Coupled Coincidence and Coupled Common Fixed Point Results of Mixed Monotone Mappings in the setting of Partially Ordered Metric Spaces

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

By:

Nigus Baye

Advisor: Kidane Koyas (Ph.D.)

Co-Advisor: Solomon Gebregiorgis (M.Sc.)

February, 2020

Jimma, Ethiopia

Declaration

I, the undersigned declare that, the thesis entitled Coupled Coincidence and Coupled Common Fixed Point results of mixed monotone mappings in the setting of partially ordered metric spaces is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged.
Name: Nigus Baye
Signature: ——————————
Date: ——————————
The work has been done under the supervision of:
Name: Kidane Koyas (Ph.D.)
Signature: ——————————
Date: ——————————
Name: Solomon Gebregiorgis (M.Sc.)
Signature: ——————————
Date: ——————————

Acknowledgment

First of all, I am indebted to the almighty God who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my Special heartfelt thanks go to my advisor, Dr. Kidane Koyas and co-adviser Mr. Solomon Gebregiorgis for their unreserved support, advice, and guidance throughout the preparation of this research work. Also, I would like to thank all my friends who gave me initiation while I was preparing this research work. Last but not least, I would like to thank my family for their encouragement through out my study period.

Abstract

The purpose of this research is to establish the existence and uniqueness of coupled coincidence and coupled common fixed point results of mixed monotone mappings in the setting of partially ordered metric spaces. The study procedure we used was that of Liu et al. (2018). Our results extend and generalize several well-known comparable results in literature. We also provided an example in support of our main result.

Contents

Chapter 1

Introduction

1.1 Background of the study

Definition 1.1 *Let X be a non-empty set and $T : X \to X$ be a self map. A point x is said to be fixed point of T if $Tx = x$.*

Fixed point theory is a powerful tool in modern mathematics. The origin of fixed point theory lies in the method of successive approximations used for proving existence of solutions of differential equations introduced independently by Joseph Liouville in 1837 and Charles Emile Picard in 1890. But formally it was started in the beginning of twentieth century as an important part of analysis. The abstraction of this classical theory is the pioneering work of the great Polish mathematician Stefan Banach published in 1922 which provides a constructive method to find the fixed points of a map. He developed a theorem called Banach Contraction Principle which states as follows. Let X be a complete metric space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point. This principle is one of a very power test for existence and uniqueness of the solution of considerable problems arising in mathematics. The Banach contraction mapping is one of the pivotal results of analysis. Fixed-point theory is an important tool in the study in functional analysis. It is also considered to be the key connection between pure and applied mathematics. Its application is not limited to various branches of mathematics but also in many fields such as, economics, Biology, Chemistry, Physics, Statistics, Computer Science, engineering etc.

Banach contraction principle has been extended and generalized in different directions. One of the generalizations is in the setting of partially ordered metric spaces given by Ran and Reurings (2004). They generalized Banach contraction principle in partially ordered sets with some applications to matrix equations. Also, Nieto and Lopez (2007) and Agarwal *et al*.(2008) presented some new results for contractions in partially ordered metric spaces.
Bhaskar and Lakshmikantham (2006) initiated the concept of coupled fixed point for non-linear contractions in partially ordered metric spaces. Lakshmikantham and Ciric (2009) established coupled coincidence and coupled common fixed point theorems for two mappings F and g where F has the mixed g-monotone property. Inspired and motivated by the research works of Liu *et al*.(2018), the purpose of this research is to establish new coupled coincidence and coupled common fixed point results for a pair of mixed monotone mappings in the frame work of partially ordered complete metric spaces.

1.2 Statement of the Problem

In this study, we concentrated on establishing coupled coincidence and coupled common fixed point results for a pair of mixed monotone maps satisfying certain contractive condition in the stetting of partially ordered metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this research was to establish Coupled Coincidence and Coupled Common Fixed Point results of mixed monotone mappings in the framework of partially ordered Metric Spaces.

1.3.2 Specific objectives

This study has the following specific objectives

- To prove the existence of coupled coincidence and coupled common fixed point of mixed monotone mappings in partially ordered metric spaces.
- To prove the uniqueness of coupled coincidence and coupled common fixed point of mixed monotone mappings in partially ordered metric spaces.
- To provide an examples in support of the main result.

1.4 Significance of the study

The result of this study may have the following importance

- It may give basic research skill to the researcher.
- It may be used as a reference for any researcher who has interest in doing research in the area.
- It may be applied to solve existence of solution of some integral and differential equations.

1.5 Delimitation of the Study

This study was delimited to finding coupled coincidence and coupled common fixed point results of mixed monotone mappings in partially ordered metric Spaces.

Chapter 2

Review of Related literatures

Historically, the study of fixed point theory began in 1912 with a theorem given by famous Dutch mathematician L. E. J. Brouwer. This is the most famous and important theorem on the topological fixed point property. Brouwer proved fixed point theorem as a solution of the equation $f(x) = x$. Later, he proved fixed point theorem for a square, a circle, a sphere, a cube etc. An important generalization of Brouwer is discovered in 1930 by J.Schauder which states that a continuous map on a convex compact subspace of a Banach space has a fixed point. However, Banach has given an abstract frame work for broad application well beyond the scope of elementary deferential and integral equations, in which he recognized the fundamental role of metric completeness. The amount of research and investigation of fixed point theory greatly increased in 1970's. Fixed point theorem using more generalized contractive mappings were done by several authors namely Abbas *et al*. (2012), Agarwal *et al*.(2008), Bhaskar and Lakshmikantham (2006), Choudhury and Kundu (2010), Luong and Thuan (2010), Nieto and Lopez (2007) and others. Fixed point theory has a wide application in all fields of quantitative science. Therefore, it is quite natural to consider various generalizations of metric space in order to address the needs in various fields of quantitative science.

