STABILITY AND BIFURCATION ANALYSIS OF MAXWELL- BLOCH EQUATIONS

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science (M.Sc.) in Mathematics.

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DECLARATION

I, here submit the thesis entitled by "**Stability and Bifurcation Analysis of Maxwell-Bloch Equations**" for the award of degree of Master of Science in Mathematics. I, the undersigned declare that, this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or the like, where other sources of information have been used, they have been acknowledge.

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Abstract

In this thesis, stability and bifurcation analysis of Maxwell-Bloch equations were considered. By the aid of divergence test, it was proved that the system is dissipative. Steady state points of the equations were determined. The equations were linearized using Jacobian matrix about each equilibrium points. The local stability condition of each critical point was proved by using Routh- Huwertiz stability criteria. By the aid of Lyapunov theorem, equilibrium point one was proved to be globally asymptotically stable with some specific condition on pumping energy parameter. It is impossible to speak global stability property of the two remaining equilibrium points in the sense of Lyapunov due to the fact that one of the criteria to apply the theorem is not satisfied. Furthermore, the result of Hopf bifurcation revealed that the system doesn't undergo Hopf bifurcation at equilibrium point one by any choice of pumping energy parameter and with some specific conditions the system undergoes Hopf bifurcation about the two remaining equilibrium points for a certain value of pumping energy parameter. Finally, in order to verify the applicability of the result two numerical examples were solved and MATLAB simulation was implemented to support the findings of the study.

Key words*: Maxwell -Bloch equation, Local stability, global stability, Routh- Huwertiz stability criteria, Lyapunov theorem, Hopf bifurcation.*

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CHAPTER ONE

1. INTRODUCTION

1.1 Background of Study

Mathematical model is a description of a system using mathematical concepts and language. Mathematical modeling is the application of mathematics to describe real-world problems and investigate important questions that arise from it. Mathematical models are used in Physics, Chemistry, Biology, Engineering disciplines, Medicine, Ecology as well as in the social sciences. Mathematical model is a powerful tool for understanding historical, practical and the fundamental phenomena of physics which cannot be understood by verbal reasoning alone (Alder, 2001).

Maxwell-Bloch equations are set of coupled ordinary differential equations, which form the foundation of classical electromagnetism, classical optics and electric circuits together with the Lorenz force law. The equations also provide mathematical model for electric, optical and radio technologies, such as power generation, electric motors, wireless communication, etc
(Maxwell, 1892).

The Maxwell-Bloch equations also called the optical Bloch equations which were first derived by Tito Arecchi and Rodolfo Bonifacio of Milan (Arecchi and Bonifacio, 1965).They described the dynamics of a two-state quantum system interacting with the electromagnetic mode of an optical resonator. The Maxwell- Bloch equations first appeared in quantum optics in the context of the phenomenon called self-induced transparency (SIT). In particular, the Maxwell-Bloch equations describe the interaction between a two-level quantum mechanical system and an electromagnetic wave.

The Maxwell-Bloch equations widely used in non-linear optics in general and to model quantum cascade lasers (QCL) (Jirauschek and Kubis, 2014). These model equations are a system of non-linear ordinary differential equations which plays a prominent role in the field of non-linear optics. This system models the resonant interaction between light and optically active medium consisting of two-level atoms. Non-linear evolution equations have attracted a lot of attentions since they are able to describe the non-linear phenomena in many fields of sciences and Engineering (Ablowity and Clarkson, 2004). The Self-induced transparency (SIT) phenomenon plays a role in overcoming the attenuation in the optical communication systems.

Therefore, scholars or researchers have pointed out the reduced Maxwell-Bloch equations can be applied to get for the phenomenon of self-induced transparency (Hao and Zhang, 2015).

In general, mathematical models of Maxwell-Bloch equations are used in Physics, Chemistry, Biology, Engineering disciplines and others related sciences. In 1965, Tito Arecchi and Rodolfo Bonifacio of Milan discovered the Maxwell-Bloch equations which is a system of non-linear ordinary differential equations of the form:

$$\begin{aligned} \frac{dx}{dt} &= k\left(y-x\right) \\ \frac{dy}{dt} &= r_1\left(xz-y\right) \\ \frac{dz}{dt} &= r_2\left(\lambda+1-z-\lambda xy\right) \end{aligned} \tag{1.1}$$

Where the parameter λ may be positive, negative or zero, k, r_1 and r_2 are positive parameters. λ is a pumping energy parameter, k is the decay rate in the laser cavity due to beam transmission, r_1 is the decay rate of the atomic polarization, r_2 is the decay rate of the population inversion, x is the dynamics of the electric field, y is Atomic polarization and z is the population inversion.

Non-linear mathematical models of real-world phenomena that are formulated in terms of ordinary differential equations as in Eq. (1.1) are not easy to directly solve for their solution and hence it is necessary to use qualitative approaches, such as stability and bifurcation analysis, to investigate their solution behaviors. Stability theory plays a central role in system engineering, especially in the field of control systems and automation with regard to both dynamics and control. Bifurcation occurs when a small change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior (Blanchard *et al.*, 2006). In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoretical ecology, there is the application of what is referred to as bifurcation analysis; the analysis of a system of non-linear ordinary differential equations under parameter variation.

Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex eigenvalues of linearized system crosses the imaginary axis of the complex plane.

A dissipative system is defined as a system whose phase space volumes shrink whereas in a conservative system phase space volume is conserved. Conservative systems have constant entities (usually, energy). Physically, we mean systems with no influx and no production of energy/matter.

Dissipative systems lose energy with time. In order to maintain persistent behaviors the dissipative system must have influx of energy/matter. If a dissipative system starts at its stable equilibrium point, it stays there for arbitrarily long and one cannot see the basin of attraction and compression of the volume (Strogatz, 1994).

Hassard*et al.* (1981). studied the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions by applying the normal form theory and the center manifold theorem. In 2019, Makwata*et al*. investigated stability and bifurcation analysis of fishery model with allee effects and they obtained the three different equilibrium solutions as one being stable and with two being saddles. Pijush*et al*.(2018). investigated the stability and bifurcation analysis of three-species food chain model with fear and they concluded that chaotic dynamics can be controlled by the fear factors. Yang *et al*.(2017). described chamostat model which involve control strategy with threshold window are analyzed. They investigated the qualitative analysis such as existence and stability of equilibrium points of the system and proved that pseudo-equilibrium cannot coexist. In 2017, Algaba*et al*. studied the local bifurcations of equilibrium in the Lorenz system, when the parameters are allowed to take any real value has been successfully completed in the case of the pitchfork and Hopf bifurcations.

Tee and Salleh.(2016). Investigated Hopf bifurcation of non-linear modified Lorenz system using normal form theory that was the technique used in Hassard*et al.*(1981). In 2015, Nijamuddin Ali and Santabrata Chakravarty proposed the stability and bifurcation analysis of three species competitive food chain model system incorporating prey-refuge and this study showed that competition among predators could be beneficial for predators.

However, Stability and Bifurcation analysis of Maxwell-Bloch equations is not yet investigated in the existing literature. Therefore, the main objective of this study is to analysis Stability and Bifurcation of Maxwell-Bloch equations given by equation (1.1).

1.2 Statement of the Problem

This study focuses on the following problems.

- System property in relation to dissipative, conservative or neither.
- Local stability conditions of Maxwell- Bloch equations.
- Global stability conditions of Maxwell- Bloch equations.
- Hopf bifurcation conditions of Maxwell -Bloch equations.

1.3 Objectives of the Study

1.3.1 General Objective of the Study

The general objective of this study is to analysis Stability and Bifurcation of Maxwell-Bloch equations given by Eq. (1.1).

1. 3.2 Specific Objectives of the Study

The specific objectives of the study are:

- To check whether the system is dissipative, conservative or neither.
- To determine local stability conditions of Maxwell- Bloch equations.
- To determine global stability conditions of Maxwell- Bloch equations.
- To determine Hopf bifurcation conditions of Maxwell -Bloch equations.

1.4 Significance of the Study

This study helps others professionals working on the area of electricity and magnetism by providing them appropriate conditions for well transfer of message in electric circuit, optics and radio technologies, such as power generation, electric motors and wireless communication.

1.5 Delimitation of the Study

This study is delimited to Stability and Bifurcation analysis of Maxwell-Bloch equations given by Eq. (1.1).

CHAPTER TWO

2. LITERATURE REVIEW

Mathematical physics refers to the development of mathematical methods for the application of mathematics to solve the problems in physics. Mathematical physics is the application of mathematical modeling to solve problems in physics and physics phenomena. It is one of the fastest growing research areas in mathematics and it contributing significantly to our understanding of the real-world and also, mathematical physics is an interdisciplinary subject where theoretical physics and mathematics intersect (Jan Philip, 1960).

The Maxwell-Bloch equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism and from them one can develop most of the working relationships in the field. Because of their concise statement, they embody a high level of mathematical sophistication and therefore not generally introduced in an introductory treatment of the subject (Maxwell, 1892).The quantum Maxwell-Bloch equations for spatially in homogeneous semi-conductor lasers are derived from fully quantum mechanical operator dynamics described the interaction of the light field with the quantum states of the electrons and the holes near the band gap (Holger and Hess, 1999).

Maxwell-Bloch equations and the light field equations correspond exactly to the classical Maxwell’s equations it is possible to focus only on the local light-matter interaction (Hess and Kuhn, 1996).The Maxwell-Bloch equations have more parameters than the Lorenz system; this justifies a more detailed parameter study and Chaotic behavior has been experimentally observed in laser systems and also, studying the range of parameters for which this can occur is important for controlling chaos in possible applications (Haken, 1985).

The theory of non-linear dynamics systems or non-linear control systems if control inputs are involved has been greatly advanced since the 19^{th} century. Today, non-linear control systems are used to describe a great variety of scientific and engineering phenomena ranging from social, life and physical sciences to engineering and technology. Stability of a dynamical system, with or without control and disturbance inputs is a fundamental requirement for its practical value, particular in most real- world (Merkin, 1997).

Narducci and Squicciarini.(1986). investigated the exact linear stability analysis of the plane-wave Maxwell-Bloch equations for a ring laser, then they obtained the linearized Maxwell-Bloch equations and show how one can derive and exact characteristic equation that holds for arbitrary values. The Linear stability analysis and investigation of the periodic and chaotic self-pulsing behavior are presented for the Maxwell-Bloch equations of a bi stable model in contact with asqueezed vacuum field (Hassan *et al*, 2000).

Puta, M.(2000). studied the stability problem of the Maxwell-Bloch equations from laser- matter dynamics with one control applied on the OX_1 axis of the closed-Loop system. In 2003, Hacinliyan and Aybar investigated the non-linear stability and Hopf bifurcation of Maxwell-Bloch equations and determined the full parameters space for chaotic behavior in laser models. In 2008, Thair and Azzawi studied the stability of the non-linear ordinary differential systems and they concluded that the stability conditions of Lorenz non-linear ordinary differential equations at its critical points by depending on the parameters.

In 2009, Arnold investigated the discretization of Maxwell-Bloch dynamical systems and he determined the stationary states of the cavity field which are subjected to stability analysis. The Maxwell-Bloch equations dynamical system consists of the Maxwell's equation for an electromagnetic field coupled to a quantum electronic system to represent a resonant or near resonant polarization induced in the propagation medium. Hacinliyan *et al*.(2010). studied the stability and Chaotic behavior of the approximate solutions of Maxwell-Bloch equations based on the LotkaVolterra system. In 2012, Nurul Huda Gazi studied the stability analysis and Hopf bifurcation of dynamical behavior of fish and mussel population in a fish farm. In 2015, Robert studied the linearization and stability analysis of non-linear ordinary differential equations and he analyzed the theory of non-linear ordinary differential equations.

CHAPTER THREE

3. METHODOLOGY

3.1 Study Period

This study was conducted from September, 2018 to February, 2020.

3.2 Study Design

The study employed mixed design (analytical and experimental approaches).

3.3 Source of Information

The sources of information for the study were journals, published article and related information from internet.

3.4 Mathematical Procedures

This study was conducted based on the following procedures:-

1. Checking whether the system is dissipative, conservative or neither;
2. Determining the equilibrium point of the system;
3. Linearizing Maxwell- Bloch equations about equilibrium points;
4. Determining the local stability conditions of the system;
5. Analyzing the global stability of the system;
6. Determining Hopf bifurcation conditions of the system;
7. Verifying the result using numerical simulation.

