GM}{r}+\frac{2GMh^2}{c^2r^3}$ Let $u=\frac{1}{r}$, that is usually employed in Newtonian orbit calculation, we find that:

$(\frac{du}{d\phi})^2 + u^2 = \frac{x^2}{h^2}(k^2 - 1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}$ By differentiating this equation with respect to ϕ finally we get,

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2 \tag{4.1.16}$$

In Newtonian gravity, the equations of motion of a particle of mass m in the equatorial plane $\theta = \frac{\pi}{2}$ mainly determined from the Lagrangian as,

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GMm}{r}$

From the Euler-Lagrangian equations we have

$r^2\dot{\phi} = h,$

$\ddot{r} = \frac{h}{r^3} - \frac{GM}{r^2}$

where the integration constant h is the specific angular momentum of the particle. If we now substitute $u = \frac{1}{r}$ and eliminate the time variable the Newtonian equation of motion for planetary orbit is obtained,

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} \tag{4.1.17}$$

In this equation $u = \frac{1}{r}$ where r is the radial distance from the mass, where as in eq.(4.1.16) r is a radial coordinate that is related to distance through the metric.

4.1.2 Radial motion of massive particle

For radial motion ϕ is constant which implies that $h = 0$. Thus Eq.(4.1.15) reduces to:

$$\dot{r}^2 = c^2(k^2 - 1) + \frac{2GM}{r} \tag{4.1.18}$$

Differentiating this equation with respect to τ and dividing through by $\dot{r}$ gives:

$$\ddot{r} = \frac{-GM}{r^2} \tag{4.1.19}$$

Consider a particle dropped from rest at $r = R$. From Eq.(4.1.18) we see that,

$k^2 = 1 - \frac{2GM}{c^2R}$, so Eq.(4.1.18) can be written as

$$\frac{\dot{r}^2}{2} = GM(\frac{1}{r} - \frac{1}{R}) \tag{4.1.20}$$

This has the same as the Newtonian formula equating the gain in kinetic energy to the loss in gravitational potential energy for a particle (of unit mass) falling from rest at $r = R$. By considering a particle dropped from rest at infinity, and setting $k = 1$ in the geodesic equation (4.1.12) and (4.1.18) we get,

$$\frac{dt}{d\tau} = (1 - \frac{2GM}{c^2r})^{-1} \tag{4.1.21}$$

$$\frac{dr}{d\tau} = -(\frac{2GM}{r})^{\frac{1}{2}} \tag{4.1.22}$$

where in Eq.(4.1.22) we take the negative square root.This equation form the basis of our discussion of a radially infalling particle dropped from rest at infinity.

From these we see that the components of the 4-velocity of the particle in the(t,r,θ, ϕ) coordinate system are,

$[u^\mu] = [\frac{dx^\mu}{d\tau}] = ((1 - \frac{2GM}{c^2r})^{-1}, -(\frac{2GM}{r})^{\frac{1}{2}}, 0, 0)$ Eq.(4.1.22) determines the trajectory of particle r(τ).Integrating this equation gives

$\tau = \frac{2}{3}\sqrt{\frac{r_0^3}{2GM}} - \frac{2}{3}\sqrt{\frac{r^3}{2GM}}$, where we have written the integration constant in a form such that τ and $r = r_0$. Thus τ is the proper time experienced by the particle in falling from $r = r_0$ to a coordinate radius r. In stead of parametrise the worldline interms of proper time τ, alternatively we describe the path as r(t). So

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -(\frac{2GM}{r})^{\frac{1}{2}}(1 - \frac{2GM}{c^2r}) \tag{4.1.23}$$

Integrating again gives

$t = \frac{2}{3}(\sqrt{\frac{r_0^3}{2GM}} - \sqrt{\frac{r^3}{2GM}}) + \frac{4GM}{c^3}(\sqrt{\frac{r_0 c^2}{2GM}} - \sqrt{\frac{rc^2}{2GM}}) + \frac{2GM}{c^3}\ln|(\frac{\sqrt{\frac{r_c^2}{(2GM)}+1}}{\sqrt{\frac{r_c^2}{(2GM)}-1}})(\frac{\sqrt{\frac{r_c^2}{(2GM)}-1}}{\sqrt{\frac{r_c^2}{(2GM)}+1}})|$

The choice of the integration constant gives $t = 0$ at $r = r_0$.