Existence of coupled fixed points in partially ordered metric spaces was first investigated in 1987 by Guo and Lakshmikantham. Also, Bhaskar and Lakshmikantham (2006) established some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces. Lakshmikantham and C̀iric̀ introduced the notions of mixed g-monotone mapping and coupled coincidence point and proved some coupled coincidence point and coupled common fixed point theorems in partially ordered complete metric spaces which are more general than the result of Bhaskar and Lakshmikantham. Furthermore, their works have been extended and generalized by many authors in different type of spaces, we refer Fadila & Ahmad (2013), Imdad *et al.*(2009), and Jhade and Khan (2015).

A number of articles on coupled fixed point, coupled coincidence point, and coupled common fixed point theorems have been published. Abbas *et al.* (2010) introduced the concept of w-compatible mappings to obtain coupled coincidence point and coupled common fixed point for non-linear contractive mappings in cone metric spaces. Common fixed point results for commuting mappings in metric spaces were first deduced by Jungck (1988). The concept of commuting has been weakened in various directions and in several ways over the years. One such notion which is weaker than commuting is the concept of compatibility introduced by Jungck (1988).

Inspired and motivated by the research works of Liu *et al.* (2018), the purpose of this research is to establish new coupled coincidence and coupled common fixed point results for a pair of mixed monotone mappings in the frame work of partially ordered complete metric spaces.

Chapter 3

Methodology

3.1 Study period and site

The study was conducted at Jimma University under the department of mathematics from September 2018 to February 2020.

3.2 Study Design

In this research work we employed analytical design.

3.3 Source of Information

The relevant sources of information for this study were books, published articles and related studies from internet.

3.4 Mathematical Procedure of the Study

In this research under taking, we followed the standard procedures. The procedures are:

1. Establishing theorems
2. Constructing sequences

3. Showing that constructed sequences are Cauchy
4. Showing the convergences of the sequences
5. Proving the existence and uniqueness of coupled coincidence points
6. Proving the existence of coupled common fixed points
7. Proving the uniqueness of the coupled common fixed points
8. Giving examples in support of the main findings

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Definition 4.1 *(Khan et al., 1984). A function ϕ: $R^+ \to R^+$ is called an altering distance function if the following conditions are satisfied.*

(i) ϕ *is continuous and non-decreasing.*

(ii) $\phi(t) = 0$ *if and only if t=0.*

Throughout this thesis :
R denotes the set of real numbers;
$R^+ = [0, +\infty)$.
ϕ denotes all altering distance functions.
Ψ denotes the set of continuous functions such that:
$\Psi = \{\psi \in C(R^+, R^+) | \psi(0) = 0$, and for any $t > 0$, $\psi(t) > 0\}$.

Definition 4.2 *Let (X,d) be a metric space and $T : X \to X$ be a self-map, then T is said to be a contraction mapping if there exists a constant $k \in [0,1)$ called a contraction factor such that*

$$d(Tx,Ty) \leq kd(x,y)$$

for all $x, y \in X$.

Definition 4.3 *A set M is said to be partially ordered set if there is a binary relation "$\preceq$" defined on it such that:*

(i) *If $a \preccurlyeq a$ for all $a \in M$ (Reflexivity);*

(ii) *If $a \preccurlyeq b$ and $b \preccurlyeq a$ for all $a, b \in M$ then $a = b$ (anti-symmetry);*

(iii) *If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$ for all $a, b, c \in M$ (Transitivity).*

The pair $(M, \preccurlyeq)$ is called partially ordered set.

Note:Two elements $a \in M$ and $b \in M$ are said to be comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$ or both.

Definition 4.4 *Let X be a nonempty set, then $(X, d, \preccurlyeq)$ is said be partially ordered metric space if:*

(i) *(X,d) is a metric space and*

(ii) *$(X, \preccurlyeq)$ is a partially ordered set.*

Definition 4.5 *(Bhaskar & Lakshmikantham, 2006). Let X be a partially ordered set. A mapping $F : X \times X \to X$ is said to have a mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$;*
$x_1, x_2 \in X$, $x_1 \preccurlyeq x_2 \Rightarrow F(x_1, y) \preccurlyeq F(x_2, y)$ and
$y_1, y_2 \in X$, $y_1 \preccurlyeq y_2 \Rightarrow F(x, y_1) \succcurlyeq F(x, y_2)$.

