EXISTENCE OF THREE POSITIVE SOLUTIONS FOR SECOND ORDER UNDAMPED THREE POINT BOUNDARY VALUE PROBLEMS IN CONE BANACH SPACE

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

Prepared by:
JEMAL NESRU
Under the supervision of:
Wesen Legesse (Ph.D.)
Girma Kebede (M.Sc.)

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Jimma, Ethiopia

Declaration

I, the undersigned declared that, the thesis entitled **"existence of three positive solutions for second order undamped three point boundary value problem"** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged.
Name: Jemal Nesru
Signature: ——————————
Date: ——————————
The work has been done under the supervision of:
Name: Wesen Legesse (Ph.D.)
Signature: ——————————
Date: ——————————
Name: Girma Kebede (M.Sc.)
Signature: ——————————
Date: ——————————

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Abstract

This thesis is concerned with second-order three-point undamped non-linear boundary value problems. It also focused on constructing Green's function for corresponding homogeneous boundary value problems by using Green's function properties. Under the suitable conditions, we established the existence of three positive solution by applying Avery and Peterson fixed point theorem. To illustrate the result examples are provided. This study was mostly dependent on secondary source of data such as journals and books which related to our study area.

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Chapter 1

Introduction

1.1 Background of the study

Boundary value problems associated with linear as well as non-linear ordinary differential equations have created a great deal of interest and play an important role in many fields of applied mathematics such as engineering design and manufacturing. Major industries like automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications as well as emerging technologies like biotechnology and nanotechnology rely on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high-technological products. In these applied setting, positive solutions are meaningful.

In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A boundary condition is a condition that is required to be satisfied at all or part of the boundary of a region in which a set of differential condition is to be solved. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions [Zhang 2017].

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. Boundary value problems arise in applications where some physical process involves knowledge of information at the edges. For example, it may be possible to measure the electric potential around the edge of a semi-conductor and then use this information to infer the potential distribution near the middle HELM(2008).

Some theories such as the Krasnoselskii's fixed point theorem, the Leggett-Williams fixed point theorem, Avery's generalization, of the Leggett-Williams fixed point theorem and Avery-Henderson fixed point theorem have given a decisive impetus for the development of the modern theory of differential equations. The advantage of these techniques lies in that they do not demand the knowledge of solution, but have great power in application, in finding positive solutions, multiple positive solutions, and eigenvalue intervals for which there exists one or more positive solutions.

In analyzing nonlinear phenomena many mathematical models give rise to problems for which only positive solutions make sense. Therefore, since the publication of the monograph positive solutions of Operator Equations in the year 1964 by academician M.A. Krasnoselskii, hundreds of research articles on the theory of positive solutions of nonlinear problems have appeared. In this vast field of research, we are focused on the second order un damped three point boundary value problem. Most results so far have been obtained mainly by using the fixed-point theorems in cones, such as the Guo-Krasnoselskii's fixed point theorem [Krasnoselskii 1964, Leggett 1979, Ma 2001], and so on.

The existence of positive solutions for BVPs is very important, especially in ecological and population biology models. Some existence theorems give explicit formulas for solutions (e.g., Cramer's rule). Some theorems construct computational solutions (e.g., Bolzano-Weierstrass theorem). Other theorems are settled by no constructive proofs which simply deduce the necessity of solutions without indicating any method for determining them (e.g., the Brouwer fixed point theorem shows that the non existence would lead to a contradiction) [Yanlei Zhang ,2017].

The existence of positive solutions of boundary value problems was studied by many researchers. We list down few of them which are related to our particular problem. In the past few years, there has been increasing interest in studying certain three-point boundary value problems for nonlinear ordinary differential equations; to identify a few.

He, in 2002 [He 2002], established the existence of at least three positive solutions to the second order three point boundary value problems.

$$\begin{aligned} u'' + f(t,u) = 0, 0 < t < 1 \\ u(0) = 0, \alpha u(\eta) = u(1) \\ \alpha > 0, 0 < \eta < 1. \end{aligned}$$

Using the Leggett-Williams fixed-point theorem.

Liu,in 2014 [Liu, 2014], established the existence, multiplicity, and nonexistence of positive solutions,

$$u''(t) + \beta^2 u(t) + \lambda q(t) f(t, u(t)) = 0, 0 < t < 1$$
$$u(0) = 0, u(t) = \sigma u(\eta)$$
$$\text{where}, \beta \in (0, \frac{\pi}{2}), \eta \in (0, 1)$$

λ is a posetive constant, by using the fixed point index theorem, degree theory, and fixed point theorem in cones.

Neito, in 2013 [Nieto 2013], established the existence of a solution for a three-point boundary value problem for a second order differential equation at resonance

$$-u''(t) = f(t, u(t)), 0 \leq t \leq T$$
$$u(0) = 0, \alpha u(\eta) = u(T)$$
$$\text{where } T > 0, f : [u, T] \times \mathbb{R} \to \mathbb{R} \text{ is continuous function}$$
$$\alpha \in \mathbb{R}, \eta \in (0, T).$$

Li, in 2017 [Li 2017], investigated the existence of nontrivial solutions for some super linear second order three-point boundary value problems.

$$-u''(t) = f(t, u(t)), 0 \leq t \leq 1$$
$$u'(0) = 0, u(1) = \alpha u(\eta)$$
$$\text{where } 0 < \alpha < 1, 0 < \eta < 1.$$

by applying new fixed point theorems in ordered Banach spaces with the lattice structure derived by Sun and Liu.

Motivated by the above mentioned results, we established the existence of three positive solutions for second order undamped three point boundary value problems

$$-u''(t) + k^2 u(t) = f(t, u(t)) \tag{1.1}$$

$$u(0) = 0, u(1) = \alpha u(\eta), 0 < \eta < 1, k > 0. \tag{1.2}$$

by applying Avery and Peterson fixed point theorems in cone Banach space and some examples will be demonstrated for the applicability of our main result.

