STABILITY AND BIFURCATION ANALYSIS OF RIKITAKE MODEL

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE MASTERS OF SCIENCES (M.Sc.) IN MATHEMATICS.

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DECLARATION

I, here submit the thesis entitled by "**Stability and Bifurcation Analysis of RIKITAKE Model**" for the award of degree of Master of Science in Mathematics. I, the undersigned declare that, this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or the like, where other sources of information have been used, they have been acknowledge.

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Abstract

In this thesis, stability and bifurcation analysis of Rikitake model was considered. By the aid of divergence, the system is proved to be dissipative. Two Steady state points of the equations were determined. The equations were linearized using Jacobian matrix about each equilibrium points and yield the same characteristic equation. The local stability condition of each critical point was proved by using Routh Huwertiz stability criteria. It is impossible to generalize the global stability property of the two equilibrium point in sense of Lyapunov as one of the condtion is failed to be satisfied. Furthermore, the result of Hopf bifurcation revealed that the system undergoes Hopf bifurcation at the two equilibrium points. Finally, in order to verify the applicability of the result two numerical examples were solved and MATLAB simulation was implemented to support the findings of the study.

Key words: Rikitake model, Local stability, global stability, Routh Huwertiz stability criteria, Lyapunov theorem, Hopf bifurcation.

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CHAPTER ONE

1. INTRODUCTION

1.1 Background of the Study

Mathematical modeling is the application of mathematics to describe real world problems and investigating important questions that arise from it. Using mathematical tools, the real-world problem is translated to a mathematical problem, which mimics the real-world problem. A solution to the mathematical problem is obtained, which is interpreted in the language of real-world problem to make predictions about the real world. Mathematical modeling can be described as an activity which allows a mathematician to be biologist, chemist, ecologist, economist depending on the problem under consideration. The primary aim of a modeler is to undertake experiments on the mathematical representation of a real-world problem, instead of undertaking experiments in the real world (Sandip, 2008). Models describe our beliefs about how the world functions. In mathematical modeling, we translate those beliefs into the language of mathematics (Abramowitz and Stwgum, 1968).

The Rikitake system is simplified dynamic model which attempts to explain the irregular polarity switching of the earth's geomagnetic field. The physics of the Rikitake system has been studied by various authors like Japanese geophysicist Rikitake (Rikitake, 1958) and (Llibre and Messias, 2009). The Rikitake system is a mathematical model obtained from a simple mechanical system used to study the reversals of the Earth's magnetic field in a two-disc dynamo model (Denis *et al.,* 2010). The system has a 3-dimentional Lorenz type chaotic attractor around its two singular points. However, this attractor is not bounded by any ellipsoidal surface as in the Lorenz attractor. The model consists of two identical single Faraday-disk dynamos of the Bullard type coupled together. The Rikitake dynamo is composed of two conducting rotating disks which are connected to two coils so that the current in each coil feeds the magnetic field of the other. The Rikitake dynamo consists of two coupled disc dynamos of Bullard type, each disc dynamo is made up of an axis of rotation, a disc and a wire, all made from the same conductive materials.

The earth's magnetic field generates the magnetosphere which shields the earth from the stream of charged particles in solar wind. The geomagnetic field has reversed repeatedly during the Earth's history. The mechanism behind the reversal and the wide variation in reversal interval is not completely understood. The reversal of the Earth's magnetic field is the Rikitake system (Rikitake, 1958) which is given by system of nonlinear first order differential equation given as:

$$\begin{aligned} \frac{dx}{dt} &= -vx + zy \\ \frac{dy}{dt} &= -vy + (z-a)x \\ \frac{dz}{dt} &= 1 - xy \end{aligned} \quad , \qquad (1.1)$$

where a and v are control parameters which is assumed to be positive. v represent the resistive dissipation and a represents the difference in the angular velocities of the two disks. x, y and z represents the electric current flowing in disk 1, the electric current flowing in disk 2 and the corresponding angular velocity respectively.

Nonlinear Mathematical models of real-world phenomena that are formulated in terms of ODEs as in Eq. (1.1) are not easy to directly solve for their solution and hence it is necessary to use qualitative approaches, such as stability and bifurcation analysis, to investigate their solution behaviors. Stability analysis of a system is important in control theory. The asymptotical stability is based on the inhibition and the coexistence factors between the two competing species. Bifurcation analysis is the analysis of a system of ordinary differential equations (ODE's) under parameter variation. Performing a local bifurcation analysis is often a powerful way to analysis the properties of such systems, since it predicts what kind of behavior occurs in the system when there is change in parameter.

A dissipative system is defined as a system whose phase space volumes shrink where as in a conservative system phase space volume is conserved. Conservative systems have constant entities (usually, energy). Physically, we mean systems with no influx and no production of energy/matter. Dissipative systems lose energy with time. In order to maintain persistent behaviours the dissipative system must have influx of energy/matter.

If a dissipative system starts at its stable equilibrium point, it stays there for arbitrarily long and one cannot see the basin of attraction and compression of the volume (Strogatz, 1994).

In 2008, Liu *et al*. analyzed the dynamics of Rikitake two disk dynamos to explain the reversals of the Earth's magnetic field. They concluded that the chaotic behavior of the system can be used to simulate the reversals of the geomagnetic field. The Rikitake chaotic attractor was studied by several authors (Millen, 1999 and Mohammad *et al.*, 2013).In 2009, Llibre and Messias used the Poincare compactification to study the dynamics of the Rikitake system at infinity, showing that there are orbits which escape to or come from infinity, instead of going towards the attractor. Wanga *et al. (*2015) investigated about the stability analysis of susceptible-vaccines-exposed-infectious-recovered model with continuous age structure in the exposed and infectious classes. They investigate the global dynamics of this model in the sense of basic reproduction number via constructing Lyapunov functions. Gideon *et al. (*2014) investigated the stability analysis of model of cooperative and competitive species and they obtained the cooperative system was found to be stable at one of the two equilibrium points presents and unstable (Saddle) at the other. Four equilibrium points existed for the competitive species model for which the system is stable at one point and locally asymptotically stable at the other three points.

In (2019) Jinming *et al.* investigated stability and bifurcation analysis of a predator-prey system with the week allee effect. They analyzed the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. In 2018, another paper regarding chemostat model is presented a 4-dimensional food chain in a chemistat with removal rates (Sheikh, 2004 and Mahrouf, 2005). They studied local and global stability of equilibria along with elementary properties including boundedness of solutions, invariance of non-negativity, dissipativity and persistence analysis. In 2017, Xiao *et al.* investigated stability analysis of the system equilibria and existence of Hopf bifurcation by using the Hopf bifurcation conditions. In 2016, Wang *et al.* investigated stability and Hopf bifurcation analysis of an epidemic model by using the method of multiple scales. Then, the amplitude of bifurcating periodic solution and the conditions which determine the stability of the bifurcating periodic solution are obtained. The validity of analytical results is shown by their consistency with numerical simulations.

In 2015, Nijamudin Ali and Santabrata Chakrravarty investigated the stability analysis of three species competitive food chain model incorporate prey refuge and they analyzed local and global stability of the equilibria in order to examine the behavior of the system. In 2014, Mandal *et al.* investigated stability and bifurcation of switched dynamical systems and verified using several hybrid dynamics system.

However, stability and bifurcation analysis of Rikitake model given by Eq. (1.1) is not yet investigated in the existing literature. Consequently, the main goal of this study is to investigate stability and bifurcation analysis of Rikitake model represented by Eq. (1.1).

1.2 Statement of the Problem

This Study focuses on the following problems

- System property in relation to dissipative, conservative or neither.
- Local stability conditions of Rikitake system.
- Global stability conditions of Rikitake system .
- Hopf bifurcation conditions of Rikitake system.

1.3 Objective of the Study

1.3.1 General objective

The general objective of this study is to investigate stability and bifurcation analysis of Rikitake model given by Eq. (1.1).

1.3.2 Specific objectives of the study

The specific objectives of the study are:

- To check whether the system is dissipative, conservative or neither.
- To determine local stability conditions of Rikitake system.
- To determine global stability conditions of Rikitake system.
- To determine Hopf bifurcation conditions of Rikitake system.

1.4 Significance of the Study

This study helps Geophysicists to keep track of appropriate conditions for the reversals of earth's magnetic field polarity so that the mechanism behind the reversal and the wide variation in reversal interval can be understood.

1.5 Delimitation of the Study

This study is delimited to stability and bifurcation analysis of Rikitake system given by Eq. (1.1).