Definition 4.6 *(Bhaskar & Lakshmikantham, 2006). An element $(x,y) \in X \times X$ where X is any non-empty set is called a coupled fixed point of the mapping*
$F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 4.7 *(Lkshmikantham and Ćirić, 2009). An element $(x,y) \in X \times X$ is called:*

(i) *a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if F(x, y) = g(x) and F(y, x) = g(y), and (gx, gy) is called coupled point of coincidence.*

(ii) *a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = g(x) = x$ and $F(y,x) = g(y) = y$.*

Definition 4.8 *(Lkshmikantham and Ciric, 2009). Let X be a partially ordered set. A mapping $F : X \times X \to X$ and $g : X \to X$ be two mappings;*

(i) We say that F has the g -mixed monotone property if $F(x,y)$ is g monotone non-decreasing in x and non-increasing in y. That is, for any $x_1, x_2 \in X$ and $y_1, y_2 \in X$;
$g(x_1) \preccurlyeq g(x_2) \Rightarrow F(x_1,y) \preccurlyeq F(x_2,y)$ *and*
$g(y_1) \preccurlyeq g(y_2) \Rightarrow F(x,y_1) \succcurlyeq F(x,y_2)$.

(ii) Let (X,d) be a metric spaces the mappings $F : X \times X \to X$ and $g : X \to X$ are called compatible if

$$\lim_{n\to\infty} d\Big(gF(x_n,y_n), F(gx_n,gy_n)\Big) = 0 \text{ and } \lim_{n\to\infty} d\Big(gF(y_n,x_n), F(gy_n,gx_n)\Big) = 0.$$

when ever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} gx_n \text{ and } \lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} gy_n.$$

Definition 4.9 *(Lkshmikantham and Ciric, 2009). Suppose X is a non-empty set. The mappings $F : X \times X \to X$ and $g : X \to X$ are called commutative if $g(F(x,y)) = F(gx,gy)$ for all $x, y \in X$.*

Definition 4.10 *(Abbas et al., 2010). The mappings $F : X \times X \to X$ and $g : X \to X$ are called weakly Compatible if;*

$$g(F(x,y)) = F(gx,gy) \text{ and } g(F(y,x)) = F(gy,gx)$$

whenever gx = F(x, y) and gy = F(y, x).

Theorem 4.1 *(Liu, Mao & Shi, 2018). Assume:*
(H_1) $\psi \in \Psi$.
(H_2) *Let X be a partially ordered metric space and a mapping $F : X \times X \to X$ being*

a mixed monotone mapping, there exists a constant $k \in (0,1)$ *such that:*

$$\begin{aligned}\phi[d(F(u,v),F(x,y))+d(F(v,u),F(y,x))] &\leq k\phi(d(u,x)+d(v,y))-\psi(k[d(u,x)\\ &\quad +d(v,y)])\end{aligned}$$

for all $x,y,u,v \in X$ *and for each* $u \preccurlyeq x$ *and* $v \succcurlyeq y$*.* ϕ *satisfies*
$\phi(t+s) \leq \phi(t)+\phi(s)$*, for all* $t,s \in [0,+\infty)$*.*
(H_3) *There exists* $(u_0,v_0) \in X \times X$ *such that* $u_0 \preccurlyeq F(u_0,v_0)$ *and* $v_0 \succcurlyeq F(u_0,v_0)$*. And*
(H_4) *One of the following condition holds.*

(a) F is continuous

(b) X has the following properties

- *(i) If a non decreasing sequence* $\{u_n\} \to u$*, then* $u_n \preccurlyeq u$ *for all n;*
- *(ii) If a non increasing sequence* $\{v_n\} \to v$*, then* $v_n \succcurlyeq v$ *for all n.*

Then there exist $u,v \in X$ *such that* $u = F(u,v)$ *and* $F(v,u) = v$*.*

4.2 Main Results

Theorem 4.2 *Let $(X,d,\preccurlyeq)$ be a partially ordered complete metric space. Suppose $F : X\times X \to X$ and $g : X \to X$ are continuous such that F has the mixed g-monotone property and commutes with g on X such that there exists $x_0, y_0 \in X$ with $gx_0 \preccurlyeq F(x_0,y_0)$ and $gy_0 \succcurlyeq F(y_0,x_0)$. The following conditions are satisfied.*

(i) $F(X\times X) \subseteq g(X)$.

(ii) *F and g are weakly compatible.*

(iii) *There exists $k \in (0,1)$ such that*

$$\phi[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))] \leq k\phi(d(gx,gu)+d(gy,gv))-\psi(k[d(gx,gu) + d(gy,gv)]) \quad (4.1)$$

for all $x,y,u,v \in X$ and for each $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$ and ϕ satisfies $\phi(t+s) \leq \phi(t)+\phi(s)$, for all $t,s \in [0,+\infty)$.

(iv) *F have a comparable property such that $(F(x,y),F(y,x)) \preccurlyeq (F(u,v),F(v,u))$ and $(F(z,t),F(t,z)) \preccurlyeq (F(u,v),F(v,u))$ for all $x,y,u,v,t,z \in X$.*

Then F and g have a unique coupled coincidence point. Consecutively F and g have a unique coupled common fixed point.

Proof: By hypothesis there exists x_0 and $y_0 \in X$ such that $gx_0 \preccurlyeq F(x_0,y_0)$ and $gy_0 \succcurlyeq F(y_0,x_0)$.
Since $F(X\times X) \subseteq g(X)$, there exists $x_1, y_1 \in X$ such that:

$$gx_1 = F(x_0,y_0)$$

and

$$gy_1 = F(y_0,x_0).$$

Again from $F(X\times X) \subseteq g(X)$, there exists $x_2, y_2 \in X$ such that $gx_2 = F(x_1,y_1)$ and $gy_2 = F(y_1,x_1)$.