In particular

$\tau \to \frac{2}{3}\sqrt{\frac{r_0^3}{2GM}}$, as $r \to 0$

$t \to \infty$, as $r \to \frac{2GM}{c^2}$

Evidently the particle takes a finite proper time to reach $r = 0$, when the worldline is expressed in the form r(t), however, we see that asymptotically approaches $\frac{2GM}{c^2}$ as $t \to \infty$. What velocity a stationary observer at r measures for the infalling particle as it passes. From the Schwarzschild metrics, (4.1.1), we see that,for stationary observer at coordinate radius r, a coordinate time interval dt corresponds to a proper time interval

$dt' = (1 - \frac{2GM}{c^2 r})^{\frac{1}{2}} dt$ Similarly,a radial coordinate separation dr corresponds to a proper radial distance measured by the observer equal to

$dr' = (1 - \frac{2GM}{c^2 r})^{\frac{-1}{2}} dr$.

Thus the velocity of the radially infalling particle , as measured by a stationary observer at r, is given by

$$\frac{dr'}{dt'} = (1 - \frac{2GM}{c^2 r})^{-1}\frac{dr}{dt} = -(\frac{2GM}{r})^{\frac{1}{2}} \tag{4.1.24}$$

Thus as the particle approaches $r = \frac{2GM}{c^2}$, a stationary observer at that radius observes that the particles velocity tends to c.

Equation (4.1.24) valid for $r > \frac{2GM}{c^2}$, It is impossible to have a stationary observer at $r \leq \frac{2GM}{c^2}$.

4.1.3 Circular motion of massive particles

For circular motion in the equatorial plane, r=constant. and so $\dot{r} = \ddot{r} = 0$.

setting $u = \frac{1}{r} = constant$ in the "shape" Eq.(4.1.16) we have the following,

$u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$ .From which

$h^2 = \frac{GMr^2}{r - \frac{3GM}{c^2}}$.

Putting $\dot{r} = 0$ in the energy equation (4.1.15) and substituting the above expression for h^2 allows us to identify the constant k,

$$k = \left(\frac{1 - \frac{2GM}{c^2 r}}{1 - \frac{3GM}{c^2 r}}\right)^{\frac{1}{2}} \tag{4.1.25}$$

The energy of a particle of rest mass m_o in a circular of radius r is given by;

$E = km_oc^2$. We use this result to determine which circular orbits are bound. For this we require $E < m_oc^2$, so the limits on r for the orbit to be bound are given by $k = 1$. This gives : $(1 - \frac{2GM}{c^2 r})^2 = 1 - \frac{3GM}{c^2 r}$ Which is satisfied when $r = \frac{4GM}{c^2}$ or $r = \infty$. Thus over the range $\frac{4GM}{c^2} < r < \infty$, circular orbits are bound.

4.1.4 Stability of massive particle orbits

The above analysis appears to suggest that the closest bound circular orbit around a massive spherical body is at $r = \frac{4GM}{c^2}$. However, we have not yet determine whether this orbit is stable or not.

In Newtonian dynamics the equation of motion of a particle in a central potential can be written

$\frac{1}{2}(\frac{dr}{dt})^2 + V_{eff}(r) = E$,

where:

$V_{eff}(r)$- is the effective potential and

E- is the total energy of the particle per unit mass. For an orbit around a spherical mass M, the effective potential is:

$$V_{eff}(r) = \frac{-GM}{r} + \frac{h^2}{2r^2} \tag{4.1.26}$$

Where h is the specific angular momentum of the particle. In general relativity, the energy equation (4.1.15) for the motion of a particle around a central mass can be written as,

$\frac{1}{2}(\frac{dr}{d\tau})^2 + \frac{h^2}{2r^2}(1 - \frac{2GM}{c^2 r}) - \frac{GM}{r} = \frac{x^2}{2}(k^2 - 1)$, where the constant $k = \frac{E}{(m_o c^2)}$.

Thus in general relativity we identify the effective potential per unit mass as follows;

$$V_{eff}(r) = \frac{-GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{c^2 r^3} \tag{4.1.27}$$

Differentiating Eq.(4.1.27) gives,

$\frac{dV_{eff}}{dr} = \frac{GM}{r^2} - \frac{h^2}{r^3} + \frac{3GMh^2}{c^2 r^4}$, and so the exterema of the effective potential are located at the solutions of the quadratic equation.