By a positive solution of (1.1)-(1.2) we mean a function $u(t)$ which is positive on $0 \leq t \leq 1$ and satisfies the differential equation (1.1) for $0 \leq t \leq 1$ and three-point boundary conditions (1.2).

The rest of this thesis was organized as follows: We first presented some definitions and theorem which are needed throughout this work and construct the Green's function for the corresponding homogeneous boundary value problem and state fixed point result by using the Avery and peterson fixed point theorem in a cone Banach space. We investigating the existence of three positive solution for second order un- damped three point boundary value problem (1.1), (1.2). Finally as an application, examples will be included to verify the illustrative result.

1.2 Statement of the problem

In this study we focused on establishing the existence of three positive solutions for second-order undamped three-point boundary value problems by using Avery and peterson fixed point theorem (1.1), (1.2).

1.3 Objectives of the study

1.3.1 General objective

General Objective of this research is to investigate the existence of three positive solutions by applying Avery and Peterson fixed point theorem.

1.3.2 Specific objectives

The specific objectives of the present study are:

1. To construct Green's function of the boundary value problem (1.1), (1.2).

2. To formulate the equivalent operator equations of the given boundary value problems.

3. To determine the fixed point of the operator equations.

4. To illustrate the result by using particular examples.

1.4 Significance of the study

The outcomes of this study have the following importance:

- The outcome of this study may contribute to research activities on study area.
- It may provide basic research skill to researcher.
- May have application in studying the existence of positive solution to second order undamped three point boundary value problem.

1.5 Delimitation of the Study

This study was delimited to finding the existence of three positive solution of second order undamped three point boundary value problem.

Chapter 2

Review of Related literatures

2.1 Over view of the Study

The existence of positive solution for second order three point boundary value problems has been studied extensively, which can be seen by the work of many researchers. In this section we list down few of them which are related to our particular problem.

Ma, in 1999[Ma, 1999], established the existence of positive solutions to the boundary-value problem

$$\begin{aligned} u'' + \alpha(t)f(u) &= 0, 0 < t < 1 \\ u(0) = 0, \alpha u(\eta) &= u(1) \end{aligned}$$

where $0 < \eta < 1, 0 < \alpha < \frac{1}{\eta}$, by applying the fixed point theorem in cones.

Zima, in 2004 [Zima 2004], established the existence of positive solution of second order three-point boundary value problem

$$\begin{aligned} x''(t) + f(t, x(t)) &= 0, 0 \leq t \leq 1 \\ x(0) = 0, \alpha x(\eta) &= x(1) \\ 0 < \eta < 1, \alpha &\geq 0 \end{aligned}$$

By establishing a norm-type cone expansion and compression fixed point theorem for completely continuous operator.

Naceri, in 2013 [Naceri 2013], establish the existence of at least three positive solutions of the boundary value problems for systems of second-order ordinary differential equations of the form .

$$\begin{aligned}-u''(t)+k^2u(t)&=f(t,u(t),v(t)),0<t<1\\ -v''(t)+\omega^2v(t)&=g(t,u(t),v(t)),0<t<1\\ u(0)&=0,v(0)=0\\ u(1)&=\beta u(\eta),v(1)=\lambda v(\eta)\end{aligned}$$

where $f:[0,1]\times[0,\infty)\times[0,+\infty)\to[0,+\infty)$, $g:[0,1]\times[0,\infty)\times[0,+\infty)\to[0,+\infty)$ k, ω are posetive constant $0<\eta<1, 0<\beta<\beta_0, 0<\lambda<\lambda_0$, by applying the Leggett-Williams fixed point theorems.

Sveikate, in 2016 [Sveikate 2016], established the existence of solutions.

$$\begin{aligned}x''+k^2x&=f(t,x)\\ x(0)=0,x(1)&=\alpha x(\eta)\\ 0<\eta<1,&\alpha>0\end{aligned}$$

by using the quasilinearization approach.

We, in 2019[We, 2019], established existence of positive solutions of second order three-point boundary value problem with dependence on the first-order derivative

$$\begin{aligned}x''(t)+f(t,x(t),x'(t))&=0,0<t<1\\ x(0)&=0,x(1)=\mu x(\eta)\end{aligned}$$

where f:$[0,1]\times[0,\infty)\times\mathbb{R}\to[0,\infty)$ is continuous and $\eta>0,\eta\in(0,1),\mu\eta<1$ by using fixed point theorem in a cone and some inequalities of the Green's functions.

2.2 Preliminaries

In this section we recall some known definitions, theorems and basic concepts on Green's function that will be used in the proof of our main results.

Definition 2.2.1. *A differential equation together with its boundary conditions is referred to as boundary value problem.*

Definition 2.2.2. *A differential equation together with three point boundary conditions is referred to as three point boundary value problem.*

Definition 2.2.3. *Let X be a non-empty set. A map said to be a self-map with domain of $T = D(T) = X$ and range of $T = R(T) \subset X$*

Definition 2.2.4. *Let X is a non-empty set and $T : X \to X$ be self-map. A point x in X is called a fixed point of T if $Tx = x$.*

Definition 2.2.5 (Agarwal 2008)**.** *We consider the second-order linear DE.*

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = r(x), x \in J = [\alpha, \beta], \tag{2.1}$$

where the functions $p_0(x), p_1(x), p_2(x)$ and $r(x)$ are continuous in J and boundary conditions of the form

$$\begin{aligned} l_1[y] &= a_0y(\alpha) + a_1y'(\alpha) + b_0y(\beta) + b_1y'(\beta) = A \\ l_2[y] &= c_0y(\alpha) + c_1y'(\alpha) + d_0y(\beta) + d_1y'(\beta) = B \end{aligned} \tag{2.2}$$

where $a_i, b_i, c_i, d_i, i = 0, 1$ and A and B are given constants and l is differential operator.
The boundary value problems (2.1), (2.2) are called nonhomogeneous two-point linear boundary value problems, where as the homogeneous DE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0 \tag{2.3}$$

together with the homogeneous boundary conditions

$$l_1[y] = 0, l_2[y] = 0 \tag{2.4}$$

be called a homogeneous two-point linear boundary value problem.