Continuing this process we can construct sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for $n = 0, 1, 2, \cdots$ and since F has g-monotone property, we have:

$$gx_0 \preccurlyeq F(x_0, y_0) = gx_1 \preccurlyeq gx_2 \preccurlyeq \ldots \preccurlyeq F(x_n, y_n) = gx_{n+1} \preccurlyeq \cdots .$$

Similarly

$$gy_0 \succcurlyeq F(y_0, x_0) = gy_1 \succcurlyeq gy_2 \ldots \succcurlyeq F(y_n, x_n) = gy_{n+1} \succcurlyeq \cdots .$$

If $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some n, then $gx_n = F(x_n, y_n)$ and $gy_n = F(y_n, x_n)$, i.e., (x_n, y_n) is a coupled coincidence point of F and g and this completes the proof.
So, from now on, we assume that $gx_n \neq gx_{n+1}$ and $gy_n \neq gy_{n+1}$ for all n.
Since $gx_{n-1} \preccurlyeq gx_n$ and $gy_{n-1} \succcurlyeq gy_n$, then from (4.1), we have

$$\begin{aligned}
\phi[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] &= \phi[d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))] \\
&\leq k\phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) - \psi(k[d(gx_n, gx_{n-1}) + \\
&\quad d(gy_n, gy_{n-1})]) \\
&\leq k\phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})).
\end{aligned}$$

Since $k \in (0, 1)$ and ϕ is continuous and non-decreasing, we have:

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}). \tag{4.2}$$

Thus (4.2) holds for each $n \in N$.
Let $\delta_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)$.
It follows that the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $\delta \geq 0$ such that:

$$\lim_{n\to\infty} \delta_n = \delta.$$

Now, We want to show that $\delta = 0$.
Suppose on the contrary, that $\delta > 0$.

Since ϕ is continuous, $\phi(\delta) = \lim_{n\to\infty} \phi(\delta_n)$.

$$\begin{aligned}
\phi(\delta) &= \lim_{n\to\infty} \phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) \\
&\leq k \lim_{n\to\infty} \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) - \lim_{n\to\infty} \psi(k[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]) \\
&\leq k\phi(\delta) - \lim_{n\to\infty} \psi(k\delta_{n-1}) \leq k\phi(\delta) \\
&< \phi(\delta)(\text{ since } k \in (0,1)).
\end{aligned}$$

Which is a contradiction. Hence $\delta = 0$.
Now, we want to show $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.
Suppose at least $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequences, then there exists a positive constant ε such that for any $k > 0$, there exists $n_k > m_k > k$ such that

$$s_k = d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \geq \varepsilon. \tag{4.3}$$

where $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ are subsequences of $\{gx_n\}$, and $\{gy_{n_k}\}$ and $\{gy_{m_k}\}$ are subsequences of $\{gy_n\}$.
Let n_k be the smallest integer satisfying $n_k > m_k > k$ and (4.3) holds. Thus

$$d(gx_{n_{k-1}}, gx_{m_k}) + d(gy_{n_{k-1}}, gy_{m_k}) < \varepsilon \tag{4.4}$$

From (4.3), (4.3) and by the triangle inequality, we have:

$$\begin{aligned}
\varepsilon \leq s_k = d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) &\leq d(gx_{n_k}, gx_{n_{k-1}}) + d(gx_{n_{k-1}}, gx_{m_k}) + \\
&\quad\ d(gy_{n_k}, gy_{n_{k-1}}) + d(gy_{n_{k-1}}, gy_{m_k}) \\
&= d(gx_{n_{k-1}}, gx_{m_k}) + d(gy_{n_{k-1}}, gy_{m_k}) + \delta_{n_{k-1}} \\
\varepsilon \leq s_k &< \varepsilon + \delta_{n_{k-1}}. \qquad (4.5)
\end{aligned}$$

Setting $k \to \infty$ in (4.5) we get :

$$\varepsilon \leq \lim_{k\to\infty} s_k < \varepsilon + \lim_{k\to\infty} \delta_{n_{k-1}}.$$

Since $\lim_{n\to\infty} \delta_n = 0$, it implies that $\lim_{k\to\infty} \delta_{n_{k-1}} = 0$. And hence $\lim_{k\to\infty} s_k = \varepsilon$.

Again by the triangle inequality:

$$\begin{aligned} s_k &\leq d(gx_{n_k}, gx_{n_{k+1}}) + d(gx_{n_{k+1}}, gx_{m_{k+1}}) + d(gx_{m_{k+1}}, gx_{m_k}) + d(gy_{n_k}, gy_{n_{k+1}}) + d(gy_{n_{k+1}}, gy_{m_{k+1}}) \\ &\quad + d(gy_{m_{k+1}}, gy_{m_k}) \\ &= \delta_{n_k} + \delta_{m_k} + d(gx_{n_{k+1}}, gx_{m_{k+1}}) + d(gy_{n_{k+1}}, gx_{m_{k+1}}) \end{aligned}$$

and then by sub-additivity of ϕ, we have:

$$\phi(s_k) \leq \phi(\delta_{n_k} + \delta_{m_k}) + \phi(d(gx_{n_{k+1}}, gx_{m_{k+1}})) + \phi(d(gy_{n_{k+1}}, gx_{m_{k+1}})).$$

