Accelerated Non-Standard Finite Difference Method for Singularly Perturbed Reaction -Diffusion Boundary Value Problems.

A thesis submitted to department of mathematics, Jimma university in partial fulfilment for the requirement of the degree of master science in mathematics (Numerical analysis).

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Declaration

I hereby declare that this work which is being presented in this research thesis entitled **"Accelerated non-standard finite difference method for singularly perturbed reaction-diffusion boundary value problems"** is an authentic record of my own work. It has not been submitted elsewhere university for the award of any other degree, and that all the sources I have used or quoted have been indicated and acknowledge as complete references.

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Abstract

In this study, accelerated non-standard finite difference method is presented for solving singularly perturbed reaction-diffusion boundary value problems. Richardson extrapolation technique helps to improve accuracy of the solution and accelerates its rate of convergence from second order to fourth order. Consistency and stability of the present method established very well to guarantee the convergence of the method. Model examples were considered to illustrate the conformation of theoretical description with experiential results. In a net shell, the presented method is formulated for the class of singularly perturbed reaction- diffusion boundary value problems which is stable, convergent and gives more accurate solution than some methods existing in the literature.

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Acronyms

BVP - Boundary value problem

FDM - Finite difference method

NSFDM - Non-standard finite difference method

ODE - Ordinary differential equation

PDE - Partial differential equation

SP - Singularly perturbed

SPP - Singularly perturbed problem

SPBVP - Singularly perturbed boundary value problem

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CHAPTER ONE: INTRODUCTION

1.1 Background of the study

Numerical analysis is a mathematics, a technique and used to obtain the approximant solution of a mathematical problems on computer. Numerical analysis is widely used by applied mathematician, scientists, engineers and computer scientists to solve their problems. The advantage of the numerical analysis is that it gives numerical solution, even when a problem has no analytical solution. It is important to realize that a numerical solution is always numerical, but analytical methods usually give a result in terms of mathematical functions that can be evaluated for specific instances. However, numerical results can be plotted to show some of the behavior of the solution of a problem (Bella, 2014).

The numerical analysis result is an approximation, but the results can be made as accurate as desired. Ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in tractable forms. Such problems may each be formulated, for examples internes of algebraic or transcendental equation an ODE or PDE of a set of such equations (Akram, 2013).

Many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving large or small parameters. A differential equation in which the highest order derivative is multiplied by a small positive parameter is called perturbed problem and the parameter is known as the perturbation parameter. Singularly perturbed problems occur in a number of areas of applied mathematics, sciences, engineering, fluid mechanics, elasticity and quantum mechanics (Lie J, 2008).

The mathematical manifestation of boundary layers is due to the presence of small parameters multiplying the coefficients of some, or all of the terms with the higher order derivatives in the differential equations, and such equations are said to be singularly perturbed. Singular perturbation is that the nature of the differential equations changes completely in the limit case, when the singular perturbation parameter is equal to zero (Roos *et al*, 2010).

The concept of an ε uniform numerical method is clearly, define where ε denotes the singular perturbation parameter and the essential idea is that the convergence properties, the accuracy and computational cost of the numerical method should be independent of the value of the singular perturbation parameter ε. Singular perturbation problems we address contain a small parameter ε that reflects the ratio between the slow and the fast time scale (Sabyasachi *et al*, 2017).

Reaction-diffusion describes the process, in which multiple participating chemicals or agents react with each other, while simultaneously diffusing or spreading through a liquid or gaseous medium. A reaction diffusion equation comprises a reaction term and a diffusion term (Christina, 2011). A reaction-diffusion model is a system of mathematical equations that describe how the concentration of one or more substances are affected by reaction and diffusion processes. Boundary value problems for system of singularly perturbed differential equation often occur, for example, in modeling and analysis of heat and mass transfer processes when the thermal conductivity and diffusion coefficients are small and or the rate of reactions is large, the processes of mathematical modeling of chemical reactions with point sources taken into account, there arises a boundary value problem on an infinite interval for a system of reaction diffusion with point source. There is thin translation layer where solution varies rapidly with narrow region called boundary layer (Miller *et al*, 2010). Most of the existing classical finite difference methods which have been used in solving different order of singularly perturbed problems of reaction diffusion equation give good result only when the mesh size h is much less than the perturbation parameter ε (Phaneendra *et al*, 2015).

Recently, different scholars proposed different numerical method to solve the singularly perturbed boundary value problems of the reaction diffusion problems, for instance,

Fasika *et al*, (2016) have proposed fourth order of compact finite difference method for singularly perturbed 1D reaction diffusion problems, explains higher accuracy but the reaction term is only a constant. Feyisa *et al*, (2017) have proposed six order of compact finite difference method for singularly perturbed 1D reaction diffusion problems, the authors explain the rate of convergence and tries to validate their schemes from examples but the examples of their schemes perturbation parameter and mesh size much too close each other and not uniformly convergent. Phaneendra *et al,* (2015) have proposed a fitted arithmetic average

three-point finite difference method and even if their scheme validated of maximum absolute error applying on some numerical examples that give more accurate numerical solution but, there is no detailed explanation of analysis method on uniform convergent and stability. Terefe *et al*, (2016) proposed that forth order stable central difference method for self- adjoint and Yitbarek *et al*, (2017) proposed that six order stable central difference method for self-adjoint singularly perturbed 1D reaction diffusion problems, Subburayan, (2013) proposed that an initial value technique for singularly perturbed reaction diffusion problems with a negative shift, Gracia *et al*, (2015), proposed that numerical approximation of solution derivatives in the case of singularly perturbed time dependent reaction diffusion problems, Clavero *et al*, (2010) proposed that the uniformly convergent of a finite difference schemes for singularly perturbed reaction diffusion problems, Zhou *et al*, (2016) proposed that a variation perturbation method for solving singularly perturbed reaction diffusion problems. But, still it is observed that there is lack of accuracy and convergence because the treatment of singular perturbation problems is not trivial distribution and the solution profile depends on perturbation parameter ε and mesh size h (Doolan *et al*, 1980) and (Roos *et al*, 1942).