CHAPTER FOUR

4. RESULT AND DISCUSSIONS

4.1 Preliminaries

Definition 1: Let f be any vector field of the system, then $\frac{dD}{dt} = \int \nabla . f dD$

Where $\nabla . f\left(x, y, z\right)$ is divergence of the vector field f. If D is decreasing exponentially, then the system is dissipative. If D is increasing exponentially, then the system is expansive and if D is constant, then the system is conservative (Strogatz, 1994).

Definition 2: Consider non-linear system $\frac{dx}{dt} = f\left(x\right)$, where $f : R^n \to R^n$. A point $x^* \in R^n$ is an equilibrium point if $\frac{dx}{dt}\left(x^*\right) = f\left(x^*\right) = 0$ (Khalil, 2003).

Definition 3: A linear system of ordinary differential equation is given in the form

$\frac{dx}{dt} = AX$, $X(t) \in R^n$ where the constant coefficient matrix A or the Jacobian matrix is $\left(nxn\right)$

Square matrix and $\frac{dx}{dt} = \begin{pmatrix} \frac{dx_i}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$ where $i = 1, 2, 3, 4, ..., n$

Definition 4: For a linear system $\frac{dx}{dt} = AX$. The stability of the system at equilibrium point can be determined by location of eigenvalues of Jacobian matrix A. This is expressed as follows;

I. If the all eigenvalues of the Jacobian matrix have real parts less than zero, then the linear system is asymptotically stable and

II. If at least one of the eigenvalue of Jacobian matrix has real part greater than zero, then the system is unstable (Khalil, 2003).

Definition 5: Routh-Hurwitz Stability Criterion (Katsuhiko, 1970)

The local stability of the equilibrium points of the system is applying the Routh's stability criterion for the given characteristic polynomial of the form $a_0 m^n + a_1 m^{n-1} + \ldots a_n = 0$

Where $a_0 \neq 0$ and $a_n > 0$, then the Routh-Hurwitz array or table is given as follows.

$$
\begin{array}{c|ccccc}
m^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
m^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
m^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
m^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
m^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \\
m^2 & e_1 & e_2 & & & \\
m^1 & f_1 & & & & \\
m^0 & g_0 & & & &
\end{array}
$$

Where $b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$, $b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$, $b_3 = \frac{a_1a_6 - a_0a_7}{a_1}$, $c_1 = \frac{b_1a_3 - a_1b_2}{b_1}$, $c_2 = \frac{b_1a_5 - a_1b_3}{b_1}$, $c_3 = \frac{b_1a_7 - a_1b_4}{b_1}$, $d_1 = \frac{c_1b_2 - b_1c_2}{c_1}$, $d_2 = \frac{c_1b_3 - b_1c_3}{c_1}$

The equilibrium point is stable if there is no sign change in the first column and the equilibrium point is unstable if there is sign change in the first column of the Routh table above (Khalil, 2003).

Definition 6: Let x^* is an equilibrium point and a scalar function $v : D \to R$ is said to be:

1. Positive definite function if $v(x^*) = 0$ and $v(x) > 0$ for all $x \in D - \{x^*\}$
2. Negative definite function if $v(x^*) = 0$ and $v(x) < 0$ for all $x \in D - \{x^*\}$

Definition 7: **Leading Principal Minors of Matrix (Michalis, 2017)**

The leading principal minor of a matrix A of order k is the minor of order k obtained by deleting the last n-k rows and columns.

Consider $3x3$ matrix A

$$
A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
$$

All leading principal minors are:

$$D_1 = |a_{11}|,\ D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Definition 8: Let A be asymmetric nxn matrix. Then A is positive definite matrix if and only if $D_k > 0$ for all leading principal minors and A is negative definite matrix if and only if $(-1)^k D_k > 0$ for all leading principal minors. Where $1 \le k \le n$ (Michalis, 2017)

Definition 9: **Hessian Matrix (Michalis, 2017)**

Let $f(x)$ be a scalar function in n variables, then the Hessian Matrix of f is the matrix consisting of all the second order partial derivatives of f.

The Hessian matrix of f at the point x is the nxn matrix such that

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & & \frac{\partial^2 f}{\partial^2 x_n} \end{pmatrix}$$

Hessian matrix is used to determine whether a point on a surface of image is local maximum or local minimum. If Hessian matrix is a positive definite at the equilibrium point, then the function f has local minimum at the equilibrium point.

Theorem 4.1 Lyapunov Stability Theorem (Khalil, 2003)

Let $x = x^*$ be an equilibrium point of non-linear system of $\frac{dx}{dt} = f(x),\ f: D \to R^n$.

Suppose $v: D \to R$ be continuously differentiable function such that:-

I. $v(x^*) = 0$

II. $v(x) > 0$ for all $x \in D - \{x^*\}$

III. $\frac{dv(x)}{dt} \le 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*).Then $x = x^*$ is stable.

Theorem 4.2 (Globally asymptotically stable)

Let $x = x^*$ be an equilibrium point of non-linear system of $\frac{dx}{dt} = f(x)$, $f : D \to R^n$.

Let $v : D \to R$ be continuously differentiable function such that:-

1. $v(x^*) = 0$
2. $v(x) > 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*)
3. $\frac{dv(x)}{dt} \leq 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*)
4. $v(x)$ is radially unbounded. Then $x = x^*$ is globally asymptotically stable.

Theorem 4.3 Hopf Bifurcation

Let J_0 be a Jacobian matrix of a continuous parametric dynamical system evaluated at equilibrium point. Suppose that all eigenvalues of J_0 have a negative real parts except one conjugate non-zero purely imaginary pair $\pm i\omega$. A Hopf bifurcation arises when these two eigenvalues cross the imaginary axis because of a variation of the system.

4.2 Dissipative or Conservative of the System

Consider system (1.1) given by

$$\frac{dx}{dt} = k(y - x)$$
$$\frac{dy}{dt} = r_1(xz - y)$$
$$\frac{dz}{dt} = r_2(\lambda + 1 - z - \lambda xy)$$

Let
$$\begin{aligned} f_1 &= k(y - x) \\ f_2 &= r_1(xz - y) \\ f_3 &= r_2(\lambda + 1 - z - \lambda xy) \end{aligned} \qquad (4.1)$$

From the system (4.1), $\frac{\partial f_1(x,y,z)}{\partial x} = -k$, $\frac{\partial f_2(x,y,z)}{\partial y} = -r_1$, $\frac{\partial f_3(x,y,z)}{\partial z} = -r_2$

$$\nabla . f(x,y,z) = \frac{\partial f_1(x,y,z)}{\partial x} + \frac{\partial f_2(x,y,z)}{\partial y} + \frac{\partial f_3(x,y,z)}{\partial z}$$
$$= -(k + r_1 + r_2)$$

$$\frac{dD}{dt} = \int \nabla . f dD = \int -(k + r_1 + r_2) dD$$

$$\frac{dD}{dt} = -(k + r_1 + r_2) D$$

$$\frac{1}{D} dD = -\left(k + r_1 + r_2\right) dt$$

$$\int \frac{1}{D} dD = -\int \left(k + r_1 + r_2\right) dt$$

$$\ln D = -\left(k + r_1 + r_2\right) t + c$$

$$e^{\ln D} = e^{-\left(k + r_1 + r_2\right) t + c}$$

$$D = e^{-\left(k + r_1 + r_2\right) t + c}$$

$$D = e^{-(k + r_1 + r_2) t} x e^{c}$$

$$D = D_0 e^{-\left(k + r_1 + r_2\right) t} \quad \left(D_0 = e^{c}\right)$$

D is decreasing exponentially

Therefore, the system (1.1) is dissipative.

4.3 Equilibrium Points of System

The dynamic of the system is characterized by the existence and number of equilibrium points as well as their types of stability. The mathematical model under consideration from system (1.1)

$$\frac{dx}{dt} = k\left(y - x\right)$$

$$\frac{dy}{dt} = r_1\left(xz - y\right)$$

$$\frac{dz}{dt} = r_2\left(\lambda + 1 - z - \lambda xy\right)$$

To find the equilibrium point, equate the system (1.1) with zero

$$k(y - x) = 0 \qquad (4.2)$$

$$r_1(xz - y) = 0 \qquad (4.3)$$

$$r_2(\lambda + 1 - z - \lambda xy) = 0 \qquad (4.4)$$

From equation (4.2)

$$y - x = 0 \quad \left(k \neq 0\right)$$

$$x = y \tag{4.5}$$

From equation (4.3)

$$xz - y = 0 \; (r_1 \neq 0)$$

$$y - xz = 0 \tag{4.6}$$

From equation (4.4)

$$\lambda + 1 - z - \lambda xy = 0 \; (r_2 \neq 0)$$

$$z = -\lambda xy + \lambda + 1 \tag{4.7}$$

Plugging equation (4.5) into equation (4.6)

$$x(1-z) = 0, \; x = 0 \text{ or } z = 1,$$

If $x = 0$, then $y = 0$, substituting $x = 0$ or $y = 0$ into equation (4.7) it gives $z = \lambda + 1$, then the first equilibrium point is $E_1 = (0, 0, \lambda + 1)$

If $z = 1$, then solve for x and y from equation (4.7)

$$0 = \lambda(-xy + 1) \;, \; \lambda = 0 \; or \; -xy = -1$$

$$xy = 1 \tag{4.8}$$

Plugging equation (4.5) into equation (4.8)

$$x^2 = 1$$

$$x = y = -1, \; x = y = 1$$

Then the second equilibrium point is $E_2 = (-1, -1, 1)$ and the third equilibrium point is $E_3 = (1,1,1)$.

Therefore, we end up with three equilibrium points of the system as follows:-

$$E_1 = (0, 0, \lambda + 1), E_2 = (-1, -1, 1) \text{ and } E_3 = (1,1,1)$$

4.4 Local Stability Analysis

4.4.1 Linearizing Maxwell-Bloch Equations

Linearize the system (1.1) at each equilibrium points and state the local stability conditions of the system. From system (4.1)

$$\begin{aligned}\frac{\partial f_1}{\partial x} &= -k \\ \frac{\partial f_1}{\partial y} &= k \\ \frac{\partial f_1}{\partial z} &= 0\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{\partial f_2}{\partial x} &= r_1 z \\ \frac{\partial f_2}{\partial y} &= -r_1 \\ \frac{\partial f_2}{\partial z} &= r_1 x\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial f_3}{\partial x} &= -r_2 \lambda y \\ \frac{\partial f_3}{\partial y} &= -r_2 \lambda x \\ \frac{\partial f_3}{\partial z} &= -r_2\end{aligned} \tag{4.11}$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} \tag{4.12}$$

Subsisting equations (4.9), (4.10) and (4.11) into equation (4.12)

$$A=\begin{pmatrix} -k & k & 0 \\ r_1 z & -r_1 & r_1 x \\ -r_2\lambda y & -r_2\lambda x & -r_2 \end{pmatrix}$$

The Jacobian matrix evaluated at the first equilibrium point

$E_1=(0,0,\lambda+1)$ is $J=A|E_1=(0,0,\lambda+1)$

$$J=\begin{pmatrix} -k & k & 0 \\ r_1(\lambda+1) & -r_1 & 0 \\ 0 & 0 & -r_2 \end{pmatrix} \qquad (4.13)$$

The characteristic equation for equation (4.13) is $|J-mI|=0$

$$|J-mI|=\begin{vmatrix} -k-m & k & 0 \\ r_1(\lambda+1) & -r_1-m & 0 \\ 0 & 0 & -r_2-m \end{vmatrix}=0$$

$$(-k-m)\begin{vmatrix} -r_1-m & 0 \\ 0 & -r_2-m \end{vmatrix}-k\begin{vmatrix} r_1(\lambda+1) & 0 \\ 0 & -r_2-m \end{vmatrix}+0\begin{vmatrix} r_1(\lambda+1) & -r_1-m \\ 0 & 0 \end{vmatrix}=0$$

$$-m^3-km^2-r_1m^2-r_2m^2-kr_2m-r_1r_2m+kr_1\lambda m+kr_1r_2\lambda=0$$

$$m^3+(k+r_1+r_2)m^2+(kr_2+r_1r_2-kr_1\lambda)m-kr_1r_2\lambda=0$$

$$m^3+a_1m^2+a_2m+a_3=0 \qquad (4.14)$$

$a_1=k+r_1+r_2$

Where $a_2=kr_2+r_1r_2-kr_1\lambda$ (4.15)

$a_3=-kr_1r_2\lambda$

The Routh array or Routh-Hurwitz table is

$$\begin{array}{c|ccc} m^3 & 1 & a_2 & 0 \\ m^2 & a_1 & a_3 & 0 \\ m^1 & \dfrac{a_1a_2-a_3}{a_1} & 0 & \\ m^0 & a_3 & & \end{array}$$

Applying Routh-Hurwitz stability criterion for characteristic equation (4.14)

$1>0$

$a_1=k+r_1+r_2>0$, since, k,r_1,r_2 are positive parameters

$$a_1a_2 - a_3 = kr_2\left(k + 2r_1 + r_2\right) + r_1r_2\left(r_1 + r_2\right) - k\lambda r_1\left(k + r_1\right) > 0$$

$$a_3 = -kr_1r_2\lambda > 0 \text{ if } \lambda < 0 \tag{4.16}$$

As a result, the system (1.1) is locally asymptotically stable at the equilibrium point $E_1 = (0,0,\lambda+1)$ provided that condition (4.16) is satisfied.