Then

$$\begin{aligned} \phi(d(gx_{n_{k+1}}, gx_{m_{k+1}})) + \phi(d(gy_{n_{k+1}}, gy_{m_{k+1}})) &= \phi(d(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k}))) \\ &\quad + \phi(d(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k}))) \\ &\leq k\phi(d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})) \\ &\quad - \psi(k[d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})]) \\ \phi(s_k) &\leq k\phi(s_k) - \psi(ks_k) + \phi(\delta_{n_k} + \delta_{m_k}) \text{ (since } k \in (0,1)) \\ \phi(s_k) &\leq \phi(s_k) - \psi(s_k) + \phi(\delta_{n_k} + \delta_{m_k}) \qquad (4.6) \end{aligned}$$

Setting $k \to \infty$ in (4.6) we get :

$$\lim_{k\to\infty} \phi(s_k) \leq \lim_{k\to\infty} \phi(s_k) - \lim_{k\to\infty} \psi(s_k) + \lim_{k\to\infty} [\phi(\delta_{n_k} + \delta_{m_k})]$$

since $\delta_n \to 0$, $s_k \to \varepsilon$, and ψ is continuous, we have:

$$\begin{aligned} \phi(\varepsilon) &\leq \phi(0) + \phi(\varepsilon) - \lim_{k\to\infty} \psi(s_k) \\ \phi(\varepsilon) &\leq \phi(\varepsilon) - \lim_{k\to\infty} \psi(s_k) < \phi(\varepsilon), \end{aligned}$$

which is a contradiction.
Therefore $\{gx_n\}$and $\{gy_n)$ are Cauchy sequences. So,

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 0, \lim_{n,m\to\infty} d(gy_n, gy_m) = 0.$$

Since X is a complete partially ordered metric space, there exist $x, y \in X$ such that:

$$\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = x, \lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = y.$$

From the commutativity of F and g, we have:

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n) \tag{4.7}$$
$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n) \tag{4.8}$$

Now, our claim is $gx = F(x,y)$ and $gy = F(y,x)$.
Since F is continuous and letting $n \to \infty$ in (4.7) and (4.8), we get:

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) = F(x,y).$$
$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n) = F(y,x).$$

Hence (gx, gy) is a coupled point of coincidence and (x,y) is a coupled coincidence point.
Now, we want to show the uniqueness of coupled coincidence point of F and g.
Let (x, y) and (z,t) are coupled coincidence point of F and g that is:
$F(x,y) = gx, F(z,t) = gz, F(y,x) = gy, F(t,z) = gt$.
Claim: $gx = gz$ and $gy = gt$.
By assumption there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$.
Without loss of generality, we can assume that:

$$(F(x,y), F(y,x)) \preccurlyeq (F(u, v), F(v, u))$$

and

$$(F(z,t),F(t,z)) \preccurlyeq (F(u,v),F(v,u)).$$

Put $u_0 = u$ and $v_0 = v$ and by hypothesis there exists $(u_1, v_1) \in X \times X$ such that $gu_1 = F(u_0, v_0), gv_1 = F(v_0, u_0)$.
For $n \geq 1$, continuing the process we construct sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = F(u_n, v_n) \text{ and } gv_{n+1} = F(v_n, u_n)$$

for all n.
Further set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $t_0 = t$, then on the same way we define sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ and $\{gt_n\}$.
Since $(gx, gy) = (F(x,y), F(y,x)) = (gx_1, gy_1)$ and $(F(u,v), F(v,u)) = (gu_1, gv_1)$ are comparable, we have:

$$(gx, gy) \preccurlyeq (gu, gv).$$

By induction $(gx_n, gy_n) \preccurlyeq (gu_n, gv_n)$ for all n.
Then

$$\begin{aligned}
\phi(d(gx, gu_{n+1}) + d(gy, gv_{n+1})) &= \phi(d(F(x,y), F(u_n, v_n)) + d(F(y,x), F(v_n, u_n))). \\
&\leq \phi k(d(gx, gu_n) + d(gy, gv_n)) - \\
&\quad \psi(k[d(gx, gu_n) + d(gy, gv_n)]) \qquad (4.9) \\
&\leq \phi k(d(gx, gu_n) + d(gy, gv_n)) \\
\phi(d(gx, gu_{n+1}) + d(gy, gv_{n+1})) &\leq \phi k(d(gx, gu_n) + d(gy, gv_n)).
\end{aligned}$$

Since ϕ is non-decreasing and $k \in (0,1)$:

$$d(gx, gu_{n+1}) + d(gy, gv_{n+1}) \leq d(gx, gu_n) + d(gy, gv_n).$$

which implies $d(gx, gu_n) + d(gy, gv_n)$ is a non-increasing sequence.
Then there exists $r \geq 0$ such that $\lim_{n\to\infty}[d(gx, gu_n) + d(gy, gv_n)] = r$.

