# DYNAMICS OF NON-DEGENERATE THREE LEVEL LASER IN A CLOSED CAVITY AND COUPLED WITH TWO MODE VACUUM RESERVOIR

A Thesis Submitted to the School of Graduate Studies Department of Physics **Jimma University** In Partial Fulfillment of the Requirements for the Degree of Masters of Science in **Physics** (***Quantum Optics and information***)

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented in any other university, and that all sources of material used for the thesis have been dully acknowledged.

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# Abstract

In this research, we have studied the squeezing and statistical properties of the cavity light beams produced by a dynamics of non-degenerate three-level laser in a closed cavity and coupled with a two-mode vacuum reservoir for single atom via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we have calculated the global and local mean and variance of the photon number as well as the quadrature squeezing of the cavity light single-mode and two-modes.Furthermore we determined the photon entanglement as well as the atom-cavity entanglement. It is found to be the maximum quadrature squeezing for the values of stimulated emission decay constant(i.e $\gamma_c = 0.4$ and $0.2$ ). The maximum quadrature squeezing is found to be $43.43\%$ below the vacuum-state level. Moreover we have found that both the mean photon number for a two-mode laser light beam is the sum of the mean photon numbers and the single-mode light beams. On the other hand, the quadrature squeezing is due to the correlation of the two light beams. In view of this correlations the two mode cavity light is entangled. The degree of entanglement increases with the increase in stimulated emission decay constant.

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# 1

# Introduction

The interaction of a three-level laser with a cavity mode has attracted a great deal of interest in recent years[1-7]. It is believed that atomic coherence is found to be responsible for various important quantum features of the emitted photons. In general, the atomic coherence can be induced in a three-level atom by preparing the atom initially in a coherent superposition of the top and bottom levels or by coupling these levels by coherent light after it is injected in to the cavity [2,4,5,6, 8,9]. The superposition or the coupling of the top and bottom levels is responsible for the interesting non classical features of the emitted photons. In a three-level laser the top, intermediate, and the bottom levels are denoted by $|a\rangle, |b\rangle$ , and $|c\rangle$ in which the transitions between levels $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ are assumed to be dipole allowed, with direct transition between levels $|a\rangle \rightarrow |c\rangle$ to be dipole forbidden. When the atom makes a transition from the top to the intermediate level and then from the intermediate to the bottom level, two photons are emitted. If the two photons have different frequencies, then the three-level atom is called anon-degenerate three-level atom otherwise it is called degenerate. Some authors have studied the statistical and the squeezing properties of the light produced by a three-level atom in which the crucial role is played by the superposition of the top and bottom levels [2,3,4,5,10,11,12,13]. It is found that the cavity modes exhibit squeezing under certain conditions. On the other hand, a three-level atom in which the top and bottom levels coupled by a coherent light have been studied by different authors [3, 5,7,14]. They have predicted that such a system can generate squeezed light over a large-range of the amplitude of the coherent light. The squeezing in this case is due to the coupling of the top and bottom levels[15].

Entanglement is one of the fundamental tools for quantum information processing and communication protocols.The generation and manipulation of entanglement has attracted a great deal of interest with wide applications in quantum teleportation, quantum dense coding, quantum computation, quantum error correction, and quantum cryptography [16]. Recently, much attention is given on the generation of continuous-variable entanglement to manipulate the discrete counterparts, quantum bits, to perform quantum information processing. In general, the degree of entanglement decreases when it interacts with the environment. But, the efficiency of quantum information processing highly depends on the degree of entanglement. Therefore, it is necessary to generate strongly entangled states which can survive from external noise. In general, due to the result of the strong correlation between the cavity modes, a two-mode squeezed state violates certain classical inequalities and then can be used in preparing Einstein-Podolsky-Rosen (EPR)-type entanglement [17].

Recently, Tesfa [18] has studied the squeezing property of the cavity modes produced by a non-degenerate three-level laser applying the solutions of stochastic differential equations.