CHAPTER TWO

LITERATURE REVIEW

In physics, a dynamical system is described as a particle whose state varies over time and thus obeys differential equations involving time derivatives. The study of dynamical systems is the focus of dynamical systems theory, which has applications to a wide variety of fields such as mathematics, physics, biology, chemistry, engineering, economics, and medicine. Dynamical systems are a fundamental part of chaos theory, logistic map dynamics, bifurcation theory, the self-assembly and self-organization processes, and the chaos concept. Nonlinear dynamical systems, describing changes in variables over time, may appear chaotic, unpredictable, or counterintuitive, contrasting with much simpler linear systems (Boeing, 2016).

Mathematical physics refers to the development of mathematical methods for the application of mathematics to problems in physics. The term mathematical physics is used to denote research aimed at studying and solving problems inspired by physics or thought experiments within a mathematically rigorous framework. Mathematical physics covers a very broad academic realm distinguished only by the blending of pure mathematics and physics. Although related to theoretical physics, mathematical physics in this sense emphasizes the mathematical rigour of the same type as found in mathematics (Frenkel and Fillippov,2000).

Mathematical models can take many forms, including dynamical systems, statistical models, differential equations, or game theoretic models. Mathematical models are usually composed of relationships and variables. Relationships can be described by operators, such as algebraic operators, functions, differential operators, etc. Variables are abstractions of system parameters of interest that can be quantified. A dynamic model accounts for time-dependent changes in the state of the system. Mathematical models are of great importance in the natural sciences, particularly in physics. Physical theories are almost invariably expressed using mathematical models (Aris and Rutherford, 1994).

Among the studied topics related with the Rikitake system, the stability of the equilibrium points, the chaotic behavior, integrals and invariant manifolds, Hamiltonian dynamics and many others(Valls,2005).

It is known by geophysicists that the Earth's magnetic field has reversed its polarity many times along geological history (Glatzmaiers and Roberts, 1995).Ahmad Harb and Nabil Ayoub, 2014 analyzed the dynamics of the Rikitake two disk dynamo system and they showed that under certain value of control parameter, the system experiences a chaotic behavior. The study showed that the designed controller was so effective in controlling the unstable chaotic oscillations.

The concepts of equilibrium and stability come from Classical Mechanics (Arrowsmith and Place, 1992). A state where a system is in balance with the external forcing so that it does not change in time is called an equilibrium position. However, any equilibrium position may be either stable or unstable. If released near a stable equilibrium position, the system will evolve towards such a position. On the contrary, if released near an unstable equilibrium position, it will go far away from this position. The stability problem of nonlinear systems has been extensively studied.

Most dynamical systems are described by ordinary differential equations or difference equations. In general, these systems are nonlinear and include many parameters. Small changes in the values of their parameters may have large effects on the behaviors of the system. Determining a way to analyze such a dynamical system is an important problem. So far, a large number of methods of analyzing nonlinear dynamical systems have been proposed. It is well known that the solution to most nonlinear dynamical systems cannot be obtained analytically. This means we must conduct numerous simulations using the different fixed sets of parameter values and initial conditions. However, such simulations only give information about one stable solution at a time, and they tend to take a long time to reach a solution. The topological properties of the solutions to a dynamical system may change when a parameter of the system changes slightly. This phenomenon is called bifurcation. Bifurcation analysis, which is the investigation of bifurcations depending on the system parameters, is a way to gain deep insights into the fundamental properties of dynamical systems. Furthermore, bifurcation analysis enables us to identify the range of a parameter over which a system behaves stably, the total behavior of the solution in the large and the transition mechanisms of the dynamic responses (Yoshinaga and Kawakami,1995).

CHAPTER THREE

3. METHODOLOGY

3.1 Study Period

This study was conducted from September, 2018 to January, 2020.

3.2 Study Design

The study employed mixed design (analytical and experimental approaches).

3.3 Source of Information

The sources of information for the study were journals, published article and related information from internet.

3.4 Mathematical Procedures

This study was conducted based on the following procedures

1. Checking whether the system is dissipative, conservative or neither;
2. Determining the equilibrium point of the system;
3. Linearizing Rikitake system about positive equilibrium point;
4. Determining the local stability conditions of the system;
5. Analyzing the global stability of the system;
6. Determining Hopf bifurcation conditions of the system;
7. Verifying the result using numerical simulation.

CHAPTER FOUR

4. RESULT AND DISCUSSION

4.1 Preliminaries

Definition 1: For any three dimensional system: $x' = f(x)$, the volume involves $D' = \int \nabla . fdD$ where ∇ is divergence operator and D is volume in phase space. If D decreasing exponentially, then the system is dissipative. If D increasing exponentially, then the system is expansive. If D is constant, then the system is conservative.

Definition 2: Equilibrium point

Consider non-linear system $\frac{dx}{dt} = f(x)$, where $f : R^n \to R^n$. A point $x^* \in R^n$ is an equilibrium point if $\frac{dx}{dt}(x^*) = f(x^*) = 0$

Definition 3: For a linear system $\frac{dx}{dt} = AX$ the stability of equilibrium point $\frac{dx}{dt} = 0$ can be completely determined by location of eigenvalues of A. This is expressed as follows;

I.If all the eigenvalues of the Jacobian matrix have real parts less than zero, then the linear system is asymptotically stable and,

II.If at least one of the eigenvalue of Jacobian matrix has real part greater than zero, then the system is unstable. (Khalil, 1996).

Definition 4: Routh-Hurwitz Stability Criterion (RH-Criterion) (Katsuhiko, 1970)

The local stability of the equilibrium points of the system can be determined by applying the Routh's stability criterion for the given characteristic polynomial of the form:

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$$

where $a_0 \neq 0$ and $a_n > 0$, then the Routh-Hurwitz array or table is given as follows.

$$
\begin{array}{c|ccccc}
\lambda^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
\lambda^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
\lambda^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\lambda^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\lambda^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \\
\lambda^2 & e_1 & e_2 & & & \\
\lambda^1 & f_1 & & & & \\
\lambda^0 & g_0 & & & &
\end{array}
$$

Where $b_1 = \dfrac{a_1a_2 - a_0a_3}{a_1}$, $b_2 = \dfrac{a_1a_4 - a_0a_5}{a_1}$, $b_3 = \dfrac{a_1a_6 - a_0a_7}{a_1}$, $c_1 = \dfrac{b_1a_3 - a_1b_2}{b_1}$, $c_2 = \dfrac{b_1a_5 - a_1b_3}{b_1}$, $c_3 = \dfrac{b_1a_7 - a_1b_4}{b_1}$, $d_1 = \dfrac{c_1b_2 - b_1c_2}{c_1}$, $d_2 = \dfrac{c_1b_3 - b_1c_3}{c_1}$

The equilibrium point is stable if there is no sign change in the first column of the Routh table above.

Dealing with zero row

Routh Hurwitz procedure provides an "auxiliary polynomial", $a(\lambda)$, that contains the roots of interest as factors. The auxiliary polynomial is a factor of the original polynomial. The coefficients of the auxiliary polynomial appear in the row above the zero row. Finally, we replace the zero row by the coefficients of the derivative of the auxiliary polynomial.

Definition 5: **Leading Principal Minors of Matrix (Eriksen, 2010)**

The leading principal minor of a matrix A of order k is the minor of order k obtained by deleting the last n-k rows and columns.

Consider $3x3$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

First order leading principal minors is: $D_1 = |a_{11}|$

Second order leading principal minor is: $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Third order leading principal minor is: $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Definition 6: Let A be a symmetric nxn matrix. Then A is positive definite matrix if and only if $D_k > 0$ for all leading principal minors and A is negative definite matrix if and only if $(-1)^k D_k > 0$ for all leading principal minors, where $1 \le k \le n$

Theorem 4.1 Lyapunov Stability Theorem (Strogatz,1994)

Let $x = x^*$ be an equilibrium point of non-linear system of $\frac{dx}{dt} = f(x)$, $f : D \to R^n$.

Suppose $V : D \to R$ be continuously differentiable function such that:-

I. $V(x^*) = 0$

II. $V(x) > 0$ for all $x \in D - \{x^*\}$

III. $\frac{dV(x)}{dt} < 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*).Then $x = x^*$ is stable.

Theorem 4.2 Globally asymptotically stable

Let $x = x^*$ be an equilibrium point of non-linear system of $\frac{dx}{dt} = f(x)$, $f : D \to R^n$.

Let $V : D \to R$ be continuously differentiable function such that:-

1. $V(x^*) = 0$
2. $V(x) > 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*)
3. $\frac{dV(x)}{dt} < 0$ for all $x \in D - \{x^*\}$ (Domain D excluding x^*)
4. $V(x)$ is radially unbounded. Then $x = x^*$ is globally asymptotically stable

Definition 7: Hessian Matrix

Let $f(x)$ be a scalar function in n variables, then the Hessian Matrix of f is the matrix consisting of all the second order partial derivatives of f.

