# SCHWARZSCHILD BLACK HOLE RADIATION WITH QUANTUM SEMI-CLASSICAL APPROACH

By

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# Table of Contents

# List of Figures

# Abstract

Probably, one of the earliest outstanding theoretical prediction of General Relativity is the existence of black holes resulted from stellar collapse on self gravitation. However, more than a dozen of decades has passed without any plausible observation. As a result, almost interest was lost except its abstract mathematical existence. Nevertheless, the 1970's paradigm shift by the Bekenstein-Hawking black hole quantum mechanical radiation theory has lead to the indirect discovery of the object. In fact, the direct observation of the black holes, the late noble prize in physics is a celebrity and an encouraging progress that opens a window to look further about our universe. This has motivated us to rework on the current issue of the black hole radiation mechanism pertaining to the semi-classical quantum approach. Especially, the Painleve metric adopted in the present decade for this purpose is re-derived from Einstein field equations to elucidate its intrinsic nature. Then, we detailed the geometry around the hole considering trip of particles into and out of it where the co-moving Friedmann-Lemaiter-Robertson and the static Schwarzschild standard metrics be assumed. Finally, we did reproduce the black hole radiation in agreement with the results of the recent literatures.

# Acknowledgements

The realization that my graduate years are finally coming to an end has afforded me the opportunity to reflect back on how I actually reached this point. It has become more than obvious to me that I couldn't have come this far without the support and guidance of the many people that I have crossed paths with over the years.

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# Chapter 1

# General Introduction

## 1.1 Thesis Scheme

In this introductory chapter we provide the background, literature review, statement of the problem, objectives and methods of the work. In chapter 2 we introduce Einstein General Relativity (GR) theory . The purpose of this chapter is to provide the necessary boundary issues contained in Einstein Field Equations (EFEs). In fact it is the background physics we have used in our work. In Chapter 3, we derive the appropriate black hole (BH) radiation via quantum tunneling in the Wentzel-Kramers-Brillouin (WKB) approximation. The detailing of the background physics, scheme of the method implemented, boundary conditions issues thereof are being throughways provided. In chapter 4 we discuss the results of our work. In the final chapter, chapter 5 we give our summary and conclusions.

## 1.2 Background of the problem and Literature review

### 1.2.1 Background

Black holes are the most mysterious and fascinating objects of our universe. They were predicted by the Einstein's theory of general relativity and their existence is one of the triumph of this theory. From the birth of GR, its solutions had already implied the theoretical existence of such objects. However, their direct observational discovery almost have taken a century ago [1],[2]. Today, this unique objects are revealed by observation and has given window to the universe for more astronomical discoveries and progress in science. The detection of gravitational waves has given great hope to obtain vital information on the nature and properties of black holes [3]. In the early works, before sophisticated astronomical instrumentations were developed, the existence of BHs were alarmingly being lost interest, except its mathematical existence. But, the great extension of astronomical observations began early in the 1960's brought a revival of interest in the classical theory of general relativity. Then many of the new phenomena such as quasars, pulsars and compact X-ray sources were being discovered which indicates the existence of very strong gravitational fields. These developments have to do with certain recently discovered quantum effects associated with black holes that provide a remarkable connection between black holes and the laws of thermodynamics [4].

During the past 40 years, researches in the physics of black holes has brought strong hints of a very deep and fundamental relationship between gravitation, thermodynamics, and quantum theory [5]. The great theoretical efforts are made to predict in detail the waveforms of gravitational waves emitted by black holes. Mainly, the discovery of Hawking radiation and its derivation confirmed indirect the existence of black hole entropy and led to the formulation of black hole thermodynamics. But it has led to the infamous information loss problem and makes black hole physics became a particularly fascinating area of the study [6].

Even today there remain open questions such as the information paradox, the microscopic origin of entropy and the final state of evaporation. A complete understanding of those problems is only possible within a consistent quantum theory of gravity. In recent years promising progress in this direction has been made within loop quantum gravity and string theory. However, at least for very large black holes, quantum effects can be studied within semi-classical theory as well [7]. A feature of the Schwarzschild solution is not emphasized in the early days, but given great prominence since the presence of the Schwarzschild radius, which is the signature for the phenomenon of black holes [8].

### 1.2.2 Literature review

The possibility that stars could collapse to form black holes was first theoretically discovered in l939 by J. Robert Oppenheimer and H.Snyder, who were manipulating the equations of the collapse of a homogeneous sphere of pressureless gas in Einstein's general relativity [9]. They found that the sphere eventually becomes cut off from all communication with the rest of the Universe. This was assumed as the first rigorous calculation demonstrating the formation of a black hole. However, in the late 1950s, J. A. Wheeler who's first coincide the name black hole in 1968 and his collaborators began a serious investigation of the problem of collapse [10]. Moreover if the collapse is spherically symmetric then it produce the Schwarzschild black hole.

Some written histories of black hole indicate as the black holes were predicted long before the beginning of the space age and they were perceived as byproducts of mathematical theories, existed only in the imagination of a few scientists. Related to this the idea of dark stars can be traced back to the late 18th century, when John Michell 1768 (English philosopher and geologist) and some years later to Pierre-Simon Laplace in 1796 (French mathematician and astronomer) speculated that, if a planet or a star were dense enough, their escape velocity would equal to the speed of light [11]. The more papers states as investigation of the properties of a black hole and the possibility of their existence arises from the idea of gravitational collapse [12].

Many researchers states as; the first real black hole solution where light could not even escape a region of spacetime, was first published 1958 by David Finkelstein. At this time black holes were only theoretical objects. It wasn't until the discovery of a pulsar 1967 that studying of gravitational collapsed compact objects became of interest. In the early 1970 Jacob Bekenstein and Stephen Hawking formulated black hole thermodynamics [13]. The most intriguing aspect of black hole radiation is that it contains elements from quantum theory, thermodynamics and general relativity. Thus, one may say that in black hole radiation all the three foundational theories of physics meet for the first time [14].

Since Hawking proved the existence of black hole radiation, much effort has been devoted to the study of the black hole radiation. Several years ago, Parikh and Wilczek proposed a method to calculate the emission rate of particles tunneling through the event horizon of black hole. They treat Hawking radiation as a tunneling process and think that the barrier is created by the outgoing particle itself. Their key insight is to find a coordinate system, which well behaves at the horizon. They calculated the corrected emission spectrum of the spherically symmetric black holes, such as Schwarzschild black holes and Reissner Nordstrom black holes [15].

Still, in principle the black hole could lose all of its mass to Hawking radiation and shrink to nothing in the process. As a result the radiation itself contains less information than the information that was originally in the spacetime. But such a process violates the conservation of information that is implicit in general relativity and quantum field theory, the two theories that led to the prediction. This paradox is considered a big deal these days, and there are a number of efforts to understand how the information can somehow be retrieved. A currently popular explanation relies on string theory, and basically says that black holes have a lot of hair, in the form of virtual stringy states living near the event horizon. Generally, it is an area of active research these time [16].

## 1.3 Statement of the problem

Generally, in classical theory black holes can only absorb any objects and not emit even light. But the quantum mechanical effects cause black holes to create and emit energy in the form of radiation. Then, debates among the scientific communities come up whether there is such information loss or not. This issue is still open and unsettled. Recently, researches are proposing a new model that will compromise the debates, i.e. a way to reconcile the classical issue and the quantum gravity, a semi-classical approach model. So in this work we study this issue.

### 1.3.1 Research questions

- What is the implication of Einstein Field Equation to end product of sufficiently massive stellar object?
- Is the currently and widely considered Painleve metric in the semi-classical approach of BH radiation a new and independent metric than the Schwarzschild metric? And how this metric is connected to the co-moving, Friedman-Lemaiter-Robertson-Walker (FLRW) metric?
- What are the unique properties of Schwarzschild black hole radiation?
- What is the problem related to Hawking quantum radiation?