Tamirat [19] has studied the analysis of the quantum properties of cavity light produced by a coherently driven non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. A three-level laser with the top and bottom levels of the atoms injected into the cavity coupled by a strong coherent light can also generate light in a squeezed state [20]. The steady state entanglement in a non-degenerate three-level laser has been studied when the atomic coherence is induced by initially preparing atoms in coherent superposition of the top and bottom levels [20-24] and when the top and bottom levels of the three-level atoms injected into a cavity are coupled by coherent light [24-26]. Moreover, Fesseha has studied the quantum properties of the light emitted by the three level atoms available in a closed cavity and pumped to the top level at a constant rate by means of electron bombardment [26]. More recently, Eyob [27] has studied continuous-variable entanglement in non-degenerate three-level laser with a parametric amplifier. In this model the injected atomic coherence introduced by initially preparing the atoms in a coherent superposition of the top and bottom levels. In addition, to exhibiting a two-mode squeezed light, this combined system produces light in an entangled state. In one model of such a laser, three-level atoms initially in the upper level are injected at a constant rate into the cavity and removed after they have decayed due to spontaneous emission. It appears to be quite difficult to prepare the atoms in a coherent super position of the top and bottom levels before they are injected into the laser cavity. Besides, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity.

In this thesis, we study the quantum properties of the light generated by a dynamics of non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. In order to carry out our calculation, we put the noise operators associated with the vacuum reservoir in normal order. Thus, first we obtain the quantum Langevin equations for the cavity mode operators. Then, employing the large-time approximation scheme, we calculate the equations of evolution of the expectation values of atomic operators with the aid of Heisenberg picture. Moreover, we determine the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators. And applying the resulting solutions, we obtain the global as well as the local mean photon number, photon number variance, and quadrature variances for light mode a and light mode b. Moreover, employing the resulting solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators, we determine the global mean photon number, photon number variance, photon number correlations, intensity difference fluctuations, quadrature variance, quadrature squeezing, and photon entanglement of the two-mode cavity light.

# 2

# Operator Dynamics

In this chapter we consider a dynamics of non degenerate three-level laser driven by coherent light and with the cavity modes coupled to a two-mode vacuum reservoir via a single-port mirror as shown in Fig.2.1. We first set up the interaction Hamiltonian for a dynamics of non-degenerate three-level atom with the cavity modes and the quantum Langevin equations for the cavity mode operators. In addition, employing the Hamiltonian and the Heisenberg equation, we drive the equations of evolution of the expectation values of the atomic operators. Finally, we determine the steady-state solutions of the resulting equations of evolution. Here we carry out our calculation by putting the noise operators associated with the two-mode vacuum reservoir in normal order.

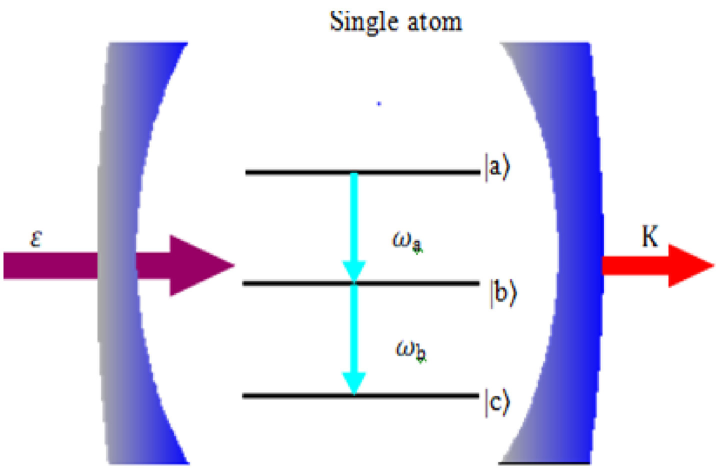


Figure 2.1: Scheme of dynamics of non-degenerate three level laser in a closed cavity and coupled to two-mode vacuum reservoir.

## 2.1 The interaction Hamiltonian

We consider here the case in which the dynamics of non-degenerate three-level atom in cascade configuration are available in a closed cavity. We denote the top, intermediate, and bottom levels of the three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. As shown in Fig. (2.1) for non-degenerate cascade configuration, when the atom makes a transition from level $|a\rangle$ to $|b\rangle$ and from levels $|b\rangle$ to $|c\rangle$ two photons with different frequencies are emitted. The emission of light when the atoms makes the transition from the top level to the intermediate level is light mode $a$ and the emission of light when the atom makes the transition from the intermediate level to the bottom level is light mode $b$. We assume that the cavity mode $a$ is at resonance with transition $|a\rangle \rightarrow |b\rangle$ and the cavity mode $b$ is at resonance with the transition $|b\rangle \rightarrow |c\rangle$, with top and bottom levels of the three-level atom coupled by coherent light. The coupling of the top and bottom levels of a non-degenerate three-level atom by coherent light can be described by the Hamiltonian expressed [8]