The Hessian Matrix of f at the point x is the nxn matrix such that

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1{}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2{}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & & \frac{\partial^2 f}{\partial^2 x_n} \end{pmatrix}$$

Hessian matrix is used to determine the points of global maximum and minimum. If Hessian Matrix is a positive definite at the equilibrium point, then the function f has global minimum at the equilibrium point.

Hopf Bifurcation

Theorem 4.3: Le J_0 be a Jacobian matrix of a continuous parametric dynamical system evaluated at equilibrium point. Suppose that all eigenvalues of J_0 have a negative real parts except one conjugate nonzero purely imaginary pair $\pm i\omega$. Hopf Bifurcation arises when these two eigenvalues cross the imaginary axis because of a variation of the system.

4.2 Dissipative or conservativeness of the system

Consider the system (1.1) under consideration;

$$\begin{aligned} \frac{dx}{dt} &= -vx + zy \\ \frac{dy}{dt} &= -vy + (z-a)x \\ \frac{dz}{dt} &= 1 - xy \end{aligned} \quad , \qquad (1.1)$$

Let $\begin{aligned} f_1 &= -vx + zy \\ f_2 &= -vy + (z-a)x \\ f_3 &= 1 - xy \end{aligned}$

Divergence of the system

$$\nabla .\ f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(-vx + zy) + \frac{\partial}{\partial y}(-vy + (z-a)x) + \frac{\partial}{\partial z}(1 - xy)$$

$$= -v - v + 0$$

$$= -2v < 0$$

$$D' = \int \nabla .\ f dD$$

$$D' = \int -2v dD = -2vD$$

$$\frac{dD}{dt} = -2vD$$

$$\frac{1}{D} dD = -2v dt$$

Integrating both sides, we get

$\ln D = -2vt + c$, where c is constant value

$D = e^{-2vt+c} = D_0 e^{-2vt}$, where $D_0 = e^c$

Hence, D is decreasing exponentially ,then the system is dissipative.

4.3 Equilibrium point

To find the equilibrium point of system (1.1) , equate the right hand side with zero.

$$-vx + zy = 0 \tag{4.1}$$

$$-vy + (z-a)x = 0 \tag{4.2}$$

$$1 - xy = 0 \tag{4.3}$$

From Eq. (4.1) ; $z = \frac{vx}{y}$ (4.4)

From Eq. (4,3) , $y = \frac{1}{x}$ (4.5)

Plugging equations (4.4) and (4.5) in to equation (4.2) becomes

$$-vy + (z - a)x = 0$$

$$-v(\frac{1}{x}) + (\frac{vx}{y} - a)x = 0$$

$$-\frac{v}{x} + (vx^2 - a)x = 0$$

$$vx^4 - ax^2 - v = 0$$

Let $x^2 = w$

$vw^2 - aw - v = 0$ it's form of quadratic equation. We can solve using quadratic formula.

$$w = \frac{a \pm \sqrt{a^2 - 4(v)(-v)}}{2v} = \frac{a \pm \sqrt{a^2 + 4v^2}}{2v}$$

$$w = \frac{a + \sqrt{a^2 + 4v^2}}{2v} \text{ or } w = \frac{a - \sqrt{a^2 + 4v^2}}{2v}$$

Since $x^2 = w$, then

$$x_1 = \sqrt{\frac{a + \sqrt{a^2 + 4v^2}}{2v}} \text{ or } x_2 = -\sqrt{\frac{a + \sqrt{a^2 + 4v^2}}{2v}}$$

Since, $y = \frac{1}{x}$, then

$$y_1 = \sqrt{\frac{2v}{a + \sqrt{a^2 + 4v^2}}} \text{ or } y_2 = -\sqrt{\frac{2v}{a + \sqrt{a^2 + 4v^2}}}$$

Again, $z = \frac{vx}{y} = vx^2$

$$z = v\ (\sqrt{\frac{a+\sqrt{a^2+4v^2}}{2v}})^2$$

$$z_1 = \frac{a+\sqrt{a^2+4v^2}}{2} \text{ or } z_2 = \frac{a+\sqrt{a^2+4v^2}}{2}$$

Therefore, the equilibrium points of the system are

$$E_1 = (\sqrt{\frac{a+\sqrt{a^2+4v^2}}{2v}}, \sqrt{\frac{2v}{a+\sqrt{a^2+4v^2}}}, \frac{a+\sqrt{a^2+4v^2}}{2})$$

$$E_2 = (-\sqrt{\frac{a+\sqrt{a^2+4v^2}}{2v}}, -\sqrt{\frac{2v}{a+\sqrt{a^2+4v^2}}}, \frac{a+\sqrt{a^2+4v^2}}{2})$$

4.4 Local Stability Analysis

Linearization

$$f_1 = \frac{dx}{dt} = -vx + zy$$

Let $f_2 = \frac{dy}{dt} = -vy + (z-a)x$

$$f_3 = \frac{dz}{dt} = 1 - xy$$

Let $A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{pmatrix}$

Let J_1 be Jacobian matrix evaluated at equilibrium point E_1

$$J_1 = \begin{pmatrix} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{pmatrix}_{E_1} = \begin{pmatrix} -v & \frac{a+\sqrt{a^2+4v^2}}{2} & \sqrt{\frac{2v}{a+\sqrt{a^2+4v^2}}} \\ \frac{a+\sqrt{a^2+4v^2}}{2}-a & -v & \sqrt{\frac{a+\sqrt{a^2+4v^2}}{2v}} \\ -\sqrt{\frac{2v}{a+\sqrt{a^2+4v^2}}} & -\sqrt{\frac{a+\sqrt{a^2+4v^2}}{2v}} & 0 \end{pmatrix} \qquad (4,7)$$

For the sake of simplicity, let $k^2 = \frac{a+\sqrt{a^2+4v^2}}{2v}$ then J_1 becomes

$$J_1 = \begin{pmatrix} -v & vk^2 & \frac{1}{k} \\ vk^2-a & -v & k \\ -\frac{1}{k} & -k & 0 \end{pmatrix} \qquad (4.8)$$

To find the characteristics equation of equation (4,7) , we use $\det(J_1 - \lambda I) = 0$

$$\begin{vmatrix} -v-\lambda & vk^2 & \frac{1}{k} \\ vk^2-a & -v-\lambda & k \\ -\frac{1}{k} & -k & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} v+\lambda & -vk^2 & -\frac{1}{k} \\ -vk^2+a & v+\lambda & -k \\ \frac{1}{k} & k & \lambda \end{vmatrix} = 0$$

$$\lambda^3 + 2v\lambda^2 + (v^2 + k^2 - v^2k^4 + vk^2a + \frac{1}{k^2})\lambda + 3vk^2 + \frac{v}{k^2} - a = 0$$

By substituting value of k' and simplifying leads to:

$$\lambda^3 + 2v\lambda^2 + \frac{\sqrt{a^2+4v^2}}{v}\lambda + 2\sqrt{a^2+4v^2} = 0$$ is the characteristics equation

To solve the values of λ is very difficult. So, by applying RH – criterion we can determine the stability of E_1.

λ^3	1	$\dfrac{\sqrt{a^2+4v^2}}{v}$	0
λ^2	$2v$	$2\sqrt{a^2+4v^2}$	0
λ^1	b_1	0	0
λ^0	c_1	0	0

To find $b_1 = \dfrac{a_1a_2 - a_0a_3}{a_1} = \dfrac{2v\left(\dfrac{\sqrt{a^2+4v^2}}{v}\right) - 1\left(2\sqrt{a^2+4v^2}\right)}{2v} = 0$

RH-criterion table becomes

λ^3	1	$\dfrac{\sqrt{a^2+4v^2}}{v}$	0
λ^2	$2v$	$2\sqrt{a^2+4v^2}$	0
λ^1	0	0	0
λ^0	c_1	0	0

The auxiliary polynomial:

$$a(\lambda) = 2v\lambda^2 + 2\sqrt{a^2+4v^2}$$

The derivative of this auxiliary polynomial with respect to λ becomes

$$\frac{da(\lambda)}{d\lambda} = 4v\lambda + 0$$

The RH-criterion table becomes

λ^3	1	$\dfrac{\sqrt{a^2+4v^2}}{v}$	0
λ^2	$2v$	$2\sqrt{a^2+4v^2}$	0
λ^1	$4v$	0	0
λ^0	c_1	0	0

$$c_1=\frac{b_1a_3-a_2b_2}{b_1} \text{ where, } b_2=\frac{a_1a_4-a_0a_5}{a_1}=\frac{2v(0)-1(0)}{2v}=0$$

$$c_1=\frac{b_1a_3-a_2(0)}{b_1}=a_3$$

$$=2\sqrt{a^2+4v^2}$$

Finally RH-criterion table becomes

λ^3	1	$\dfrac{\sqrt{a^2+4v^2}}{v}$	0
λ^2	$2v$	$2\sqrt{a^2+4v^2}$	0
λ^1	$4v$	0	0
λ^0	$2\sqrt{a^2+4v^2}$	0	0

As a result the positive equilibrium point of the system (1,1) is locally asymptotically stable by Routh Hurwitz criterion.