## 1.4 Objectives

### 1.4.1 General objective

- To study Schwarzschild Black hole radiation with quantum semi-classical approach.

### 1.4.2 Specific objectives

- To study and describe the implication of Einstein Field Equation to end product of sufficiently massive stellar object.

- To work out the intrinsic implication of the currently and widely considered Painleve metric in the semi-classical approach of BH radiation with that of the well established Schwarzschild metric and hence its connection to FLRW metric.

- To explain the unique properties of Schwarzschild black hole radiation.

- To describe the problem related to the Hawking quantum radiation.

## 1.5 Methodology

The ”Dynamic Einstein Field Equation” is used to drive black hole solution. Then the WKB semi-classical technique is used to drive relevant parameter to study the contents of black hole radiation. Objectively, we assume the Painleve coordinates that will bridge the quantum and classical phenomena at black hole horizon. Then, with appropriate boundary conditions and reasonable approximations we extract numerical data from the analytically derived equations for discussions and comments. For the numerical data extraction the latest standard version of MATHEMATICA numerical software is used.

# Chapter 2

# Introduction to General Relativity and its application

General relativity is a relativistic theory of gravitation. The physical interpretation of general relativity depends on the concept of local inertial frames. Relativity teaches us that all forms of energy are equivalent to mass, so that a relativistic theory of gravity would presumably have all forms of energy as sources of the gravitational field. Also, general relativity describes the force of gravity as the curvature of the fabric of space-time. This curvature is caused by matter (energy) the two being equivalent in relativity due to Einstein's famous equation $E = mc^2$ and is governed by Einstein's equations of general relativity, see the next following section.

## 2.1 Einstein Field Equation

All solar system observations and almost all other observations related to gravity are perfectly described within Einstein's General Relativity [17]. The first solution presented to the Einstein field equations was published by Karl Schwarzschild in 1916, which thought as allowed us to make many physical predictions with increased precision. It is the unique solution for the field outside a static, spherically symmetric body [18]. The gravitational field of a simple Einstein model star consists of the interior and exterior the Schwarzschild solutions. They are joined together at the surface of the star [19]. The Einstein equations can be put as usually the form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.1.1}$$

The quantity $G_{\mu\nu}$ is called the Einstein tensor, $T_{\mu\nu}$ is stress-energy tensor and $g_{\mu\nu}$ is metric components . Also the Ricci tensor ($R_{\mu\nu}$) can be wrrtten as

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\lambda\mu}. \tag{2.1.2}$$

is symmetric. In empty space, it takes the form

$$R_{\mu\nu} = 0. \tag{2.1.3}$$

Also the scalar curvature $R$,

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.1.4}$$

The Christoffel symbol $\Gamma^\lambda_{\mu\nu}$ is,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{2.1.5}$$

In view of this result it is symmetric with respect to two lower indices.

Einstein's equations determine the geometry of spacetime based on the matter content in that spacetime and motion of matter is determined by this geometry. Moreover, the motion of matter is determined by properties of geometry and is built in the Einstein equations. Simply put, matter tells geometry how to curve while, geometry tells matter how to move. In this way geometry ceases to be just the main where physics happens.

## 2.2 Black Holes

In modern cosmology it is believed that black holes forms from the collapse of stars. These formations are described as follows: that as long as stars emitting heat and light into space, they are able to support themselves against their own inward gravity with the outward pressure generated by heat from nuclear reactions in their deep interiors. As they exhaust nuclear fuel, its unbalanced self gravitational attraction causes it to collapse. Then, if a burned out star has a mass larger than about twice the mass of our sun no amount of additional pressure can stave off total gravitational collapse. For a non rotating collapsed star, the size of the resulting black hole is proportional to the mass of the parent star [20].

Furthermore, more standard theorems are governs the properties of four dimensional black hole solutions of general relativity in either a vacuum or coupled to an electromagnetic field. Some of these solutions are Schwarzschild Black Hole, Reissner-Nordstrom Black Hole and Kerr Black Hole. In particular, such black holes are either static and spherically symmetric, or rotating and axisymmetric [21].

### 2.2.1 Schwarzschild solution and its implication

The Schwarzschild spacetime is one of the unique solutions to the Einstein equations corresponds to a metric that describes the gravitational field exterior to a static, spherical and uncharged mass without angular momentum.

**Birkhoffs Theorem**

The Birkhoffs Theorem is a theorem of general relativity which states that all spherical gravitational fields whether from a star or from a black hole are indistinguishable at large distances. A consequence of this assumed as the purely radial changes in a spherical star do not affect its external gravitational field. That the Schwarzschild geometry is relevant to gravitational collapse follows from Birkhoffs theorem. The geometry of a given region of spacetime be spherically symmetric and done as a solution to the Einstein field equations in vacuum. This geometry is necessarily a piece of the Schwarzschild geometry. In particular, Birkhoffs theorem described as which implies, if a spherically symmetric source like a star changes its size, however does so always remaining spherically symmetric, then it cannot propagate any disturbances into the surrounding space [22].

### Event Horizon

The central, even astonishing property of the Schwarzschild horizon is anything that crosses it cannot get back outside it. The definition of a general horizon (called an event horizon) focuses on this property [23]. An event horizon described as the boundary in spacetime between events that can communicate with distant observers and events that cannot, see figure (2.1). This definition assumes that distant observers exist, the spacetime is asymptotically flat. And it permits the communication to take an arbitrarily long time. An event is considered to be outside the horizon provided it can emit a photon in even just one special direction that eventually makes it out to a distant observer. The most important part of the definition to think about is that the horizon is a boundary in spacetime, not just in the space defined by one moment of time.

Event horizon is a three dimensional surface that separates the events of spacetime into two regions: trapped events inside the horizon and untrapped events outside.

Since no form of communication can go faster than light, the test of whether events can communicate with distant observers is whether they can send light rays, that is whether there are null rays that can get arbitrarily far away. As the boundary between null rays that can escape and null rays that are trapped, the horizon itself is assumed as composed of null world lines. These are the marginally trapped null rays, the ones that neither move away to infinity nor fall inwards. By definition these marginal null rays stay on the horizon forever, because if a ray were to leave it toward the exterior or interior, then it would not mark the horizon.

This definition fits the Schwarzschild horizon which is static and unchanging, but when we consider dynamical situations there are some surprises. The formation of a horizon from a situation where there is initially no black hole illustrates well the dynamical nature of the horizon. Considering the collapse of a spherical star is thought to form a black hole. In the end there is a static Schwarzschild horizon, but before that there is an intermediate period of time in which the horizon is growing from zero radiuses to its full size.

**Singularity theorem**

In the early 1960s, Penrose applied global geometrical techniques to prove a famous series of singularity theorems. These showed that in realistic situations an event horizon (a closed trapped surface) will be formed and that there must exist a singularity within this surface, i.e. a point at which the curvature diverges and general relativity ceases to be valid. The singularity theorems were important in convincing people that black holes must form in nature [24].