$GMr^2 - h^2 r + \frac{3GMh^2}{c^2} = 0$ Which occur at

$r = \frac{h}{2GM}(h \pm \sqrt{h^2 - \frac{12G^2M^2}{c^2}})$. We note that if $h = \sqrt{12}\frac{GM}{c} = 2\sqrt{3}\frac{GM}{c}$ then there is only one extremum and, there are no turning points in the orbit for lower values of h.The significance of this result is that the inner most stable circular orbit has

$r_{min} = \frac{6GM}{c^2}$ This orbit, with $r = \frac{6GM}{c^2}$ and $\frac{hc}{(GM)} = h\sqrt{3}$, is unique in satisfying both

$\frac{dV_{eff}}{dr} = 0$ and

$\frac{d^2V_{eff}}{dr^2} = 0$

4.2 Massless Particles

The trajectory of a photon and any other particle having zero rest mass is a null geodesic [19].

From the previous equations (4.1.7),(4.1.8),and (4.1.9) by the condition $g_{\mu\nu}\dot{x^\mu}\dot{x^\nu}$, we have

$$(1-\frac{2GM}{c^2r})\dot{t}=k \tag{4.2.1}$$

$$c^2(1-\frac{2GM}{c^2r})\dot{t^2}-(1-\frac{2GM}{c^2r})^{-1}\dot{r^2}-r^2\dot{\phi^2}=0 \tag{4.2.2}$$

$$r^2\dot{\phi}=h \tag{4.2.3}$$

For photon trajectories, an analogue of the energy equation (4.1.15) can be obtained by substituting (4.2.1) and (4.2.3) in to (4.2.2) which gives,

$$\dot{r^2}+\frac{h^2}{r^2}(1-\frac{2GM}{c^2r})=x^2k^2 \tag{4.2.4}$$

Similarly, the analogue for photons of the 'shape'equation (4.1.16) is obtained by substituting $h=r^2\dot{\phi}$ in to Eq.(4.2.2) and using the fact that

$\frac{dr}{d\lambda}=\frac{dr}{d\phi}\frac{d\phi}{d\lambda}=\frac{h}{r^2}\frac{dr}{d\phi}$ Using the substitution $u=\frac{1}{r}$ and differentiating with respect to ϕ we get,

$$\frac{d^2u}{d\phi^2}+u=\frac{3GM}{c^2}u^2, \tag{4.2.5}$$

4.2.1 Stability of photon orbits

We can rewrite the 'energy' equation (4.2.4) for photon orbits as:

$$\frac{\dot{r}^2}{h^2} + V_{eff}(r) = \frac{1}{b^2} \tag{4.2.6}$$

Where $b = \frac{h}{ck}$ and the effective potential

$$V_{eff}(r) = \frac{1}{r^2}(1 - \frac{2GM}{c^2 r}) \tag{4.2.7}$$

4.3 Orbital Equations and Bound Orbits

This follows the method presented in [2].

In discussing the exact solutions for the orbital motion in the equatorial plane by considering r as a function of ϕ instead of τ we get,[20]

$$(\frac{dr}{d\phi})^2 = (E^2 - 1)\frac{r^4}{h^2} + \frac{2M}{h^2}r^3 - r^2 + 2Mr \tag{4.3.1}$$

If we introduce the variable $u = \frac{1}{r}$, as in the analysis of the Keplerian orbits in the Newtonian theory.Now by replacing this the fundamental equation becomes;

$$(\frac{du}{d\phi})^2 = 2GMu^3 - u^2 + \frac{2M}{h^2}u - \frac{1 - E^2}{h^2} \tag{4.3.2}$$

This equation determines the geometry of the geodesics in the invariant plane.Once it have been solved for $u = u(\phi)$.

$$\frac{d\tau}{d\phi} = \frac{1}{hu^2} \tag{4.3.3}$$

$$\frac{dt}{d\phi} = \frac{E}{hu^2(1 - 2Mu)} \tag{4.3.4}$$

Bound Orbits

This solutions of Eq.(4.3.2) will depend on $E^2 < 1$ or $E^2 \geq 1$ This distinction are between bound orbits and unbound orbits. Bound orbits are governed by an equation:

$$\frac{du}{d\phi} = f(u) \tag{4.3.5}$$

where f(u) is given by,

$$f(u) = 2Mu^3 - u^2 + \frac{2M}{h^2}u - \frac{1-E^2}{h^2} \tag{4.3.6}$$

It is clear that the geometry of geodesics will be determined by the positions of the roots $f(u) = 0$. Since f(u) is cubic in u, there are two possibilities: either all roots are all, or one of them is real and the two remaining are complex conjugate ones. Let u_1, u_2, u_3 denote the roots of $f(u) = 0$. Then we have,

$$u_1 u_2 u_3 = \frac{(1-E^2)}{2Mh^2} \tag{4.3.7}$$

and

$$u_1 + u_2 + u_3 = \frac{1}{2}M \tag{4.3.8}$$

Since $1 - E^2 > 0$,it must allow for one positive real root. From the further facts that $f < 0$ for $f(u) \to \pm\infty$, for $u \to \pm\infty$