Let J_2 be Jacobian matrix evaluated at equilibrium point E_2

$$J_2=\begin{pmatrix} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{pmatrix}_{E_2}=\begin{pmatrix} -v & \dfrac{a+\sqrt{a^2+4v^2}}{2} & -\sqrt{\dfrac{2v}{a+\sqrt{a^2+4v^2}}} \\ \dfrac{a+\sqrt{a^2+4v^2}}{2}-a & -v & -\sqrt{\dfrac{a+\sqrt{a^2+4v^2}}{2v}} \\ \sqrt{\dfrac{2v}{a+\sqrt{a^2+4v^2}}} & \sqrt{\dfrac{a+\sqrt{a^2+4v^2}}{2v}} & 0 \end{pmatrix}$$

Again let $k^2 = \frac{a+\sqrt{a^2+4v^2}}{2v}$ then J_2 becomes

$$J_2 = \begin{pmatrix} -v & vk^2 & -\frac{1}{k} \\ vk^2-a & -v & -k \\ \frac{1}{k} & k & 0 \end{pmatrix}$$

Then find the characteristic equation of J_2, $\det\left(J_2-\lambda I\right)=0$

$$\begin{vmatrix} -v-\lambda & vk^2 & -\frac{1}{k} \\ vk^2-a & -v-\lambda & -k \\ \frac{1}{k} & k & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} v+\lambda & -vk^2 & \frac{1}{k} \\ -vk^2+a & v+\lambda & k \\ -\frac{1}{k} & -k & \lambda \end{vmatrix} = 0$$

$$\lambda^3 + 2v\lambda^2 + (v^2+k^2-v^2k^4+vk^2a+\frac{1}{k^2})\lambda + 3vk^2 + \frac{v}{k^2} - a = 0$$

Substitute the value of k^2 and simplifying leads to:

$$\lambda^3 + 2v\lambda^2 + \frac{\sqrt{a^2+4v^2}}{v}\lambda + 2\sqrt{a^2+4v^2} = 0$$

Which is the same characteristics equation with that of characteristics equation at equilibrium point E_1.

By the same analysis equilibrium point E_2 of the system (1,1) is locally asymptotically stable by Rourth-Hurwitz criterion.

4.5 Global stability analysis

Let V be appropriate candidate Lyapunov function such that:

$V\left(x,y,z\right) = \frac{1}{2}(x-x^*)^2 + \frac{1}{2}(y-y^*)^2 + \frac{1}{2}(z-z^*)^2$, where $\left(x^*, y^*, z^*\right)$ equilibrium point E_1

$i.V(x^*, y^*, z^*) = 0$

The function V at equilibrium point E_1 is zero

$ii.V(x, y, z) > o$

The function V is positive definite

$iii.\lim_{(x,y,z)\to\infty} V(x, y, z) \to \infty$

The function V is radially unbounded

iv.The derivative of V with respect to t:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

$$= (x - x^*)\ \frac{dx}{dt} + (y - y^*)\ \frac{dy}{dt} + (z - z^*)\ \frac{dz}{dt}$$

$$= (x - x^*)\ \left(-vx + zy\right) + (y - y^*)\ \left(-vy + (z - a)x\right) + (z - z^*)\ \left(1 - xy\right)$$

$$= -\left[\left(x^* - x\right)\left(-vx + zy\right) + \left(y^* - y\right)\left(-vy + (z - a)x\right) + \left(z^* - z\right)\left(1 - xy\right)\right]$$

$$= -\left[-vxx^* + zyx^* + vx^2 - vyy^* + zxy^* - axy^* + vy^2 - xyz + axy + z^* - xyz^* - z\right]$$

Let $g\left(x, y, z\right) = -vxx^* + zyx^* + vx^2 - vyy^* + zxy^* - axy^* + vy^2 - xyz + axy + z^* - xyz^* - z$

$$g_x = -vx^* + 2vx + zy^* - ay^* - yz + ay - yz^*$$

$$g_{xx} = 2v$$

$$g_{xy} = -z + a - z^* = g_{yx}$$

$$g_{xz} = y^* - y = g_{zx}$$

$$g_y = zx^* - vy^* + 2vy - xz + ax - xz^*$$

$$g_{yy} = 2v$$

$$g_{yz} = x^* - x = g_{zy}$$

$$g_z = yx^* + xy^* - xy - 1$$

$$g_{zz} = 0$$

Let H be Hessian matrix evaluated at equilibrium point E_1

$$H = \begin{pmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{pmatrix}_{E_1(x^*,y^*,z^*)} = \begin{pmatrix} 2v & -z+a-z^* & y^*-y \\ -z+a-z^* & 2v & x^*-x \\ y^*-y & x^*-x & 0 \end{pmatrix}_{E_1(x^*,y^*,z^*)}$$

$$= \begin{pmatrix} 2v & -z^*+a-z^* & y^*-y^* \\ -z^*+a-z^* & 2v & x^*-x^* \\ y^*-y^* & x^*-x^* & 0 \end{pmatrix} = \begin{pmatrix} 2v & -2z^*+a & 0 \\ -2z^*+a & 2v & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, first order leading principal minor

$D_1 = 2v > 0$ since $v > 0$

Second order principal minor

$D_2 = 4v^2 - \left(-2z^* + a\right)^2$, formed by deleting third row and third column

$$= 4v^2 - 4(z^*)^2 + 4az^* - a^2$$

$$= 4v^2 - 4\left(\frac{a+\sqrt{a^2+4v^2}}{2}\right)^2 + 4a\left(\frac{a+\sqrt{a^2+4v^2}}{2}\right) - a^2$$

$$= 4v^2 - a^2 - 2a\sqrt{a^2+4v^2} - a^2 - 4v^2 + 2a^2 + 2a\sqrt{a^2+4v^2} - a^2$$

$$= -a^2 < 0$$

Third ordered principal minor

$D_3 = 0$, because the third row and column are zero

The Hessian matrix is indefinite.

As a result, it is impossible to deal with global stability of the equilibrium point E_1 in the sense of Lyapunov as one of the conditions is failed to be satisfied.

*Let V be appropriate candidate Lyapunov function such that:
$V(x,y,z)=\frac{1}{2}(x-x^*)^2+\frac{1}{2}(y-y^*)^2+\frac{1}{2}(z-z^*)^2$,where (x^*,y^*,z^*) equilibrium point E_2 ,which is the same Lyapunov function with equilibrium point E_1. By the same analysis, it is impossible to deal with global stability of the equilibrium point E_2 in the sense of Lyapunov as one of the condition is failed to be satisfied.

4.6 Hopf Bifurcation analysis

1. Suppose $\lambda = i\omega$ is a pure imaginary eigenvalue of the characteristic equation $\omega > 0$;

$$\lambda^3+2v\lambda^2+\frac{\sqrt{a^2+4v^2}}{v}\lambda+2\sqrt{a^2+4v^2}=0$$

Substitute $i\omega$ in the above equation, we get;

$$(i\omega)^3+2v(i\omega)^2+\frac{\sqrt{a^2+4v^2}}{v}(i\omega)+2\sqrt{a^2+4v^2}=0$$

$$\left(-\omega^3+\frac{\sqrt{a^2+4v^2}}{v}\omega\right)i-2v\omega^2+2\sqrt{a^2+4v^2}=0 \tag{4.9}$$

From Eq. (4.9) , we have;

$$-\omega^3+\frac{\sqrt{a^2+4v^2}}{v}\omega=0 \tag{4.10}$$

$$-2v\omega^2+2\sqrt{a^2+4v^2}=0 \tag{4.11}$$

From Eq. (4.10) , $\omega(-\omega^2+\frac{\sqrt{a^2+4v^2}}{v})=0$

$$\omega^2=\frac{\sqrt{a^2+4v^2}}{v} \tag{4.12}$$

From Eq. (4.11) , $\omega^2=\frac{\sqrt{a^2+4v^2}}{v}$ $\qquad (4.13)$

Equating equations (4.12) and (4.13) , leads to:

$$\frac{\sqrt{a^2+4v^2}}{v}=\frac{\sqrt{a^2+4v^2}}{v}$$

$0=0$, which is always true

Hence, it is possible to take any value of v in terms of a as a bifurcation parameter.