Generally horizon and singularity are the identity of black hole which describe most

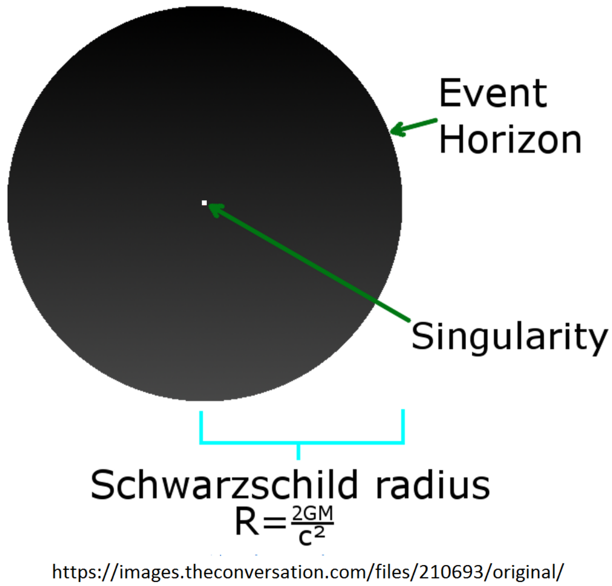


Figure 2.1: Black hole horizon and singularity

properties of black hole. The geometry of a spherical symmetric vacuum, i.e. vacuum spacetime outside the spherical black hole is the Schwarzschild geometry can be described in terms of the Schwarzschild metric,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.2.1}$$

which was derived originally as the external field of a static star, with non zero components of,

$$g_{tt} = -f(r),\ g_{rr} = f(r)^{-1},\ g_{\theta\theta} = r^2,\ and\ g_{\varphi\varphi} = r^2 \sin^2\theta. \tag{2.2.2}$$

Where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \tag{2.2.3}$$

is the metric on the unit two-sphere,

$$f(r) = (1 - \frac{2GM}{r}), \tag{2.2.4}$$

where $G$ is the gravitational constant, $M$ is the black hole mass, and (c = 1) is the speed of light in vacuum. The metric equation (2.2.1) is a static solution and the metric is asymptotically flat.

The Schwarzschild geometry illustrates clearly the highly non-Euclidean character of spacetime geometry when gravity becomes strong. Furthermore, it illustrates many of techniques one can use to analyze strong gravitational fields. The Schwarzschild spacetime, whose metric is given in equation (2.2.1), shows the failure of coordinates which have an obvious interpretation in one region of the spacetime (the region for which $r > 2GM$ ), but not in another (the region for which $r < 2GM$).

Thus the metric coefficients diverge at $r = 0$ and $r = 2GM$, $f(r)$ is obviously coordinate dependent. So a metric divergence may just be a coordinate singularity, originating from the breakdown of the employed coordinate system. The singularity at $r = 0$ turns out to be a true curvature singularity. A sign of this fact is that the coordinate-independent scalars can be constructed from the Riemann tensor diverges. But at $r = 2GM$ is not true singularity. By transforming to the Eddington-Finkelstein coordinates we can show that at $r = 2GM$ the spacetime is perfectly regular and overcoming this obstacle. In these coordinates it also becomes that the hypersurface located at $r = 2GM$ is an event horizon and we will see about this in the following section.

### 2.2.2 Other metrics from Schwarzschild metric

#### Eddington Felkilstein coordinate

As mentioned in section [2.2.1], at first the metric was thought to be singular at the Schwarzschild radius ($R_s$)

$$R_s = \frac{2GM}{r}, \tag{2.2.5}$$

but the coordinate transformation found by Eddington showed that it is possible to move the singularity on the Schwarzschild radius. However Eddington shown as which noted that $R_s$ seems smaller than a radius of any astronomical object and hence plays no role in nature [25]. By consider a light ray propagating in the radial direction that with $\theta$ and $\varphi$ are constant and $ds^2 = 0$. Then the Schwarzschild metric equation (2.2.1) can be write in the form

$$ds^2 = 0 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2, \tag{2.2.6}$$

and on account of this result, one can readily get

$$\frac{dt}{dr} = \pm\,(1 - \frac{2GM}{r})^{-1}\,. \tag{2.2.7}$$

This equation shows for large $r$ the slope $\frac{dt}{dr} = \pm 1$, as it would be in flat space, while as $r$ approached to the $r = 2GM$ we get $\frac{dt}{dr} = \pm\infty$. Thus, shows a light ray never seems to get there, at least in this coordinate system; instead it seems to asymptote to this radius. This is really an illusion; the light ray (a massive particle) actually has no trouble reaching $r = 2GM$. But an observer far away would never be able to tell. If we stayed outside while an intrepid observational general relativist dove into the black hole, sending back signals all the time, we would simply see the signals reach us more and more slowly.

As infilling astronauts approach $r = 2GM$, any fixed interval $\triangle\tau_1$ of their proper time corresponds to a longer and longer interval $\triangle\tau_2$ from our point of view. This continues forever; we would never see the astronauts cross $r = 2GM$, we would just see them move more and more slowly (become redder and redder, almost as if they were embarrassed to have done something as stupid as diving into a black hole). Indeed, we would never be able to see the in falling astronauts reach $r = 2GM$. But the astronauts will still be there. To show this, the best way is to switch to a different coordinate system, which is better behaved at $r = 2GM$. Equation (2.2.7) shows that the problem with the current coordinates is that $\frac{dt}{dr} \to \infty$ along radial null geodesics which approach $r = 2GM$. Thus, progress in the $r$ direction becomes slower and slower with respect to the coordinate time $t$. This suggests a way to fix it replace the $t$ with a coordinate which moves more slowly along null geodesics. The first thing solving equation (2.2.7), and gives

$$t = r^* + C\,, \tag{2.2.8}$$

where $C$ is constant, $r^*$ is known as the tortoise coordinate and it defined as

$$dr^* = \frac{dr}{1 - \frac{2GM}{r}}\,, \tag{2.2.9}$$

where $r$ is the Schwarzschild radial coordinate. Equation (2.2.9) indicates that $dr^* \to dr$ in the limit of large $r$. So the tortoise coordinate is approximately equal to the Schwarzschild radial coordinate at distances far from the black hole. Equation (2.2.9) also indicates that $dr^*$ blows up near $r = 2GM$. Integrating both sides of equation (2.2.9),

$$r^* = r + 2GMln|\frac{r}{2GM} - 1| + C\,. \tag{2.2.10}$$

Note that the natural logarithm is undefined for $r \leq 2GM$, which means that the tortoise coordinate only describes the spacetime geometry outside of the black hole. This gives $r^* = r^*(r)$ which indicate that the tortoise coordinate ranges over $r^* = (-\infty, +\infty)$, the tortoise coordinate is defined non-uniquely. And since $r^*$ can take on any value on the real line, It is free to make $C$, in equation (2.2.9) equal to any real number that wanted. The best way is to choose $C = 0$,

$$r^* = r + 2GMln[\frac{r}{2GM} - 1]\,. \tag{2.2.11}$$

The reason for defining $r^*$ such that, to satisfies equation (2.2.9), because it can be used to put the Schwarzschild metric's time-radial component in a form that is conformally flat. Squaring both sides of equation (2.2.9) allows us to rewrite the Schwarzschild metric the radial component as

$$g_{rr} = \frac{dr}{1 - \frac{2GM}{r}} = (1 - \frac{2Gm}{r})dr*^2 \tag{2.2.12}$$

Substituting this into the Schwarzschild metric in equation (2.2.1) gives

$$ds^2 = -(1 - \frac{2GM}{r})(dt^2 - dr*^2) + r^2 d\Omega^2 \,. \tag{2.2.13}$$

The Eddington-Finkelstein coordinates $u$ and $v$ are lightcone coordinates defined with respect to Schwarzschild coordinate time $t$ and tortoise coordinate $r^*$,

$$u = t - r^* \Rightarrow t = u + r^* \tag{2.2.14}$$

and

$$v = t + r^* \Rightarrow t = v - r^* \,, \tag{2.2.15}$$

with $u$ and $v$ corresponding to outgoing and ingoing radial lightlike geodesics, respectfully. By Squaring both sides, taking differentials of the equation (2.2.16) and by taking the value of $dr^*$ from equation (2.2.9),

$$ds^2 = -(1 - \frac{2GM}{r})du^2 + 2dudr + r^2 d\Omega^2, \tag{2.2.16}$$

is the line element for the outgoing Eddington-Finkelstein coordinates.