Case

$\Rightarrow$ If the three roots are all different,

There exists two distinct orbits confined to the interval $u_1 < u < u_3$ and $u > u_3$, i.e an orbit that oscillates b/n two extreme values for r and an orbit , starting at a certain aphelion distance given by $\frac{1}{u_3}$ plunges in to the singularity at $r = 0$, i.e $u \to \infty$. These two classes of orbits are called orbits of the first kind and the second kinds.Orbits of both kinds are most conveniently parameterized by an eccentricity e and a latus rectum l,similar to Newtonian orbits.

Orbits of the first kind

For this all three roots are positive, and we can write them as;

$$u_1 = \frac{1}{l}(1-e) \tag{4.3.9}$$

$$u_2 = \frac{1}{l}(1+e) \tag{4.3.10}$$

$$u_3 = \frac{1}{2M} - \frac{2}{l} \tag{4.3.11}$$

The semilatus rectum l is some positive constant and the eccentricity $e < 1$ for $u_1 > 0$, as required by the condition $E^2 < 1$.

The conformity with the ordering $u_1 < u_2 < u_3$ requires

$\frac{1}{2M} - \frac{2}{l} \geq \frac{1+e}{l}$

$$l \geq 2M(3+e) \tag{4.3.12}$$

Let $\mu \equiv \frac{M}{l}$

The inequality becomes,

$$\mu \leq \frac{1}{2(3+e)}, or, 1 - 6\mu - 2\mu e \geq 0 \tag{4.3.13}$$

In this parameter now f(u) is written as

$$f(u) = 2M(u - \frac{1+e}{l})(u - \frac{1}{2M} + \frac{2}{l}) \tag{4.3.14}$$

For a Keplerian ellipse, the semilatus rectum l is the distance measured from a focus such that;

$$\frac{1}{l} = \frac{1}{2}(\frac{1}{r_+} + \frac{1}{r_-}) \tag{4.3.15}$$

where $r_+ = a(1+e)$ and $r_- = a(1-e)$ are the aphelion and perihelion positions of the orbit respectively. Substituting the values of r_+ and r_- in to Eq.(4.3.15) for l it gives:

$$\frac{1}{l} = \frac{1}{a(1-e^2)} \tag{4.3.16}$$

The values of the two becomes,

$$r_+ = \frac{l}{1-e}, and, r_- = \frac{l}{1+e} \tag{4.3.17}$$

This justifies for the roots u_1 and u_2. This has to agree with the original form of the function, giving the relations;

$$\frac{M}{h^2} = \frac{1}{l^2}[l - M(3+e^2)] \tag{4.3.18}$$

$$\frac{1-E^2}{h^2} = \frac{1}{l^3}[(l-4M)(3-e^2)] \tag{4.3.19}$$

If expressed in terms of μ

$$\frac{1}{h^2} = \frac{1}{lM}[1 - \mu(3+e^2)] \tag{4.3.20}$$

$$\frac{1-E^2}{h^2} = \frac{1}{h^2}[(1-4\mu)(1-e^2)] \tag{4.3.21}$$

From this equation it follows that $\mu < \frac{1}{3+e^2}$ and $\mu < \frac{1}{4}$

As in the Keplerian problem, we now make [21]

$$u = \frac{1}{l}(1 + ecos\theta) \tag{4.3.22}$$

θ is now a kind of relativistic anomaly.