Let $v=\frac{a}{2}$ be a bifurcation parameter

Substitute $v=\frac{a}{2}$ into the characteristic equation and solving for λ

$$\lambda^3+2(\frac{a}{2})\lambda^2+\frac{\sqrt{a^2+4\frac{a^2}{4}}}{\frac{a}{2}}\lambda+2\sqrt{a^2+4\frac{a^2}{4}}=0$$

$$\lambda^3+a\lambda^2+\frac{2\sqrt{2a^2}}{a}\lambda+2\sqrt{2a^2}=0$$

$$\lambda^3+a\lambda^2+2\sqrt{2}\lambda+2\sqrt{2}a=0$$

$$(\lambda+a)\left(\lambda^2+2\sqrt{2}\right)=0$$

$$\lambda=-a<0 \text{ and } \lambda^2=-2\sqrt{2}$$

$$\lambda=\sqrt{-2\sqrt{2}}$$

$$\lambda=\pm i\sqrt{2\sqrt{2}}$$

$$\lambda=\pm i\omega$$

Hence condition (1) is satisfied.

2. Let $p(\lambda)$ be the characteristic equation of the Jacobian matrix of the equilibrium points

$$p(\lambda):\lambda^3+2v\lambda^2+\frac{\sqrt{a^2+4v^2}}{v}\lambda+2\sqrt{a^2+4v^2}=0$$

Differentiate both sides with respect to v:

$$3\lambda^2\frac{d\lambda}{dv}+4v\lambda\frac{d\lambda}{dv}+2\lambda^2+\frac{\sqrt{a^2+4v^2}}{v}\frac{d\lambda}{dv}-\frac{a^2}{v^2\sqrt{a^2+4v^2}}+\frac{8v}{\sqrt{a^2+4v^2}}=0$$

$$\left(3\lambda^2+4v\lambda+\frac{\sqrt{a^2+4v^2}}{v}\right)\frac{d\lambda}{dv}=-2\lambda^2+\frac{a^2}{v^2\sqrt{a^2+4v^2}}-\frac{8v}{\sqrt{a^2+4v^2}}$$

$$\left(\frac{3v\lambda^2+4v^2\lambda+\sqrt{a^2+4v^2}}{v}\right)\frac{d\lambda}{dv}=\frac{-2\lambda^2v^2\sqrt{a^2+4v^2}+a^2-8v^3}{v^2\sqrt{a^2+4v^2}}$$

$$\frac{d\lambda}{dv}=\frac{-2\lambda^2v^2\sqrt{a^2+4v^2}+a^2-8v^3}{v^2\sqrt{a^2+4v^2}}\times\frac{v}{3v\lambda^2+4v^2\lambda+\sqrt{a^2+4v^2}}$$

$$\frac{d\lambda}{dv}=\frac{-2\lambda^2v^2\sqrt{a^2+4v^2}+a^2-8v^3}{3\lambda^2v^2\sqrt{a^2+4v^2}+4v^3\lambda\sqrt{a^2+4v^2}+va^2+4v^3}$$

$$\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{3\lambda^2v^2\sqrt{a^2+4v^2}+4v^3\lambda\sqrt{a^2+4v^2}+va^2+4v^3}{-2\lambda^2v^2\sqrt{a^2+4v^2}+a^2-8v^3}$$

Evaluate $\left(\frac{d\lambda}{dv}\right)^{-1}$ at pure imaginary eigenvalue $\lambda=i\omega$

$$\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{3v^2\left(i\omega\right)^2\sqrt{a^2+4v^2}+4v^3\left(i\omega\right)\sqrt{a^2+4v^2}+va^2+4v^3}{-2v^2\left(i\omega\right)^2\sqrt{a^2+4v^2}+a^2-8v^3}$$

$$\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{-3v^2\omega^2\sqrt{a^2+4v^2}+4v^3\left(i\omega\right)\sqrt{a^2+4v^2}+va^2+4v^3}{2v^2\omega^2\sqrt{a^2+4v^2}+a^2-8v^3}$$

$$\mathrm{Re}\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{-3v^2\omega^2\sqrt{a^2+4v^2}+va^2+4v^3}{2v^2\omega^2\sqrt{a^2+4v^2}+a^2-8v^3}$$

Evaluate $\mathrm{Re}\left(\frac{d\lambda}{dv}\right)^{-1}$ at $v=\frac{a}{2}$

$$\mathrm{Re}\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{-3\frac{a^2}{4}\omega^2\sqrt{a^2+4\frac{a^2}{4}}+\frac{a}{2}a^2+4\frac{a^3}{8}}{2\frac{a^2}{4}\omega^2\sqrt{a^2+4\frac{a^2}{4}}+a^2-8\frac{a^3}{8}}$$

$$\mathrm{Re}\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{\frac{-3\sqrt{2}}{4}a^3\omega^2+a^3}{\frac{\sqrt{2}}{2}a^3\omega^2+a^2-a^3}$$

$$\text{Re}\left(\frac{d\lambda}{dv}\right)^{-1} = \frac{-3a^3 + a^3}{2a^3 + a^2 - a^3} = \frac{-2a^3}{a^3 + a^2}$$

Hence $\text{Re}\left(\frac{d\lambda}{dv}\right)^{-1} = \frac{-2a^3}{a^3 + a^2} = \frac{-2a}{a+1} \neq 0$

$$sign\left[\text{Re}\left(\frac{d\lambda}{dv}\right)^{-1}\right] = -1$$

Condition (2) is satisfied.

As the result the system (1.1) undergoes Hopf Bifurcation at $v = \frac{a}{2}$

4.7 Numerical Examples

Example 1.Consider parameter values of a and v given to be:

$a=2$ and $v=1$

The system (1.1) becomes:

$$f_1=\frac{dx}{dt}=-x+zy$$

$$f_2=\frac{dy}{dt}=-y+(z-2)x$$

$$f_3=\frac{dz}{dt}=1-xy$$

Dissipative or conservativeness of the system

$$\nabla.f=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$$\nabla.f=-1-1=-2<0$$

Then $D^{'}=\int\nabla.fdD=\int-2dD=-2D$

$$\frac{D^{'}}{D}=-2$$

$$\frac{1}{D}\frac{dD}{dt}=-2$$

$$\frac{1}{D}dD=-2dt$$

Integrate both sides

$$\ln D=-2t+c$$

$$D=e^{-2t+c}$$

$$D=D_0e^{-2t},\quad D_0=e^c \text{ is constant}$$

Hence, D is decreasing exponentially. Therefore, the system is dissipative.

Equilibrium point

To find the equilibrium point, equate the right hand side equal to zero

$$-x+zy=0 \qquad (4.14)$$

$$-y+(z-2)x=0 \qquad (4.15)$$

$$1-xy=0 \qquad (4.16)$$

From Eq.(4.14), $z=\frac{x}{y}$ $\qquad (4.17)$

From Eq.(4.16) $y=\frac{1}{x}$ $\qquad (4.18)$

Plugging equations (4.17) and (4.18) into (4.15)

$$-\frac{1}{x}+x^3-2x=0$$

$$x^4-2x^2-1=0$$

Let $x^2=w$, then the equation becomes

$$w^2-2w-1=0$$

$$w=1\pm\sqrt{2}$$

Since $x^2=1\pm\sqrt{2}$

$$x^2=1+\sqrt{2}$$

$$x_1=\sqrt{1+\sqrt{2}} \text{ and } x_2=-\sqrt{1+\sqrt{2}}$$

From Eq.(4.18): $y=\frac{1}{x}$

$$y_1=\sqrt{\frac{1}{1+\sqrt{2}}} \text{ and } y_2=-\sqrt{\frac{1}{1+\sqrt{2}}}$$

From Eq.(4.17): $z=\frac{x}{y}$

$$z=1+\sqrt{2}$$

The equilibrium points are:

$$E_1=\left(\sqrt{1+\sqrt{2}},\sqrt{\frac{1}{1+\sqrt{2}}},1+\sqrt{2}\right)$$

$$E_2=\left(-\sqrt{1+\sqrt{2}},-\sqrt{\frac{1}{1+\sqrt{2}}},1+\sqrt{2}\right)$$

Local stability

Linearization

$$A=\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}=\begin{pmatrix} -1 & z & y \\ z-2 & -1 & x \\ -y & -x & 0 \end{pmatrix}$$

Let J_1 be Jacobian matrix evaluated at E_1

$$J_1=\begin{pmatrix} -1 & 1+\sqrt{2} & \sqrt{\frac{1}{1+\sqrt{2}}} \\ -1+\sqrt{2} & -1 & \sqrt{1+\sqrt{2}} \\ -\sqrt{\frac{1}{1+\sqrt{2}}} & -\sqrt{1+\sqrt{2}} & 0 \end{pmatrix}$$