Also, by applying the same procedure to equation (2.2.15) for ($dt = dv + dr^*$) we can get,

$$ds^2 = -(1 - \frac{2M}{r})dv^2 - 2dvdr + r^2 d\Omega^2, \tag{2.2.17}$$

is the line element for the ingoing Eddington-Finkelstein coordinates.

In both equations (2.2.16 and 2.2.17) of the Eddington-Finkelstein coordinates the light cones do not fold up at $r = 2GM$ but tilt over, so that for $r > 2GM$ and $r < 2GM$ only movement in the direction of increasing and decreasing $r$, towards the singularity at $r = 0$ is allowed.

### 2.2.3 Hawking Radiation and its implication

In classical theory thought as black holes can only absorb and not emit even light. In 1974 Hawking startled the physics communities by providing that black holes are not black and they radiate energy continuously [26]. The relation between black hole entropy and horizon area together with the first law of black hole mechanics described as indicates that black holes do have a temperature ($T$) that should be proportional to surface gravity ($\kappa$). As

$$T = \frac{\hbar\kappa}{2\pi k_B}, \tag{2.2.18}$$

where is $\hbar$ the reduced Planck constant and $k_B$ is the Boltzmann constant. If the black hole is immersed in black body radiation of lower temperature then the generalized second law is violated, unless the black hole also emits radiation.

The quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with temperature. So, we accepted that the black hole can radiate with wavelength ($\lambda$) and the only length-scale in the problem is the size of the horizon. Assume for photon with wavelength equal to the radius of the black hole has (ignoring the curvature of the spacetime) an energy equal to the

$$E = hv = h\frac{c}{\lambda} = hc\frac{c^2}{2GM}, \tag{2.2.19}$$

where, $E, h, v$ are energy, Planks constant, frequency of photon respectively, and $\lambda = R = 2GM$ is Schwarzschild radius, see section [2.2.1], which is approximately to the wavelength. When black holes are absorbing every that falls on them, then their temperature ($T$) should be at least approximately related to the energy, by setting

$$E = k_B T \tag{2.2.20}$$

On account of Equation (2.2.19) and (2.2.20) we see that

$$T = \frac{hc^3}{2GMk_B} \tag{2.2.21}$$

This thermal emission leads to show a decrease in the mass of the black hole and to its eventual disappearance [27]. By using the value of $\kappa$ in equation (2.2.18) the Hawking temperature ($T_H$) of a black hole is

$$T_H = \frac{\hbar c^3}{8\pi GMk_B} \tag{2.2.22}$$

Then, the black hole temperature is straight forward to calculate the black hole entropy that from the first law of the BH mechanics, which is essentially the energy conservation relation, related to the change of BH mass ($M$) with the change of its entropy ($S_{BH}$), electric charge ($Q$), and angular momentum ($J$) as

$$dM = T_h dS_{BH} + \Phi dQ + \Omega dJ, \tag{2.2.23}$$

$\Omega$ is the angular velocity and $\Phi$ is the electrostatic potential. So, for nonrotating uncharged BHs, the entropy ($S$) has the simple form

$$dS_{BH} = \frac{dM}{T_h} = 8\pi GMdM \tag{2.2.24}$$

Therefore,

$$dS_{BH} = 8\pi GMdM = d(4\pi GM^2), \tag{2.2.25}$$

where $R = 2GM$ then

$$dS_{BH} = d(4\pi GM^2) \tag{2.2.26}$$

$$S_{BH} = \frac{\pi R^2}{4} = \frac{A}{4} \tag{2.2.27}$$

is known as Bekenstein-Hakwing entropy, where $A = \pi R^2$ is surface area of black hole. Thus, shows black hole thermodynamics describes the behavior of a black hole in terms of the laws of thermodynamics by relating mass to energy, horizon to entropy and surface gravity to temperature. Hawking radiation would only occur if the black hole is warmer than the environment surrounding it; because the black hole would need to radiate away energy in order to reach thermal equilibrium and these black holes are hence unstable.

It is described in the presence of an event horizon, thought occasionally one member of a virtual pair will fall into the black hole while its partner escapes to infinity. The particle that reaches infinity will have a positive energy, but the total energy is conserved; therefore the black hole has to lose mass. This the escaping particles assumed as Hawking radiation. It is not a very big effect and the temperature goes up as the mass goes down [28].

Specifically, Hawking's calculations indicated that black hole evaporation via Hawking radiation does not preserve information. Because Hawking assumed a fixed, curved background spacetime geometry and that the black hole's mass remains constant as it radiates. Today, many physicists believe that the holographic principle demonstrates that Hawking's conclusion was incorrect, and that information is in fact preserved. Hawking derivation employed field modes of arbitrarily high frequency near the black hole horizon, although these do not appear in the final result.

# Chapter 3

# Black Hole Radiation via Quantum Semi-Classical Approach

After the discovery of quantum black hole thermal radiance by Hawking it became pretty clear to concern the interface of gravity, quantum theory and thermodynamics. This describes as a radiating black hole loses its' energy and therefore shrinks, evaporating away to a fate which is still debated. It is a long debated question how the thermal nature of Hawking radiation can be reconciled with unitarity (information loss puzzle), unitarity is a milestone of classical and quantum physics. Also many new ideas came out from the recognition that quantum field theory implied a thermal spectrum and the principle of black hole complementarity aimed to reconcile the apparent loss of unitarity implied by the Hawking process with the rest of physics as seen by external observers.

Nevertheless, in the semi-classical result that the radiation caused by the changing metric of the collapsing star approaches a steady outgoing flux at large times, implying a drastic violation of energy conservation. Energy conservation requires fixing the total energy of the space-time before and after particle emission. Since black hole mass and volume are linked together, a mass reduction due to the emission of a particle translates into a size contraction; so it is some worry how to deal with quantum fluctuations of the metric originating from such contraction.

In this case no graviton quantization is involved or, said in other words, passing from different spherically symmetric configurations does not produce gravitational waves. As a consequence, the only degree of freedom remained in the problem is the position of the emitted particle (actually a thin shell). Thus, to keep things as simple as possible, it adapted restrict to consider uncharged, static, spherically symmetric black holes emitting neutral matter.

The quantum field theory in curved spacetime is a semiclassical theory, in which used to study quantum fields on a fixed (i.e. classical) background. The semiclassical theory of black holes and in particular the Hawking effect is elegant but, it is full of conceptual problems. Therefore, the study of semiclassical black holes is necessary to reveals the tension and conflicts between the theories of general relativity and quantum mechanics. Somebody could be briefly understand two of these issues through black hole radiation [29].

## 3.1 Particles trip to the black hole in the FLRW Background

The method of describing a spherical contraction of a uniformly distributed dust star is making a physically reasonable junction of the two different spacetimes corresponding to the interior and exterior regions of the collapsing body. The interior and exterior solutions are given by the Freidmann-Lemaire-Roberttos-Walker (FLRW) metric and the Schwarzschild metric described in different coordinate systems [30].

### 3.1.1 Painleve metric

Assume from far distance to the black hole we need background metric that is from Freidmann-Roberttos-Walker (FRW) metric. Consider a generic FRW space-time, namely one with constant curvature spatial sections. The FRW line element can be written as

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1+kr^2} + r^2 d\Omega^2] \qquad (3.1.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and $r$ is measured in units of the curvature radius and as usual, $k = 0, -1, +1$ labels flat, open and closed three-geometries, respectively. The important case $k = 0$ is deserves as special attention in this study with the case of flat geometry. Then the above FRW metric written as

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2 d\Omega^2]\,, \qquad (3.1.2)$$

where $a(t)$ is the scale factor.