At aphelion,$\theta = \pi$, we found $u = \frac{(1-e)}{l}$ and

At perihelion, $\theta = 0, u = \frac{(1+e)}{l}$

This substitution leads to the equation;

$(\frac{d\theta}{d\phi})^2 = 1 - 2\mu(3 + ecos\theta)$

$$= (1 - 6\mu + 2\mu e) - 4\mu ecos^2(\frac{\theta}{2}) \tag{4.3.23}$$

or,

$\pm\frac{d\theta}{d\phi} = \sqrt{1 - 6\mu + 2\mu e}\sqrt{1 - k^2cos^2(\frac{\theta}{2})}$ where

$k^2 = \frac{4\mu e}{1-6\mu+2\mu e}$

The solution for ϕ can be expressed in terms of the Jacobian integral as,

$$F(\psi, k) = \int_0^{\psi} \frac{d\gamma}{\sqrt{1 - k^2sin^2\gamma}} \tag{4.3.24}$$

where $\psi = \frac{1}{2}(\pi - \theta)$,thus finally written as

$$\phi = \frac{2}{\sqrt{1 - 6\mu + 2\mu e}}F(\frac{\pi}{2}, \frac{\theta}{2}, k) \tag{4.3.25}$$

where the origin of ϕ has been chosen at aphelion passage where $\theta = \pi$.

The perihelion passage occurs at $\theta = 0$, where $\psi = \frac{\pi}{2}$.

The solution can be completed by the expressions for the proper time and the coordinate time as;

$$\tau = \frac{1}{h}\int \frac{d\phi}{u^2} = \frac{1}{h}\int \frac{d\phi}{d\theta}\frac{d\theta}{u^2} \tag{4.3.26}$$

and

$$t = \frac{E}{h}\int \frac{d\phi}{d\theta}\frac{d\theta}{u^2(1 - 2Mu)} \tag{4.3.27}$$

The first-order corrections to the Keplerian orbits of the Newtonian theory can readily be deduced from Eq.(4.3.23)

Under normal conditions, the parameter $\frac{\mu}{h}$ is a very small quantity. It is essentially the ratio of the gravitational radius M to the major axis of a planetary orbit or a binary star orbit. So expanding Eq.(4.3.23) to the first order in μ to obtain

$$-d\phi = d\theta(1 + 3\mu + \mu e cos\theta) \tag{4.3.28}$$

integrating this gives

$$-\phi = (1 + 3\mu)\theta + \mu e sin\theta) \tag{4.3.29}$$

From this we understand that the change in ϕ after one complete revolution during which θ changes by 2π is $2\pi(1 + 3\mu)$. Therefore,the advance of the perihelion

$\Delta\phi$, per revolution is,

$$\Delta\phi = \frac{6\pi M}{l} = \frac{6\pi GM}{a(1 - e^2)c^2} \tag{4.3.30}$$

where

a- is the semi major axis of the particle's orbit.

l- is semilatus rectum and

e- is eccentricity of particle's orbit.

From Eq.(4.3.22) replacing $u = \frac{1}{r}$ one can have,

$\frac{1}{r} = \frac{1}{l}(1 + ecos\theta)$ from $l = r(1 + ecos\theta)$ and

$l = a(1 - e^2)$ finally this gives

$$r(\theta) = \frac{a(1 - e^2)}{(1 + ecos(\theta))} \tag{4.3.31}$$

4.3.1 Unbound Orbit

Both massless light ray and massive objects(particles)experience trajectory bending in a gravitational field [22]. A massive object can have a significant effect on the propagation of photons. Photons can travel in a circular orbit at $r = \frac{3GM}{c^2}$. we do not expect to observe this effect directly, but a more modest bending of light can be observed.For investigating the slight deflection of light, we follow an approximation technique as we use before. The shape equation for photon trajectory in the equatorial plane of the Schwarzschild geometry is;

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2u^2} \tag{4.3.32}$$

where $u = \frac{1}{r}$

In the absence of matter, the right-hand side vanishes and the solution equals

$$u = \frac{sin\phi}{b} \tag{4.3.33}$$

which represents a straight line path with impact parameter b. we treat Eq. (4.3.32) as the zeroth order solution to the equation of motion.Thus, we write the general relativistic solution

$$u = \frac{sin\phi}{b} + \Delta u \tag{4.3.34}$$

where Δu is a perturbation. By substituting this expression in to Eq. (4.3.31) , we get to the first order in Δu. which is

$$\frac{d^2u}{d\phi^2} + \Delta u = \frac{3GM}{c^2b^2} sin^2\phi \tag{4.3.35}$$

Integrating this

$$\Delta u = \frac{3GM}{2c^2b^2}(1 + \frac{1}{3}cos2\phi) \tag{4.3.36}$$

Adding Eq.(4.3.34) and Eq.(4.3.35) together gives

$$u = \frac{sin\phi}{b} + \frac{3GM}{2c^2b^2}(1 + \frac{1}{3}cos2\phi) \tag{4.3.37}$$

Now consider the limit $r \to \infty$, $u \to \infty$ For a slight deflection we take $sin\phi \approx \phi$, and $cos2\phi \approx 1$.