Thus, from the above investigation, Fasika *et al*, (2016), presents well accuracy, but, the reaction term is only constant. Terefe *et al*, (2016) and Yitbarek *et al*, (2017) presents self - adjoint higher order reaction diffusion boundary value problems with a reaction term is a continuous function presents well convergence and stability but not more accurate. Zhou, (2016) and Clavero, (2010), have proposed time dependent reaction diffusion problems for different methods and presents well accuracy. So, these study have been interested to improve the accuracy of reaction-diffusion boundary value problems.

Therefore, the main objective of this study is to develop a stable, convergent and gives more accurate numerical solution than some of the existing methods for solving singularly perturbed reaction-diffusion boundary value problems.

1.2 Objective of the study

1.2.1 General objective

The general objective of this study is to develop accelerated non-standard finite difference method for solving singularly perturbed reaction-diffusion boundary value problems.

1.2.2 Specific objectives

- ❖ To formulate second order non-standard finite difference schemes for solving singularly perturbed reaction-diffusion boundary value problems.
- ❖ To apply Richardson extrapolation method to accelerate the rate of convergence.
- ❖ To establish the convergence of the proposed method.

1.3. Significance of the study

The results obtained in this research may:

- Provide some background information for other researchers who work on this area.
- Introduce the application of numerical methods in different field of studies.
- Help the graduate students to acquire research skills and scientific procedures.

1.4. Delimitation of the study

This study was delimited to accelerated non- standard finite difference method for solving singularly perturbed reaction-diffusion boundary value problems of the form:

$$-\varepsilon y''(x)+b(x)y(x)=f(x), \qquad x\in\Omega=(0,1)$$

subject to the boundary conditions:

$$y(0)=\alpha \text{ and } y(1)=\beta$$

where, ε is perturbation parameter that satisfies $0<\varepsilon<<1$, α, β are given constants and the functions $b(x)$, $b(x)\geq b>0$, b is a constant and $f(x)$ are assumed to be sufficiently continuous differentiable functions.

CHAPTER TWO: REVIEW OF RELATED LITERATURES

2.1 Singularly perturbed boundary value problems

Singular perturbation problems containing a small parameter ε multiplying to their highest derivative term arise in many field, such as fluid mechanics fluid dynamics, chemical reactor theory and elasticity, which have received significant attention (Nayfeh, 1981). Singularly perturbed problems are differential equations (ODE or PDE) that depend on a small positive parameter ε and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches to zero, such problems are said to be singularly perturbed, where we regard ε as perturbation parameter (Roos *et al*, 1942).

Singular perturbations appear in various fields of science and engineering. It is known that these problems depend on a small positive parameter ε in such a way that the solution exhibits a multiscale character, i.e, there are thin layers where the solutions changes rapidly, while a way from the layers it behaves regularly, and hence numerical solutions of singularly perturbed problems usually presents difficulties that we have to be careful when choosing numerical methods. There exist a variety of methods for solving singularly perturbed boundary value problems, such as methods based on the initial value technique, as to research works on numerical solution of singularly perturbed boundary value problems (Phaneendra *et al*, 2015). Science and technology develops many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving small parameters have become increasingly complex and therefore require the use of asymptotic methods. The term singular perturbations were first used by (Friedrichs, 1946) in a paper presented at a seminar on non-linear vibrations at New York University. Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, civil engineering, optimal control, etc. The classification of singularly perturbed higher order problems depend on how the order of the original equation is affected if one sets, where is a small positive parameter multiplying the highest derivative occurring in the differential equation. If the order is reduced by one, we say that the problem is of convection-diffusion type and of reaction-diffusion type

if the order is reduced by two. It is well known that the solution of singularly perturbed boundary value problems is described by slowly and rapidly varying parts. So there are thin transition layers where the solution can jump suddenly, while away from the layers the solution varies slowly and behaves regularly (Akram and Afia, 2013).

The order of differential equation is reduced by one the singular perturbation problem is said to be convection diffusion equation. But if the order of the differential equation is reduced by two, then the singularly perturbed differential equation is said to be reaction diffusion problem (Phaneendral *et al*, 2012). Basically, the problem of ineffectiveness for solving singularly perturbed problems has been associated with the perturbation parameter. Accordingly, more efficient and simpler numerical methods are required to solve singularly perturbed reaction diffusion boundary value problems. In recent years a large number of methods have been established to provide accurate result. (Phaneendra *et al*, 2015; Terefe *et al*, 2016; Jalilian *et al*, 2015; Fasika *et al*, 2016 ; Yitbarek *et al*, 2017; Ghazala and Imran, 2014; Sonali and Hradysh, 2015). (O'Malley, 1974, 1991; Nayfeh, 1973, 1981; Cole and Kevorkian, 1979; Bender and Orszag, 1978; Eckhaus, 1973; Vandyke, 1975; Bellman, 1964), and have the details of numerical and asymptotic solutions in (Doolan *et al*, 1980; Goering *et al*, 1983; Hemker, 1977; Hemker and Miller, 1979; Miller, 1993; Miller *et al*, 1996). Those show that a considerable amount of work has been done for the development of numerical methods to boundary value problems using various methods.