$$\hat{H}' = \frac{i\Omega}{2}\big[\hat{\sigma}_c^\dagger - \hat{\sigma}_c\big], \tag{2.1}$$

where

$$\hat{\sigma}_c = |c\rangle\ \langle a| \tag{2.2}$$

is lowering atomic operator and

$$\Omega = 2\varepsilon\lambda. \tag{2.3}$$

Here $\varepsilon$, considered to be real and constant, is the amplitude of the driving coherent light and $\lambda$ is the coupling constant between the driving coherent light and the three-level atom. In addition, the interaction of a three-level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H}'' = ig\big[\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b\big], \tag{2.4}$$

where

$$\hat{\sigma}_a = |b\rangle\ \langle a|, \tag{2.5}$$

$$\hat{\sigma}_b = |c\rangle\ \langle b|, \tag{2.6}$$

$g$ is the coupling constant between the atom and cavity mode $a$ or $b$, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b. Thus up on combining eqs. (2.1) and (2.4), the interaction of the three-level atom with the driving coherent light and cavity mode $\hat{a}$ and $\hat{b}$ is described by the Hamiltonian as

$$\hat{H}_S(t) = ig\big[\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b\big] + \frac{i\Omega}{2}\big[\hat{\sigma}_c^\dagger - \hat{\sigma}_c\big] \tag{2.7}$$

## 2.2 Quantum Langevin Equations

We recall that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [7,8]. We can therefore, drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}] \tag{2.8}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}], \tag{2.9}$$

where $\kappa$ is the cavity damping constant. Then in view of Eq. (2.7), the quantum Langevin equations for cavity mode operators $\hat{a}$ and $\hat{b}$ turns out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a, \tag{2.10}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b. \tag{2.11}$$

## 2.3 Equations of evolution of the atomic oprators

Here we seek to derive the equations of evolution of the expectation values of the atomic operators by applying the Heisenberg equation. Moreover, we find the steady-state solutions of the equations of evolution of the atomic operators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = -i\langle[\hat{A}, \hat{H}]\rangle \tag{2.12}$$

along with the interaction Hamiltonian of the system described by (2.7), one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = g[\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle] + \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle \tag{2.13}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = g[\langle\hat{\eta}_c\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c\rangle - \langle\hat{\eta}_b\hat{b}\rangle] - \frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle \tag{2.14}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = g[\langle\hat{\sigma}_b\hat{a}\rangle - \langle\hat{\sigma}_a\hat{b}\rangle] + \frac{\Omega}{2}\left[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle\right] \tag{2.15}$$

where

$$\hat{\eta}_a = |a\rangle\ \langle a| \tag{2.16}$$

$$\hat{\eta}_b = |b\rangle\ \langle b| \tag{2.17}$$

$$\hat{\eta}_c = |c\rangle\ \langle c| \tag{2.18}$$

Next we seek to calculate the probability for the three-level atom to be in the top, intermediate, and bottom level by using Heisenberg equation

$$\frac{d}{dt}\langle|a\rangle\langle a|\rangle = -i\langle[|a\rangle\langle a|, \hat{H}]\rangle, \tag{2.19}$$

Since $\langle|a\rangle\langle a|\rangle = \rho_{aa}$, this can be rewritten as

$$\frac{d}{dt}\rho_{aa} = -i\langle[|a\rangle\langle a|, \hat{H}]\rangle, \tag{2.20}$$

so that with the aid of Eq.(2.7), we have

$$\frac{d}{dt}\rho_{aa} = g[\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle] + \frac{\Omega}{2}\big[\rho_{ac} + \rho_{ca}\big] \tag{2.21}$$

$$\frac{d}{dt}\rho_{bb} = g[\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b\rangle - \langle\hat{\sigma}_a^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a\rangle] \tag{2.22}$$

$$\frac{d}{dt}\rho_{cc} = -g(\langle\hat{\sigma}_b^\dagger\hat{a}\rangle - \langle\hat{b}^\dagger\hat{\sigma}_b\rangle) - \frac{\Omega}{2}\big[\rho_{ac} + \rho_{ca}\big] \tag{2.23}$$

$$\frac{d}{dt}\rho_{ac} = -g(\langle\hat{a}^\dagger\hat{\sigma}_b^\dagger\rangle - \langle\hat{b}^\dagger\hat{\sigma}_a^\dagger\rangle) - \frac{\Omega}{2}\big[\rho_{cc} - \rho_{aa}\big]. \tag{2.24}$$