Finally the total deflection is

$$\Delta\phi = \frac{4GM}{c^2 b} \tag{4.3.38}$$

where b is the impact parameter.

Chapter 5

Result and Discussion

Result

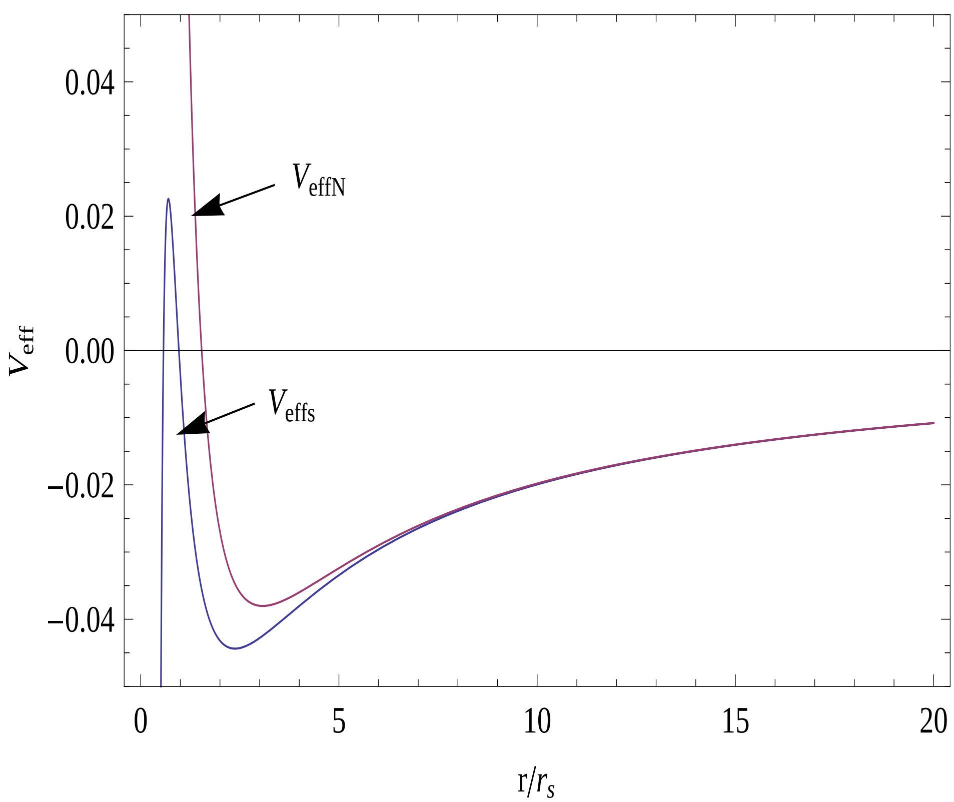

Figure 5.1: Effective potential as a function of radius ($\frac{r}{r_s}$)for various values of the angular momentum

For Eq.4.1.27 figure (5.1)we find that:

In Newtonian gravity, the circular orbits appear at $r_c = \frac{h^2}{GM}$, and

In general relativity the situation is different only for r is sufficiently small and the difference resides in the term $\frac{-GMh^2}{r^3}$, the behaviors of the two graphs are similar.

But as $r \to 0$,the potential goes to ∞, in the Newtonian case.

At $r = \frac{2GM}{c^2}$, the potential is always zero; inside this radius is the black hole.

In Newtonian case there is only one stable circular orbit,but in General Relativity there are two, which are one unstable at the upper and one stable at the lower part.