2.2 Exact scheme of finite difference

An exact finite difference scheme is one for which the solution to the difference equation has the same general solution as the associated differential equation. the solution of the difference scheme is exactly equal to the solution of the ODE on the computational grid for fixed, but, arbitrary step-size h difference scheme exists. If the solution is known, then the scheme can be constructed (Micken, 1999). The situation regarding exact finite difference schemes is more problematic for PDEs. as for the case of ODE's, the question as to whether exact schemes can be constructed for PDE is very dependent upon the existence of known solutions to the PDE of interest. Another difficulty is the problem of defining precisely what

is to be understood as the general solutions of the given PDE (Jean, 2016). A detailed examination and analysis of the various exact finite difference schemes of the following conclusion: first discrete models for the functional dependence on the step size than those given by conventional methods, Second, a major characteristics of exact schemes is the discrete modeling of non-linear terms by nonlocal representation on the computational grid. a major advantage of having exact finite difference schemes for deferential equations is that various questions related to the issues of consistency, stability and convergence do not arise (Jean *et al*, 2016).

2.3 Non-standard finite difference

Non-standard numerical methods were introduced by Mickens (1994) as a viable tool that provides approximate solution to differential equations and retain the qualitative properties of the equation. In Mickens (1994, 2000), valuable reasons for numerical instabilities were given in some particular investigated cases. The preservation of the qualitative properties of the considered differential equation with respect to these schemes is of great interest in finite difference methods of solving differential equations. The major consequence of this result is that such scheme does not allow numerical instabilities to occur. He proposed a new method of construction of discrete models whose solution have the same qualitative properties as that of the corresponding differential equations for all step-sizes and thus eliminate the elementary numerical instabilities that can arise (Obyomi and Olabode, 2013).

Non-standard finite difference schemes are generalization of the usual discreet models of deferential equations, their most important property is in many cases, the complete absence of the elementary numerical instabilities which plague the usual finite difference schemes (Mickens, 1999). The non-standard scheme is elementary stable in the limit case of the space independent variable. furthermore, the scheme is stable with respect to conservation of energy in the stationary case. The scheme is stable with respect to the boundedness and positivity property, this scheme is elementary stable in the limit case of the space independent variable and it is also stable with respect to the conservation of energy in the stationary case (Jean, Lubuma, 2016).

Non-standard method is more stable than the standard finite methods and the domain of h for stability in the non-standard is larger than those of the standard method, if the denominator functions are chosen in appropriate from the non-standard methods produce better results (Yoghoubi, 2015). The numerical methods are expected to define discrete dynamical system that are required to preserve the essential properties of the exact solution. The short coming of the classical numerical methods specifically the theta methods, for being reliable discrete dynamical systems is that the step size s subjected to a constraint. The time step size should be small enough if the schemes wear to replicate qualitatively properties of the exact solutions. the schemes we study are non-standard variants of the theta method, the non-standard finite difference method aims at preserving the qualitative properties at no cost with regard to the value of time step size. Non-standard finite difference schemes that have no spurious fixed points comparing to the dynamical system under consideration, the linear stability/ instability, property of the fixed points, being the same for both discrete and continuous system obtains a sharper condition for the elementary stability of the schemes (Phumezile Kama, 2009).

As introduced in the literature, most researchers have been tried to find approximate solution for singularly perturbed boundary value reaction-diffusion problems. Some of the researchers who have done used to a constant coefficients and some of them who have done used a variable coefficient. But, some researchers have not been get more accurate solutions. Owing this, we find on more accurate and stable numerical method for solving singularly perturbed reaction-diffusion boundary value problem by using accelerated non-standard finite difference method.

CHAPTER THREE: METHODOLOGY

This chapter consists: study area and period, study design, source of information and mathematical procedures.

3.1. Study area and period

This study is conducted at Jimma University department of mathematics from January 2019 to February 2020 G.C. Conceptually, the study was focused on accelerate non-standard finite difference method for solving singularly perturbed reaction-diffusion boundary value problem.

3.2. Study design

The study was employed mixed design (i.e. documentary review and numerical experimental design).

3.3. Source of information

The relevant sources of the information for this study are books, journals and related studies from internet.

3.4 Mathematical procedure

In order to achieved the stated objectives, the study followed the following procedures:

- Describing the problem.
- Discretizing the solution domain/ interval.
- Formulating non-standard finite difference scheme
- Apply Richardson extrapolation method.
- Establish the consistency and stability of the scheme.
- Writing MATLAB code for the obtained scheme.
- Illustrate with numerical examples and results to support theoretical descriptions.

CHAPTER FOUR: DESCRIPTION OF THE METHOD

4.1 Formulation of the method

Consider the singular perturbed reaction-diffusion boundary value problem of the form:

$$-\varepsilon y''(x)+b(x)y(x)=f(x), \qquad x\in\Omega=(0,1) \qquad (1)$$

subject to the boundary conditions:

$$y(0)=\alpha \text{ and } y(1)=\beta \qquad (2)$$

where, ε is a perturbation parameter that satisfies $0<\varepsilon<<1$, α , β are given constants and the functions $b(x)$, b(x) ≥ b > 0, b is a constant and $f(x)$ are assumed to be sufficiently continuous differentiable functions.

Now we divide the interval $\overline{\Omega}=[0,1]$ in to N equal parts with constant mesh length $h=\frac{1}{N}$.

Let $0=x_0<x_1<.....x_N=1$ be the mesh points. Then we have $x_i=ih, \quad i=1,2,...,N$.