Now to study the black hole radiation nobody can stay near the horizon. So at large distance from the black hole (for any mass) the metric is given approximately by the

FWR metric and when expansion is ignored one obtain Schwarzschild metric which is isotropic coordinate (can be used to model spacetime outside a black hole). Here we have to produce the mix of non-comoving radial coordinate and a comoving radial coordinate by combining the Schwarzschild metric and FWR metric to get suitable metric. Let take the wave front velocity whose radius is

$$R(t,r) = a(t)r\,, \tag{3.1.3}$$

where the expansion is ignored $a(t) = a(o)$, one obtain the Schwarzschild metric. Then,

$$dR = \dot{a}rdt + adr, \tag{3.1.4}$$

from which follows

$$dR = R(\frac{\dot{a}}{a})dt + adr = RHdt + adr, \tag{3.1.5}$$

where $H = \frac{\dot{a}}{a}$ and known as Hubble constant. It then follows that

$$dr = \frac{dR - RHdt}{a}\,. \tag{3.1.6}$$

From this result, one can readily get

$$dr^2 = \frac{1}{a^2}[dR^2 + R^2H^2dt^2 - 2RHdtdr]. \tag{3.1.7}$$

By substitute the value of $r^2$ and $dr^2$ from (3.1.3) and (3.1.7) in equation (3.1.2) we get

$$ds^2 = -[1 - R^2H^2]dt^2 + [dR^2 - 2RHdtdR + R^2d\Omega^2], \tag{3.1.8}$$

is FRW metric in comoving coordinates. Also from the FLRW model of cosmology the equation with non cosmological constant their is,

$$H^2 = \frac{8\pi G}{3}\rho \tag{3.1.9}$$

where $\rho$ is energy density which is $\rho = \frac{M}{\frac{4\pi}{3}R^3}$ and for Schwarzschild black hole that static and without cosmological constant ($\Lambda = 0$). In view of this, we have

$$H^2 = \frac{2GM}{c^2 R^3}, \tag{3.1.10}$$

from which follows

$$R^2 H^2 = \frac{2GM}{c^2 R}, \tag{3.1.11}$$

It then follows that

$$2RH = 2\sqrt{\frac{2GM}{c^2 R}} \tag{3.1.12}$$

Now the mixed metric become

$$ds^2 = -(1 - \frac{2GM}{c^2 R})dt^2 + dR^2 - 2\sqrt{\frac{2GM}{c^2 R}}dRdt + R^2 d\Omega^2 \tag{3.1.13}$$

is the Painleve metric and it can readily give static metric. This mixed metric is being used to drive the intended black hole radiation. In fact this metric is further worked out to be the most spherically static Einstein field equation solution in the absence of cosmology. In equation (3.1.13) by setting $ds^2 = 0$ to make the geodesics lightlike, and $d\Omega = 0$ to make the geodesics radial, this equation can be written as,

$$0 = -(1 - \frac{2GM}{c^2 R})dt^2 + dR^2 - 2\sqrt{\frac{2GM}{c^2 R}}dRdt. \tag{3.1.14}$$

Then by dividing equation (3.1.14) by $dt^2$ gives a function that is quadratic in $\frac{dR}{dt}$

$$(\frac{dR}{dt})^2 - 2\sqrt{\frac{2GM}{R}}\frac{dR}{dt} - (1 - \frac{2GM}{R}) = 0, \tag{3.1.15}$$

which can be solved using the quadratic formula to give

$$\frac{dR}{dt} = \pm 1 + \sqrt{\frac{2GM}{R}} \tag{3.1.16}$$

where $\pm$ corresponds to outgoing/ingoing radial null geodesics.

### 3.1.2 WKB Approximation

The Wentzel, Kramers, and Brillouin (WKB) approximation is used to find approximate general solutions to linear differential equations and allows the step of solving the differential equation to be skipped. The WKB approximation can be applied to differential equations that have solutions with either constant, or slowly varying coefficients. The assumption of slowly varying coefficients in the present derivation is a result of truncating the action after a first order approximation. So for derivative of the wavefunction's one can drop a second or derivative of the wavefunction's when approximation is made. This very straightforwardly explains why the coefficient is assumed to be slowly varying. It is adapted to drop the time-dependence of the wavefunction because the spacelike contribution to the tunneling event occurs instantaneously. Therefore the wavefunction is written as

$$\psi(x) = Ae^{\frac{i\Im(x)}{\hbar}} \tag{3.1.17}$$

where $\Im$ is classical action. The general plane wave solutions in equation (3.1.17) can be inserted into the time-independent Schrodinger equation (TISE),

$$-\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{3.1.18}$$

which describes the probability distribution of $\Psi$ in the presence of a potential energy distribution $V(x)$. Solutions to equation (3.1.18) is the form $\psi \sim e^{\frac{i}{\hbar}\Im(x)}$. Differentiating $\psi$ with respect to position,

$$\psi^{'} = \frac{i}{\hbar}\Im^{'}\psi,\ \psi^{''} = (\frac{i}{\hbar}\Im^{''} - \frac{1}{\hbar^2}\Im^{'2})\psi \tag{3.1.19}$$

and substituting its derivatives into the TISE (3.1.18) it like,

$$-\frac{\hbar^2}{2m}[\frac{i}{\hbar}\Im^{''} - \frac{i}{\hbar^2}\Im^{\prime 2}] - [E - V(x)] = 0. \tag{3.1.20}$$

Then, by substituting the $E = \frac{p^2}{2m} + V(x)$ can be rearranged to obtain $p^2 = 2m(E - V(x))$, which can then be substituted into equation (3.1.20) to give

$$-\frac{\hbar^2}{2m}[\frac{i}{\hbar}\Im'' - \frac{i}{\hbar^2}\Im'^2] - [\frac{p^2}{2m}] = 0. \quad (3.1.21)$$

It then follows that

$$i\hbar\Im'' - \Im'^2 - p^2 = 0. \quad (3.1.22)$$

From this point, the WKB approximation can be derived by Taylor expanding the classical action $\Im(x)$ in powers of $\hbar$ and then truncating the power series after linear order. This method of approximation is called semiclassical, because the quantum mechanical effects are retained only to linear order in $\hbar$. It's worth pointing out that this is a very reasonable approximation $\hbar \sim 10^{-34}, \hbar^2 \sim 10^{-64}, \hbar^3 \sim 10^{-102}, ...$ by Taylor expanding the classical action in powers of $\hbar$ and collecting like-powers equation 3.1.22, as

$$-(p^2 - \Im_0^{'})^2 + (i\Im_0^{''} - 2\Im_0^{'}\Im_1^{'})\hbar(i\Im_1^{''} - \Im_2^{'} - 2\Im_O^{'}\Im_2^{'})\hbar^{'2} + ... = 0 \quad (3.1.23)$$

Since the right-hand side of equation (3.1.23) is equal to zero, the coefficient of each power in $\hbar$ on the left-hand side must also be equal to zero. Taking the zeroth order term in equation (3.1.22), one can readily obtain

$$p^2 = -\Im_0^{'2} \Rightarrow \Im_{0(x)} = \pm \int_{x_0}^{x} p(x)dx. \quad (3.1.24)$$

The first order term in equation (3.1.23), takes the form

$$\frac{i}{2}\Im_0^{''} = \Im_0^{'}\Im_1^{''}, \quad (3.1.25)$$

$$\frac{i}{2}p^{'} = p\Im_1^{'} \quad (3.1.26)$$

and

$$\frac{i}{2}\int\frac{dp}{p}=\int dS_1 \Rightarrow \Im_1(x)=\frac{i}{2}\ln|p|. \tag{3.1.27}$$