in which

$$\rho_{aa} = \langle a|\rho|a\rangle, \tag{2.25}$$

$$\rho_{bb} = \langle b|\rho|b\rangle, \tag{2.26}$$

$$\rho_{cc} = \langle c|\rho|c\rangle, \tag{2.27}$$

$$\rho_{ac} = \langle a|\rho|c\rangle, \tag{2.28}$$

with $\rho_{aa}$, $\rho_{bb}$, and $\rho_{cc}$ being the probability for the atom to be the top, intermediate, and bottom levels, respectively. We see that Eqs. (2.13)-(2.15) and (2.21)-(2.24) are nonlinear and coupled differential equations and hence it is not possible to obtain exact time-dependent solution of these equations. We intend to overcome this problem by applying the large-time approximation [15]. Thus applying the large-time approximation scheme, we obtain from Eqs. (2.10) and (2.11) the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a, \tag{2.29}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b. \tag{2.30}$$

Evidently, these turn out to be exact relations at steady-state. Now introducing Eqs. (2.29) and (2.30) into Eqs. (2.13), (2.14), (2.15), (2.21), (2.22), (2.23), and (2.24), the equations of evolution of the atomic operators take the form

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\gamma_c\langle\hat{\sigma}_a\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.31}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_b\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.32}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -\frac{\gamma_c}{2}\langle\hat{\sigma}_c\rangle + \frac{\Omega}{2}\big[\rho_{cc} - \rho_{aa}\big], \tag{2.33}$$

$$\frac{d}{dt}\rho_{aa} = -\gamma_c\rho_{aa} + \frac{\Omega}{2}\big[\rho_{ac} + \rho_{ca}\big], \tag{2.34}$$

$$\frac{d}{dt}\rho_{bb} = -\gamma_c[\rho_{bb} - \rho_{aa}], \tag{2.35}$$

$$\frac{d}{dt}\rho_{cc} = \gamma_c\rho_{bb} - \frac{\Omega}{2}\big[\rho_{ca} + \rho_{ac}], \tag{2.36}$$

$$\frac{d}{dt}\rho_{ac} = -\frac{\gamma_c}{2}\rho_{ac} + \frac{\Omega}{2}\big[\rho_{cc} - \rho_{aa}\big], \tag{2.37}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.38}$$

is the stimulated emission decay constant. Based on the definition of this decay constant, we infer that an atom in the top level and inside a closed cavity emits photons due to its interaction with the cavity modes. We certainly identify this process to be stimulated photon emission. The operators $\rho_{aa}$, $\rho_{bb}$, and $\rho_{cc}$ representing the number of atoms in the top, intermediate, and bottom levels, respectively. We easily find the steady-state solutions of Eqs. (2.31)-(2.37) to be

$$\langle\hat{\sigma}_a\rangle = \frac{\Omega}{2\gamma_c}\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.39}$$

$$\langle\hat{\sigma}_b\rangle = -\frac{\Omega}{\gamma_c}\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.40}$$

$$\rho_{aa} = \frac{\Omega}{2\gamma_c}\big[\rho_{ac} + \rho_{ca}\big], \tag{2.41}$$

$$\rho_{bb} = \rho_{aa}, \tag{2.42}$$

$$\rho_{ac} = \frac{\Omega}{2\gamma_c}\big[\rho_{cc} - \rho_{aa}\big]. \tag{2.43}$$

With the aid of the identity

$$\rho_{aa} + \rho_{bb} + \rho_{cc} = 1, \tag{2.44}$$

along with Eq. (2.43), we obtain

$$\rho_{ac} = \frac{\Omega}{\gamma_c} - \frac{3\Omega^2}{2\gamma_c^2}\big[\rho_{ac} + \rho_{ca}\big]. \tag{2.45}$$

Since $\rho_{ac}$ is real, we see that $\rho_{ac} = \rho_{ca}$. In view of this, Eq. (2.45) can be put in the form

$$\rho_{ac} = \frac{\gamma_c\Omega}{\gamma_c^2 + 3\Omega^2}, \tag{2.46}$$

In view of this result, we see that

$$\rho_{aa} = \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}, \tag{2.47}$$

$$\rho_{bb} = \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}, \tag{2.48}$$

$$\rho_{cc} = \frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2}. \tag{2.49}$$