For Eq.4.2.7

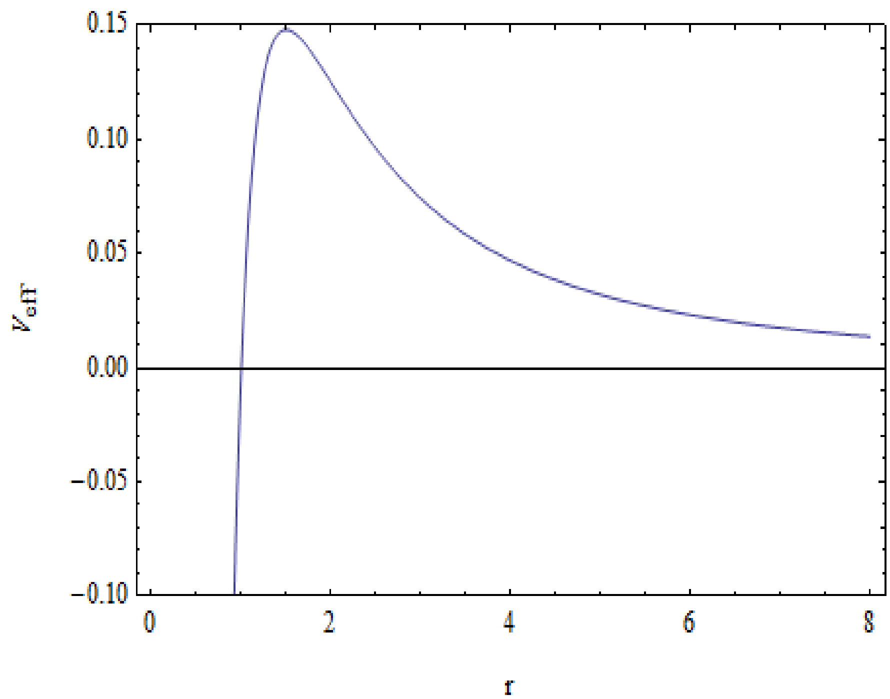

Figure 5.2: Effective potential for photon orbits

From figure (5.2) we see that the effective potential has a single maximum at $r = \frac{3GM}{c^2}$, where the value of the potential is, $\frac{c^4}{27G^2M^2}$. Therefore, the circular orbit at $r = \frac{3GM}{c^2}$ is unstable. From these ,we conclude that there are no stable circular orbits of photon in the schwarzschild geometry.It comes from the infinity and then deflects when it reaches event horizon then goes to infinity.

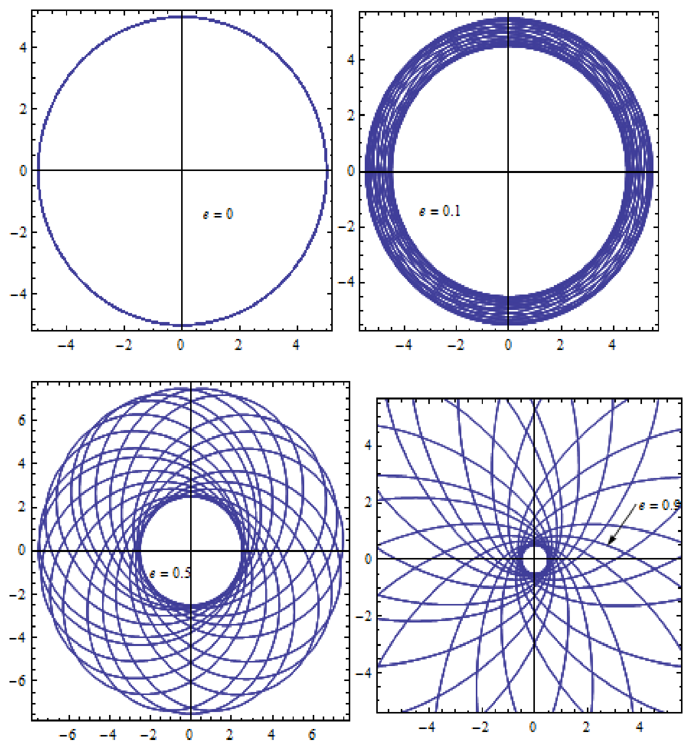

Figure 5.3: Ellipses with different eccentricities

For Eq.4.3.31,From figure (5.3) for some values of eccentricities($e < 1$),i.e (e=0,0.1,0.5,and 0.9) the particles have different elliptical shapes.The particle is closest to the black hole when $\theta = 0$ and this minimum distance is at $r_- = \frac{l}{1+e}$ and, again the greatest distance occurs at aphelion when $\theta = \pi$ then $r_+ = \frac{l}{1-e}$. when the particle revolves around