Taking the limit on (4.9) as $n \to \infty$, we get:

$$\phi(r) \leq \phi(r) - \psi(r).$$

It follows that

$$\psi(r) \leq 0.$$

From the property of ψ we have:

$$\psi(r) = 0$$

and

$$r = 0.$$

Therefore $\lim_{n\to\infty}[d(gx, gu_n) + d(gy, gv_n)] = 0$
which in turn implies that:

$$\lim_{n\to\infty} d(gx, gu_n) = 0 \text{ and } \lim_{n\to\infty} d(gy, gv_n) = 0. \tag{4.10}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(gz, gu_n) = 0 \text{ and } \lim_{n\to\infty} d(gt, gv_n) = 0. \tag{4.11}$$

From (4.10), (4.11) and by the uniqueness of the limit, it follows that $gx = gz$ and $gy = gt$.
Hence, (gx, gy) is a unique coupled point of coincidence.
Since $gx = F(x, y)$ and $gy = F(y, x)$, by weakly compatible of F and g, we have:

$$g(gx) = g(F(x, y)) = F(gx, gy).$$

and

$$g(gy) = g(F(y, x)) = F(gy, gx).$$

Denote $gx = a$ and $gy = b$, then

$$g(a) = F(a,b) \text{ and } g(b) = F(b,a). \tag{4.12}$$

Thus, (a,b) is a coupled coincidence point.
Then with $z = a$ and $t = b$, it follows that $ga = gx$ and $gb = gy$.
That is

$$g(a) = a \text{ and } g(b) = b. \tag{4.13}$$

From (4.12) and (4.13), we have:

$$a = g(a) = F(a,b).$$

$$b = g(b) = F(b,a).$$

Therefore (a,b) is a coupled common fixed point of F and g.
To prove the uniqueness of the point (a,b), assume that (c,d) is another coupled common fixed point of F and g. Then, we have:

$$c = gc = F(c,d), d = gd = F(d,c).$$

Since (c,d) is a coupled coincidence point of F and g, ,we have $gc = gx = a$ and $gd = gy = b$. Thus:

$$c = gc = ga = a.$$

and

$$d = gd = gb = b.$$

Hence, the coupled common fixed point is unique. □

Theorem 4.3 *Let (X,d) be a partially ordered complete metric space and $F : X \times X \to X$ and $g : X \to X$ are maps where F has the mixed g-monotone property* *and for $k \in (0,1)$ satisfying:*

$$
\begin{aligned}
\phi[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))] \quad \leq \quad & k\phi(d(gx,gu)+d(gy,gv))-\psi(k[d(gx,gu) \\
& +d(gy,gv)]) \qquad (4.14)
\end{aligned}
$$

for all $x,y,u,v \in X$ and $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose X has the following properties:

(a) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all n.

(b) If a non-increasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all n.

If there exists x_0, $y_0 \in X$ such that $gx_0 \preccurlyeq F(x_0,y_0)$ and $gy_0 \succcurlyeq F(y_0,x_0)$,
then F and g have a coupled coincidence pint.

Proof: In the previous Theorem (4.2), we have proved $\{gx_n\}$and $\{gy_n\}$ are Cauchy sequences and since X is a complete partially ordered metric space, there exists $x,y \in X$ such that $\lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} gy_n = y$. And from the continuity of g we have:

$$\lim_{n\to\infty} g(gx_n) = gx \text{ and } \lim_{n\to\infty} g(gy_n) = gy.$$

Since F and g commute to each other, we have:
$g(gx_{n+1}) = g(F(x_n,y_n)) = F(gx_n,gy_n)$ and $g(gy_{n+1}) = g(F(y_n,x_n)) = F(gy_n,gx_n)$.
Again $\{gx_n\}$ is a non-decreasing and $gx_n \to x$, and $\{gy_n\}$ is a non-increasing and $gy_n \to y$. So we have $gx_n \preccurlyeq x$ and $gy_n \succcurlyeq y$.
Then by the triangle inequality we have:

$$
\begin{aligned}
\phi(d(gx,F(x,y)) \quad \leq \quad & \phi(d(gx,g(gx_{n+1}))+d(g(gx_{n+1}),F(x,y))) \\
= \quad & \phi(d(gx,g(gx_{n+1}))+d(F(gx_n,gy_n),F(x,y)) \\
\leq \quad & \phi(d(gx,g(gx_{n+1})))+\phi(d(F(gx_n,gy_n),F(x,y))) \\
\leq \quad & \phi(d(gx,g(gx_{n+1})))+\phi(d(F(gx_n,gy_n),F(x,y)) \\
+ \quad & d(F(gy_n,gx_n),F(y,x))) \\
\leq \quad & \phi(d(gx,g(gx_{n+1})))+k\phi(d(g(gx_n),gx)+d(g(gy_n),gy)) \\
- \quad & \psi(k[d(g(gx_n),gx)+d(g(gy_n),gy)]). \qquad (4.15)
\end{aligned}
$$

Taking $n \to \infty$ in (4.15), we get:

$$\phi(d(gx, F(x,y)) \quad \leq \quad 0.$$

This implies that $F(x,y) = gx$.
Similarly

$$\begin{aligned}
\phi(d(gy, F(y,x)) &\leq \phi(d(gy, g(gy_{n+1})) + d(g(gy_{n+1}), F(y,x))), \\
&= \phi(d(gy, g(gy_{n+1})) + d(F(gy_n, gx_n), F(y,x)) \\
&\leq \phi(d(gy, g(gy_{n+1}))) + \phi(d(F(gy_n, gx_n), F(y,x))) \\
&\leq \phi(d(gy, g(gy_{n+1}))) + \phi(d(F(yx_n, gx_n), F(y,x)) \\
&+ d(F(gx_n, gy_n), F(x,y))) \\
&\leq \phi(d(gy, g(gy_{n+1}))) + k\phi(d(g(gy_n), gy) + d(g(gx_n), gx)) \\
&- \psi(k[d(g(gy_n), gy) + d(g(gx_n), gx)]). \qquad (4.16)
\end{aligned}$$

Taking $n \to \infty$ in (4.16), we get;

$$\phi(d(gy, F(y,x)) \quad \leq \quad 0.$$

This implies that $F(y,x) = gy$. □

Therefore, (gx, gy) is a coupled point of coincidence and (x, y) a coupled coincidence point.