The Jacobian matrix evaluated at the second equilibrium point $E_2 = (-1,-1,1)$ is

$$J = A\big|E_2 = (-1,-1,1)$$

$$J = \begin{pmatrix} -k & k & 0 \\ r_1 & -r_1 & -r_1 \\ r_2\lambda & r_2\lambda & -r_2 \end{pmatrix} \tag{4.17}$$

The characteristic equation for equation (4.17) is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -k-m & k & 0 \\ r_1 & -r_1-m & -r_1 \\ r_2\lambda & r_2\lambda & -r_2-m \end{vmatrix} = 0$$

$$(-k-m)\begin{vmatrix} -r_1-m & -r_1 \\ r_2\lambda & -r_2-m \end{vmatrix} - k\begin{vmatrix} r_1 & -r_1 \\ r_2\lambda & -r_2-m \end{vmatrix} + 0\begin{vmatrix} r_1 & -r_1-m \\ r_2\lambda & r_2\lambda \end{vmatrix} = 0$$

$$-m^3 - km^2 - r_1m^2 - r_2m^2 - kr_2m - r_1r_2m - r_1r_2\lambda m - 2kr_1r_2\lambda = 0$$
$$m^3 + \left(k + r_1 + r_2\right)m^2 + \left(kr_2 + r_1r_2 + r_1r_2\lambda\right)m + 2kr_1r_2\lambda = 0$$
$$m^3 + b_1m^2 + b_2m + b_3 = 0 \tag{4.18}$$

$b_1 = k + r_1 + r_2$

Where $b_2 = kr_2 + r_1r_2 + r_1r_2\lambda$ $\quad(4.19)$

$b_3 = 2kr_1r_2\lambda$

The Routh array or Routh-Hurwitz table is

$$\begin{array}{c|ccc} m^3 & 1 & b_2 & 0 \\ m^2 & b_1 & b_3 & 0 \\ m^1 & \dfrac{b_1b_2 - b_3}{b_1} & 0 & \\ m^0 & b_3 & & \end{array}$$

Applying Routh-Hurwitz stability criterion for characteristic equation (4.18)

$$1>0$$

$b_1 = k + r_1 + r_2 > 0$, since, k, r_1, r_2 are positive parameters

$$b_1 b_2 - b_3 = k r_2 \left(k + 2r_1 + r_2\right) + r_1 r_2 \left(r_1 + r_2 + \lambda r_1 + \lambda r_2\right) - k r_1 r_2 \lambda > 0$$

$$b_3 = 2k r_1 r_2 \lambda > 0 \text{ if } \lambda > 0 \qquad (4.20)$$

Therefore, the system (1.1) is locally asymptotically stable at the equilibrium point $E_2 = (-1,-1,1)$ provided that condition (4.20) is satisfied.

The Jacobian matrix evaluated at the third equilibrium point $E_3 = (1,1,1)$ is $J = A|E_3 = (1,1,1)$

$$J = \begin{pmatrix} -k & k & 0 \\ r_1 & -r_1 & r_1 \\ -r_2\lambda & -r_2\lambda & -r_2 \end{pmatrix} \qquad (4.21)$$

The characteristic equation for equation (4.21) is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -k-m & k & 0 \\ r_1 & -r_1 - m & r_1 \\ -r_2\lambda & -r_2\lambda & -r_2 - m \end{vmatrix} = 0$$

$$(-k-m)\begin{vmatrix} -r_1 - m & r_1 \\ -r_2\lambda & -r_2 - m \end{vmatrix} - k\begin{vmatrix} r_1 & r_1 \\ -r_2\lambda & -r_2 - m \end{vmatrix} + 0\begin{vmatrix} r_1 & -r_1 - m \\ -r_2\lambda & -r_2\lambda \end{vmatrix} = 0$$

$$-m^3 - km^2 - r_1 m^2 - r_2 m^2 - k r_2 m - r_1 r_2 m - r_1 r_2 \lambda m - 2k r_1 r_2 \lambda = 0$$
$$m^3 + \left(k + r_1 + r_2\right) m^2 + \left(k r_2 + r_1 r_2 + r_1 r_2 \lambda\right) m + 2k r_1 r_2 \lambda = 0$$

$$m^3 + c_1 m^2 + c_2 m + c_3 = 0 \qquad (4.22)$$

Where
$$c_1 = k + r_1 + r_2$$
$$c_2 = k r_2 + r_1 r_2 + r_1 r_2 \lambda \qquad (4.23)$$
$$c_3 = 2k r_1 r_2 \lambda$$

The Routh array or Routh-Hurwitz table is

$$\begin{array}{c|ccc} m^3 & 1 & c_2 & 0 \\ m^2 & c_1 & c_3 & 0 \\ m^1 & \dfrac{c_1 c_2 - c_3}{c_1} & 0 & \\ m^0 & c_3 & & \end{array}$$

Applying Routh-Hurwitz stability criterion for characteristic equation (4.22)

$$1>0$$

$c_1=k+r_1+r_2>0$, since, k,r_1,r_2 are positive parameters

$$c_1c_2-c_3=kr_2\left(k+2r_1+r_2\right)+r_1r_2\left(r_1+r_2+\lambda r_1+\lambda r_2\right)-kr_1r_2\lambda>0$$

$$c_3=2kr_1r_2\lambda>0 \text{ if } \lambda>0 \tag{4.24}$$

As a result, the system (1.1) is locally asymptotically stable at the equilibrium point $E_3=(1,1,1)$ provided that condition (4.24) is satisfied.

4.5 Global Stability Analysis of the System

To analyze the global asymptotic stability of non-linear system (1.1)

Let $v_1(x,y,z)=\frac{1}{k}x^2+\frac{1}{r_1}y^2+\frac{1}{r_2}(z-\lambda-1)^2$ be candidate Lyapunov function at equilibrium point $E_1=(0,0,\lambda+1)$, then:-

1. $v_1\left(x^*,y^*,z^*\right)=v_1(0,0,\lambda+1)=0$
2. $v_1(x,y,z)>0$ for all $(x,y,z)\in D-\left\{\left(x^*,y^*,z^*\right)\right\}$

   Hence, $v_1(x,y,z)$ is positive definite function.

3. $\frac{dv_1}{dt}(x,y,z)=\frac{\partial v_1}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_1}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_1}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2x}{k}\frac{dx}{dt}+\frac{2y}{r_1}\frac{dy}{dt}+\frac{2(z-\lambda-1)}{r_2}\frac{dz}{dt}$$
$$=2x(y-x)+2y(xz-y)+2(z-\lambda-1)(\lambda+1-z-\lambda xy)$$
$$=-2x^2-2y^2-2z^2-2\lambda^2+2\lambda^2xy-2\lambda xyz+2xyz+2\lambda xy+2xy+4\lambda z+4z-4\lambda-2$$
$$=-2\left(x^2+y^2+z^2+\lambda^2-\lambda^2xy+\lambda xyz-xyz-\lambda xy-xy-2\lambda z-2z+2\lambda+1\right)$$

$$\frac{dv_1}{dt}(x,y,z)=-2g(x,y,z)$$

Where $g(x,y,z)=x^2+y^2+z^2+\lambda^2-\lambda^2xy+\lambda xyz-xyz-\lambda xy-xy-2\lambda z-2z+2\lambda+1$

$$g_x = 2x - \lambda^2 y + \lambda yz - yz - \lambda y - y \quad g_y = 2y - \lambda^2 x + \lambda xz - xz - \lambda x - x \quad g_z = 2z + \lambda xy - xy - \lambda - 2$$

$$g_{xy} = -\lambda^2 + \lambda z - z - \lambda - 1 \quad g_{yx} = -\lambda^2 + \lambda z - z - \lambda - 1 \quad g_{zx} = \lambda y - y$$

$$g_{xz} = \lambda y - y \quad g_{yz} = \lambda x - x \quad g_{zy} = \lambda x - x$$

$$g_{xx} = 2 \quad g_{yy} = 2 \quad g_{zz} = 2$$

Construct Hessian matrix for $g(x, y, z)$ at the first equilibrium point $E_1 = (0, 0, \lambda + 1)$ to check whether Hessian matrix is positive definite.

$$H = \begin{pmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -\lambda^2 + \lambda z - z - \lambda - 1 & \lambda y - y \\ -\lambda^2 + \lambda z - z - \lambda - 1 & 2 & \lambda x - x \\ \lambda y - y & \lambda x - x & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -\lambda - 2 & 0 \\ -\lambda - 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\Bigg|_{E_1 = (0, 0, \lambda + 1)}$$

All leading principal minors of Hessian matrix at $E_1 = (0, 0, \lambda + 1)$ are :-

$$D_1 = 2 \ , \ D_2 = \begin{vmatrix} 2 & -\lambda - 2 \\ -\lambda - 2 & 2 \end{vmatrix} = -\lambda(\lambda + 4) \ \text{ and } \ D_3 = \begin{vmatrix} 2 & -\lambda - 2 & 0 \\ -\lambda - 2 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2\lambda(\lambda + 4)$$

The leading principal minors are:

$$D_1 = 2 > 0,$$

$$D_2 = -\lambda(\lambda + 4) > 0,$$

$$D_3 = -2\lambda(\lambda + 4) > 0 \ \text{ if } \ \lambda \in (-4, 0) \qquad (4.25)$$

The Hessian matrix is positive definite if condition (4.25) is satisfied.

So that $\frac{dv_1}{dt}(x, y, z)$ is negative definite function when condition (4.25) is satisfied.

4. $\lim\limits_{(x,y,z)\to\infty} v_1(x,y,z) = \lim\limits_{(x,y,z)\to\infty}\left[\frac{x^2}{k}+\frac{y^2}{r_1}+\frac{1}{r_2}(z-\lambda-1)^2\right]=\infty$

$v_1(x,y,z)$ is radially unbounded.

As a result, the equilibrium point $E_1=(0,0,\lambda+1)$ is globally asymptotically stable by Lyapunov Stability theorem if condition (4.25) is satisfied.

Suppose $v_2(x,y,z)=\frac{1}{k}(x+1)^2+\frac{1}{r_1}(y+1)^2+\frac{1}{r_2}(z-1)^2$ be appropriate Lyapunov function at the Second equilibrium point $E_2=(-1,-1,1)$, then:-

1. $v_2(x^*,y^*,z^*)=v_2(-1,-1,1)=0$
2. $v_2(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_2(x,y,z)$ is positive definite function.