The function called a Green's function $G(x, t)$ for the homogeneous boundary value problems (2.3)-(2.4) and the solution of the non homogeneous boundary value problem (2.1)-(2.2) can be explicitly expressed in terms of $G(x, t)$. Obviously, for the homogeneous problem (2.3)-(2.4) the trivial solution always exists. Green's function for the boundary value problem (2.3)-(2.4) is defined in the square $[\alpha, \beta] \times [\alpha, \beta]$ and possesses the following fundamental properties:

i. $G(x, t)$ is continuous $[\alpha, \beta] \times [\alpha, \beta]$.

ii. $\frac{\partial G(x,t)}{\partial x}$ is continuous in each of the triangles $\alpha \leq x \leq t \leq \beta$ and $\alpha \leq t \leq x \leq \beta$.
Moreover, $\frac{\partial G(t^+,t)}{\partial x} - \frac{\partial G(t^-,t)}{\partial x} = -\frac{1}{p_0(t)}$
where, $\frac{\partial G(t^+,t)}{\partial x} = \lim_{x \to t, x > t} \frac{\partial G(x,t)}{\partial x}$, $\frac{\partial G(t^-,t)}{\partial x} = \lim_{x \to t, x < t} \frac{\partial G(x,t)}{\partial x}$.

iii. for every $t \in [\alpha, \beta], z(x) = G(x,t)$ is a solution of the differential equation(2.3) in each of the intervals $[\alpha, t)$ and $(t, \beta]$.

iv. for every $t \in [\alpha, \beta], z(x) = G(x,t)$ satisfies the boundary conditions (2.4).

These properties completely characterize Green's function $G(x,t)$.

Definition 2.2.6. *Let $-\infty < a < b < \infty$ a collection of real valued functions $A = f_i : [a, b] \to R$ is said to be*

(i) Uniformly bounded, if there exists a constant $M > 0$ with $|f_i(t)| \le M$, for all $t \in [a, b]$ and for all $f_i \in A$, and

(ii) Equi continuous, if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies $|f_i(t_1) - f_i(t_2)| < \epsilon$, for all $t_1, t_2 \in [a, b]$ and for every $f_i \in A$.

Definition 2.2.7. *A normed linear space is a linear space X in which for each vector x,there corresponds a real number, denoted by $||x||$ called the norm of x and has the following properties:*

i. $\|x\| \ge 0$, for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$,

ii. $||x + y|| \le ||x|| + ||y||$ for all $x, y \in X$,

iii. $||\alpha x|| = |\alpha| ||x||$, for all $x \in X$ and α be a scalar.

Definition 2.2.8. *A Banach space is a complete normed space*

Definition 2.2.9. *Let E be a real Banach space. A nonempty closed convex set P is called a cone, if it satisfies the following two conditions:*

(i) $u \in P, \alpha \ge 0$ implies $\alpha u \in P$, and

(ii) $u \in P$ and $-u \in P$ implies $u = 0$.

Definition 2.2.10. *Let X and Y be two metric spaces. A map $T : X \to Y$ is said to be completely continuous, if it is continuous and maps bounded sets into precompact sets.*

Definition 2.2.11. *Let X and Y be Banach Spaces and $T : X \to Y$. An operator $T$is said to be completely continuous, if T is continuous and for each bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.*

Definition 2.2.12. *Let E be a real Banach space with cone P. A map $f : P \to [0, \infty)$ is said to be a nonnegative continuous convex functional on P, if f is continuous and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, for all $x, y \in P$ and $\lambda \in [0, 1]$.*

Definition 2.2.13. *Let E be a real Banach space with cone P. A map $f : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P if f is continuous and $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$,for all $x, y \in P$ and $\lambda \in [0, 1]$.*

The function $u(t) \in C^2[0, 1]$ is a positive solution of the boundary value problem,

$$\begin{aligned} -u''(t) + k^2 u(t) = f(t, u(t)) \\ u(0) = 0, u(1) = \alpha u(\eta), 0 < \eta < 1, k > 0. \end{aligned} \tag{2.5}$$

If $u(t)$ is positive on the given interval and satisfies both the differential equation and the boundary conditions.

Let α, γ, θ, ψ be maps on P with α a nonnegative continuous concave functional ;γ, θ nonnegative continuous convex functional , and ψ a nonnegative continuous functional.Then for positive real numbers a, b, c and d we define the following subset of p.

$$\begin{aligned} \overline{p(\gamma, d)} &= \{u \in p | \gamma(u) \leq d\} \\ P(\alpha, \gamma, b, d) &= \{u \in \overline{P(\gamma, d)} | \alpha(u) \geq b\}, \\ P(\alpha, \theta, \gamma, b, c, d) &= \{u \in \overline{P(\gamma, d)} | \alpha(u) \geq b, \theta(u) \leq c\}, \\ P(\psi, \gamma, a, d) &= \{u \in \overline{P(\gamma, d)} | \psi(u) \geq a\}, \\ R(\gamma, \psi, a, d) &= \{u \in \overline{P(\gamma, d)} | a \leq \psi(u), \gamma(u) \leq d\}. \end{aligned}$$

Theorem 2.2.14 (Avery and Peterson, 2001)**.** *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functional on P. Let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda \psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $M \overline{P(\gamma, d), d}, \alpha(x) \leq \psi(x)$ and $\|x\| \leq M\gamma(x)$ for $x \in \overline{P(\gamma, d)}$. Suppose that $T : \overline{P(\gamma, d)} \to \overline{(P(\gamma, d)}$ is completely continuous operator and there exist positive numbers a, b and c with $a < b$ such that*