To describe a finite difference scheme for equation (1), we consider the homogeneous equation:

$$-\varepsilon y''(x)+b(x)y(x)=0 \qquad (3)$$

where $b(x)\geq b>0$, b is a constant.

Here equation (3) has two linear independent solutions, namely, $e^{\rho x}$ and $e^{-\rho x}$ with $\rho=\sqrt{\frac{b}{\varepsilon}}$

We denote the approximate solution to $y(x)$ at the grid points x_i by y_i. The theory of difference given by Jean *et al*, (2006), shows that the second order linear difference equation

$$\begin{vmatrix} y_{i-1} & e^{\rho x_{i-1}} & e^{-\rho x_{i-1}} \\ y_i & e^{\rho x_i} & e^{-\rho x_i} \\ y_{i+1} & e^{\rho x_{i+1}} & e^{-\rho x_{i+1}} \end{vmatrix}=0$$

Since $x_{i\pm1} = x_i \pm h$ we have

$$y_{i-1}\left(e^{-\rho h} - e^{\rho h}\right) - y_i\left(e^{-2\rho h} - e^{2\rho h}\right) + y_{i+1}\left(e^{-\rho h} - e^{\rho h}\right) = 0$$

Because of the relations $e^{-2\rho h} - e^{2\rho h} = \left(e^{-\rho h} - e^{\rho h}\right)\left(e^{\rho h} + e^{-\rho h}\right)$ and also $2\cos(\rho h) = e^{\rho h} + e^{-\rho h}$, the above equation can be re-written as

$$-y_{i-1} + 2\cosh(\rho h)\, y_i - y_{i+1} = 0 \qquad (4)$$

Now, equation (4) is the exact difference scheme of equation (3) and in the sense that the difference equation in equation (4) has the same general solution, which is the form $y_i = c_1 e^{\rho x_i} + c_2 e^{-\rho x_i}$ as the differential equation given in equation (3) Jean *et al*, (2006).

Using the identity $\cosh(\rho h) = 1 + 2\left(\sinh\left(\frac{\rho h}{2}\right)\right)^2$, equation (4) can be transformed to

$$-\varepsilon \frac{y_{i-1} - 2y_i + y_{i+1}}{\frac{4}{\rho^2}\left(\sinh\left(\frac{\rho h}{2}\right)\right)^2} + by_i = 0$$

This implies that the exact scheme of the non-homogeneous equation

$$-\varepsilon y''(x) + b(x)y(x) = f(x) \qquad (5)$$

Equation (5) given by

$$-\varepsilon \frac{y_{i-1} - 2y_i + y_{i+1}}{\frac{4}{\rho^2}\left(\sinh\left(\frac{\rho h}{2}\right)\right)^2} + b_i y_i = f_i \qquad (6)$$

4.2 Non-standard finite difference scheme

A difference equation to determine approximate solution y_i to the solution $y(x)$ of the given governing equation is called a non-standard finite difference method if the classical

denominator h or h^2 of the discrete first order or second order derivative is replaced by a non- negative function ϕ such that

$$\phi(h) = h + O\left(h^2\right) \text{ or } \quad \phi(h) = h^2 + O(h^3) \quad \text{as } 0 < h \to 0$$

As Jean *et al*, (2006), provided the important observation that the complex structure of the denominator of the discrete derivative in equation (6) constitutes a general property of these schemes, which is useful while designing reliable schemes for such problems. To demonstrate the procedure, we consider equation (1) which at a fixed node x_i, reads as

$$-\varepsilon y_i'' + b_i y_i = f_i \qquad (7)$$

Interested by equation (6),we may approximate by equation (7) by the non-standard scheme as

$$-\varepsilon \frac{y_{i-1} - 2y_i + y_{i+1}}{\phi_i^2} + by_i = f_i \qquad (8)$$

where $\phi_i = \phi_i(h, \varepsilon) = \frac{2}{\rho}\sinh\left(\frac{\rho h}{2}\right) = h^2 + O\left(h^3\right)$, and $\rho = \sqrt{\frac{b_i}{\varepsilon}}$

The non-standard scheme in equation (8) can be written in three term recurrence relation

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i, \qquad i = 1, 2, 3, ..., N-1 \qquad (9)$$

where $E_i = \frac{\varepsilon}{\phi_i^2} = G_i$, $F_i = b_i + \frac{2\varepsilon}{\phi_i^2}$ and $H_i = f_i$,

Now, the system given in equation (9) can be re- write in matrix form as

$$MY = H^* \qquad (10)$$

where M is the $(N-1)\times(N-1)$ coefficient matrix, Y and H^* are column matrices as given below

$$M = \frac{\varepsilon}{\phi_i^2}\begin{bmatrix} 2+b_i\frac{\phi_i^2}{\varepsilon} & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2+b_i\frac{\phi_i^2}{\varepsilon} & -1 & 0 & \vdots & \vdots \\ 0 & -1 & 2+b_i\frac{\phi_i^2}{\varepsilon} & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \vdots & 0 & -1 & 2+b_i\frac{\phi_i^2}{\varepsilon} & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2+b_i\frac{\phi_i^2}{\varepsilon} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix} \quad \text{and} \quad H^* = \begin{bmatrix} H_1 + E_1 y_0 \\ H_2 \\ \vdots \\ H_{N-2} \\ H_{N-1} + G_1 y_N \end{bmatrix}$$

4.3 Richardson extrapolation

The basic idea behind extrapolation is that whenever the leading term in the error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of the parameter mesh sizes h and $0.5h$ to obtain a higher-order approximation and the technique is known as Richardson extrapolation. This procedure is a convergence acceleration technique which consists of considering a linear combination of two computed approximations of a solution (on two nested meshes). The linear combination turns out to be a better approximation.