3. $\frac{dv_2}{dt}(x,y,z)=\frac{\partial v_2}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_2}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_2}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2(x+1)}{k}\frac{dx}{dt}+\frac{2(y+1)}{r_1}\frac{dy}{dt}+\frac{2(z-1)}{r_2}\frac{dz}{dt}$$
$$=(2x+2)(y-x)+(2y+2)(xz-y)+(2z-2)(\lambda+1-z-\lambda xy)$$
$$=-2x^2-2y^2-2z^2-2\lambda xyz+2xyz+2\lambda xy+2xy+2xz+2\lambda z-2x+4z-2\lambda-2$$
$$=-2\left(x^2+y^2+z^2+\lambda xyz-xyz-\lambda xy-xy-xz-\lambda z+x-2z+\lambda+1\right)$$

$$\frac{dv_2}{dt}(x,y,z)=-2f(x,y,z)$$

Where $f(x,y,z)=x^2+y^2+z^2+\lambda xyz-xyz-\lambda xy-xy-xz-\lambda z+x-2z+\lambda+1$

$f_x=2x+\lambda yz-yz-y-z+1$ $\quad f_y=2y+\lambda xz-xz-\lambda x-x$ $\quad f_z=2z+\lambda xy-xy-x-\lambda-2$

$f_{xy}=\lambda z-z-\lambda-1$ $\quad f_{yx}=\lambda z-z-\lambda-1$ $\quad f_{zx}=\lambda y-y-1$

$f_{xz}=\lambda y-y-1$ $\quad f_{yz}=\lambda x-x$ $\quad f_{zy}=\lambda x-x$

$f_{xx}=2$ $\quad f_{yy}=2$ $\quad f_{zz}=2$

Construct Hessian matrix for $f(x,y,z)$ at the second equilibrium point $E_2=(-1,-1,1)$ to check whether Hessian matrix is positive definite.

$$H=\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & \lambda z-z-\lambda-1 & \lambda y-y-1 \\ \lambda z-z-\lambda-1 & 2 & \lambda x-x \\ \lambda y-y-1 & \lambda x-x & 2 \end{pmatrix}$$

$$=\left.\begin{pmatrix} 2 & -2 & -\lambda \\ -2 & 2 & 1-\lambda \\ -\lambda & 1-\lambda & 2 \end{pmatrix}\right|_{E_2=(-1,-1,1)}$$

All leading principal minors of Hessian matrix at $E_2=(-1,-1,1)$ are :-

$$D_1=2,\quad D_2=\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}=0 \text{ and } D_3=\begin{vmatrix} 2 & -2 & -\lambda \\ -2 & 2 & 1-\lambda \\ -\lambda & 1-\lambda & 2 \end{vmatrix}=-2(2\lambda-1)^2$$

Since the leading principal minors $D_1=2>0,\quad D_2=0$ and $D_3=-2(2\lambda-1)^2<0$, then the Hessian matrix is neither positive nor negative definite or Hessian matrix is indefinite.

So that it is impossible to identify the algebraic sign of $\frac{dv_2}{dt}(x,y,z)$.

Therefore, It is impossible to deal with global stability condition of the equilibrium point $E_2=(-1,-1,1)$ in the sense of Lyapunov stability theorem.

Let $v_3(x,y,z)=\frac{1}{k}(x-1)^2+\frac{1}{r_1}(y-1)^2+\frac{1}{r_2}(z-1)^2$ be appropriate Lyapunov function at the third equilibrium point $E_3=(1,1,1)$, then:-

1. $v_3(x^*,y^*,z^*)=v_3(1,1,1)=0$
2. $v_3(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_3(x,y,z)$ is positive definite function.

3. $\frac{dv_3}{dt}(x,y,z)=\frac{\partial v_3}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_3}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_3}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2(x-1)}{k}\frac{dx}{dt}+\frac{2(y-1)}{r_1}\frac{dy}{dt}+\frac{2(z-1)}{r_2}\frac{dz}{dt}$$

$$=(2x-2)(y-x)+(2y-2)(xz-y)+(2z-2)(\lambda+1-z-\lambda xy)$$

$$=-2x^2-2y^2-2z^2-2\lambda xyz+2xyz+2\lambda xy+2xy-2xz+2\lambda z+2x+4z-2\lambda-2$$

$$=-2\left(x^2+y^2+z^2+\lambda xyz-xyz-\lambda xy-xy+xz-\lambda z-x-2z+\lambda+1\right)$$

$$\frac{dv_3}{dt}(x,y,z)=-2h(x,y,z)$$

Where $h(x,y,z)=x^2+y^2+z^2+\lambda xyz-xyz-\lambda xy-xy+xz-\lambda z-x-2z+\lambda+1$

$h_x=2x+\lambda yz-yz-\lambda y+z-1$ $\quad h_y=2y+\lambda xz-xz-\lambda x-x$ $\quad h_z=2z+\lambda xy-xy+x-\lambda-2$

$h_{xy}=\lambda z-z-\lambda-1$ $\quad h_{yx}=\lambda z-z-\lambda-1$ $\quad h_{zx}=\lambda y-y+1$

$h_{xz}=\lambda y-y+1$ $\quad h_{yz}=\lambda x-x$ $\quad h_{zy}=\lambda x-x$

$h_{xx}=2$ $\quad h_{yy}=2$ $\quad h_{zz}=2$

Construct Hessian matrix for $h(x,y,z)$ at the third equilibrium point $E_3=(1,1,1)$ to check whether Hessian matrix is positive definite.

$$H=\begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & \lambda z-z-\lambda-1 & \lambda y-y+1 \\ \lambda z-z-\lambda-1 & 2 & \lambda x-x \\ \lambda y-y+1 & \lambda x-x & 2 \end{pmatrix}$$

$$=\left.\begin{pmatrix} 2 & -2 & \lambda \\ -2 & 2 & \lambda-1 \\ \lambda & \lambda-1 & 2 \end{pmatrix}\right|_{E_3=(1,1,1)}$$

All the leading principal minors of Hessian matrix at $E_3=(1,1,1)$ are:

$$D_1=2\,,\ D_2=\begin{vmatrix}2 & -2\\ -2 & 2\end{vmatrix}=0 \text{ and } D_3=\begin{vmatrix}2 & -2 & \lambda\\ -2 & 2 & \lambda-1\\ \lambda & \lambda-1 & 2\end{vmatrix}=-2(2\lambda-1)^2$$

Since the leading principal minors $D_1=2>0,\ \ D_2=0$ and $D_3=-2(2\lambda-1)^2<0$, then the Hessian matrix is indefinite.

So that it impossible to identify the algebraic sign of $\frac{dv_3}{dt}(x,y,z)$.

As a result, it is impossible to deal with global stability condition of the equilibrium point $E_3=(1,1,1)$ in the sense of Lyapunov stability theorem.

4.6 Hopf Bifurcation Analysis of the System

Suppose that the system (1.1) has critical point for some parameter $\lambda=\lambda_0$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real part. Furthermore, Let $\text{Re}\left(\frac{d\lambda}{dm}\right)\neq 0$, then the Hopf bifurcation occurs at $\lambda=\lambda_0$.

Let the characteristic equation (4.14) has pure imaginary eigenvalues $m=\pm i\omega(\omega>0)$ at $\lambda=\lambda_0$

$$m^3+a_1m^2+a_2m+a_3=0$$
$$(\omega i)^3+a_1(\omega i)^2+a_2(\omega i)+a_3=0$$
$$(-\omega^3+a_2\omega)i-a_1\omega^2+a_3=0$$

Equating the real and imaginary parts with zeros yields.

$$-\omega^3+a_2\omega=0 \tag{4.26}$$
$$-a_1\omega^2+a_3=0 \tag{4.27}$$

From equation (4.26) Since $\omega>0$

$$\omega^2=a_2 \tag{4.28}$$

From equation (4.27)

$$\omega^2=\frac{a_3}{a_1} \tag{4.29}$$

Equating equations (4.28) and (4.29)

$$a_1 a_2 - a_3 = 0 \tag{4.30}$$

Substituting equation (4.15) into equation (4.30) to compute for parameter λ yield

$$(k + r_1 + r_2)(kr_2 + r_1 r_2 - kr_1 \lambda) + kr_1 r_2 \lambda = 0$$

$$-\lambda\left(k^2 r_1 + kr_1^2\right) = -\left(k^2 r_2 + 2kr_1 r_2 + r_2 r_1^2 + kr_2^2 + r_1 r_2^2\right)$$

$$\lambda = \frac{r_2 (k + r_1 + r_2)}{kr_1} \tag{4.31}$$

Plugging equation (4.31) into equation (4.15)

$$\begin{aligned} a_1 &= k + r_1 + r_2 \\ a_2 &= -r_2^2 \\ a_3 &= -kr_1 r_2 \left(\frac{r_2 (k + r_1 + r_2)}{kr_1} \right) \\ &= -r_2^2 (k + r_1 + r_2) \end{aligned} \tag{4.32}$$

From equation (4.28)

$$\omega^2 = -r_2^2$$

$$\omega = \pm i r_2,$$

Which contradicts the fact that $\omega > 0$.

Substituting equation (4.32) into equation (4.14) yields.

$$m^3 + (k + r_1 + r_2) m^2 - r_2^2 m - r_2^2 (k + r_1 + r_2) = 0$$

$$\left[m + (k + r_1 + r_2)\right]\left(m^2 - r_2^2\right) = 0$$

$$\left[m + (k + r_1 + r_2)\right](m - r_2)(m - r_2) = 0$$

$$m_1 = -(k + r_1 + r_2) \text{ or } m^2 = r_2^2$$

$$m_{2,3} = \pm r_2$$

Since $m_{2,3}$ are not pure imaginary eigenvalues, then one of Hopf bifurcation condition is not satisfied.

As a result, the system (1.1) does not undergo Hopf bifurcation at $\lambda = \dfrac{r_2 (k + r_1 + r_2)}{kr_1}$.

Let the characteristic equation (4.18) has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$

$$m^3 + b_1 m^2 + b_2 m + b_3 = 0$$

$$(i\omega)^3 + b_1 (i\omega)^2 + b_2 (i\omega) + b_3 = 0$$

$$(-\omega^3 + b_2\omega) i - b_1\omega^2 + b_3 = 0$$

Equating the real and imaginary parts with zeros yields.

$$-\omega^3 + b_2\omega = 0 \tag{4.33}$$

$$-b_1\omega^2 + b_3 = 0 \tag{4.34}$$

From equation (4.33) Since $\omega > 0$

$$\omega^2 = b_2 \tag{4.35}$$

From equation (4.34)

$$\omega^2 = \frac{b_3}{b_1} \tag{4.36}$$

Equating equations (4.35) and (4.36)

$$b_1 b_2 - b_3 = 0 \tag{4.37}$$

Plugging equation (4.19) into equation (4.37) to calculate for λ

$$(k + r_1 + r_2)(kr_2 + r_1 r_2 + r_1 r_2 \lambda) - 2kr_1 r_2 \lambda = 0$$

$$k^2 r_2 + 2kr_1 r_2 + kr_2^2 + r_2 r_1^2 + r_1 r_2^2 + kr_1 r_2 \lambda + r_2 r_1^2 \lambda + r_1 r_2^2 \lambda - 2kr_1 r_2 \lambda = 0$$

$$r_2 (k + r_1)(k + r_1 + r_2) + \lambda r_1 r_2 (r_1 + r_2 - k) = 0$$

$$\lambda = \frac{-(k + r_1)(k + r_1 + r_2)}{r_1 (r_1 + r_2 - k)} \tag{4.38}$$

Substituting equation (4.38) into equation (4.19)

$$b_1 = k + r_1 + r_2$$

$$b_2 = \frac{-2kr_2 (k + r_1)}{r_1 + r_2 - k} \tag{4.39}$$

$$b_3 = \frac{-2kr_2 (k + r_1)(k + r_1 + r_2)}{r_1 + r_2 - k}$$

From equation (4.35)

$$\omega^2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k}$$

$$\omega = \pm\sqrt{\frac{2kr_2(k+r_1)}{k-(r_1+r_2)}} \text{ if } k > r_1+r_2 \tag{4.40}$$

Substituting equation (4.39) into equation (4.18)

$$(r_1+r_2-k)m^3+(r_1+r_2-k)(k+r_1+r_2)m^2-2kr_2(k+r_1)m-2kr_2(k+r_1)(k+r_1+r_2)=0$$

$$\left[m+(k+r_1+r_2)\right]\left[(r_1+r_2-k)m^2-2kr_2(k+r_1)\right]=0$$

$$m_1 = -(k+r_1+r_2) \text{ or } m^2 = \frac{2kr_2(k+r_1)}{r_1+r_2-k}$$

$$m_{2,3} = \pm\sqrt{\frac{2kr_2(k+r_1)}{r_1+r_2-k}}$$

Since $k > r_1+r_2$

$$m_{2,3} = \pm\sqrt{\frac{-2kr_2(k+r_1)}{k-(r_1+r_2)}}$$

$$= \pm i\sqrt{\frac{2kr_2(k+r_1)}{k-(r_1+r_2)}}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied if condition (4.40) is satisfied.