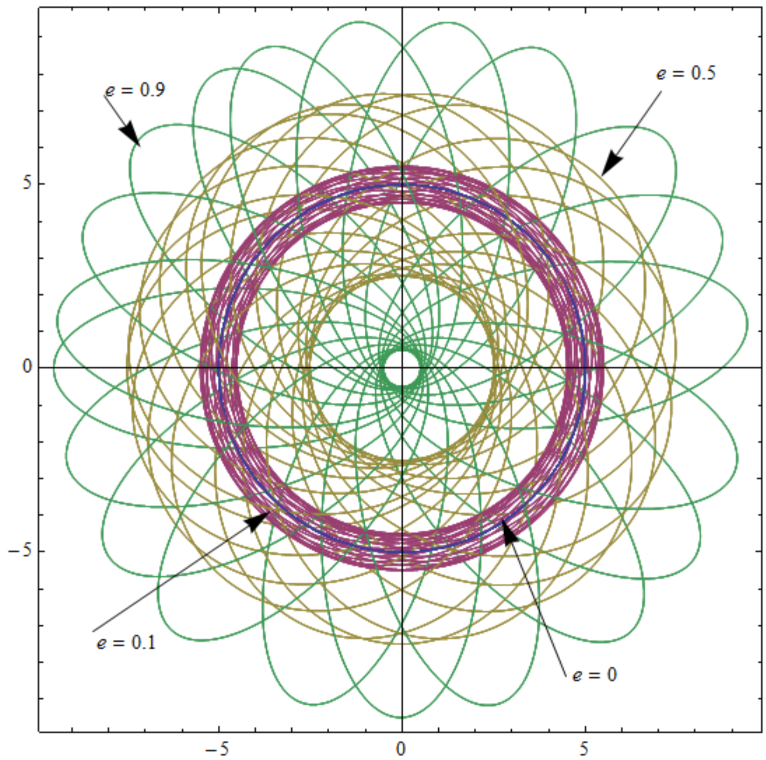

Figure 5.4: Ellipses with different eccentricities all in one

a black hole, the velocity of the particles becomes slow at the two points,perihelion and aphelion.For small eccentricity, in the case $e = 0$ the semi-major and semi-minor axis are equal, then the orbit is circle.

Discussion

Bound orbits which are not circular will oscillate around the radius of the stable circular orbit.

For massive particles,

The circular orbits are at;

$r_c = \frac{h^2 \pm \sqrt{h^4 - 12G^2M^2h^2}}{2GM}$

For large h,there are two circular orbits, one stable and one unstable.

$$r = \frac{h^2 \pm h^2(\frac{1-6G^2M^2}{h^2})}{2GM} = (\frac{h^2}{GM}, 3GM)$$

In this limit, the stable circular orbit becomes further and further away, while the unstable one approaches $3GM$.

As we decrease h, the two circular orbits come closer together,they coincide when the discriminant vanishes.i.e $h = \sqrt{12}GM$ for which $r_{min} = 3R_s$ It disappears entirely for smaller h. Thus $\frac{6GM}{c^2}$ is the smallest possible radius of a stable circular orbit in the Schwarzschild metric.There also unbound orbits, which come in from infinity and turn around, and bound but non circular ones, which oscillate around the stable circular radius.

Therefore, Schwarzschild solution possesses stable circular orbits for $r > \frac{6GM}{c^2}$ and unstable circular orbits for $\frac{3GM}{c^2} < r < \frac{6GM}{c^2}$.

For massless particles there are no circular orbits. Massless particles actually move in straight line, since the Newtonian gravitational force on a massless particle is zero. Interms of the effective potential a photon with a given energy will come in from $r = \infty$ and gradually slow down, but the speed of light is not changing until it reaches the turning point,then it will start moving away back to $r = \infty$. The smallest value of h for which the photon will come closer before it starts moving away, is those trajectories which are initially aimed closer to the gravitating body.

At last when we come to the particles trajectory of bounded orbits round the black hole, since the eccentricity we have taken during our work is $0 \leq e < 1$, the trajectory that happened while the particle rotates round the black hole is ellipse.The points in the trajectory which are closest to the focus and furthest away from the focus are called the perihelion and aphelion respectively.

Chapter 6

Conclusion and summary

When we study about the dynamics of particles around stellar black holes on the basis of general relativity,we used Einstein's field equations and we derived the Schwarzschild metric solutions. Using this metric we did drive different relevant dynamical equations. Also starting from geodesic equation in connection with Lagrangian equation we derived equations for the trajectories of both massive and massless particles, like equation for effective potential of massive and massless particles, polar equations of ellipse and the behavior of the trajectories have been studied. In our analytical derivations particles motion would be considered and using these equations we generate the numerical data by MATHEMATICA and produce different graphs(figures).As a result during the motion of particles around stellar black holes those particles with weak gravity (Newtonian)have stable circular orbits while particles with strong gravity have both stable and unstable orbits. when these particles approach to the Schwarzschild radius they will trapped in to a black hole. There are also no stable circular photon orbits in the Schwarzschild geometry. They have only one unstable circular orbits.

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