From the definition of non-standard finite difference approximation, the approximate differential term is second order that indicates the truncation error of the formulated method is $O(h^2)$. Hence, we have

$$\left|y(x_i) - Y_N\right| \le C\left(h^2\right) \qquad (11)$$

where $y(x_i)$ and Y_N are exact and approximate solutions respectively, C is constant independent of mesh sizes h.

Let Ω^{2N} be the mesh obtained by bisecting each mesh interval in Ω^N and denote the approximation of the solution on Ω^{2N} by Y_{2N}. Consider equation (11) works for any $h \neq 0$, which implies:

$$y(x_i) - Y_N \leq C\left(h^2\right) + R^N, \qquad x_i \in \Omega^N \tag{12}$$

So that, it works for any $\frac{h}{2} \neq 0$ yields:

$$y(x_i) - Y_{2N} \leq C\left(\left(\frac{h}{2}\right)^2\right) + R^{2N}, \qquad x_i \in \Omega^{2N} \tag{13}$$

where the remainders, R^N and R^{2N} are $O(h^4)$. A combination of inequalities in equation (12) and (13) leads to $3y(x_i) - \left(4Y_{2N} - Y_N\right) = O(h^4)$, which suggests that

$$\left(Y_N\right)^{ext} = \frac{1}{3}\left(4Y_{2N} - Y_N\right) \tag{14}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$\left| y(x_i) - \left(Y_N\right)^{ext} \right| \leq Ch^4 \tag{15}$$

Now, using these two different solutions which are obtained by the same scheme given by equation (9), we get another third solution in terms of the two by equation (15). This is Richardson extrapolation method for the second order non-standard finite difference scheme only to accelerate the rate of convergence to fourth order.

4.4 Consistency of the method

Local truncation errors refer to the differences between the original differential equation and its finite difference approximations at grid points and measure how well a finite difference discretization approximates the differential equation (Zhilin *et al*, 2018). In the present method, the truncation error given in equation (11) written as

$$|TE| \leq Ch^2 \tag{16}$$

A finite difference scheme is called consistent if the limit of truncation error *TE* is equal to zero as the mesh size *h* goes to zero (Siraj *et al*, 2019). Now, by this definition the consistency of present scheme given in equation (9) with the local truncation error in equations (11) and (15) satisfied as:

$$\lim_{h\to 0} TE = \lim_{h\to 0} Ch^2 = \lim_{h\to 0} Ch^4 = 0 \qquad (17)$$

Thus, the proposed method is consistent.

4.5. Stability of the method

If we multiply both sides of equation (9) by ϕ_i^2 and consider the limit as $\phi_i^2 \to 0$, because ϕ_i^2 is a function of mesh size *h*, we get:

$$E_i = G_i = \varepsilon \quad \text{and} \quad F_i = 2\varepsilon \qquad (18)$$

This leads to the coefficient matrix *M* of the system becomes

$$M = \begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & \cdots & \cdots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \vdots & \vdots \\ 0 & -\varepsilon & 2\varepsilon & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & -\varepsilon & 0 \\ \vdots & \vdots & 0 & -\varepsilon & 2\varepsilon & -\varepsilon \\ 0 & \cdots & 0 & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

Here, M is a tri-diagonal matrix. The co-diagonal contains E_i, G_i with $E_i \neq 0$ and $G_i \neq 0$, $\forall\, i = 1, 2, \cdots, N-1$. Hence, M is irreducible. Also, $|E_i| > 0$ and $|G_i| > 0$ and in each row of M, the sum of the two off-diagonal elements less than or equal to the modulus of the diagonal element.

$$\Rightarrow |F_i| \geq |E_i| + |G_i|$$

This proves the diagonal dominant of M. Under these conditions the Thomas algorithm is stable.

As proved by (Smith, 1985), the eigenvalues of a tri-diagonal matrix $(N-1)\times(N-1)$ of matrix *M* are:

$$\lambda_s = F_i - 2\sqrt{E_i G_i}\cos\frac{s\pi}{N}, \quad s = 1, 2, \ldots, N-1$$
$$= 2\varepsilon - 2\sqrt{\varepsilon^2}\cos\frac{s\pi}{N} = 2\varepsilon\left(1-\cos\frac{s\pi}{N}\right)$$

Moreover, from trigonometric identity, we have: $1-\cos\frac{s\pi}{N} = 2\sin^2\frac{s\pi}{2N}$.

Hence, the eigenvalues of matrix *M* re-written as:

$$\lambda_s = 2\varepsilon\left(2\sin^2\frac{s\pi}{2N}\right) = 4\varepsilon\left(\sin\frac{s\pi}{2N}\right)^2 \le 4\varepsilon \qquad (19)$$

A finite difference method for the boundary value problem is stable if *M* is invertible and

$$\left\|M^{-1}\right\| \le C, \quad \forall\ 0 < h < h_0 \qquad (20)$$

where *C* and h_0 are two constants that are independent of *h*,(Zhilin *et al*, 2018).

Since, matrix *M* is symmetric, its inverse M^{-1} is symmetric and the eigenvalues M^{-1} is given by $\frac{1}{\lambda_s}$. Thus, by the definition, we have:

$$\left\|M^{-1}\right\| = \frac{1}{\lambda_s} = \frac{1}{4\varepsilon} \le C$$

where *C* is independent of *h*. Hence, the developed scheme in equation (9) is *stable*.