C1. $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq 0$ *and* $\alpha(Tx) > b$ *for* $x \in P(\alpha, b; \theta; c; \gamma, d)$ *;*

C2. $\alpha(Tx) > b$ *for* $x \in P(\alpha, b; \gamma, d)$ *with* $\theta(Tx) > c$ *;*

C3. $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$ *.*

Then T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ *such that* $\gamma(x_i) \leq d$ *for* $i = 1, 2, 3; b \leq \alpha(x_1); a \leq \psi(x_2)$ *with* $\alpha(x_2) \leq b; a \leq \psi(x_3) \leq a$*.*

2.3 Green's Functions and Bounds

In this section, we construct Green's function for the corresponding homogeneous boundary value problem to (1.1). Before formulation of Green's function for three-point boundary value problem, first we construct Green's function for two point homogeneous boundary value problem,

$$-u'' + k^2 u = 0 \tag{2.6}$$

$$u(0) = 0, u(1) = 0 \tag{2.7}$$

For equation (2.6) two linearly independent solution are $u_1(t) = -\sinh kt + \cosh kt$ and $u_2(t) = \sinh kt + \cosh kt$. Hence, the problem (2.6)-(2.7) has only trivial solution if and only if

$$\Delta = \begin{vmatrix} u_1(0) & u_2(0) \\ u_1(1) & u_2(1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\sinh k + \cosh k & \sinh k + \cosh k \end{vmatrix} = 2 \sinh k \neq 0, k > 0$$

To show this $u_1(t)$ and $u_2(t)$ be two linearly independent solution if the differential equation (2.6). Green's function for second- order two- point boundary value can be written in the form of

$$H(t,s) = \begin{cases} u_1(t)\lambda_1(s) + u_2(t)\lambda_2(s), & \text{if} \quad 0 \leq t \leq s \leq 1 \\ u_1(t)\mu_1(s) + u_2(t)\mu_2(s), & \text{if} \quad 0 \leq s \leq t \leq 1 \end{cases} \tag{2.8}$$

Where, $\lambda_1(s), \lambda_2(s), \mu_1(s)$, and $\mu_2(s)$ are functions. By applying properties of (i) and (ii) ,we obtain

$$\begin{cases} u_1(t)(\mu_1(s) - \lambda_1(s)) + u_2(t)(\mu_2(s) - \lambda_2(s)) = 0, \\ u_1'(t)(\mu_1(s) - \lambda_1(s)) + u_2'(t)(\mu_2(s) - \lambda_2(s)) = -1. \end{cases} \tag{2.9}$$

Let $v_1(s) = \mu_1(s) - \lambda_1(s)$ and $v_2(s) = \mu_2(s) - \lambda_2(s)$ ————-(*)
Then

$$\begin{cases} u_1(s)V_1(s) + u_2(s)v_2(s) = 0, \\ u_1'(s)V_1(s) + u_2'(s)v_2(s) = -1. \end{cases}$$

From this we get

$$v_1(s) = \frac{1}{2k(-\sinh ks + \cosh ks)} \text{and } v_2(s) = \frac{-1}{2k(\sinh ks + \cosh ks)}$$

Frome $(*)$, we have $\mu_1(s) = v_1(s) + \lambda_1(s)$ and $\mu_2(s) = v_2(s) + \lambda_2(s)$
By using the boundary condition of (4.2) , we obtain

$$\begin{cases} u_1(0)\lambda_1(s) + u_2(0)\lambda_2(s) = 0, \\ u_1(1)(V_1(s) + \lambda_1(s)) + u_2(1)(v_2(s) + \lambda_2(s)) = 0. \end{cases}$$

$$\begin{cases} \lambda_1(s) + \lambda_2(s) = 0, \\ (-\sinh k + \cosh k)(\lambda_1(s) + \frac{1}{2k(-\sinh ks+\cosh ks)}) \\ +(\sinh k + \cosh k)(\lambda_2(s) + \frac{1}{2k(\sinh ks+\cosh ks)}) = \quad 0. \end{cases}$$

By applying Cramer's rule ,we find the value of $\lambda_1(s)$ and $\lambda_2(s)$.
For $\lambda_1(s)$,

$$\lambda_1(s) = \begin{vmatrix} 0 & 1 \\ \frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{k} & \sinh k + \cosh k \end{vmatrix} = \frac{\cosh k.\sinh ks - \sinh k.\cosh ks}{2k\sinh k}$$

For $\lambda_2(s)$,

$$\lambda_2(s) = \begin{vmatrix} 1 & 0 \\ -\sinh k + \cosh k & \frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{k} \end{vmatrix} = \frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{2k\sinh k}$$

Hence ,

$$u_1(t)\mu_1(s) + u_2(t)\mu_2(s) = u_1(t)(\lambda_1(s) + v_1(s)) + u_2(t)(\lambda_2(s) + v_2(s))$$
$$= (-\sinh kt + \cosh kt)(\frac{\cosh k.\sinh ks - \sinh k.\cosh ks}{2k\sinh k} + \frac{1}{2k(-\sinh ks + \cosh ks)})$$
$$+(\sinh kt + \cosh kt)(\frac{\cosh ks.\sinh k - \sinh ks.\cosh k}{2k\sinh k} - \frac{1}{2k(\sinh ks + \cosh ks)})$$
$$= \frac{\sinh kt \sinh k(1-t)}{k\sinh k}, 0 \le t \le s \le 1$$

Therefore,

$$H(t,s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k\sinh k}, & 0 \le t \le s \le 1 \\ \frac{\sinh ks \sinh k(1-t)}{k\sinh k}, & 0 \le s \le t \le 1 \end{cases} \tag{2.10}$$