Equations (3.1.24) and (3.1.25) can be used to find a first order semiclassical approximation for $\psi(x)$, that

$$\psi(x)=exp[\frac{i}{\hbar}\Im(x)]=exp[\pm\frac{i}{\hbar}\int_{x0}^{x}p(x)dx-\frac{1}{2}\ln|p|] \tag{3.1.28}$$

after taking the second term as a coefficient, gives approximate general solutions to the time-independent Schrodinger equation, and the WKB approximation for $\psi_(x)$, is found to be

$$\psi(x)\approx\frac{C\pm}{\sqrt{|p(x)|}}e^{\pm\frac{i}{\hbar}\int_{x0}^{x}p(x)dx},\ p(x)\equiv\sqrt{2m(E-V(x))} \tag{3.1.29}$$

where, $m, E$ and $p(x)$ are the mass, total energy, and classical momentum of the tunneling particle. $V(x)$ is the potential barrier that the particle must tunnel through. For the case of a particle tunneling across the event horizon, $V(x)$ can be thought of as being associated with the gravitational potential energy barrier that the particle must overcome in order to escape to future lightlike infinity. For tunneling model of Hawking radiation we assume the particle must tunnel against an energy barrier that is determined by the particle's own total energy. Note that the momentum $p(x)$ in equation (3.1.29) implies that both energy and momentum are conserved if $V(x)$ is made to be equal to the particle's own self-energy. The integral in the exponent of $\psi(x)$, from equation (3.1.29), is equal to the classical action

$$\Im(x)=\int_{x_0}^{x}p(x^{'})dx^{'} \tag{3.1.30}$$

$\Im(x)$ is real-valued when the particle is in a classically allowed region, because $E > \Im(x)$ implies $p(x)$ is real. $\Im(x)$ is imaginary when the particle is in a region with

$V(x) > E$, since $\Im(x) > E$ implies $p(x)$ is imaginary. If the particle is massless then $p(x)$ must be expressed in a way that does not explicitly assume the particle to have a mass $m$. This can be done by recalling that quantum mechanics has a Hamiltonian formalism. This means that $p(x)$ can be viewed under the more general context as the particle's canonical momentum

$$p(x) = \frac{\partial L}{\partial \dot{x}}, \tag{3.1.31}$$

where $L$ is the Lagrangian.

## 3.2 Black Hole Radiation with the WKB approximation

### 3.2.1 Radial null geodesic method

The quantum description of a black hole (BH), namely the Hawking radiation (HR) is closely related to the existence of an event horizon to the BH. The derivation of Hawking that BH evaporates particles was based on quantum field theory. Hartle and Hawking subsequently derived the BH temperature at the semiclassical level using the Feynmann path integral. From the other, Hawking radiation is the most reliable result of quantum gravity derived with semi-classical techniques. The mathematical complexity forces to develop semi-classical approaches for studying BH radiation. However, these semi-classical techniques were classified into two approaches [31]

i. the tunneling approach of Parikh and Wilczek, which referred to as the radial null geodesic method and

ii. the standard Hamilton-Jacobi (HJ) method (known as complex path integral formalism) by Padmanabhan et al.

In this study we used the first approach (radial null geodesic method), because the second approach is deal with the tunnelling of massless particles beyond the semi-classical approximation by HamiltonJacobi (HJ) method. In semiclassical tunneling analysis the radial null geodesic method is a common way of evaluating Hawking radiation. The energy conservation in tunneling of a thin shell from the hole is the main ingredient for this approach. The imaginary part of the action from the s-wave emission is connected to the Boltzmann factor for emission to relate with Hawking temperature (HT).

As mentioned at the end of section [3.1.2] equation (3.1.31) does not make any assumptions about the particle's mass. A particle tunneling across the event horizon is assumed to travel along the radial coordinate axis. Indicating this with the replacement $x^{'} \rightarrow r$ into equation (3.1.30) and relabeling the momentum to be interpreted as a component of a momentum 4-vector,

$$\Im = \int_{r_i}^{r_f} p(r)dr \tag{3.2.1}$$

During radiation a decrease in a black hole's mass from $M \rightarrow M - \omega$ is necessarily accompanied by a corresponding decrease in its Schwarzschild radius, from $R_s = 2M \rightarrow 2(M - \omega)$,(here, $\omega$ represent the change in mass of BH during radiation). Consistent with this total change in radius, the present model treats the outgoing particle as initially occupying the finite and well-defined region of space between

$$r_{in} = 2GM - \omega \quad \text{and} \quad r_{out} = 2GM$$

in an s-wave configuration. The classically forbidden region is taken to be the spherical surface at $r = 2(M - \omega)$, thus having an infinitesimal width.

When the outgoing wave is traced back towards the horizon its wavelength as measured by local fiducial observers is ever-increasingly blue-shifted. Near the horizon the radial wavenumber approaches infinity and the point particle or WKB approximation is justified. The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from $r_{in}$ to $r_{out}$ can be expressed as

$$Im\Im = Im\int_{r_{in}}^{r_{out}} p(r)dr = Im\int_{r_{in}}^{r_{out}}\int p^{'}(r)dr \tag{3.2.2}$$

For computational convenience we can treat the particle as a rigid object, which undergoes an infinitesimal rigid motion ($\ni$) during integration. Because, treating the particle as a rigid object makes sense computationally that when one point on the particle moves, all of the rest of the points on the particle have to move with it. Then $r = 2G(M-\omega)-\ni$ is smallest $r$ value occupied by the outgoing particle before undergoing a rigid motion.

When the particle is treated as a spherical shell of width $2\omega$ this is the largest $r$ coordinate the particle can be located at while still objectively be contained inside of the black hole. During integration, this point is treated as the particle's location, and the particle is translated via rigid motion from

$$r = 2G(M-\omega)-\ni, \quad \text{to} \quad r = 2G(M-\omega)+\ni\ ,$$

correspond to the radial point along the s-wave's radial width that has the smallest $r$ value (i.e. the inner most radial coordinate of the outgoing particle). To change variable of equation (3.2.2) from momentum to energy, and switch the order of an other equation we used Hamilton's equations that,

$$r^{'} = \frac{\partial H}{\partial p(r)} = \frac{\partial}{\partial p(r)}[K(pr)+U(r)] = \frac{\partial K}{\partial p(r)} \tag{3.2.3}$$

which follows

$$\tfrac{\partial U}{\partial p(r)} = 0.$$

where $K$ and $U$ represents kinetic and potential energy respectively. Then,

$$p^{'}(r) = -\frac{\partial H}{\partial r}. \tag{3.2.4}$$

Since $H$ is conserved, the total differential

$$dH = 0 \ , \quad \text{and}$$
$$dp(r) \neq \tfrac{dH}{r^{'}}$$

depend on this, one can define

$$\tfrac{\partial K}{\partial p(r)} = \tfrac{dK}{dp(r)} \ ,$$

because the potential energy is independent of $p(r)$. This means that equation (3.2.3) rearranged for

$$dp(r) = \frac{dT}{r^{'}} \tag{3.2.5}$$

The Hamiltonian can be expressed in terms of the system's scalar quantities as

$$H = K + U = \omega + (M - \omega) \tag{3.2.6}$$

Using the Dirac-sea-like interpretation of pair creation, the $\omega$ in $U = M - \omega$ should be thought of as a correction to the black hole's mass that is extracted by the positive energy particle's creation. Then equation (3.2.6) shows

$$dp(r) = \frac{d\omega}{r^{'}} \tag{3.2.7}$$

which can be substituted into equation (3.2.2) to give

$$\Im = \int_{r_{in}}^{r_{out}} \int_{0}^{\omega} \frac{d\omega^{'}}{\dot{r}} dr \tag{3.2.8}$$

From equation(3.1.16) of mixed metric $r \to R$ and for our case $M \to M - \omega$

$$\Im = \int_{R_{in}}^{R_{out}} \int_0^{\omega} \frac{d\omega^{'}}{1 + \sqrt{\frac{2G(M-\omega)}{R}}} dR\,. \tag{3.2.9}$$