Next compute the $\frac{dm}{d\lambda}$ from the characteristic equation of the Jacobian matrix for equation (4.18)

$$\left(3m^2+2b_1m+b_2\right)\frac{dm}{d\lambda}+\left(\frac{db_1}{d\lambda}m^2+\frac{db_2}{d\lambda}m+\frac{db_3}{d\lambda}\right)=0$$

$$\frac{dm}{d\lambda} = \frac{-\left(\frac{db_1}{d\lambda}m^2+\frac{db_2}{d\lambda}m+\frac{db_3}{d\lambda}\right)}{3m^2+2b_1m+b_2} \tag{4.41}$$

Determine the derivatives of equation (4.19) with respect to λ

$$\frac{db_1}{d\lambda}=0$$
$$\frac{db_2}{d\lambda}=r_1r_2 \qquad (4.42)$$
$$\frac{db_3}{d\lambda}=2kr_1r_2$$

Plugging equation (4.42) into equation (4.41) yield.

$$\frac{dm}{d\lambda}=-\left(\frac{r_1r_2m+2kr_1r_2}{3m^2+2b_1m+b_2}\right)$$

$$\frac{d\lambda}{dm}=\left(\frac{dm}{d\lambda}\right)^{-1}=-\left(\frac{3m^2+2b_1m+b_2}{2kr_1r_2+r_1r_2m}\right)$$

$$=-\left(\frac{3(i\omega)^2+2b_1(i\omega)+b_2}{2kr_1r_2+r_1r_2(i\omega)}\right)$$

$$=\frac{3\omega^2-b_2-2b_1\omega i}{2kr_1r_2+r_1r_2\omega i}\left(\frac{2kr_1r_2-r_1r_2\omega i}{2kr_1r_2-r_1r_2\omega i}\right)$$

$$=\frac{2\left(3k\omega^2-b_2k-b_1\omega^2\right)}{r_1r_2\left(4k^2+\omega^2\right)}-\frac{\left(3\omega^2-b_2+4b_1k\right)\omega i}{r_1r_2\left(4k^2+\omega^2\right)}$$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right)=\frac{2\left(3k\omega^2-b_2k-b_1\omega^2\right)}{r_1r_2\left(4k^2+\omega^2\right)}=\frac{b_2\left(4k-2b_1\right)}{r_1r_2\left(4k^2+b_2\right)}\neq 0$$

Since $\operatorname{Re}\left(\frac{d\lambda}{dm}\right)\neq 0$, then second condition of Hopf bifurcation is satisfied if condition (4.40) is satisfied.

As a result, the system (1.1) under goes Hopf bifurcation at $\lambda=\frac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)}$ when condition (4.40) is satisfied.

Suppose the characteristic equation (4.22) has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$

$$m^3 + c_1 m^2 + c_2 m + c_3 = 0$$

$$(i\omega)^3 + c_1(i\omega)^2 + c_2(i\omega) + c_3 = 0$$

$$(-\omega^3 + c_2\omega)i - c_1\omega^2 + c_3 = 0$$

Equating the real and imaginary parts with zeros yields.

$$-\omega^3 + c_2\omega = 0 \tag{4.43}$$

$$-c_1\omega^2 + c_3 = 0 \tag{4.44}$$

From equation (4.43) Since $\omega > 0$

$$\omega^2 = c_2 \tag{4.45}$$

From equation (4.44)

$$\omega^2 = \frac{c_3}{c_1} \tag{4.46}$$

Equating equations (4.45) and (4.46)

$$c_1 c_2 - c_3 = 0 \tag{4.47}$$

Plugging equation (4.23) into equation (4.47) to compute for λ

$$(k + r_1 + r_2)(kr_2 + r_1 r_2 + r_1 r_2 \lambda) - 2kr_1 r_2 \lambda = 0$$

$$k^2 r_2 + 2kr_1 r_2 + kr_2{}^2 + r_2 r_1{}^2 + r_1 r_2{}^2 + kr_1 r_2 \lambda + r_2 r_1{}^2 \lambda + r_1 r_2{}^2 \lambda - 2kr_1 r_2 \lambda = 0$$

$$r_2(k + r_1)(k + r_1 + r_2) + \lambda r_1 r_2 (r_1 + r_2 - k) = 0$$

$$\lambda = \frac{-(k + r_1)(k + r_1 + r_2)}{r_1(r_1 + r_2 - k)} \tag{4.48}$$

Substituting equation (4.48) into equation (4.23)

$$c_1 = k + r_1 + r_2$$

$$c_2 = \frac{-2kr_2(k + r_1)}{r_1 + r_2 - k} \tag{4.49}$$

$$c_3 = \frac{-2kr_2(k + r_1)(k + r_1 + r_2)}{r_1 + r_2 - k}$$

From equation (4.44)

$$\omega^2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k}$$

$$\omega = \pm\sqrt{\frac{2kr_2(k+r_1)}{k-(r_1+r_2)}} \text{ if } k > r_1+r_2 \tag{4.50}$$

Substituting equation (4.49) into equation (4.22)

$$(r_1+r_2-k)m^3+(r_1+r_2-k)(k+r_1+r_2)m^2-2kr_2(k+r_1)m-2kr_2(k+r_1)(k+r_1+r_2)=0$$

$$\left[m+(k+r_1+r_2)\right]\left[(r_1+r_2-k)m^2-2kr_2(k+r_1)\right]=0$$

$$m_1=-(k+r_1+r_2) \text{ or } m^2=\frac{2kr_2(k+r_1)}{r_1+r_2-k}$$

$$m_{2,3}=\pm\sqrt{\frac{2kr_2(k+r_1)}{r_1+r_2-k}}$$

Since $k > r_1+r_2$

$$m_{2,3}=\pm\sqrt{\frac{-2kr_2(k+r_1)}{k-(r_1+r_2)}}$$

$$=\pm i\sqrt{\frac{2kr_2(k+r_1)}{k-(r_1+r_2)}}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied if condition (4.50) is satisfied.

Next compute the $\frac{dm}{d\lambda}$ from the characteristic equation of the Jacobian matrix for equation (4.22)

$$(3m^2+2c_1m+c_2)\frac{dm}{d\lambda}+\left(\frac{dc_1}{d\lambda}m^2+\frac{dc_2}{d\lambda}m+\frac{dc_3}{d\lambda}\right)=0$$

$$\frac{dm}{d\lambda}=\frac{-\left(\frac{dc_1}{d\lambda}m^2+\frac{dc_2}{d\lambda}m+\frac{dc_3}{d\lambda}\right)}{3m^2+2c_1m+c_2} \tag{4.51}$$

Determine the derivatives of equation (4.23) with respect to λ

$$\frac{dc_1}{d\lambda}=0$$
$$\frac{dc_2}{d\lambda}=r_1r_2 \qquad (4.52)$$
$$\frac{dc_3}{d\lambda}=2kr_1r_2$$

Substituting equation (4.52) into equation (4.51) yield.

$$\frac{dm}{d\lambda}=-\left(\frac{r_1r_2m+2kr_1r_2}{3m^2+2c_1m+c_2}\right)$$

$$\frac{d\lambda}{dm}=\left(\frac{dm}{d\lambda}\right)^{-1}=-\left(\frac{3m^2+2c_1m+c_2}{2kr_1r_2+r_1r_2m}\right)$$

$$=-\left(\frac{3(i\omega)^2+2c_1(i\omega)+c_2}{2kr_1r_2+r_1r_2(i\omega)}\right)$$

$$=\frac{3\omega^2-c_2-2c_1\omega i}{2kr_1r_2+r_1r_2\omega i}\left(\frac{2kr_1r_2-r_1r_2\omega i}{2kr_1r_2-r_1r_2\omega i}\right)$$

$$=\frac{2\left(3k\omega^2-c_2k-c_1\omega^2\right)}{r_1r_2\left(4k^2+\omega^2\right)}-\frac{\left(3\omega^2-c_2+4c_1k\right)\omega i}{r_1r_2\left(4k^2+\omega^2\right)}$$

$$\mathrm{Re}\left(\frac{d\lambda}{dm}\right)=\frac{2\left(3k\omega^2-c_2k-c_1\omega^2\right)}{r_1r_2\left(4k^2+\omega^2\right)}=\frac{c_2\left(4k-2c_1\right)}{r_1r_2\left(4k^2+c_2\right)}\neq 0$$

Since $\mathrm{Re}\left(\frac{d\lambda}{dm}\right)\neq 0$, then it provided that second condition of Hopf bifurcation is satisfied if condition (4.50) is satisfied.

Therefore, the system (1.1) undergoes Hopf bifurcation at $\lambda=\frac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)}$ when condition (4.50) is satisfied.

4.7 Numerical Examples

Example 1. Consider the parameters values with $r_1 = r_2 = 1,\ k = 4,\ \ \lambda = 2$, then the system (1.1) becomes

$$\frac{dx}{dt} = 4(y - x)$$
$$\frac{dy}{dt} = (xz - y)$$
$$\frac{dz}{dt} = (3 - z - 2xy)$$

Dissipative or Conservative of the System

Let
$$f_1 = 4(y - x)$$
$$f_2 = (xz - y)$$
$$f_3 = (3 - z - \lambda xy)$$

$$\frac{\partial f_1}{\partial x} = -4\ , \qquad \frac{\partial f_2}{\partial y} = -1\ , \qquad \frac{\partial f_3}{\partial z} = -1$$

The divergence of the vector field $\nabla. f\left(x, y, z\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(k + r_1 + r_2) = -6$

$$\frac{dD}{dt} = \int \nabla. f dD = \int -(k + r_1 + r_2) dD$$
$$\frac{dD}{dt} = -6D$$
$$\frac{1}{D} dD = -6dt$$
$$\int \frac{1}{D} dD = -\int 6dt$$
$$\ln D = -6t + c$$
$$e^{\ln D} = e^{-6t+c}$$
$$D = e^{-6t+c}$$
$$D = e^{-6t} x e^{c}$$
$$D = D_0 e^{-6t} \quad \left(D_0 = e^{c}\right)$$

D is decreasing exponentially

Therefore, the system (1.1) is dissipative.

Equilibrium Points of System

We end up with three equilibrium points of the system as follows:-

$$E_1 = (0,0,\lambda+1) = (0,0,3)$$
$$E_2 = (-1,-1,1) = (-1,-1,1)$$
$$E_3 = (1,1,1) = (1,1,1)$$

Local Stability Analysis

Linearizing Maxwell-Bloch Equations

Linearize the system (1.1) at each equilibrium points and state the local stability conditions of the system. Consider the above system

$$\frac{\partial f_1}{\partial x} = -4 \qquad \frac{\partial f_2}{\partial x} = z \qquad \frac{\partial f_3}{\partial x} = -2y$$
$$\frac{\partial f_1}{\partial y} = 4 \qquad \frac{\partial f_2}{\partial y} = -1 \qquad \frac{\partial f_3}{\partial y} = -2x$$
$$\frac{\partial f_1}{\partial z} = 0 \qquad \frac{\partial f_2}{\partial z} = x \qquad \frac{\partial f_3}{\partial z} = -1$$

Jacobian matrix $A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -4 & 4 & 0 \\ z & -1 & x \\ -2y & -2x & -1 \end{pmatrix}$

I. The Jacobian matrix evaluated at the first equilibrium point

$E_1 = (0,0,3)$ is $J = A|E_1 = (0,0,3)$

$$J = \begin{pmatrix} -4 & 4 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at $E_1 = (0,0,3)$ is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -4-m & 4 & 0 \\ 3 & -1-m & 0 \\ 0 & 0 & -1-m \end{vmatrix} = 0$$

$$(-4-m)\begin{vmatrix} -1-m & 0 \\ 0 & -1-m \end{vmatrix} - 4\begin{vmatrix} 3 & 0 \\ 0 & -1-m \end{vmatrix} + 0\begin{vmatrix} 3 & -1-m \\ 0 & 0 \end{vmatrix} = 0$$

$$m^3 + 6m^2 - 3m - 8 = 0 \quad \text{where} \quad a_1 = 6, \quad a_2 = -3, \; a_3 = -8$$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_1 = (0,0,3)$

$$\begin{array}{c|ccc} m^3 & 1 & -3 & 0 \\ m^2 & 6 & -8 & 0 \\ m^1 & \frac{-5}{3} & 0 & \\ m^0 & -8 & & \end{array}$$

From the above table, Since there is sign changes in the first column of Routh- Hurwitz table, equilibrium point one is unstable.

II. The Jacobian matrix evaluated at the second equilibrium point

$E_2 = (-1,-1,1)$ is $J = A|E_2 = (-1,-1,1)$

$$J = \begin{pmatrix} -4 & 4 & 0 \\ 1 & -1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$ is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -4-m & 4 & 0 \\ 1 & -1-m & -1 \\ 2 & 2 & -1-m \end{vmatrix} = 0$$

$$(-4-m)\begin{vmatrix} -1-m & -1 \\ 2 & -1-m \end{vmatrix} - 4\begin{vmatrix} 1 & -1 \\ 2 & -1-m \end{vmatrix} + 0\begin{vmatrix} 1 & -1-m \\ 2 & 2 \end{vmatrix} = 0$$

$$m^3 + 6m^2 + 7m + 16 = 0 \quad \text{where} \quad b_1 = 6, \quad b_2 = 7, \quad b_3 = 16$$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_2 = (-1, -1, 1)$

$$\begin{array}{c|ccc} m^3 & 1 & 7 & 0 \\ m^2 & 6 & 16 & 0 \\ m^1 & \frac{13}{3} & 0 & \\ m^0 & 16 & & \end{array}$$

From RH table, Since there is no sign changes in the first column of Routh- Hurwitz table, equilibrium point two is locally asymptotically stable .