And the solution of (4.1) is given by

$$w(t) = \int_0^1 H(t,s)f(s)ds \tag{2.11}$$

And

$$w(0) = 0, w(1) = 0, w(\eta) = \int_0^1 H(\eta, s) f(s) ds \tag{2.12}$$

Lemma 2.3.1. *$H(t, s)$ has the following properties:*

(i) $H(t, s) \leq H(s, s)$, for all $t, s \in [0, 1]$;

(ii) $H(t, s) \geq NH(s, s)$, for all $t \in [\delta, 1-\delta]$, $s \in [0, 1]$, $N = \frac{\sinh k\delta}{\sinh k}$

Proof. i. $H(t, s)$ is positive for all $t, s \in [0, 1]$.
For $0 \leq s \leq t \leq 1$, we have

$$\frac{H(t, s)}{H(s, s)} = \frac{\sinh ks. \sinh k(1-t)}{\sinh ks. \sinh k(1-s)} = \frac{\sinh k(1-t)}{\sinh k(1-s)} \leq 1.$$

$\Longrightarrow H(t, s) \leq H(s, s), t, s \in [0, 1]$
For $0 \leq t \leq s \leq 1$, we have

$$\frac{H(t, s)}{H(s, s)} = \frac{\sinh kt. \sinh k(1-s)}{\sinh ks. \sinh k(1-s)} = \frac{\sinh kt}{\sinh ks} \leq 1.$$

$\Longrightarrow H(t, s) \leq H(s, s), t, s \in [0, 1]$
Therefore, $H(t, s) \leq H(s, s)$, for all $t, s \in [0, 1]$.

ii. If $s \leq t$ for $t \in [\delta, 1-\delta]$, $s \in [0, 1]$, we have

$$\frac{H(t, s)}{H(s, s)} = \frac{\sinh ks \sinh k(1-t)}{\sinh ks \sinh k(1-s)} \geq \frac{\sinh k\delta}{\sinh k}.$$
$$\Longrightarrow H(t, s) \geq NH(s, s)$$

If $t \leq s$ for $t \in [\delta, 1-\delta]$, $s \in [0, 1]$, we have

$$\frac{H(t, s)}{H(s, s)} = \frac{\sinh kt \sinh k(1-s)}{\sinh ks \sinh k(1-s)} \geq \frac{\sinh k\delta}{\sinh k}.$$
$$\Longrightarrow H(t, s) \geq NH(s, s)$$

Thus, the Lemma follows. □

The three point boundary value problem (1.1),(1.2) can be obtained by replacing $u(1) = 0$ for $u(1) = \alpha u(\eta)$ in (2.7), thus we suppose the solution of the three-point boundary value problem (1.1),(1.2) can be expressed by

$$u(t) = w(t) + B_1 \sinh kt + B_2 \sinh k(1-t) \tag{2.13}$$

Where B_1 and B_2 are constants that will be determined . From (2.11), (2.12), we know that

$$\begin{cases} u(0) = w(0) + B_1 \sinh k(0) + B_2 \sinh k(1-0) \\ u(1) = w(1) + B_1 \sinh k(1) + B_2 \sinh k(1-1) \\ u(\eta) = w(\eta) + B_1 \sinh k(\eta) + B_2 \sinh k(1-\eta) \end{cases}$$

$$\Rightarrow \begin{cases} 0 = 0 + B_1.0 + B_2 \sinh k(1) \\ u(1) = 0 + B_1 \sinh k(1) + B_2 \sinh k(0) \\ u(\eta) = w(\eta) + B_1 \sinh k(\eta) + B_2 \sinh k(1-\eta) \end{cases}$$

$$\Rightarrow B_2 = 0$$

$u(1) = \alpha u(\eta)$ we have $B_1 \sinh k = \alpha(w(\eta) + B_1 \sinh k(\eta))$

$$B_1 = \frac{\alpha w(\eta)}{\sinh k - \alpha \sinh k(\eta)}, \qquad \frac{\sinh k}{\sinh k\eta} > \alpha. \tag{2.14}$$

Therefore,

$$u(t) = w(t) + B_1 \sinh kt + B_2 \sinh k(1-t)$$

$$G(t,s) = H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} H(\eta, s)$$

$$\text{Where, } H(t,s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k \sinh k}, & 0 \le t \le s \le 1 \\ \frac{\sinh st \sinh k(1-t)}{k \sinh k}, & 0 \le s \le t \le 1 \end{cases}$$

$$H(\eta, s) = \begin{cases} \frac{\sinh k\eta \sinh k(1-s)}{k \sinh k}, & 0 \le \eta \le s \le 1 \\ \frac{\sinh s \sinh k(1-\eta)}{k \sinh k}, & 0 \le s \le \eta \le 1 \end{cases}$$

$$G(t,s) = \begin{cases} \frac{\sinh kt \sinh k(1-s)}{k \sinh k} \\ \frac{\sinh ks \sinh k(1-t)}{k \sinh k} \end{cases} + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k(\eta)} \begin{cases} \frac{\sinh k\eta \sinh k(1-s)}{k \sinh k} \\ \frac{\sinh s \sinh k(1-\eta)}{k \sinh k} \end{cases} \tag{2.15}$$

$$G(t,s) = \frac{1}{k(\sinh k - \alpha \sinh k(\eta))} \begin{cases} [\sinh k(1-s) + \alpha \sinh k(s-\eta)](\sinh kt), & t \le s \le \eta, \\ [\sinh k(1-t) + \alpha \sinh k(t-\eta)](\sinh ks), & s \le t, s \le \eta \\ \sinh kt \sinh k(1-s) \qquad , & t \le s, \eta \le s, \\ \sinh ks. \sinh k(1-t) + \alpha \sinh k\eta \sinh k(t-s), & \eta \le s \le t \le 1. \end{cases} \tag{2.16}$$