By substitute the value of $R_{in}$ and $R_{out}$,

$$\Im = \lim_{\ni \to 0^+} \int_{2G(M-\omega)-\ni}^{2G(M-\omega)+\ni} \int_0^{\omega} \frac{d\omega^{'}}{1 + \sqrt{\frac{2G(M-\omega)}{R}}} dR \tag{3.2.10}$$

This integrand is singular at $R = 2G(M - \omega)$. Also notice that the bounds of integration over $R$ are separated by an infinitesimal distance $2\epsilon$. The fact that $R$ is integrated over an infinitesimal distance provides justification for $\omega$ to be treated as a constant with respect to integration over $R$. This is important because it allows us to interchange the order of integration in equation (3.2.10) without having to worry about $\omega$ appearance in the bounds of integration over $R$. Equation (3.2.10) become

$$\Im = \int_0^{\omega} (\lim_{\ni \to 0^+} \int_{2G(M-\omega)-\ni}^{2G(M-\omega)+\ni} \frac{dR}{1 - \sqrt{\frac{2G(M-\omega')}{R}}}) d\omega^{'} \tag{3.2.11}$$

Let'

$$R = u^2 \ \to u = \sqrt{2G(M - \omega')} \tag{3.2.12}$$

then,

$$\Im = \int_0^{\omega} d\omega^{'} (\lim_{\ni \to 0^+} \int_{2G(M-\omega)-\ni}^{2G(M-\omega)+\ni} \frac{2u^2 du}{u - \sqrt{2G(M - \omega')}}) d\omega^{'} \tag{3.2.13}$$

Also, let

$$u - \sqrt{2G(M - \omega')} = \ni e^{i\phi} \to du = i \ni e^{i\phi} d\phi \tag{3.2.14}$$

allows us to make a contour deformation. It can be done either as a contour integral over an open semicircule deformed into the lower quadrants of the complex plane or alternatively, we can to deform the contour into a closed and right-handed semicircle extending into the upper quadrants of the complex plane. Taking the open semicircular contour approach, the integral over $u$ in equation (3.2.13) can now be re-expressed as

$$\int_{u_{in}}^{u_{out}} \frac{2u^2 du}{u - \sqrt{2G(M-\omega')}} = (\lim_{\ni \to 0^+} 2 \int_{\pi}^{2\pi} \frac{(\ni e^{i\phi} + \sqrt{2G(M-\omega')})^2}{\ni e^{i\phi}} (i \ni e^{i\phi} d\phi)) \tag{3.2.15}$$

$$\int_{u_{in}}^{u_{out}} \frac{2u^2 du}{u - \sqrt{2G(M-\omega')}} = \lim_{\ni \to 0^+} 2i \int_{\pi}^{2\pi} (\ni e^{i\phi} + \sqrt{2G(M-\omega')})^2 d\phi \tag{3.2.16}$$

$$= 2i \int_{\pi}^{2\pi} (\sqrt{2G(M-\omega')})^2 d\phi$$

$$= 4iG(M-\omega') \int_{\pi}^{2\pi} d\phi$$

Finally, we arrive at

$$\int_{u_{in}}^{u_{out}} \frac{2u^2 du}{u - \sqrt{2G(M-\omega')}} = i4\pi G(M-\omega') \tag{3.2.17}$$

With this, equation ( 3.2.13) becomes

$$\Im = \int_0^{\omega} 4iG(M-\omega')d\omega' \; .$$

It then follow that

$$\Im = i4\pi G(M\omega - \frac{\omega^2}{2}) \,. \tag{3.2.18}$$

So the imaginary component of the action is given by

$$Im\Im = 4\pi G(M\omega - \frac{\omega^2}{2}) \,. \tag{3.2.19}$$

A negative energy particle propagating forward in time is equivalent to a positive energy particle propagating backward in time. The calculation for an ingoing negative energy particle can be made either way. The bounds of integration over $R$ for an ingoing particle are interchanged with respect to the bounds of integration for an outgoing particle. The ingoing particle starts at $R_{out} = 2GM+$ э and tunnels an infinitesimal distance $2\omega$, across the classically forbidden surface at $R = 2GM$, to $R_{in} = 2GM-$ э. A quick calculation for the negative energy ingoing particle can be made as follows. Let $p(R)^{-}$ represent the negative energy particle and $p(R)^{+}$ represent the positive energy particle. The action is

$$\Im = \int_{Rout}^{Rin} p(R)^{-} dR = \int_{Rin}^{Rout} p(R)^{+} dR, \tag{3.2.20}$$

from which follows

$$\Im = i4\pi G(M\omega - \frac{\omega^2}{2})\,. \tag{3.2.21}$$

Generally, which gives a transmission coefficient ( tunneling rate, $\Gamma$)

$$\Gamma \simeq e^{-2Im\Im} = e^{-8\pi G(M\omega - \frac{\omega^2}{2})} \tag{3.2.22}$$

First consider for small energy $\omega$, this reduces to

$$\Gamma \simeq e^{-2Im\Im} = e^{-8\pi G(M\omega)} \tag{3.2.23}$$

Secondly consider if the emitted particle takes all of the mass of the black hole with it. This would have a transmission rate of

$$\Gamma(\omega = M) \propto e^{-4G\pi M^2} \tag{3.2.24}$$

There can only be one of these outgoing states. But there are $e^{\Delta S_{BH}}$ form equation (2.2.25), where $S_{BH}$ is the Berkenstein-Hawking entropy, states in total, so the probability of finding that states is one in that number is

$$P(\omega = M) \propto e^{-\Delta S_{BH}} \tag{3.2.25}$$

$$P(\omega = M) \propto e^{\frac{-A}{4}} \tag{3.2.26}$$

$$P(\omega = M) \propto e^{-4G\pi M^2} \tag{3.2.27}$$

where $\Delta S$ is the difference of the Bekenstein-Hawking entropy between before and after the Hawking radiation. Since, the total entropy of a BH is given by

$$S_{BH} = \frac{1}{4} A_h \tag{3.2.28}$$

where $A_h = \pi R^2$, is called the area of the BH. Then

$$S_{BH} = \pi R_h^2 = 4\pi M^2 \tag{3.2.29}$$

# Chapter 4

# Result and Discussion

As we try to describe in section (2.2.1) the singular possibility of Schwarzschild metric of equation (2.2.1) the metric coefficients can be diverge at $r = 0$ and $r = 2GM$. Now we checked the true coordinate singularity as follows. The singularity at $r = 0$ turns out to be a true curvature singularity. A sign of this fact is that the coordinate-independent scalars can be constructed from the Riemann tensor diverges. As $r \to 0$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$, implying a true curvature singularity at $r = 0$. Also, let check singularity for $r = 2GM$: take for photon Equation (2.2.1), the static metric becomes

$$ds^2 = 0 = -f(r)dt^2 + f(r)^{-1}dr^2 \tag{4.0.1}$$

in which

$$f(r)^2 dt^2 = dr^2 \tag{4.0.2}$$

from which follows

$$\frac{dt}{dr} = \pm f(r)^{-1} = \pm(1 - \frac{2GM}{r})^{-1}. \tag{4.0.3}$$

This equation can be represent the slope of the light cones on a spacetime diagram of the ( $t, r$ ) plane as figure (4.1)

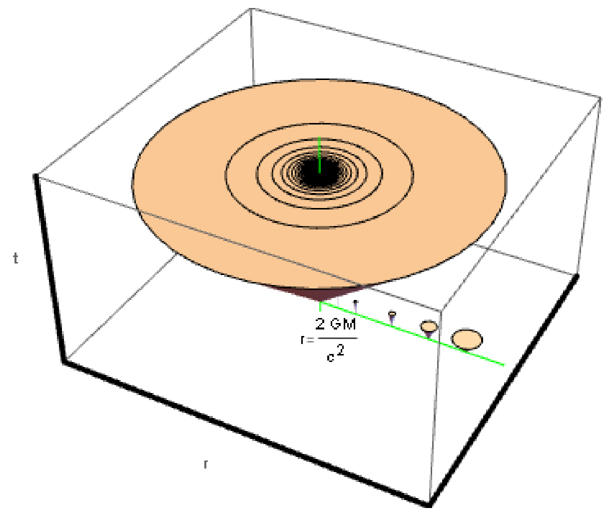


Figure 4.1: Light cones diagram depends on Schwarzschild radius.