A consistent and stable finite difference method is *convergent* by Lax's equivalence theorem (Smith, 1985). Hence, as we have shown above the proposed method is satisfying the criteria for both consistency and stability which are equivalents to convergence of the method.

4.6 Numerical examples and results

In order to test the validity of the proposed method and to demonstrate their convergence computationally, we have taken two model examples. The maximum absolute errors (*AE*) at the nodal points are given by:

$$|AE| = \max_{1\le i\le N-1} |y(x_i) - (Y_N)^{ext}|.$$

The rate of convergence (*R*) can be calculated by the formula:

$$R = \frac{\log\left(Y_N\right)^{ext} - \log\left(Y_{2N}\right)^{ext}}{\log 2}$$

where $y(x_i)$ and $\left(Y_N\right)^{ext}$ are exact solution and numerical solution respectively, at the nodal point x_i and for the rate of convergence Y_N and Y_{2N} are the numerical solutions obtained by the mesh size h and $\frac{h}{2}$ respectively.

Example 1: Consider the singularly perturbed problem

$$-\varepsilon y''(x) + 1 + x(1-x)y(x) = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions $y(0) = 0 = y(1)$

where $y(x) = 1 + x(1-x) + \left(2\sqrt{\varepsilon} - x^2 + x^3\right)\exp(\frac{-(1-x)}{\sqrt{\varepsilon}}) + \left(2\sqrt{\varepsilon} - x\left(1-x\right)^2\right)\exp(\frac{-x}{\sqrt{\varepsilon}})$

The exact solution is given by: $y(x) = 1 + (x-1)\exp(\frac{-x}{\sqrt{\varepsilon}}) - x\exp(\frac{-(1-x)}{\sqrt{\varepsilon}})$

Example 2: Consider the singular perturbation problem:

$$\varepsilon y''(x) - y(x) = \left(\cos \pi x\right)^2 + 2\varepsilon\pi^2 \cos 2\pi x, \quad 0 < x < 1$$

subject to the boundary conditions $y(0) = 0 = y(1)$.

The exact solution is given by $y(x) = \dfrac{\exp\left(\dfrac{-(1-x)}{\sqrt{\varepsilon}}\right) + \exp\left(\dfrac{-x}{\sqrt{\varepsilon}}\right)}{1 + \exp\left(\dfrac{-1}{\sqrt{\varepsilon}}\right)} - \left(\cos \pi x\right)^2$

Table 1. Comparison of maximum errors in solution for Example 1

$\varepsilon\downarrow$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Present method					
2^{-7}	2.1232e-05	1.4248e-06	9.0852e-08	5.7354e-09	3.5888e-10
2^{-8}	5.3610e-05	3.8419e-06	2.5412e-07	1.6026e-08	1.0068e-09
2^{-9}	1.0810e-04	1.0494e-05	7.0151e-07	4.4962e-08	2.8290e-09
2^{-10}	1.5101e-04	2.6881e-05	1.9157e-06	1.2574e-07	7.9518e-09
Terefe *et al*, (2016)					
2^{-7}	0.1301e-03	0.8424e-05	0.5255e-06	0.3280e-07	0.2053e-08
2^{-8}	0.5910e-03	0.3704e-04	0.2319e-05	0.1450e-06	0.9072e-08
2^{-9}	0.1331e-02	0.1444e-03	0.9916e-05	0.6241e-06	0.3905e-07
2^{-10}	0.1521e-02	0.6190e-03	0.4110e-04	0.2633e-05	0.1640e-06

Table 2. Computed rate of convergence for Example 1

$\varepsilon\downarrow$	$N=16$	$N=32$	$N=64$	$N=128$
Present method				
2^{-7}	3.8974	3.9711	3.9856	3.9983
2^{-8}	3.8026	3.9182	3.9870	3.9926
2^{-9}	3.3647	3.9030	3.9637	3.9903
2^{-10}	2.4900	3.8106	3.9294	3.9830

Table 3. Comparison of maximum errors in solution for Example 1 in the case of $\varepsilon << h$

$\varepsilon\downarrow$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
With Richardson extrapolation					
10^{-4}	2.6397e-05	4.3155e-05	3.2027e-05	3.4788e-06	2.3521e-07
10^{-5}	2.6111e-05	1.6143e-06	7.4307e-06	1.8461e-05	5.4223e-06
10^{-6}	2.6111e-05	2.5972e-14	6.8118e-09	1.8143e-06	9.8950e-07
Without Richardson extrapolation					
10^{-4}	3.7100e-04	2.7768e-04	1.0266e-04	8.6715e-05	2.4288e-05
10^{-5}	3.7100e-04	4.3110e-06	9.2615e-05	4.8464e-05	4.0917e-05
10^{-6}	3.7100e-04	2.5972e-14	1.3795e-08	6.6012e-06	3.7483e-05

Table 4. Comparison of maximum errors in solution of Example 2 for small values of $\varepsilon << h$

$\varepsilon \downarrow$	$h = 2^{-3}$	$h = 2^{-4}$	$h = 2^{-5}$	$h = 2^{-6}$
Present method				
10^{-3}	2.5862e-03	2.5391e-04	4.6771e-05	1.1642e-06
10^{-4}	1.7769e-03	7.8920e-04	1.2666e-04	1.0862e-05
10^{-5}	1.9739e-04	1.9608e-04	1.5115e-04	4.6997e-05
Phaneendra *et al*, (2015)				
10^{-3}	1.23e-2	1.10e-3	7.63e-5	4.90e-6
10^{-4}	2.35e-2	4.60e-3	5.60e-4	4.47e-5
10^{-5}	2.52e-2	6.20e-3	1.40e-3	2.39e-4