Lemma 2.3.2. *The Green's function $G(t,s)$ satisfies the following properties:*

i. $G(t,s) \ge 0, \forall t, s \in [0,1]$;

ii. $G(t,s) \le DH(s,s), \forall t, s \in [0,1]$, *iii.* $G(t,s) \geq \mathcal{M} H(s,s), \forall t, \in [\delta, 1-\delta], s \in [0,1]$

where

$$\begin{aligned} D &= 1 + \frac{\alpha . \sinh k}{\sinh k - \alpha \sinh k\eta} \\ \mathcal{M} &= \frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha . \sinh k\delta}{\sinh k - \alpha \sinh k\eta}]. \end{aligned} \tag{2.17}$$

Proof. (i) It is obvious that $G(t,s)$ is a nonnegative since $H(t,s) \geq 0$ and $\sinh k - \alpha \sinh k\eta \; > 0$

(ii) consider the following case
Case (i) if $t \leq s, \eta \leq s$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh ks}{\sinh k - \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_1 H(s,s) \end{aligned} \tag{2.18}$$

Case(ii) If $t \leq s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_2 H(s,s). \end{aligned} \tag{2.19}$$

Case(iii) if $s \leq t, s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_3 H(s,s) \end{aligned} \tag{2.20}$$

Case(iv) if $\eta \leq s \leq t \leq 1$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \sinh k\eta} H(\eta, s) \\ &\leq H(s,s) + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\leq H(s,s)[1 + \frac{\alpha \sinh k}{\sinh k - \alpha \sinh k\eta}] \\ &\leq D_4 H(s,s) \end{aligned} \tag{2.21}$$

Therefore, $G(t,s) \leq DH(s,s)$,where $D = D_1 = D_2 = D_3 = D_4$

(iii) To prove (iii) we consider the following cases:
Case(i) If $s \leq t, s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\ &\geq NH(s,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(s,s) \geq NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq M_1 H(s,s). \end{aligned} \tag{2.22}$$

Case(ii) If $t \leq s, \eta \leq s$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\ &\geq NH(s,s) + \frac{\alpha \sinh k\eta}{\sinh k - \alpha \sinh k\eta} H(s,s) \\ &\geq NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq M_2 H(s,s). \end{aligned} \tag{2.23}$$

Case(iii) If $t \leq s \leq \eta$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\ &\geq NH(s,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} NH(s,s) \\ &\geq NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq M_3 H(s,s). \end{aligned} \tag{2.24}$$

Case(iv) If $\eta \leq s \leq t \leq 1$

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} H(\eta, s) \\ &\geq NH(s,s) + \frac{\alpha \sinh kt}{\sinh k - \alpha \sinh k\eta} NH(s,s) \\ &\geq NH(s,s)[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq H(s,s)\frac{\sinh k\delta}{\sinh k}[1 + \frac{\alpha \sinh k\delta}{\sinh k - \alpha \sinh k\eta}] \\ &\geq M_4 H(s,s). \end{aligned} \tag{2.25}$$

Therefore, $G(t,s) \geq \mathcal{M} H(s,s)$, where $\mathcal{M} = M_1 = M_2 = M_3 = M_4$

□

Chapter 3

Methodology

3.1 Study period and site

The study was conducted in Jimma University under the department of Mathematics from September 2018 to February 2020 G.C.

3.2 Study Design

In this study employed analytical method of design.

3.3 Source of Information

The relevant sources of information for this study are books, published articles, journals and related studies from Internet.

3.4 Mathematical Procedure of the Study

The study followed the following steps:

1. Defining second order three-point boundary value problem.

2. Constructing the Green's function for the corresponding homogeneous equation.

3. Formulating the equivalent operator equation for the boundary value problems.

4. Determining the fixed point of the operator equation by using Avery and Peterson fixed point theorem.

5. Finally as an application, example is included to verify the theoretical result.

Chapter 4

RESULTS AND DISCUSSION

4.1 Main Result

In this section, we discuss the existence of at least three positive solutions for second-order undamped three point boundary value problems (1.1), (1.2) by applying Avery-Peterson fixed point theorem.
Obviously, $u(t) \in C^2([0,1], \mathbb{R}^+)$ is solution of (1.1)-(1.2) if and only if $u(t)$ is a solution of the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds. \tag{4.1}$$

Let $E = C([0,1])$ be a real Banach space with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and define the cone P,

$$P = \{u \in E; u(t) \geq 0 \text{ and } min_{0 \leq t \leq \eta} u(t) \geq \sigma ||u||\}$$

where $\sigma = \frac{\mathcal{M}}{D}$, then P is anon empty closed subset of E. It is obvious that E is a real Banach space and P is a cone in E. Define an operator $T : P \to E$ as

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s)) ds \tag{4.2}$$

By (4.1) $u(t)$ is a solution of boundary value problem (1.1)-(1.2) iff $u(t)$ is a fixed point of T. In this research the following condition have been assumed through out :

H1. $f \in C([0,1] \times [0,\infty), [0,\infty)), k \in (0,\infty)$

H2. $0 \leq G(t,s) < \infty, \forall t, s \in [0,1]$

H3. $\alpha < \frac{\sinh k}{\sinh k\eta}$

By applying fixed point theorem on T and putting suitable conditions on f we proved the existence of at least three fixed points in a cone.