To find characteristic equation; $\det\left(J_1-\lambda I\right)=0$

$$\begin{vmatrix} 1+\lambda & -\left(1+\sqrt{2}\right) & -\sqrt{\frac{1}{1+\sqrt{2}}} \\ 1-\sqrt{2} & 1+\lambda & -\sqrt{1+\sqrt{2}} \\ \sqrt{\frac{1}{1+\sqrt{2}}} & \sqrt{1+\sqrt{2}} & \lambda \end{vmatrix}=0$$

$$\lambda^3+2\lambda^2+2\sqrt{2}\lambda+4\sqrt{2}=0$$

To determine local stability of equilibrium point E_1 by applying RH-criterion

$$\begin{array}{c|ccc} \lambda^3 & 1 & 2\sqrt{2} & 0 \\ \lambda^2 & 2 & 4\sqrt{2} & 0 \\ \lambda^1 & b_1 & 0 & 0 \\ \lambda^0 & c_1 & 0 & 0 \end{array}$$

To find , $b_1=\frac{a_1a_2-a_0a_3}{a_1}=0$

RH-criterion table becomes

λ^3	1	$2\sqrt{2}$	0
λ^2	2	$4\sqrt{2}$	0
λ^1	0	0	0
λ^0	c_1	0	0

The auxiliary polynomial: $a(\lambda) = 2\lambda^2 + 4\sqrt{2}$

$$\frac{d}{d\lambda}a(\lambda) = 4\lambda + 0$$

The RH-criterion table becomes

λ^3	1	$2\sqrt{2}$	0
λ^2	2	$4\sqrt{2}$	0
λ^1	4	0	0
λ^0	c_1	0	0

To find $c_1 = \frac{b_1 a_3 - a_2 b_2}{b_1}$

$$c_1 = a_3$$

$c_1 = 4\sqrt{2}$,where $b_2 = 0$

Finally RH-criterion table becomes

λ^3	1	$2\sqrt{2}$	0
λ^2	2	$4\sqrt{2}$	0
λ^1	4	0	0
λ^0	$4\sqrt{2}$	0	0

As a result the equilibrium point E_1 is locally asymptotically stable.

Let J_2 be Jacobian matrix evaluated at E_2

$$J_2 = \begin{pmatrix} -1 & 1+\sqrt{2} & -\sqrt{\frac{1}{1+\sqrt{2}}} \\ -1+\sqrt{2} & -1 & -\sqrt{1+\sqrt{2}} \\ \sqrt{\frac{1}{1+\sqrt{2}}} & \sqrt{1+\sqrt{2}} & 0 \end{pmatrix}$$

To find characteristic equation, $\det\left(J_2 - \lambda I\right) = 0$

$$\begin{vmatrix} 1+\lambda & -(1+\sqrt{2}) & \sqrt{\dfrac{1}{1+\sqrt{2}}} \\ 1-\sqrt{2} & 1+\lambda & \sqrt{1+\sqrt{2}} \\ -\sqrt{\dfrac{1}{1+\sqrt{2}}} & -\sqrt{1+\sqrt{2}} & \lambda \end{vmatrix} = 0$$

$\lambda^3 + 2\lambda^2 + 2\sqrt{2}\lambda + 4\sqrt{2} = 0$, which is the same characteristic equation as equilibrium point E_1.

By the same analysis, equilibrium point E_2 is locally asymptotically stable.

MATLAB Simulation

The following diagrams indicate MATLAB simulation that shows stability of the equilibrium point.

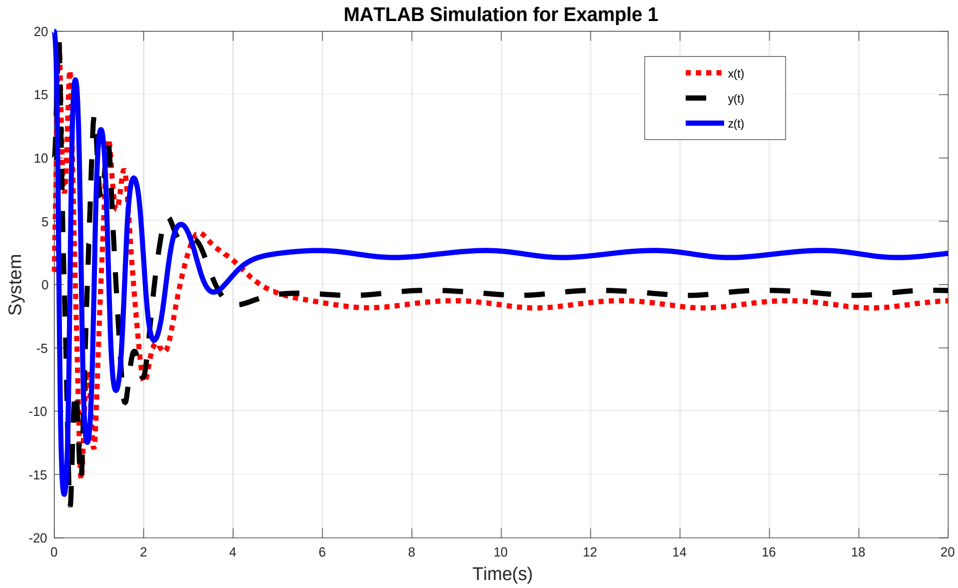

Figure 1: The graph of system versus time about equilibrium point.

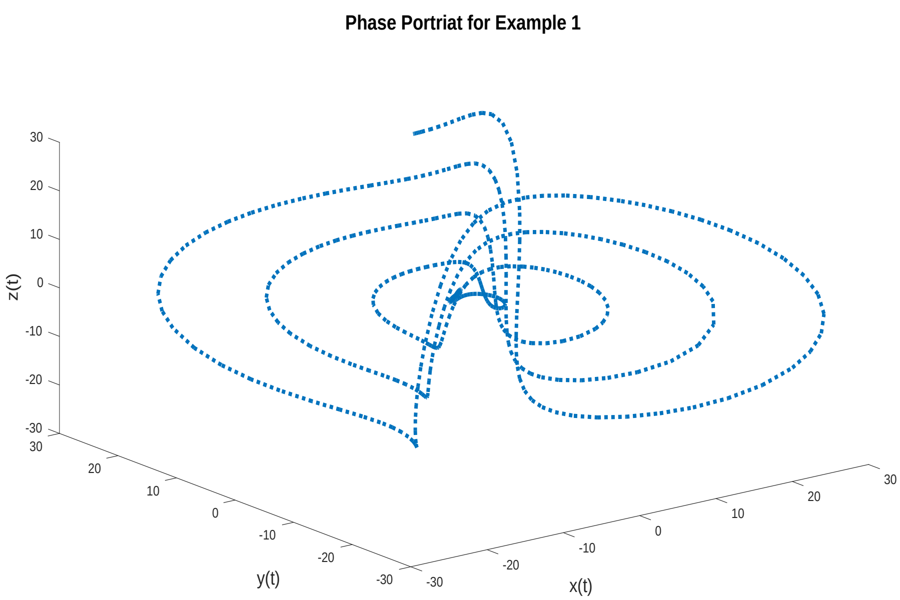

Figure 2: Phase Portrait of the system about the equilibrium point.

Figure 1 indicates that the graph of the system versus time converges to the equilibrium point. Figure 2 also revealed this fact because the trajectory of the system converges to equilibrium point which in one ways shows the stability of the equilibrium point.