Remark 4.1 *If we take $g = I$ (the identity map), then Theorem 4.2 will reduce to Theorem 4.1.*

Example 4.1 *Let $X = R$ be a set endowed with the usual order and usual metric $d(x,y) = |x-y|$ for all $x,y \in X$.*
$(R,\preccurlyeq)$ is a partially ordered set and $(R,\preccurlyeq,d)$ is a partially ordered metric space.
Define the mappings $F : X \times X \to X$ by $F(x,y) = \frac{x-2y}{8}$ for all $(x,y) \in X \times X$ and $g : X \to X$ by $g(x) = \frac{x}{2}$ for all $x \in X$. Then

(i) *F and g are continuous*

(ii) *For any $x_1,x_2 \in X$ and for all $x,y \in X$*
$gx_1 \preccurlyeq gx_2 \Rightarrow F(x_1,y) \preccurlyeq F(x_2,y)$ and $gy_1 \preccurlyeq gy_2 \Rightarrow F(x,y_1) \succcurlyeq F(x,y_2)$ which implies F has g-monotone property.

(iii) *There exists $x_0 = 0$ and $y_0 = 0$ such that $x_0,y_0 \in X$, $gx_0 \preccurlyeq F(x_0,y_0)$ and $gy_0 \succcurlyeq F(y_0,x_0)$ which means*
$g(0) = \frac{0}{2} = 0 \preccurlyeq F(0,0) = \frac{0-2(0)}{8} = 0$ and $g(0) = \frac{0}{2} = 0 \succcurlyeq F(0,0) = \frac{0-2(0)}{8} = 0$.

(iv) *$g(F(x,y)) = g(\frac{x-2y}{8}) = \frac{x-2y}{16}$ and $F(gx,gy) = F(\frac{x}{2},\frac{y}{2}) = \frac{\frac{x}{2}-2(\frac{y}{2})}{8} = \frac{x-2y}{16}$ which shows that $g(F(x,y)) = F(gx,gy)$. In addition,*
$g(F(y,x)) = g(\frac{y-2x}{8}) = \frac{y-2x}{16}$ and $F(gy,gx) = F(\frac{y}{2},\frac{x}{2}) = \frac{\frac{y}{2}-2(\frac{x}{2})}{8} = \frac{y-2x}{16}$ which shows that $g(F(y,x)) = F(gy,gx)$.
Hence F and g are commutative.

(v) *Let $\phi(t) = \frac{5t}{4}$, $\psi(t) = \frac{t}{5}$ and $k = \frac{15}{16}$, then*

$$
\begin{aligned}
d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &= \left|\frac{x-2y}{8} - (\frac{u-2v}{8})\right| + \left|\frac{y-2x}{8} - (\frac{v-2u}{8})\right| \\
&= \left|\frac{x-u}{8} + \frac{2v-2y}{8}\right| + \left|\frac{y-v}{8} + \frac{2u-2x}{8}\right| \\
&\leq \frac{1}{8}|x-u| + \frac{1}{8}|y-v| + \frac{1}{8}|y-v| + \frac{1}{4}|u-x| \\
&= \frac{3}{8}|x-u| + \frac{3}{8}|y-v| \\
&= \frac{3}{8}\left(|x-u| + |y-v|\right).
\end{aligned}
$$

and then

$$\begin{aligned}
\phi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) &= \frac{5}{4}\left(\frac{3}{8}\big(|x-u|+|y-v|\big)\right)\\
&= \frac{15}{32}\Big(|x-u|+|y-v|\Big).
\end{aligned}$$

and

$$\begin{aligned}
d(gx,gu)+d(gy,gv) &= \left|\frac{x}{2}-\frac{u}{2}\right|+\left|\frac{y}{2}-\frac{v}{2}\right|\\
&= \frac{1}{2}|x-u|+\frac{1}{2}|y-v|\\
&= \frac{1}{2}(|x-u|+|y-v|).\\
k\phi(d(gx,gu)+d(gy,gv)) &= \left(\frac{15}{16}\right)\left(\frac{5}{4}\right)\left(\frac{1}{2}(|x-u|+|y-v|)\right)\\
&= \frac{75}{128}\Big(|x-u|+|y-v|\Big).
\end{aligned}$$

Again

$$\begin{aligned}
\psi(k[d(gx,gu)+d(gy,gv)] &= \frac{1}{5}\left(\frac{15}{32}(|x-u|+|y-v|)\right)\\
&= \frac{3}{32}(|x-u|+|y-v|).\\
k\phi(d(gx,gu)+d(gy,gv))-\psi(k[d(gx,gu)+d(gy,gv)] &= \frac{75}{128}\Big(|x-u|+|y-v|\Big)-\\
&\quad \frac{3}{32}\Big(|x-u|+|y-v|\Big)\\
&= \frac{63}{128}\Big(|x-u|+|y-v|\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) &= \frac{15}{32}\Big(|x-u|+|y-v|\Big) \\
&\leq k\phi\Big(d(gx,gu)+d(gy,gv)\Big)- \\
&\quad \psi\Big(k[d(gx,gu)+d(gy,gv)]\Big) \\
&= \frac{63}{128}\Big(|x-u|+|y-v|\Big).
\end{aligned}$$

There fore, all the hypothesis of the Theorem (4.2) holds. So, F and g have a unique coupled point of coincidence and a unique coupled common fixed point which are $(g0,g0)$ and $(0,0)$ respectively.
This is because $g(F(0,0)) = F(g0,g0) = F(0,0) = 0$.
Also, F and g are commuting and weakly compatible at $(0,0)$.