For large $r$, slope $= \pm 1$, flat space and for $r = 2GM$, slope $\frac{dt}{dr} \pm \infty$, light cone closed up ( the central line of light cone ). But, here the singularity at $r = 2GM$, is turns out to be a coordinate singularity. This is failure of the standard coordinate system of Schwarzschild metric and is one of the most interesting challenges in the study of general relativity.

Therefore to study the Schwarzschild black radiation first we find the suitable metric depending on black hole event horizon properties. Through this we get Painleve metric as in Equation (3.1.13) which contains mixed metric of static and co-moving coordinate. Second in the semi-classical approach the tunneling probability is calculated directly from the principle of conservation of energy by calculating the imaginary part of the action in WKB approximation as equation (3.2.22).

Generally, When the factor $\omega^2$ is neglected in equation (3.2.22), the tunneling rate reduces to the black body radiation which is pure thermal that permits the leakage of the information from the system and expressed by a Boltzmann $e^{-\beta\omega}$ factor in which $\beta = \frac{1}{T}$. The existence of $\omega^2$ is due to the physics of energy conservation. Besides, it gives rise to a deflection from the pure thermal radiation of the BH and therefore leads to a information escaping from the BH. This phenomenon is significant on the resolution of the information loss problem.

When particles escape, the black hole loses a small amount of its mass and energy are related by Einstein's equation $E = mc^2$. The power ($P$) emitted by a black hole in the form of Hawking radiation can easily be estimated to the StefanBoltzmann power law as

$$P = A_s \sigma T_H^4, \tag{4.0.4}$$

where $P$ is the power (energy outflow), $A_s$ is Schwarzschild sphere surface area of Schwarzschild radius $R_s$, $\sigma$ is StefanBoltzmann constant and $\hbar$ is the reduced Planck constant.

Hence, from equation (2.2.22) we have

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \tag{4.0.5}$$

is Hawking radiation temperature. The Hawking temperature is defined to be the thermal temperature corresponding to the characteristic wavelength of a photon of Hawking radiation, that is detected by an observer located infinitely far away from the black hole. Combining the formulas for the Schwarzschild radius of the black hole with the equations (4.0.4) and (4.0.5)

$$\frac{dE}{dt} = -A_s \sigma (\frac{\hbar c^3}{8\pi G M(t) k_B})^4, \tag{4.0.6}$$

is Stefan Boltzmann-Schwarzschild-Hawking power law. The negative sign shows the decrement of black hole mass and $\frac{dE}{dt}$ is the energy outflow and $dE = c^2 dM$.

Moreover, we have

$$\frac{c^2 dM}{dt} = -A_s \sigma (\frac{\hbar c^3}{8\pi G M(t) k_B})^4 \tag{4.0.7}$$

$$\frac{c^2 dM}{dt} = -\pi R^2 \sigma (\frac{\hbar c^3}{8\pi G M(t) k_B})^4 \tag{4.0.8}$$

$$\frac{c^2 dM}{dt} = -\pi (\frac{2GM(t)}{c^2})^2 \sigma (\frac{\hbar c^3}{8\pi G M(t) k_B})^4 \tag{4.0.9}$$

$$M^2(t) \frac{dM}{dt} = -\sigma (\frac{\hbar c^4 c^6}{256\pi^3 G^2 k_B^4}) \tag{4.0.10}$$

Let, $\lambda = \sigma(\frac{\hbar c^4 c^6}{256\pi^3 G^2 k_B^4})$ then equation ( 4.0.10 ) becomes

$$\int_{2GM}^{2G(M-\omega)} M^2(t) dM = -\int_{t}^{t_H} \lambda dt, \tag{4.0.11}$$

is black hole mass as function of time. Where $t$ and $t_H$ is initial and final time of black hole during radiation. Also by using the value of $M(t)$ from equation (4.0.11) in equation (4.0.5)

$$T(t) = \frac{\hbar c^3}{8\pi G M(t) k_B}, \tag{4.0.12}$$

is black hole temperature as function of time.

The numerical results of equations (4.0.10) and (4.0.12) are shown as in Fig.(4.2) that shows the black hole radiation in time with respect to its mass and temperature.

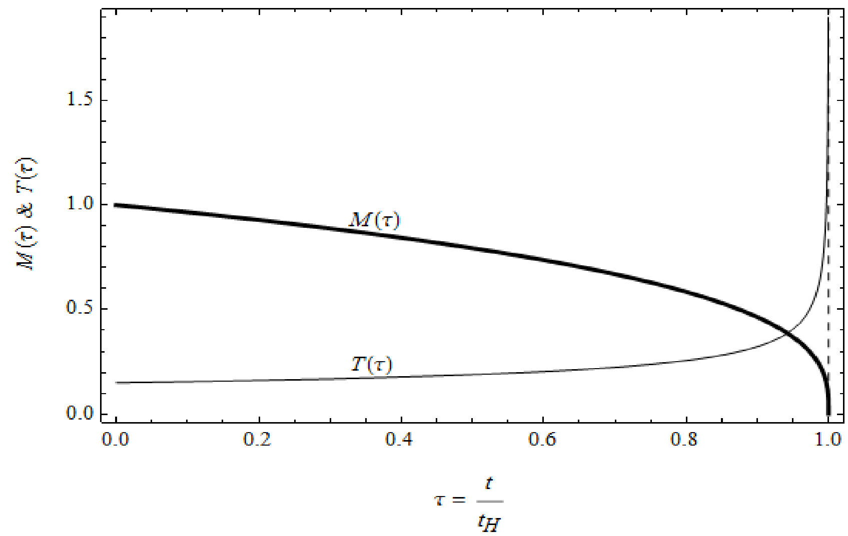


Figure 4.2: The relation between the mass and temperature of black hole as function of time. $\tau$ is represent full life-time of BH.

The Bold line represents the mass of black hole as function of time and the thin line represents temperature of the black hole as function of time. In equation (4.0.12) this thermal emission leads to show a decrease in the mass of the black hole and to its eventual disappearance. Also, the Fig.(4.2) informs the mass of the black hole continuously decreases, and at the end of black hole life time completely it can be disappeared. On other-hand, as the mass of black hole decreases inversely its temperature goes to infinity. This thermal emissions (energy as the form of radiation) are indirect not disappeared. It released to the space and increases the hole space entropy.

# Chapter 5

# Summary and Conclusion

Einstein field equation is the base one to review the Schwarzschild solution which indicates the static and spherically symmetric of black hole system without charge and cosmological constant. Objectively the painleve coordinate calculated in new technique is used to connect the classical and quantum theorem at the black hole horizon. The tunneling probability can be get directly from the principle of conservation of energy by calculating the imaginary part of the action in WKB approximation and solves the conservation problem of black hole radiation that described in Hawking quantum radiation. This radiation is propagational to the black hole temperature that black hole releases energy in the form of radiation. This causes the decrease of black hole mass and increases the black hole temperature. Imposing energy conservation means that the total space-time energy is fixed and one allows the black hole mass to fluctuate. The conclusion is that as black holes radiate the space energy is always conserved.

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