Global stability Analysis

The candidate Lyapunov function V ;

$$V(x,y,z)=\frac{1}{2}\left(x-\sqrt{1+\sqrt{2}}\right)^2+\frac{1}{2}\left(y-\sqrt{\frac{1}{1+\sqrt{2}}}\right)^2+\frac{1}{2}\left(z-\left(1+\sqrt{2}\right)\right)^2$$

$$i.\, V\left(\sqrt{1+\sqrt{2}},\sqrt{\frac{1}{1+\sqrt{2}}},1+\sqrt{2}\right)=0$$

$$ii.\, V(x,y,z)>0$$

$$iii.\, \lim_{(x,y,z)\to\infty} V(x,y,z)\to\infty$$

$iv.$ The derivative of V with respect to t:

$$\frac{dV}{dt}=\left(x-\sqrt{1+\sqrt{2}}\right)(-x+zy)+\left(y-\sqrt{\frac{1}{1+\sqrt{2}}}\right)(-y+(z-2)x)+\left(z-(1+\sqrt{2})\right)(1-xy)$$

$$\frac{dV}{dt}=-\left[-\sqrt{1+\sqrt{2}}x+\sqrt{1+\sqrt{2}}zy+x^2-\sqrt{\frac{1}{1+\sqrt{2}}}y+\sqrt{\frac{1}{1+\sqrt{2}}}zx-2\sqrt{\frac{1}{1+\sqrt{2}}}x+y^2-xyz+2xy+1-xy+\sqrt{2}-\sqrt{2}xy-z\right]$$

Let

$$g(x,y,z)=-\sqrt{1+\sqrt{2}}x+\sqrt{1+\sqrt{2}}zy+x^2-\sqrt{\frac{1}{1+\sqrt{2}}}y+\sqrt{\frac{1}{1+\sqrt{2}}}zx-2\sqrt{\frac{1}{1+\sqrt{2}}}x+y^2-xyz+2xy+1-xy+\sqrt{2}-\sqrt{2}xy-z$$

$$g_x=-\sqrt{1+\sqrt{2}}+2x+\sqrt{\frac{1}{1+\sqrt{2}}}z-2\sqrt{\frac{1}{1+\sqrt{2}}}-yz+2y-y-\sqrt{2}y$$

$$g_{xx}=2$$

$$g_{xy}=-z+1-\sqrt{2}=g_{yx}$$

$$g_{xz}=\sqrt{\frac{1}{1+\sqrt{2}}}-y=g_{zx}$$

$$g_y=\sqrt{1+\sqrt{2}}z-\sqrt{\frac{1}{1+\sqrt{2}}}+2y-xz+2x-x-\sqrt{2}x$$

$$g_{yy} = 2$$

$$g_{yz} = \sqrt{1+\sqrt{2}} - x = g_{zy}$$

$$g_z = \sqrt{1+\sqrt{2}}\,y + \sqrt{\frac{1}{1+\sqrt{2}}}\,x - xy - 1$$

$$g_{zz} = 0$$

Let H be Hessian matrix evaluated at E_1

$$H = \begin{pmatrix} 2 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_1 = 2 > 0$$

$$D_2 = 4 - (4 \times 2) = -4 < 0$$

$$D_3 = 0$$

Thus the Hessian matrix is indefinite

As a result, it is impossible to deal with global stability of the equilibrium point E_1 in the sense of Lyapunov as one of the condition is failed to be satisfied.

Hopf Bifurcation

1. Suppose $\lambda = i\omega$ be pure imaginary eigenvalue of the characteristic equation provided that $\omega > 0$

$$\lambda^3 + 2\lambda^2 + 2\sqrt{2}\lambda + 4\sqrt{2} = 0$$

$$(i\omega)^3 + 2(i\omega)^2 + 2\sqrt{2}(i\omega) + 4\sqrt{2} = 0$$

$$-i\omega^3 - 2\omega^2 + 2\sqrt{2}\omega i + 4\sqrt{2} = 0$$

From the above equations

$$-\omega^3 + 2\sqrt{2}\omega = 0 \qquad (4.19)$$

$$-2\omega^2 + 4\sqrt{2} = 0 \qquad (4.20)$$

From Eq. (4.19), $\omega\left(-\omega^2 + 2\sqrt{2}\right) = 0$

$$\omega > 0 \text{ and } \omega^2 = 2\sqrt{2}$$

From Eq.$\left(4.20\right)$, $\omega^2 = 2\sqrt{2}$

$$\omega = \sqrt{2\sqrt{2}}$$

The characteristic equation is:

$$\lambda^3 + 2\lambda^2 + 2\sqrt{2}\lambda + 4\sqrt{2} = 0$$

$$(\lambda + 2)\left(\lambda^2 + 2\sqrt{2}\right) = 0$$

$$\lambda_1 = -2 < 0 \text{ and } \lambda_{2,3} = \pm i\omega$$

Therefore, condition$\left(1\right)$ is satisfied.

2. since $\operatorname{Re}\left(\dfrac{d\lambda}{dv}\right)^{-1} = \dfrac{-2a^3}{a^3 + a^2}$

$$\operatorname{Re}\left(\frac{d\lambda}{dv}\right)^{-1} = \frac{-2\left(2\right)^3}{2^3 + 2^2} = \frac{-4}{3}$$

$$\operatorname{Re}\left(\frac{d\lambda}{dv}\right)^{-1} \neq 0$$

Therefore , condition (2) is satisfied.

As a result, the system undergoes Hopf Bifurcation at $v = \dfrac{a}{2} = 1$

Example 2. Consider parameter values given to be: $v = 0.1 = a$

The system $\left(1.1\right)$ becomes:

$$f_1 = \frac{dx}{dt} = -0.1x + zy$$

$$f_2 = \frac{dy}{dt} = -0.1y + \left(z - 0.1\right)x$$

$$f_3 = \frac{dz}{dt} = 1 - xy$$

Dissipative or conservativeness of the system

$$\nabla . f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\nabla . f = \left(-0.1\right) + \left(-0.1\right) + 0 = -0.2 < 0$$

Then $D' = \int \nabla . f dD = \int -0.2 dD = -0.2D$

$$\frac{1}{D} dD = -0.2 dt \text{, integrate both sides}$$

$$\ln D = -0.2t + c$$

$$D = D_0 e^{-0.2t} \text{, where } D_0 = e^c \text{ is constant}$$

Hence, D is decreasing exponentially, then the system is dissipative.

Equilibrium point

The equilibrium points are:

$$E_1 = \left(\sqrt{\frac{1+\sqrt{5}}{2}}, \sqrt{\frac{2}{1+\sqrt{5}}}, \frac{1+\sqrt{5}}{20} \right)$$

$$E_2 = \left(-\sqrt{\frac{1+\sqrt{5}}{2}}, -\sqrt{\frac{2}{1+\sqrt{5}}}, \frac{1+\sqrt{5}}{20} \right)$$

Local stability

Linearization

Let $A = \begin{pmatrix} -0.1 & z & y \\ z-0.1 & -0.1 & x \\ -y & -x & 0 \end{pmatrix}$

Let J_1 be Jacobian matrix evaluated at E_1

$$J_1 = \begin{pmatrix} -0.1 & \frac{1+\sqrt{5}}{20} & \sqrt{\frac{2}{1+\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{20} & -0.1 & \sqrt{\frac{1+\sqrt{5}}{2}} \\ -\sqrt{\frac{2}{1+\sqrt{5}}} & -\sqrt{\frac{1+\sqrt{5}}{2}} & 0 \end{pmatrix}$$

To find characteristic equation: $\det\left(J_1 - \lambda I\right) = 0$

$$\begin{vmatrix} -0.1-\lambda & \frac{1+\sqrt{5}}{20} & \sqrt{\frac{2}{1+\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{20} & -0.1-\lambda & \sqrt{\frac{1+\sqrt{5}}{2}} \\ -\sqrt{\frac{2}{1+\sqrt{5}}} & -\sqrt{\frac{1+\sqrt{5}}{2}} & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 + \frac{1}{5}\lambda^2 + \sqrt{5}\lambda + \frac{\sqrt{5}}{5} = 0$$

To determine local stability condition of the system, applying RH-criterion.

λ^3	1	$\sqrt{5}$	0
λ^2	$\frac{1}{5}$	$\frac{\sqrt{5}}{5}$	0
λ^1	b_1	0	0
λ^0	c_1	0	0

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = 0$$

RH-table becomes:

λ^3	1	$\sqrt{5}$	0
λ^2	$\frac{1}{5}$	$\frac{\sqrt{5}}{5}$	0
λ^1	0	0	0
λ^0	c_1	0	0

Dealing with zero row:

Find the auxiliary equation: $a(\lambda) = \frac{1}{5}\lambda^2 + \frac{\sqrt{5}}{5}$

Differentiate with respect to λ: $\frac{d\,a(\lambda)}{d\lambda} = \frac{2}{5}\lambda + 0, b_1 = \frac{2}{5}$

RH-table becomes

$$\begin{array}{c|ccc} \lambda^3 & 1 & \sqrt{5} & 0 \\ \lambda^2 & \frac{1}{5} & \frac{\sqrt{5}}{5} & 0 \\ \lambda^1 & \frac{2}{5} & 0 & 0 \\ \lambda^0 & c_1 & 0 & 0 \end{array}$$

$c_1 = \frac{b_1 a_3 - a_2 b_2}{b_1}$, where $b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = 0$

$$c_1 = a_3 = \frac{\sqrt{5}}{5}$$

RH- table becomes

$$\begin{array}{c|ccc} \lambda^3 & 1 & \sqrt{5} & 0 \\ \lambda^2 & \frac{1}{5} & \frac{\sqrt{5}}{5} & 0 \\ \lambda^1 & \frac{2}{5} & 0 & 0 \\ \lambda^0 & \frac{\sqrt{5}}{5} & 0 & 0 \end{array}$$

As a result the equilibrium point E_1 is locally asymptotically stable.