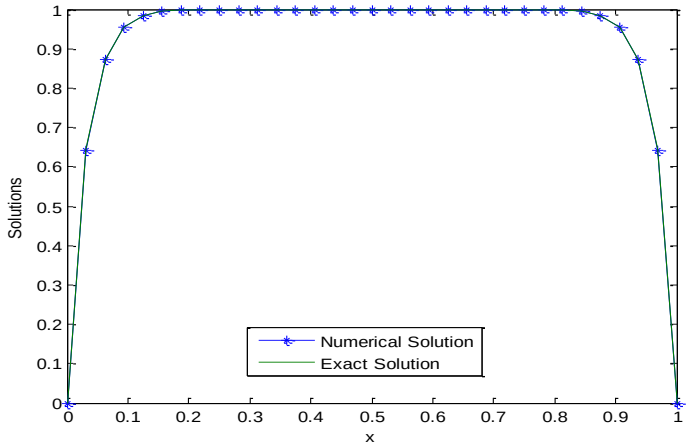

Figure 1: Physical behavior of the exact and numerical solution for Example 1 at $N = 32$ and $\varepsilon = 2^{-10}$

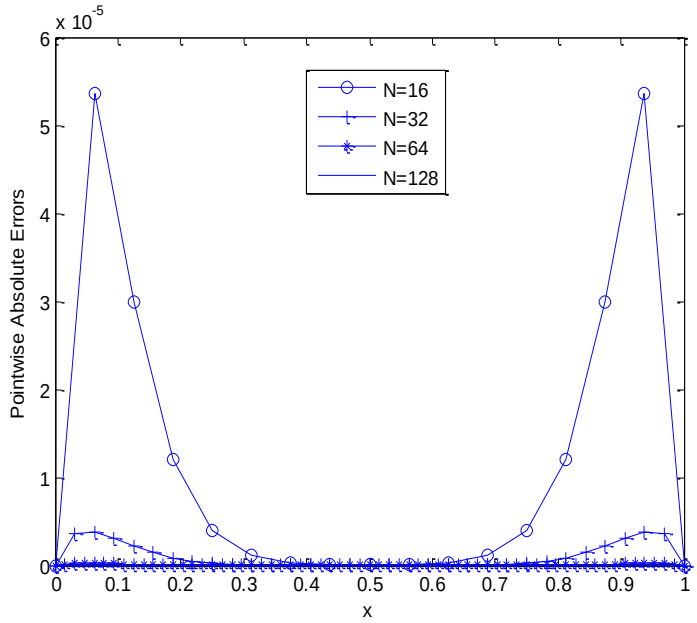

Figure 2: Graph to indicate when number of mesh size decreases at a fixed $\varepsilon = 2^{-8}$, accuracy of the numerical solution increases or errors also decreases.

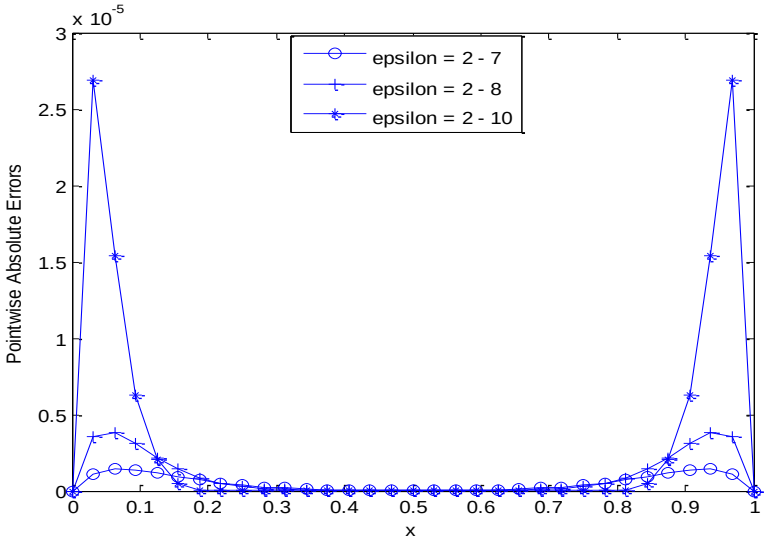

Figure 3: Graph to indicate when number of perturbation decreases at a fixed mesh number $N = 32$, accuracy of the numerical solution decreases.

CHAPTER FIVE: DISCUSSION, CONCLUSION AND SCOPE

5.1. Discussion and conclusion

In this work, accelerated non-standard finite difference method described and analyzed for solving singularly perturbed reaction-diffusion boundary value problems. Richardson extrapolation technique helps to improve accuracy of the solution and accelerate rate of convergence from second order to fourth order. Consistency and stability of the method established clearly and shortly to guarantee the convergence of the method. We consider two model examples to illustrate its numerical results in terms of maximum absolute errors and rate of convergence for different values for the perturbation parameter and mesh sizes (see Tables 1-4 and Figures 1-3). Specifically, Tables 1 and 4 used to verify the betterment of present method by producing more accurate solution that existing methods in the literature. Table 2 shows that the confirmation of fourth order of convergence in theoretical analysis with experimental results. Table 3, clearly demonstrates the effects of applying Richardson extrapolation method and improvement of the accuracy of solution. As the number of intervals N increases accuracy of the solution also increases (Table 1, 3 and 4) which implies convergence of the present method. Additionally, Figure 1 to illustrate as the problem has two (left and right) boundary layers and Figures 2 and 3 shows, the effects of mesh sizes and perturbation parameter with occurrence of maximum absolute errors in the two-layer region.