III. The Jacobian matrix evaluated at the third equilibrium point $E_3 = (1,1,1)$ is $J = A\big|E_3 = (1,1,1)$

$$J = \begin{pmatrix} -4 & 4 & 0 \\ 1 & -1 & 1 \\ -2 & -2 & -1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$ is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -4-m & 4 & 0 \\ 1 & -1-m & 1 \\ -2 & -2 & -1-m \end{vmatrix} = 0$$

$$(-4-m)\begin{vmatrix} -1-m & 1 \\ -2 & -1-m \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ -2 & -1-m \end{vmatrix} + 0\begin{vmatrix} 1 & -1-m \\ -2 & -2 \end{vmatrix} = 0$$

$m^3 + 6m^2 + 7m + 16 = 0$ where $c_1 = 6,\ c_2 = 7,\ c_3 = 16$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$

$$\begin{array}{c|ccc} m^3 & 1 & 7 & 0 \\ m^2 & 6 & 16 & 0 \\ m^1 & \frac{13}{3} & 0 & \\ m^0 & 16 & & \end{array}$$

From the above table, Since there is no sign changes in the first column of Routh -Hurwitz table, an equilibrium point three is locally asymptotically stable.

MATLAB Simulation

The following diagrams indicate MATLAB simulation that shows stability of the equilibrium point.

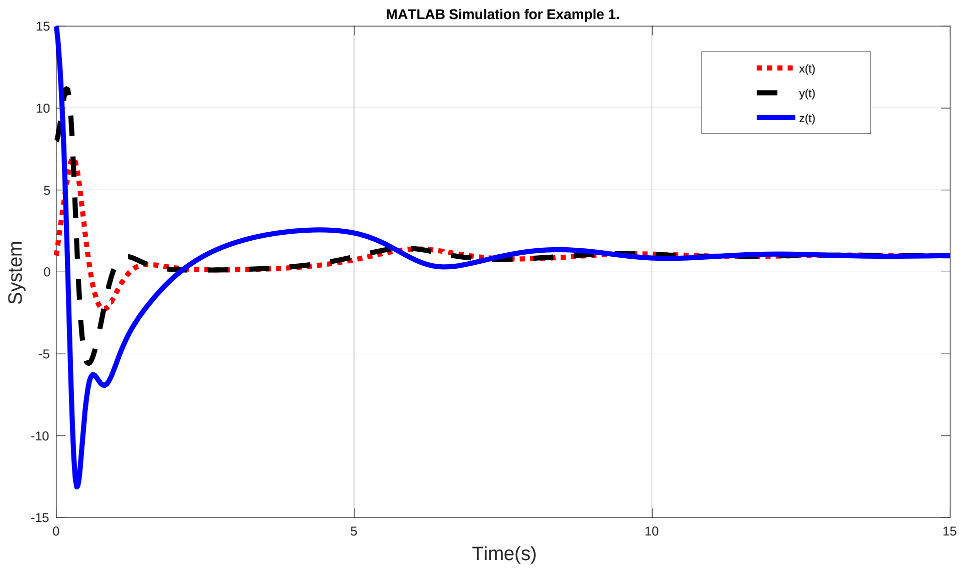

Figure 1: The graph of system versus time about equilibrium point.

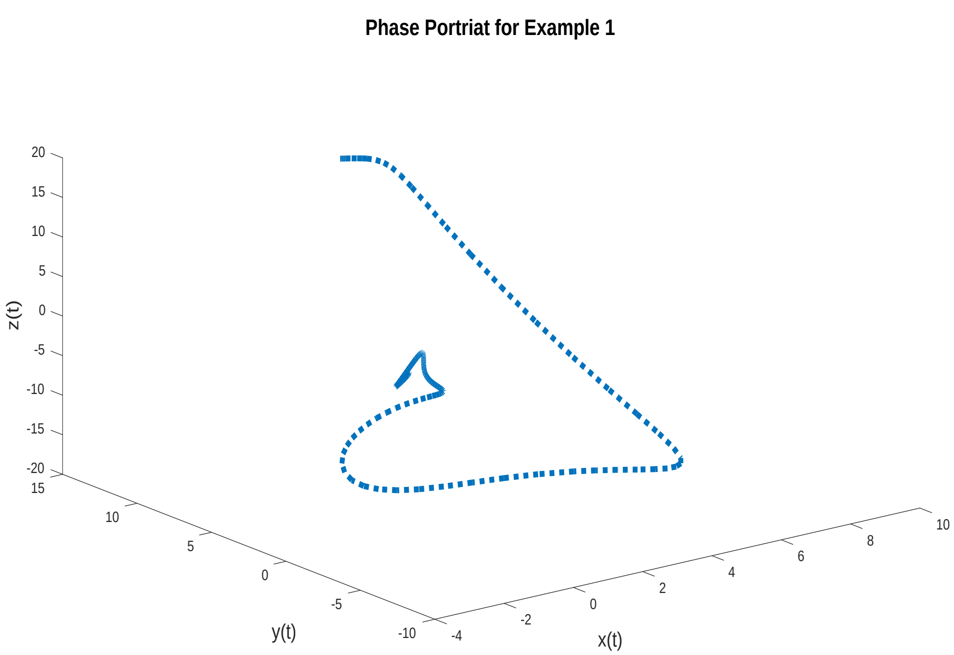

Figure 2: Phase Portrait of the system about the equilibrium point.

Discussion: Figure 1 indicates that the graph of the system versus time converges to the equilibrium point. Figure 2 also revealed this fact because the trajectory of the system converges to equilibrium point which in one ways shows the stability of the equilibrium point.

Global Stability Analysis of the System

Let $v_1(x,y,z)=\frac{1}{4}x^2+y^2+(z-3)^2$ be candidate Lyapunov function at equilibrium point $E_1=(0,0,3)$, then:-

1. $v_1(x^*,y^*,z^*)=v_1(0,0,3)=0$
2. $v_1(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_1(x,y,z)$ is positive definite function.
3. $\frac{dv_1}{dt}(x,y,z)=\frac{\partial v_1}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_1}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_1}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2x}{k}\frac{dx}{dt}+\frac{2y}{r_1}\frac{dy}{dt}+\frac{2(z-\lambda-1)}{r_2}\frac{dz}{dt}$$
$$=2x(y-x)+2y(xz-y)+2(z-\lambda-1)(\lambda+1-z-\lambda xy)$$
$$=-2(x^2+y^2+z^2+xyz-7xy-6z+9)$$
$$\frac{dv_1}{dt}(x,y,z)=-2g(x,y,z)$$

Where $g(x,y,z)=x^2+y^2+z^2+xyz-7xy-6z+9$

$g_x=2x+yz-7y$	$g_y=2y+xz-7x$	$g_z=2z+xy-6$
$g_{xy}=z-7$	$g_{yx}=z-7$	$g_{zx}=y$
$g_{xz}=y$	$g_{yz}=x$	$g_{zy}=x$
$g_{xx}=2$	$g_{yy}=2$	$g_{zz}=2$

Construct Hessian matrix for $g(x,y,z)$ at the first equilibrium point $E_1=(0,0,3)$ to check whether Hessian matrix is positive definite.

$$H=\begin{pmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & z-7 & y \\ z-7 & 2 & x \\ y & x & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\Bigg|_{E_1=(0,0,3)}$$

All the leading principal minors of Hessian matrix at $E_1=(0,0,3)$ are:-

$$D_1=2\,,\quad D_2=\begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix}=-12 \text{ and } D_3=\begin{vmatrix} 2 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=-24$$

Since the leading principal minors $D_1=2>0,\ D_2=-12<0$ and $D_3=-24<0$, then the Hessian matrix is indefinite. So that it is impossible to identify the algebraic sign of $\frac{dv_1}{dt}(x,y,z)$.

As a result, It is impossible to deal with global stability condition of the equilibrium point $E_1=(0,0,3)$ in the sense of Lyapunov stability theorem.

Suppose $v_2(x,y,z)=\frac{1}{4}(x+1)^2+(y+1)^2+(z-1)^2$ be appropriate lyapunov function at the second equilibrium point $E_2=(-1,-1,1)$, then:-

1. $v_2(x^*,y^*,z^*)=v_2(-1,-1,1)=0$
2. $v_2(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_2(x,y,z)$ is positive definite function.
3. $\frac{dv_2}{dt}(x,y,z)=\frac{\partial v_2}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_2}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_2}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2(x+1)}{k}\frac{dx}{dt}+\frac{2(y+1)}{r_1}\frac{dy}{dt}+\frac{2(z-1)}{r_2}\frac{dz}{dt}$$
$$=(2x+2)(y-x)+(2y+2)(xz-y)+(2z-2)(\lambda+1-z-\lambda xy)$$
$$=-2x^2-2y^2-2z^2-2\lambda xyz+2xyz+2\lambda xy+2xy+2xz+2\lambda z-2x+4z-2\lambda-2$$
$$=-2\left(x^2+y^2+z^2+xyz-3xy-xz+x-4z+3\right)$$

$$\frac{dv_2}{dt}(x,y,z) = -2f(x,y,z)$$

Where $f(x,y,z) = x^2 + y^2 + z^2 + xyz - 3xy - xz + x - 4z + 3$

$f_x = 2x + yz - 3y - z + 1$	$f_y = 2y + xz - 3x$	$f_z = 2z + xy - x - 4$
$f_{xy} = z - 3$	$f_{yx} = z - 3$	$f_{zx} = y - 1$
$f_{xz} = y - 1$	$f_{yz} = x$	$f_{zy} = x$
$f_{xx} = 2$	$f_{yy} = 2$	$f_{zz} = 2$

Construct Hessian matrix for $f(x,y,z)$ at the second equilibrium point $E_2 = (-1,-1,1)$ to check whether Hessian matrix is positive definite.

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & z-3 & y-1 \\ z-3 & 2 & x \\ y-1 & x & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}\Bigg|_{E_2 = (-1,-1,1)}$$

All leading principal minors of Hessian matrix at $E_2 = (-1,-1,1)$ are:-

$$D_1 = 2\,,\ \ D_2 = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0 \text{ and } D_3 = \begin{vmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & - & 2 \end{vmatrix} = -18$$

Since the leading principal minors $D_1 = 2 > 0,\ D_2 = 0$ and $D_3 = -18 < 0$, then the Hessian matrix is neither positive nor negative definite or Hessian matrix is indefinite.

So that it is impossible to identify the algebraic sign of $\frac{dv_2}{dt}(x,y,z)$.

Therefore, It is impossible to deal with global stability condition of the equilibrium point $E_2 = (-1,-1,1)$ in the sense of Lyapunov stability theorem.

Let $v_3(x,y,z)=\frac{1}{4}(x-1)^2+(y-1)^2+(z-1)^2$ be appropriate Lyapunov function at the third equilibrium point $E_3=(1,1,1)$, then:-

1. $v_3(x^*,y^*,z^*)=v_3(1,1,1)=0$
2. $v_3(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_3(x,y,z)$ is positive definite function.
3. $\frac{dv_3}{dt}(x,y,z)=\frac{\partial v_3}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_3}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_3}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2(x-1)}{k}\frac{dx}{dt}+\frac{2(y-1)}{r_1}\frac{dy}{dt}+\frac{2(z-1)}{r_2}\frac{dz}{dt}$$

$$=(2x-2)(y-x)+(2y-2)(xz-y)+(2z-2)(\lambda+1-z-\lambda xy)$$

$$=-2x^2-2y^2-2z^2-2\lambda xyz+2xyz+2\lambda xy+2xy-2xz+2\lambda z+2x+4z-2\lambda-2$$

$$=-2\left(x^2+y^2+z^2+xyz-3xy+xz-x-4z+3\right)$$

$$\frac{dv_3}{dt}(x,y,z)=-2h(x,y,z)$$

Where $h(x,y,z)=x^2+y^2+z^2+xyz-3xy+xz-x-4z+3$

$$h_x=2x+yz-3y+z-1 \qquad h_y=2y+xz-3x \qquad h_z=2z+xy+x-4$$
$$h_{xy}=z-3 \qquad h_{yx}=z-3 \qquad h_{zx}=y+1$$
$$h_{xz}=y+1 \qquad h_{yz}=x \qquad h_{zy}=x$$
$$h_{xx}=2 \qquad h_{yy}=2 \qquad h_{zz}=2$$

Construct Hessian matrix for $h(x,y,z)$ at the third equilibrium point $E_3=(1,1,1)$ to check whether Hessian matrix is positive definite or indefinite.

$$H=\begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & z-3 & y+1 \\ z-3 & 2 & x \\ y+1 & x & 2 \end{pmatrix}$$

$$=\left.\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}\right|_{E_3=(1,1,1)}$$

All the leading principal minors of Hessian matrix at $E_3=(1,1,1)$ are:-

$$D_1=2,\quad D_2=\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}=0 \text{ and } D_3=\begin{vmatrix} 2 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}=-18$$

Since the leading principal minors $D_1=2>0$, $D_2=0$ and $D_3=-18<0$, then the Hessian matrix is indefinite.