Lemma 4.1.1. *Let $H_1 - H_3$ hold the operator $T : P \to P$ is completely continuous.*

Proof. First we prove the following

1. The operator T is self map on P. Now $(1.1), (1.2)$ has a solution $u = u(t)$ if and only if u solve the operator equation,

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds = Tu(t). \tag{4.3}$$

   Now $G(t,s)$ is the Green's function for the boundary value problem, by lemma 2.3.1 and 2.3.2 we have $G(t,s) \leq DH(s,s)$, for $t,s \in [0,1]$ and $Tu \in E$, for each $u \in P$. We have

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 DH(s,s)f(s,u(s))ds \tag{4.4}$$

   and we have $Tu(t) \leq \int_0^1 DH(s,s)f(s,u(s))ds$ which implies that

$$||Tu|| \leq \int_0^1 DH(s,s)f(s,u(s))ds$$

   then,

$$\begin{aligned}
\min_{t\in[\delta,1-\delta]}(Tu)(t) &= \min_{t\in[\delta,1-\delta]} \int_0^1 G(t,s)f(s,u(s))ds \\
&\geq \mathcal{M}\int_0^1 H(s,s)f(s,u(s))ds = \frac{\mathcal{M}}{D}\int_0^1 DH(s,s)f(s,u(s))ds \\
&\geq \frac{\mathcal{M}}{D}||T(u)|| \geq \frac{\mathcal{M}}{D}||T(u)|| \\
&\geq \sigma||T(u)||.
\end{aligned}$$

   $TP \subset P$. Therefor, T is a self map on P.

2. The operator T is uniformly bounded on P. Let $u \in P$, in view of the positivity and continuity of $G(t,s), H(t,s)$ and f, we have $T : P \longrightarrow P$ is continuous .
   Let $\{u_k\}$ be a bounded sequence in P, say $||u_k|| \leq M$ for all k since f is continuous, there exist $N > 0$ such that $|f(t,u(t))| \leq N$ for all $u \in [0,\infty)$ with $0 \leq u \leq M$ then for each $t \in [0,1]$ and for each k,

$$\begin{aligned} |Tu_k(t)| &= |\int_0^1 G(t,s)f(s,u_k)ds| \\ &\leq \int_0^1 G(t,s)(N)ds \\ &\leq N\int_0^1 G(t,s)ds < +\infty \end{aligned}$$

   that is for each $t \in [0,1]$, u_k is abounded sequence of real numbers. By choosing successive subsequences for each t, there exist a subsequence $\{u_k\}$ which converges uniformly for $t \in [0,1]$.
   Hence, T is uniformly bounded.

3. The operator T is equicontinuous on P. To prove T is equicontinuous. Let $u \in P$, and $\epsilon > 0$ be given. By the continuity of $G(t,s)$, for $t \in [0,1]$, there exist $\delta > 0$ such that $|G(t_2,s) - G(t_1,s)| < \frac{\epsilon}{N}$ whenever $|t_1 - t_2| < \delta$, for t_1 ,$t_2 \in [0,1]$

$$\begin{aligned} \|Tu(t_1) - Tu(t_2)| &= |\int_0^1 (G(t_1,s) - G(t_2,s))f(s,u(s))ds| \\ &\leq \int_0^1 |G(t_2,s) - G(t_1,s)|Nds, \\ &\leq N\int_0^1 |G(t_1,s) - G(t_2,s)|ds, \\ &< \epsilon. \end{aligned}$$

   Therefore, by a standard application of the Arzela-Ascoli theorem [Royden, 2010] and the result from 1,2 and 3, T is completely continuous.

□

From above arguments, we know that the existence of at least three positive solutions of (1.1), (1.2) can be equivalent to the existence of at least three fixed points of the operator of T.

Let $\gamma(u) = ||u|| = \max_{0\leq t\leq 1}(u(t)), t \in [0,1]$. $\theta(u) = \psi(u) = \max_{0\leq t\leq 1}|u(t)|$ $\alpha(u) = \min_{\eta\leq t\leq 1}(u(t)$, where γ *and* θ are non-negative continuous convex functional, ψ is a nonnegative continuous functional α is a nonnegative continuous concave functional on the cone P.

Theorem 4.1.2. *Let the condition $(H1)-(H3)$ hold and there exist positive numbers a, b, d, with $0 < a < b < d$, such that*
$(A_1) f(t,u) \leq \frac{d}{L}, t \in [0,1], u \in [0,d],$
$(A_2) f(t,u) > \frac{b}{\mathcal{L}}, t \in [\delta, 1-\delta], u \in [b, \frac{b}{\sigma}],$
$(A_3) f(t,u) < \frac{a}{L}, t \in [0,1], u \in [0,a],$

$$\mathcal{L} = \mathcal{M}\int_{\delta}^{1-\delta} H(s,s)ds \text{ and } L = D\int_0^1 H(s,s)ds \tag{4.5}$$

Then boundary value problem (1.1)-(1.2) *has at least three positive solution* u_1, u_2 *and* $u_3 \in \overline{P(\gamma,d)}$ *satisfies* $\gamma(u_i < d)$ *for i=1,2,3 and* $\min_{\delta\leq t\leq 1-\delta} u_1(t) > b, \max_{0\leq t\leq 1} u_2(t) > a$ *with* $\min_{\delta\leq t\leq 1-\delta} u_2(t) < b$ *and* $\max_{0\leq t\leq 1} u_3(t) < a$.

Proof. We prove to satisfies all the condition of theorem which lead as the existence of at least three fixed points of T in a cone p.

$$\gamma(u) = \max_{0\leq t\leq 1} u(t)$$

Let $u \in \overline{P(\gamma,d)}$, then $\gamma(u) \leq d$.

$$\begin{aligned}\gamma(Tu)(t) &= \max_{0\leq t\leq 1}(Tu)(t) = \max_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,u(s))ds \\ &\leq D\int_0^1 H(s,s)\frac{d}{L}ds \leq \frac{dD}{L}\int_0^1 H(s,s)ds \\ &\leq d.\end{aligned}$$

Hence, $Tu(t) \in \overline{P(\gamma,d)}$

Then $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.
Assume $u(t) = \frac{d}{\sigma}, 0 \leq t \leq 1$, $\alpha(u) = \min_{\delta\leq t\leq 1-\delta} u(t) = \frac{b}{\sigma} > b$,
$\theta(u) = \frac{d}{\sigma} = \max_{0\leq t\leq 1} u(t) = \psi(u), \gamma(u) < d$.