Let J_2 be Jacobian matrix evaluated at E_2

$$J_2 = \begin{pmatrix} -0.1 & \frac{1+\sqrt{5}}{20} & -\sqrt{\frac{2}{1+\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{20} & -0.1 & -\sqrt{\frac{1+\sqrt{5}}{2}} \\ \sqrt{\frac{2}{1+\sqrt{5}}} & \sqrt{\frac{1+\sqrt{5}}{2}} & 0 \end{pmatrix}$$

To find characteristic equation: $\det\left(J_2 - \lambda I\right) = 0$

$$J_2 = \begin{pmatrix} -0.1-\lambda & \frac{1+\sqrt{5}}{20} & -\sqrt{\frac{2}{1+\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{20} & -0.1-\lambda & -\sqrt{\frac{1+\sqrt{5}}{2}} \\ \sqrt{\frac{2}{1+\sqrt{5}}} & \sqrt{\frac{1+\sqrt{5}}{2}} & -\lambda \end{pmatrix}$$

$\lambda^3 + \frac{1}{5}\lambda^2 + \sqrt{5}\lambda + \frac{\sqrt{5}}{5} = 0$, which is the same characteristic equation as equilibrium point E_1 .By the same analysis, equilibrium point E_2 is locally asymptotically stable.

Global stability

The candidate Lyapunov function:

$$V(x,y,z) = \frac{1}{2}\left(x - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^2 + \frac{1}{2}\left(y - \sqrt{\frac{2}{1+\sqrt{5}}}\right)^2 + \frac{1}{2}\left(z - (\frac{1+\sqrt{5}}{20})\right)^2$$

$i.\ V\left(\sqrt{\frac{1+\sqrt{5}}{2}}, \sqrt{\frac{2}{1+\sqrt{5}}}, \frac{1+\sqrt{5}}{20}\right) = 0$

$ii.\ V(x,y,z) > 0$

$iii.\ \lim_{(x,y,z)\to\infty} V(x,y,z) \to \infty$

$iv.$ The derivative of V with respect to t

$$\frac{dV}{dt} = \left(x - \sqrt{\frac{1+\sqrt{5}}{2}}\right)(-0.1x + zy) + \left(y - \sqrt{\frac{2}{1+\sqrt{5}}}\right)(-0.1y + (z-0.1)x) + \left(z - (\frac{1+\sqrt{5}}{20})\right)(1-xy)$$

$$\frac{dV}{dt} = -\left[-0.1\sqrt{\frac{1+\sqrt{5}}{2}}x + \sqrt{\frac{1+\sqrt{5}}{2}}zy + 0.1x^2 - 0.1\sqrt{\frac{1+\sqrt{5}}{2}}y + \sqrt{\frac{2}{1+\sqrt{5}}}zx - 0.1\sqrt{\frac{2}{1+\sqrt{5}}}x + 0.1y^2 - xyz + 0.1xy + \frac{1}{20} - \frac{1}{20}xy + \frac{\sqrt{5}}{20} - \frac{\sqrt{5}}{20}xy - z\right]$$

let

$$g(x,y,z) = -0.1\sqrt{\frac{1+\sqrt{5}}{2}}x + \sqrt{\frac{1+\sqrt{5}}{2}}zy + 0.1x^2 - 0.1\sqrt{\frac{1+\sqrt{5}}{2}}y + \sqrt{\frac{2}{1+\sqrt{5}}}zx - 0.1\sqrt{\frac{2}{1+\sqrt{5}}}x + 0.1y^2 - xyz + 0.1xy + \frac{1}{20} - \frac{1}{20}xy + \frac{\sqrt{5}}{20} - \frac{\sqrt{5}}{20}xy - z$$

$$g_x = -0.1\sqrt{\frac{1+\sqrt{5}}{2}} + 0.2x + \sqrt{\frac{2}{1+\sqrt{5}}}z - 0.1\sqrt{\frac{2}{1+\sqrt{5}}} - yz + 0.1y - \frac{1}{20}y - \frac{\sqrt{5}}{20}y$$

$$g_{xx} = 0.2$$

$$g_{xy} = -z + 0.1 - \frac{1}{20} - \frac{\sqrt{5}}{20} = g_{yx}$$

$$g_{xz} = \sqrt{\frac{2}{1+\sqrt{5}}} - y = g_{zx}$$

$$g_y = \sqrt{\frac{1+\sqrt{5}}{2}}z - 0.1\sqrt{\frac{2}{1+\sqrt{5}}} + 0.2y - xz + 0.1x - \frac{1}{20}x - \frac{\sqrt{5}}{20}x$$

$$g_{yy} = 0.2$$

$$g_{yz} = \sqrt{\frac{1+\sqrt{5}}{2}} - x = g_{zy}$$

$$g_z = \sqrt{\frac{1+\sqrt{5}}{2}}y + \sqrt{\frac{2}{1+\sqrt{5}}}x - xy - 1$$

$$z_{xx} = 0$$

Let H be Hessian matrix evaluated at E_1

$$H = \begin{pmatrix} 0.2 & \frac{-2\sqrt{5}}{20} & 0 \\ \frac{-2\sqrt{5}}{20} & 0.2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_1 = 0.2 > 0$$

$$D_2 = (0.2)^2 - \left(-\frac{2\sqrt{5}}{20}\right)^2 = 0.04 - 1 = -0.96 < 0$$

$$D_3 = 0$$

Hence, the Hessian matrix is indefinite.

As the result, it is impossible to deal with global stability of equilibrium point E_1 in the sense of Lyapunov as one of the conditions is failed to be satisfied.

Hopf Bifurcation

1. Suppose $\lambda = i\omega$ be pure imaginary eigenvalue of the characteristic equation provided that $\omega > 0$

$$\lambda^3+\frac{1}{5}\lambda^2+\sqrt{5}\lambda+\frac{\sqrt{5}}{5}=0$$

$$-i\omega^3-\frac{1}{5}\omega^2+\sqrt{5}\omega i+\frac{\sqrt{5}}{5}=0$$

From the above equation we have:

$$-\omega^3+\sqrt{5}\omega=0 \qquad (4.21)$$

$$-\frac{1}{5}\omega^2+\frac{\sqrt{5}}{5}=0 \qquad (4.22)$$

From Eq. $(4.21):\ \omega\left(-\omega^2+\sqrt{5}\right)=0$

$\omega > 0$ and $\omega^2=\sqrt{5}$

From Eq. $(4.22):\quad -\frac{1}{5}\omega^2+\frac{\sqrt{5}}{5}=0$

$\omega^2=\sqrt{5}$, $\omega=\sqrt{\sqrt{5}}$

The characteristic equation is:

$$\lambda^3+\frac{1}{5}\lambda^2+\sqrt{5}\lambda+\frac{\sqrt{5}}{5}=0$$

$$(5\lambda+1)\left(\frac{1}{5}\lambda^2+\frac{\sqrt{5}}{5}\right)=0$$

$$\lambda_1=-\frac{1}{5} \text{ and } \lambda_{2,3}=\pm i\sqrt{\sqrt{5}}$$

$$\lambda_1=-\frac{1}{5}<0 \text{ and } \lambda_{2,3}=\pm i\omega$$

Therefore, condition (1) is satisfied.

2. Since $\operatorname{Re}\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{-2a}{a+1}$

$$\operatorname{Re}\left(\frac{d\lambda}{dv}\right)^{-1}=\frac{-2(0.1)}{0.1+1}=-0.\overline{18}, \qquad \operatorname{Re}\left(\frac{d\lambda}{dt}\right)^{-1}\neq 0$$

Condition (2) is satisfied. As a result, the system undergoes Hopf Bifurcation at $v=a=0.1$

CHAPTER FIVE

CONCLUSION AND FUTURE SCOPE

5.1. Conclusions

In this study, stability and bifurcation analysis of Rikitake model was considered. The result of the study revealed that equilibrium points of the system are locally asymptotically stable by Routh Huwrtiz stability criteria. By the aid of divergence test, the system is proved to be dissipative. It is impossible to deal with the global stability of the two equilibrium points in the sense of Lyapunov. Furthermore, the result of Hopf bifurcation dictates that the system undergoes Hopf bifurcation about the two equilibrium points. Finally, in order to verify the applicability of the result two numerical examples were solved. MATLAB simulation was also implemented to support the findings of the study.

5.2. Future Scope

One can investigate stability and bifurcation analysis of Rikitake model by considering other factors like time delay and diffusion effects. Furthermore, direction and stability of Hopf bifurcation of the system is another area of future work. Moreover, qualitative analysis with regard to limit cycle, periodic solution and chaotic behavior are further area of future work.

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