So that it impossible to identify the algebraic sign of $\frac{dv_3}{dt}(x,y,z)$.

Therefore, It is impossible to deal with global stability condition of the equilibrium point $E_3=(1,1,1)$ in the sense of Lyapunov stability theorem.

Hopf Bifurcation Analysis of the System

I. Suppose the characteristic equation at $E_1=(0,0,3)$ has pure imaginary eigenvalues $m=\pm i\omega(\omega>0)$ at $\lambda=\lambda_0$, then $m^3+a_1m^2+a_2m+a_3=0$

Where $\lambda=\frac{r_2(k+r_1+r_2)}{kr_1}=\frac{3}{2}$

$$a_1=k+r_1+r_2=6$$
$$a_2=-r_2^{\ 2}=-1$$
$$a_3=-r_2^{\ 2}(k+r_1+r_2)=-6$$

$$\omega^2=-r_2^{\ 2}$$
$$\omega=\pm i1$$

Which contradicts the fact that $\omega>0$

$$m^3+6m^2-m-6=0$$
$$(m+6)(m^2-1)=0$$

$$m_1=-6 \text{ or } m_{2,3}=\pm 1$$

Since $m_{2,3}$ are not pure imaginary eigenvalues, then one of Hopf bifurcation condition is not satisfied.

As a result, the system (1.1) does not undergo Hopf bifurcation at $\lambda = \frac{3}{2}$.

II. Suppose the characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$, then $m^3 + b_1 m^2 + b_2 m + b_3 = 0$

Where $\lambda = \frac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)} = 15$

$$b_1 = k + r_1 + r_2 = 6$$
$$b_2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 20$$
$$b_3 = \frac{-2kr_2(k+r_1)(k+r_1+r_2)}{r_1+r_2-k} = 120$$
$$\omega^2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 20$$
$$\omega = \pm\sqrt{20}$$
$$m^3 + 6m^2 + 20m + 120 = 0$$
$$(m+6)(m^2+20) = 0$$
$$m_1 = -6\ , m_{2,3} = \pm i\sqrt{20}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied at $\lambda = 15$.

Next compute the $\operatorname{Re}\left(\frac{d\lambda}{dm}\right)$ from the characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{b_2(4k-2b_1)}{r_1 r_2(4k^2+b_2)} = \frac{20}{21} \neq 0$$

Since $\operatorname{Re}\left(\frac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied .

As a result, the system (1.1) under goes Hopf bifurcation at $\lambda = 15$.

III. Let the characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$, then $m^3 + c_1 m^2 + c_2 m + c_3 = 0$

Where $\lambda = \dfrac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)} = 15$

$$c_1 = k + r_1 + r_2 = 6$$

$$c_2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 20$$

$$c_3 = \frac{-2kr_2(k+r_1)(k+r_1+r_2)}{r_1+r_2-k} = 120$$

$$\omega^2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 20$$

$$\omega = \pm\sqrt{20}$$

$$m^3 + 6m^2 + 20m + 120 = 0$$

$$(m+6)(m^2+20) = 0$$

$$m_1 = -6\ , m_{2,3} = \pm i\sqrt{20}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then the condition one of the Hopf bifurcation is satisfied at $\lambda = 15$.

Next compute the $\operatorname{Re}\left(\dfrac{d\lambda}{dm}\right)$ from the characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{c_2(4k-2c_1)}{r_1 r_2(4k^2+c_2)} = \frac{20}{21} \neq 0$$

Since $\operatorname{Re}\left(\dfrac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied .

Therefore, the system (1.1) undergoes Hopf bifurcation at $\lambda = 15$.

Example 2. Consider parameters values with $k = 0.4,\quad r_1 = 0.2$, $\quad r_2 = 0.1,\quad \lambda = -0.1$, then the system(1.1) becomes

$$\frac{dx}{dt} = 0.4(y - x)$$
$$\frac{dy}{dt} = 0.2(xz - y)$$
$$\frac{dz}{dt} = 0.1(0.9 - z + 0.1xy)$$

Dissipative or Conservative of the System

Let
$$f_1 = 0.4(y - x)$$
$$f_2 = 0.2(xz - y)$$
$$f_3 = 0.1(0.9 - z + 0.1xy)$$

$$\frac{\partial f_1}{\partial x} = -0.4 , \qquad \frac{\partial f_2}{\partial y} = -0.2 \ , \qquad \frac{\partial f_3}{\partial z} = -0.1$$

The divergence of the vector field $\nabla . f\left(x, y, z\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} == -0.7$

$$\frac{dD}{dt} = \int \nabla . fdD = \int -0.7dD$$
$$\frac{dD}{dt} = -0.7D$$
$$\frac{dD}{D} = -0.7dt$$
$$\int \frac{1}{D} dD = -\int 0.7dt$$
$$\ln D = -0.7t + c$$
$$D = e^{-0.7t+c}$$
$$D = D_0 e^{-0.7t} \quad \left(D_0 = e^c\right)$$

D is decreasing exponentially

Therefore, the system(1.1) is dissipative.

Equilibrium Points of System

We end up with three equilibrium points of the system as follows:-

$$E_1 = (0,0,\lambda+1) = (0,0,0.9)$$
$$E_2 = (-1,-1,1) = (-1,-1,1)$$
$$E_3 = (1,1,1) = (1,1,1)$$

Local Stability Analysis

Linearize the system (1.1) at each equilibrium points and state the local stability conditions of the system. Consider the above system

$$\frac{\partial f_1}{\partial x} = -0.4 \ \frac{\partial f_2}{\partial x} = 0.2z \ \frac{\partial f_3}{\partial x} = 0.01y$$
$$\frac{\partial f_1}{\partial y} = 0.4 \quad \frac{\partial f_2}{\partial y} = -0.2 \ \frac{\partial f_3}{\partial y} = 0.01x$$
$$\frac{\partial f_1}{\partial z} = 0 \quad\ \frac{\partial f_2}{\partial z} = 0.2x \ \frac{\partial f_3}{\partial z} = -0.1$$

Jacobian matrix $A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.2z & -0.2 & 0.2x \\ 0.01y & 0.01x & -0.1 \end{pmatrix}$

I.The Jacobian matrix evaluated at the first equilibrium point

$E_1 = (0,0,0.9)$ is $J = A\big|E_1 = (0,0,0.9)$

$$J = \begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.18 & -0.2 & 0 \\ 0 & 0 & -0.1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at equilibrium point E_1 is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -0.4-m & 0.4 & 0 \\ 0.18 & -0.2-m & 0 \\ 0 & 0 & -0.1-m \end{vmatrix} = 0$$

$m^3 + 0.7m^2 + 0.068m + 0.0008 = 0$ where $a_1 = 0.7, \quad a_2 = 0.068, \ a_3 = 0.0008$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_1 = (0,0,0.9)$

$$\begin{array}{c|ccc} m^3 & 1 & 0.068 & 0 \\ m^2 & 0.7 & 0.0008 & 0 \\ m^1 & \frac{0.0468}{0.7} & 0 & \\ m^0 & 0.0008 & & \end{array}$$

From the above table, Since there is no sign changes in the first column of Routh- Hurwitz table, equilibrium point one is locally asymptotically stable.

II. The Jacobian matrix evaluated at the second equilibrium point

$E_2 = (-1,-1,1)$ is $J = A|E_2 = (-1,-1,1)$

$$J = \begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.2 & -0.2 & -0.2 \\ -0.01 & -0.01 & -0.1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at equilibrium point E_2 is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -0.4 - m & 0.4 & 0 \\ 0.2 & -0.2 - m & -0.2 \\ -0.01 & -0.01 & -0.1 - m \end{vmatrix} = 0$$

$$m^3 + 0.7m^2 + 0.058m - 0.0016 = 0 \text{ where } b_1 = 0.7, \quad b_2 = 0.058, \quad b_3 = -0.0016$$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$

$$\begin{array}{c|ccc} m^3 & 1 & 0.058 & 0 \\ m^2 & 0.7 & -0.0016 & 0 \\ m^1 & \frac{0.0422}{0.7} & 0 & \\ m^0 & -0.0016 & & \end{array}$$

From RH table, Since there is sign changes in the first column of Routh- Hurwitz table, equilibrium point two is unstable.

III. The Jacobian matrix evaluated at the third equilibrium point $E_3=(1,1,1)$ is $J=A|E_3=(1,1,1)$

$$J=\begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.2 & -0.2 & 0.2 \\ 0.01 & 0.01 & -0.1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at equilibrium point E_3 is $|J-mI|=0$

$$|J-mI|=\begin{vmatrix} -0.4-m & 0.4 & 0 \\ 0.2 & -0.2-m & 0.2 \\ 0.01 & 0.01 & -0.1-m \end{vmatrix}=0$$

$m^3+0.7m^2+0.058m-0.0016=0$ where $c_1=0.7,\quad c_2=0.058,\quad c_3=-0.0016$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_3=(1,1,1)$

$$\begin{array}{c|ccc} m^3 & 1 & 0.058 & 0 \\ m^2 & 0.7 & -0.0016 & 0 \\ m^1 & \frac{0.0422}{0.7} & 0 & \\ m^0 & -0.0016 & & \end{array}$$

From the above table, Since there is sign changes in the first column of Routh- Hurwitz table, equilibrium point three is unstable.

Global Stability Analysis of the System

Let $v_1(x,y,z)=\frac{x^2}{0.4}+\frac{y^2}{0.2}+\frac{(z-0.9)^2}{0.1}$ be candidate lyapunov function at equilibrium point $E_1=(0,0,0.9)$, then:-

1. $v_1(x^*,y^*,z^*)=v_1(0,0,0.9)=0$
2. $v_1(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_1(x,y,z)$ is positive definite function.

3. $$\frac{dv_1}{dt}(x,y,z)=\frac{\partial v_1}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_1}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_1}{\partial z}(x,y,z)\frac{dz}{dt}$$

$$=\frac{2x}{k}\frac{dx}{dt}+\frac{2y}{r_1}\frac{dy}{dt}+\frac{2(z-\lambda-1)}{r_2}\frac{dz}{dt}$$

$$=2x(y-x)+2y(xz-y)+2(z-\lambda-1)(\lambda+1-z-\lambda xy)$$

$$=-2\left(x^2+y^2+z^2-1.1xyz-0.91xy-1.8z+0.81\right)$$

$$\frac{dv_1}{dt}(x,y,z)=-2g(x,y,z)$$

Where $g(x,y,z)=x^2+y^2+z^2-1.1xyz-0.91xy-1.8z+0.81$

$g_x=2x-1.1yz-0.91y$	$g_y=2y-1.1xz-0.91x$	$g_z=2z-1.1xy-1.8$
$g_{xy}=-1.1z-0.91$	$g_{yx}=-1.1z-0.91$	$g_{zx}=-1.1y$
$g_{xz}=-1.1y$	$g_{yz}=-1.1x$	$g_{zy}=-1.1x$
$g_{xx}=2$	$g_{yy}=2$	$g_{zz}=2$

Construct Hessian matrix for $g(x,y,z)$ at the first equilibrium point $E_1=(0,0,0.9)$ to check whether Hessian matrix is positive definite.

$$H=\begin{pmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -1.1z-0.91 & -1.1y \\ -1.1z-0.91 & 2 & -1.1x \\ -1.1y & -1.1x & 2 \end{pmatrix}$$

$$=\left.\begin{pmatrix} 2 & -1.9 & 0 \\ -1.9 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right|_{E_1=(0,0,0.9)}$$

All leading principal minors of Hessian matrix at $E_1=(0,0,0.9)$ are:-

$$D_1=2\ ,\quad D_2=\begin{vmatrix} 2 & -1.9 \\ -1.9 & 2 \end{vmatrix}=0.39 \text{ and } D_3=\begin{vmatrix} 2 & -1.9 & 0 \\ -1.9 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=0.78$$

Since the leading principal minors $D_1=2>0,\ D_2=0.39>0$ and $D_3=0.78>0$, then the Hessian matrix is positive definite.