By definition of α

$$\begin{aligned}\alpha(Tu)(t) &= \min_{\delta\leq t\leq 1-\delta}(Tu(t))\\ &= \min_{\delta\leq t\leq 1-\delta}\int_0^1 G(t,s)f(t,u(s))ds\\ &\geq \mathcal{M}\int_0^1 H(s,s)f(s,u(s))ds\\ &\geq \mathcal{M}\int_\delta^{1-\delta} H(s,s)\frac{b}{\mathcal{L}}ds\\ &= b\frac{\mathcal{M}}{\mathcal{L}}\int_\delta^{1-\delta} H(s,s)ds\\ &> b.\end{aligned}$$

This show that condition $(C1)$ of Theorem 2.2.14 satisfied. From the second conditions of Theorem 4.1.2 $f(t,u(t))\geq \frac{b}{\mathcal{L}}, t\in[\delta,1-\delta], u\in[b,\frac{b}{\sigma}]$
If $u\in p(\gamma,\alpha,b,d)$ and $\theta(Tu)>\frac{b}{\sigma}$, then

$$\begin{aligned}\alpha(Tu)(t) &= \min_{\delta\leq t\leq 1-\delta}(Tu(t)) = \min_{t\in[\delta,1-\delta]}\int_0^1 G(t,s)f(s,u(s))ds\\ &\geq \frac{\mathcal{M}}{D}\int_0^1 DH(s,s)f(t,u(s))ds\\ &\geq \sigma\max_{0\leq t\leq 1}Tu(t) = \sigma\theta(Tu) > \frac{\sigma b}{\sigma}\\ &> b\end{aligned}$$

since $\theta(Tu)>\frac{b}{\sigma}=c$. This show that $C2$ of Theorem 2.2.14 satisfied. To verify $C3$ of Theorem 2.2.14. Obviously $\psi(0)=0<a$, so $0\notin R(\gamma,\psi,a,d)$
Suppose that $u\in R(\gamma,\psi,a,d)$ with $\psi(u)=a$, then $0\leq u(t)\leq a, t\in[0,1]$.
By $(A3)$ we get

$$\begin{aligned}\psi(Tu)(t) &= \max_{0\leq t\leq 1}Tu(t)\\ &= \max_{0\leq t\leq 1}[\int_0^1 G(t,s)f(s,u(s))ds]\\ &\leq D\int_0^1 H(s,s)f(s,u(s))ds\\ &\leq D\int_0^1 H(s,s)\frac{a}{L}ds\\ &\leq a\end{aligned}$$

Condition $C3$ of Theorem 2.2.14 also related. By Theorem 2.2.14 T has at least three fixed points u_1, u_2, u_3 such that $\min_{\delta \le t \le 1-\delta} u_1 > b, \max_{0 \le t \le 1} u_2 > a, \min_{\delta \le t \le 1-\delta} u_2 < b, \max_{0 \le t \le 1} u_3 < a$. $\square$

4.2 Example

In this section we provide example to illustrate our main result

Example 4.2.1.

$$\begin{aligned} -u''(t) + u(t) = f(t, u(t)), 0 \le t \le 1, \\ u(0) = 0, u(1) = 3u(\frac{1}{7}), \end{aligned} \tag{4.6}$$

where $k = 1, \alpha = 3, \eta = \frac{1}{7}$

$$f(t, u(t)) = \begin{cases} u^2 t, t \in [0, 1], u \in [0, 1], \\ u^2 t + (u - 1), t \in [\delta, 1 - \delta], u \in [1, 1100], \\ 1.21 \times 10^6 t + 1099, t \in [0, 1], u \in [1100, \infty]. \end{cases}$$

By the help of Equation (2.15) *and* (2.16) *the Green's function for the corresponding homogeneous BVP of* (4.6) *is*

$$G(t, s) = \frac{1}{(\sinh 1 - 3\sinh(\frac{1}{7}))} \begin{cases} [\sinh(1 - s) + 3\sinh(s - \frac{1}{7})](\sinh t), & t \le s \le \frac{1}{7}, \\ [\sinh(1 - t) + 3\sinh(t - \frac{1}{7})](\sinh s), & s \le t, s \le \frac{1}{7} \\ \sinh t \sinh(1 - s), & t \le s, \frac{1}{7} \le s, \\ \sinh s \sinh(1 - t) + 3\sinh\frac{1}{7}\sinh(t - s), & \frac{1}{7} \le s \le t \le 1. \end{cases}$$

From Equation (2.17) *and* (4.5) *by direct calculation we get*

$$L = 0.8969 \ \ \textit{and} \qquad \mathcal{L} = 0.0461 \quad \mathcal{M} = 0.433, \quad D = 5.731$$

f *is continuous and increasing on* $[0, \infty]$. *If we choose* $a = 1, b = 83$ *and* $d = 1100$, *then all the conditions of Theorem 4.1.2 are satisfied. Hence by Theorem 4.1.2, the boundary value problem* (4.6) *has at least three positive solutions.*

Chapter 5

Conclusion

Based on the obtained result the following conclusion can be derived:- In this study, we defined second-order undamped three point boundary value problems and used the properties of Green's function for constructing Green's function for homogeneous boundary value problem.

After these we formulated equivalent integral equation for the boundary value problem (1.1), (1.2) in the given interval and determine the existence of fixed point of the integral equation by applying Avery and Peterson fixed point theorem. Finally, example was provided to illustrate the result.

5.1 Future Scope

This study focused on existence of three positive solutions for second-order undamped three point boundary value problems. Any interested researcher may conduct the research on:-

- Existence of three positive solutions for n^{th}-order three point boundary value problems.
- Recently there are a number of published research papers related to this area of study .So, the researchers recommends the upcoming Post Graduate Students of the department and any other interested researchers to do their research work in area of study.

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