Generally, accelerated non-standard finite difference method is formulated for the class of singularly perturbed reaction- diffusion boundary value problems, which is stable, convergent and gives more accurate solution than some of the existing methods in the literature.

5.2 Scope of the future work

In this thesis, accelerated non-standard finite difference method is introduced for solving singularly perturbed reaction diffusion problems; hence the scheme proposed in this thesis can also be extended to fitted mesh or higher order on uniform mesh finite difference method for solving singularly perturbed reaction diffusion equations.

Reference

Akram, G. and Afia, N. (2013), Solution of fourth order singularly perturbed boundary value problem using septic spline, Middle-east journal of scientific research.

Bella M.D Hossain, (2014), Numerical solution of higher order boundary value problems using piecewise polynomial bases. university of Dhaka, Dhaka 1000,

Christina Kutter (2011), reaction diffusion equation with application, Aca. Pre.

Christy R.J. and Tamilseluan, (2014). A numerical method for singularly perturbed order differential equation of reaction diffusion type. J. AAP. Math.

Clavero, Carmelo, (2010), the uniform convergent of a finite difference schemes for singular perturbed 1D reaction diffusion problems AAP. MATH.

Doolan, E.P., Miller, J.J.H. and Schilders, W.H.A. (1980). Uniform Numerical Methodsfor Problems with Initial and Boundary Layers. Dublin, Boole Press.

Fasika Wondimu, Gemechis File, and Tesfaye Aga (2017). Sixth-order compact finite Difference method for singularly perturbed one dimensional reaction diffusion Problems Journal of Taibah University for Science 11:302–308.

Feyisa Gadosa and Gemechis File, (2017), Higher order compacts finite difference method For singularly perturbed one dimensional reaction diffusion problems. Journal of the Nigerian Mathematical Society 36(3),491-502.

Friedrichs, K.O., and Wasow, W. (1946). Singular Perturbations of Non-linear Oscillations. Duke. Mathematical Journal, 13: 367-381.

Gracia Eugene, Jose.L. O Riordan, (2015) Numerical approximation of derivative solution with case of singular perturbed time dependent reaction diffusion problems. AAP.J. MATH.

Jean M.S, Lubuma and Kailash C. patider, (2015), Contributions to the theory of non-standard finite difference methods and applications to singularly perturbed problems. World scientific publishing.

Jean. M.S Lubuma, K ailash C.patidare, (2005), uniformly convergent non-standard finite difference method for self- adjoint singularly perturbed problems.

Lewis. P. E. Ward J.P. (1991) The finite element method, principles and applications, Addison Wesley.

Lie, J. (2008) A computational method for solving singularly perturbed two-point singular boundary value problems. Int. J. Math Anal.

Mickens, J. R.E. (1990). Difference Equations: Theory and Applications. Van No strand Reinhold New York.

Mickens, J.R.E. (1994). Non-standard Finite Difference Models of differential Equations. World Scientific: Singapore.

Mickens, J.R.E. (1998). Exact solution to finite difference model of a nonlinear reaction-Advection equation applications for numerical analysis, Numerical Methods for PDE.

Mickens, J.R.E. (1999). Application of Non-standard finite differences scheme.

Mickens, J.R.E. (2000). Application of nonstandard finite difference schemes,world scientific, Singapore.

Mickens, J.R.E. (2005). Advance in the application of non-standard finite differences World Scientific, Singapore.

Nayfeh A.H, (1981), Introduction to perturbation techniques. Wiley, New York.

Obyomi A.A and B.J,Olabode, (2015), A new family of non-standard finite difference schemes for logistic equations. USA J.

Padmaja,P. Chakravarthy, p.p and Reddy, Y.N, (2012), A non-standard explicit method for solving singularly perturbed two point BVPs via method of reaction of order. International J.APP mechanics.

Phaneendra, K I.J (2015), A fitted arithmetic average three points finite deference method for singularly perturbed two-point bounder value problems with dual layers.

Phumezile Kama, (2009) Non-standard finite difference method in dynamical systems. university of Pretoria.

Roos, H.G., Zarin, H. (2010), A second order scheme for singularly perturbed Differential equations with dis-continuous source term. J. Number. Math.

Sabyasachi Chatterjee, Amit Acharya, Zvi Artstein, (2017), Computing singularly perturbed differential equations.

Siraj muslima kedir, Gemechis F.D and Tesfaye A.B, (2019), forth order stable central difference with richardson extrapolation method for second order self-adjoint Singularly perturbed boundary value problems. Journal of the Egyptian mathematics society.

Smith, GD, (1985), numerical solution of PDES, FDMs, third edition, oxford university press, NewYork.

Subburayani,V. and Ramauujian N, (2010), An initial value technique for SP reaction Diffusion Problems with a negative shift. Movi. SAD. J. MATH.

Terefe, Gemechis and Tesfaye, (2016), fourth order stable central difference method for self Adjoint singularly perturbation problems. Ethiop.J.Sci.& Techno.

Yitbarek, Gemechis and Tesfaye, (2017), six order stable central difference method for self adjoint singularly perturbed two point BVPs. Ethio. J. Edu.

Yoghoubi A.A (2015), comparison between standard and non-standard finite Difference methods for solving first and second order ODE. Int. J. APP. Math. Ethiopian Journal of Science.

Zhilin, Li, Zhonghua Q, Tang, T, (2018) Numerical solution of differential equations, introduction to finite difference and finite element methods, printed in the United Kingdom by clays,

Zhou, Guanglu and Boying Wu, (2016), A variation perturbation method for solving the time dependent singular perturbation reaction diffusion problems, thermal science vol. 20.