If the Hessian matrix is positive definite, then $\frac{dv_1}{dt}(x,y,z)$ is negative definite function.

4. $\lim_{(x,y,z)\to\infty} v_1(x,y,z) = \lim_{(x,y,z)\to\infty}\left[\frac{x^2}{0.4}+\frac{y^2}{0.2}+\frac{(z-0.9)^2}{0.1}\right]=\infty$

$v_1(x,y,z)$ is radially unbounded.

As a result, the equilibrium point $E_1=(0,0,0.9)$ is globally asymptotically stable by Lyapunov stability theorem.

Let $v_2(x,y,z)=\frac{(x+1)^2}{0.4}+\frac{(y+1)^2}{0.2}+\frac{(z-1)^2}{0.1}$ be appropriate Lyapunov function at the second equilibrium point $E_2=(-1,-1,1)$, then:-

1. $v_2(x^*,y^*,z^*)=v_2(-1,-1,1)=0$
2. $v_2(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_2(x,y,z)$ is positive definite function.

3. $\frac{dv_2}{dt}(x,y,z)=\frac{\partial v_2}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_2}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_2}{\partial z}(x,y,z)\frac{dz}{dt}$

$$=\frac{2(x+1)}{k}\frac{dx}{dt}+\frac{2(y+1)}{r_1}\frac{dy}{dt}+\frac{2(z-1)}{r_2}\frac{dz}{dt}$$
$$=(2x+2)(y-x)+(2y+2)(xz-y)+(2z-2)(\lambda+1-z-\lambda xy)$$
$$=-2x^2-2y^2-2z^2-2\lambda xyz+2xyz+2\lambda xy+2xy+2xz+2\lambda z-2x+4z-2\lambda-2$$
$$=-2\left(x^2+y^2+z^2-1.1xyz-0.9xy-xz+x-1.9z+0.9\right)$$

$$\frac{dv_2}{dt}(x,y,z)=-2f(x,y,z)$$

Where $f(x,y,z)=x^2+y^2+z^2-1.1xyz-0.9xy-xz+x-1.9z+0.9$

$f_x=2x-1.1yz-0.9y-z+1$ $\quad f_y=2y-1.1xz-0.9x$ $\quad f_z=2z-1.1xy-x-1.9$

$f_{xy}=-1.1z-0.9$ $\quad f_{yx}=-1.1z-0.9$ $\quad f_{zx}=-1.1y-1$

$f_{xz}=-1.1y-1$ $\quad f_{yz}=-1.1x$ $\quad f_{zy}=-1.1x$

$f_{xx}=2$ $\quad f_{yy}=2$ $\quad f_{zz}=2$

Construct Hessian matrix for $f(x,y,z)$ at the second equilibrium point $E_2=(-1,-1,1)$ to check whether Hessian matrix is positive definite or negative definite.

$$H=\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -1.1z-0.9 & -1.1y-1 \\ -1.1z-0.9 & 2 & -1.1x \\ -1.1y-1 & -1.1x & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -2 & 0.1 \\ -2 & 2 & 1.1 \\ 0.1 & 1.1 & 2 \end{pmatrix}\Bigg|_{E_2=(-1,-1,1)}$$

All leading principal minors of Hessian matrix at $E_2=(-1,-1,1)$ are:-

$$D_1=2\ ,\ D_2=\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}=0 \text{ and } D_3=\begin{vmatrix} 2 & -2 & 0.1 \\ -2 & 2 & 1.1 \\ 0.1 & 1.1 & 2 \end{vmatrix}=-2.88$$

Since the leading principal minors $D_1=2>0,\ D_2=0$ and $D_3=-2.88<0$, then the Hessian matrix is neither positive nor negative definite.

As a result, It is impossible to deal with global stability condition of the equilibrium point $E_2=(-1,-1,1)$ in the sense of Lyapunov stability theorem.

Suppose $v_3(x,y,z)=\dfrac{(x-1)^2}{0.4}+\dfrac{(y-1)^2}{0.2}+\dfrac{(z-1)^2}{0.1}$ be appropriate Lyapunov function at the third equilibrium point $E_3=(1,1,1)$, then:-

1. $v_3(x^*,y^*,z^*)=v_3(1,1,1)=0$
2. $v_3(x,y,z)>0$ for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

Hence, $v_3(x,y,z)$ is positive definite function.

3. $$\frac{dv_3}{dt}(x,y,z)=\frac{\partial v_3}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_3}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_3}{\partial z}(x,y,z)\frac{dz}{dt}$$

$$=\frac{2(x-1)}{k}\frac{dx}{dt}+\frac{2(y-1)}{r_1}\frac{dy}{dt}+\frac{2(z-1)}{r_2}\frac{dz}{dt}$$

$$=(2x-2)(y-x)+(2y-2)(xz-y)+(2z-2)(\lambda+1-z-\lambda xy)$$

$$=-2x^2-2y^2-2z^2-2\lambda xyz+2xyz+2\lambda xy+2xy-2xz+2\lambda z+2x+4z-2\lambda-2$$

$$=-2\left(x^2+y^2+z^2-1.1xyz-0.9xy+xz-x-1.9z+0.9\right)$$

$$\frac{dv_3}{dt}(x,y,z)=-2h(x,y,z)$$

Where $h(x,y,z)=x^2+y^2+z^2-1.1xyz-0.9xy+xz-x-1.9z+0.9$

$h_x=2x-1.1yz-0.9y+z-1$ $\quad h_y=2y-1.1xz-0.9x$ $\quad h_z=2z-1.1xy+x-1.9$

$h_{xy}=-1.1z-0.9$ $\quad h_{yx}=-1.1z-0.9$ $\quad h_{zx}=-1.1y+1$

$h_{xz}=-1.1y+1$ $\quad h_{yz}=-1.1x$ $\quad h_{zy}=-1.1x$

$h_{xx}=2$ $\quad h_{yy}=2$ $\quad h_{zz}=2$

Construct Hessian matrix for $h(x,y,z)$ at the third equilibrium point $E_3=(1,1,1)$ to check whether Hessian matrix is definite or indefinite.

$$H=\begin{pmatrix} h_{xx} & h_{xy} & h_{xz} \\ h_{yx} & h_{yy} & h_{yz} \\ h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -1.1z-0.9 & -1.1y+1 \\ -1.1z-0.9 & 2 & -1.1x \\ -1.1y+1 & -1.1x & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -2 & -0.1 \\ -2 & 2 & -1.1 \\ -0.1 & -1.1 & 2 \end{pmatrix}\Bigg|_{E_3=(1,1,1)}$$

All leading principal minors of Hessian matrix at $E_3 = (1,1,1)$ are:-

$$D_1 = 2 \quad , \quad D_2 = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0 \text{ and } D_3 = \begin{vmatrix} 2 & -2 & -0.1 \\ -2 & 2 & -1.1 \\ -0.1 & -1.1 & 2 \end{vmatrix} = -2.88$$

Since the leading principal minors $D_1 = 2 > 0,\ D_2 = 0$ and $D_3 = -2.88 < 0$, then Hessian matrix is indefinite.

Therefore, It is impossible to deal with global stability condition of the equilibrium point $E_3 = (1,1,1)$ in the sense of Lyapunov stability theorem.

Hopf Bifurcation Analysis of the System

Let the characteristic equation of the Jacobian matrix at $E_1 = (0,0,0.9)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$,then $m^3 + a_1 m^2 + a_2 m + a_3 = 0$

Where $\lambda = \dfrac{r_2(k + r_1 + r_2)}{kr_1} = 0.875$

$$a_1 = k + r_1 + r_2 = 0.7$$
$$a_2 = -r_2^{\,2} = -0.01$$
$$a_3 = -r_2^{\,2}(k + r_1 + r_2) = -0.007$$

$$\omega^2 = -0.01$$
$$\omega = \pm i\sqrt{0.01}$$

Which contradicts the fact that $\omega > 0$

$$m^3 + 0.7m^2 - 0.01m - 0.007 = 0$$
$$(m + 0.7)(m^2 - 0.01) = 0$$

$$m_1 = -0.7 \text{ or } m_{2,3} = \pm\sqrt{0.01} = \pm 0.1$$

Since $m_{2,3}$ are not pure imaginary eigenvalues, then one of Hopf bifurcation condition is not satisfied.

As a result, the system (1.1) does not undergo Hopf bifurcation at $\lambda = 0.875$.

Suppose the characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$,then $m^3 + b_1 m^2 + b_2 m + b_3 = 0$

Where $\lambda = \frac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)} = 21$

$$b_1 = k + r_1 + r_2 = 0.7$$

$$b_2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 0.48$$

$$b_3 = \frac{-2kr_2(k+r_1)(k+r_1+r_2)}{r_1+r_2-k} = 0.336$$

$$\omega^2 = 0.48$$

$$\omega = \pm\sqrt{0.48}$$

$$m^3 + 0.7m^2 + 0.48m + 0.336 = 0$$

$$(m+0.7)(m^2+0.48) = 0$$

$$m_1 = -0.7 \text{ or } m_{2,3} = \pm i\sqrt{0.48}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied at $\lambda = 21$.

Next compute the $\operatorname{Re}\left(\frac{d\lambda}{dm}\right)$ from the characteristic equation of the Jacobian matrix at

$E_2 = (-1,-1,1)$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{b_2(4k-2b_1)}{r_1 r_2(4k^2+b_2)} = \frac{0.096}{0.0224} \neq 0$$

Since $\operatorname{Re}\left(\frac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied .

As a result, the system (1.1) under goes Hopf bifurcation at $\lambda = 21$.

Assume that the characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$,then $m^3 + c_1 m^2 + c_2 m + c_3 = 0$

Where $\lambda = \dfrac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)} = 21$

$$c_1 = k + r_1 + r_2 = 0.7$$

$$c_2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 0.48$$

$$c_3 = \frac{-2kr_2(k+r_1)(k+r_1+r_2)}{r_1+r_2-k} = 0.336$$

$$\omega^2 = 0.48$$

$$\omega = \pm\sqrt{0.48}$$

$$m^3 + 0.7m^2 + 0.48m + 0.336 = 0$$

$$(m+0.7)(m^2+0.48) = 0$$

$$m_1 = -0.7 \text{ or } \quad m_{2,3} = \pm i\sqrt{0.48}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied at $\lambda = 21$.

Next compute the $\operatorname{Re}\left(\dfrac{d\lambda}{dm}\right)$ from the characteristic equation of the Jacobian matrix at

$E_3 = (1,1,1)$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{c_2(4k-2c_1)}{r_1r_2(4k^2+c_2)} = \frac{0.096}{0.0224} \neq 0$$

Since $\operatorname{Re}\left(\dfrac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied .

As a result, the system (1.1) under goes Hopf bifurcation at $\lambda = 21$.

CHAPTER FIVE

5. CONCLUSION AND FUTURE SCOPE

5.1 Conclusions

In this study, the Stability and Bifurcation analysis of Maxwell-Bloch equations were considered. Firstly, some basic definitions and theorems were discussed in the preliminary parts. The system is proved to be dissipative by the aid of divergence test. The result of the study revealed that equilibrium point one is stable and unstable for negative and positive value of pumping energy parameter respectively. The remaining two equilibrium points are stable and unstable for positive and negative value of pumping energy parameter respectively. By the aid of Lyapunov theorem, equilibrium point one was proved to be globally asymptotically stable with some specific interval of pumping energy parameter. For the remaining two equilibrium points, it is impossible to give generalization about its global asymptotical stability in the sense of Lyapunov as one of the criteria of the theorem is not satisfied. Furthermore, the result of Hopf bifurcation indicates that the system doesn't undergo Hopf bifurcation at equilibrium point one by any choice of pumping energy parameter. With some specific conditions, the system undergoes Hopf bifurcation about the two remaining equilibrium points for a certain value of pumping energy parameter. Finally, in order to verify the applicability of the result two numerical examples were solved. MATLAB simulation was also implemented to support the findings of the study.

5.2 Future Scope

One can investigate stability and Bifurcation analysis of Maxwell- Bloch equations by considering other factors like time delay and diffusion effects. Furthermore, direction and stability of Hopf bifurcation of the system is another area of future work. Moreover, qualitative analysis with regard to limit cycle, periodic solution and chaotic behavior are further area of future work.

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