Chapter 5

Conclusion and Future Scope

In 2018, Liu, Mao and shi established the existence of coupled fixed point for mapping satisfying certain contraction condition in a complete partially ordered metric space. In this thesis, we have explored the properties of partially ordered metric spaces and also discuss the difference between partially ordered metric spaces metric space. We established and proved existence and uniqueness of coupled coincidence and coupled common fixed point results for a pair of mixed monotone maps satisfying certain contractive condition in the stetting of partially ordered metric spaces. Also we provided example in support of our main result. Our work extended coupled fixed point result of a single map to coupled coincidence point and coupled common fixed point of pair of maps. The presented theorems extend and generalize several well-known comparable results in literature.

There are several published results related to existence of fixed points of self-maps defined on partially ordered metric space. There are also few results related to the existence of coupled common fixed points for a pair or more maps in this space. The researcher believes the search for the existence of coupled coincidence point and coupled common fixed points of maps satisfying different contractive conditions in partially ordered metric space is an active area of study. So, the forthcoming postgraduate students of Department of Mathematics and any researcher can exploit this opportunity and conduct their research work in this area.

References

Abbas, M., Khan, M. A., & Radenovi, S. (2010). Common coupled fixed point theorems in cone metric spaces for w -compatible mappings. *Applied Mathematics and Computation*, 217(1), 195-202.

Abbas, M., Sintunavarat, W., & Kumam, P. (2012). Coupled fixed point of generalized contractive mappings on partially ordered g-metric spaces. *Fixed Point Theory and Applications,* 2012(1), 31.

Agarwal, R. P., El-Gebeily, M. A., & ORegan, D. (2008). Generalized contractions in partially ordered metric spaces. *Appl. Anal.*, 87, 109-116.

Banach, S. (1922). Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales. *Fund. Math.*, 3, 133-181.

Bhaskar, T. & Lakshmikantham, G. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.*, 65, 1379-1393.

Brouwer, L. E. J. (1912). Uber abbildung von mannigfaltigkiten, *mathematische annalen*, 71, 97-115.

Choudhury, B. S. & Kundu, A. (2010). A coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Anal.*, 73, 2524-2531.

Ciric̀, M. A, Damjanovic, B. & Saadati, R (2011). Common fuzzy fixed point theorems in ordered metric spaces. *Math. Comput. Modelling*, 53, 1737-1741.

Fadil, Z. M. & Ahmad, A. G. B. (2010). Coupled coincidence point and common coupled fixed point results in cone b-metric spaces. *Fixed Point Theory and Applications*, 2013, 177.

Imdad, M., Ali, J., & Tanveer, M. (2009). Coincidence and common fixed point theorems for nonlinear contractions in menger PM spaces. *Chaos Solitons Fractals*, 42, 3121-3129.

Jhade, P. K. & Khan, M. S. (2014). Some coupled coincidence and common coupled fixed point theorems in complex-valued metric spaces. *Facta Universitatis, Series: mathematics and Informatics*, 29(4), 385-395.

Khan, M. S., Swaleh, M. & Sessa, S. (1984). Fixed point theorems by altering distances between the points, *Bulletin of the Australian Mathematical Society*, 30(1), 1-9.

Lakshmikantham, V. & Ciric, L. (2009). Coupled fixed point theorems for nonlinear contractive in partially ordered metric spaces. *Nonlinear Analysis:Theory, Methods & Applications*, 70(12), 4341-4349.

Liouville, J. (1837). Second mmoire sur le dveloppement des fonctions ou parties de fonctions en sries dont divers termes sont assujettis satisfaire a une młme quation diffrentielle du second ordre contenant un paramtre variable, *Math. Pure et Appi.*, 2, 16-35.

Liu, L., Mao, A. & Shi, Y. (2018). New Fixed Point theorems and application of mixed monotone mappings in partially ordered metric spaces. *Jornal of function spaces,* 2018.

Luong, N. & Thuan, N. X. (2010). Coupled fixed point theorems in partially ordered metric spaces. *Bull. Math. Anal. Appl.*, 2, 16-24.

Marr, R. D. (1965). Partially ordered space and metric spaces. *American Mathematical Monthly*, 72(6), 628-631.

Nieto, J. J., & Rodriguez-Lopez, R. (2007). Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Math. Sin. Engl. Ser.*, 23(12), 2205-2212.

Picard, E. (1890). Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. *J. Math. Pures et Appl.*, 6.

Ran, A. C. & Reurings, M. C. (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Amer. Math. Society,* 132, 1435-1443.

Schauder, J. (1930) Der fixpunktsatz in funktionalrumen. *Studia Math,* 2, 171-180.