These equations represent the steady-state solutions of the equations of evolution of the atomic operators for a dynamics of non-degenerate three-level atom in a closed cavity and coupled with two-mode vacuum reservoir. The results described by Eqs. (2.46)-(2.49) are exactly the same as those obtained by Fesseha [8]. In addition, we note that for $\Omega \gg \gamma_c$, Eqs. (2.46)-(2.49) reduce to

$$\rho_{aa} = \frac{1}{3}, \tag{2.50}$$

$$\rho_{bb} = \frac{1}{3}, \tag{2.51}$$

$$\rho_{cc} = \frac{1}{3}, \tag{2.52}$$

$$\rho_{ac} = 0. \tag{2.53}$$

Finally, in the absence of the deriving coherent light, when $\Omega = 0$, Eqs. (2.46)-(2.49) turns out to be

$$\rho_{aa} = 0, \tag{2.54}$$

$$\rho_{bb} = 0, \tag{2.55}$$

$$\rho_{cc} = 1, \tag{2.56}$$

$$\rho_{ac} = 0. \tag{2.57}$$

These results shows initially, when the deriving coherent light $(\Omega = 0)$, that all the atoms to be in bottom level. Expressions (2.47), (2.48), and (2.49) represent the probabilities for the atom to be in the top, intermediate, and bottom levels. Moreover, from Eq. (2.42), we see that at steady state $\rho_{bb} = \rho_{aa}$, which shows that the probability for the atom to be in the top level is equal to that in the intermediate level. Then introducing Eqs. (2.40) and (2.41), into Eqs. (2.32) and (2.33) respectively, we get

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\left[\gamma_c + \frac{\Omega^2}{2\gamma_c}\right]\langle\hat{\sigma}_a\rangle, \tag{2.58}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\left[\frac{\gamma_c}{2} + \frac{\Omega^2}{2\gamma_c}\right]\langle\hat{\sigma}_b\rangle. \tag{2.59}$$

At steady state, we have

$$\langle \hat{\sigma}_a \rangle = 0, \tag{2.60}$$

$$\langle \hat{\sigma}_b \rangle = 0, \tag{2.61}$$

From the above results, we note that the atomic operators $\langle \hat{\sigma}_a \rangle$ and $\langle \hat{\sigma}_b \rangle$ are Gaussian variables with zero mean. Using the definition

$$\hat{\sigma} = \hat{\sigma}_a + \hat{\sigma}_b, \tag{2.62}$$

and taking into account Eqs. (2.2), (2.5), (2.6), (2.25)-(2.28) , it can be readily established that

$$\hat{\sigma}^{\dagger}\hat{\sigma} = \rho_{aa} + \rho_{bb}, \tag{2.63}$$

$$\hat{\sigma}\hat{\sigma}^{\dagger} = \rho_{bb} + \rho_{cc}, \tag{2.64}$$

$$\hat{\sigma}^2 = \rho_{ac}. \tag{2.65}$$

We note that the steady-state solutions of Eqs. (2.10) and (2.11) are

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a, \tag{2.66}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b. \tag{2.67}$$

Now employing Eqs. (2.66) and (2.67), the commutation relations for the cavity mode operators are found to be

$$[\hat{a}, \hat{a}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\rho_{bb} - \rho_{aa}\big], \tag{2.68}$$

$$[\hat{b}, \hat{b}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\rho_{cc} - \rho_{bb}\big], \tag{2.69}$$

Now adding Eqs. (2.68) and (2.69), we get

$$[\hat{c}, \hat{c}^{\dagger}] = \frac{\gamma_c}{\kappa}\big[\rho_{cc} - \rho_{aa}\big] \tag{2.70}$$

and the sum of Eqs. (2.10) and (2.11), one can easily established that

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} - g\hat{\sigma}, \tag{2.71}$$

in which

$$\hat{c} = \hat{a} + \hat{b}. \tag{2.72}$$

We next proceed to obtain the expectation value of the cavity mode operators. The expectation value of the solution of Eq. (2.10) is expressible as

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\kappa t/2} + g\int_0^t e^{\kappa t'/2}\langle \hat{\sigma}_a(t') \rangle dt'. \tag{2.73}$$

With the help of Eq. (2.60) and the assumption that the cavity light is initially in a vacuum state, Eq. (2.73) turns out to be

$$\langle \hat{a}(t) \rangle = 0. \tag{2.74}$$

In view of the linear equation described by Eq. (2.68) and the result given by Eq. (2.74), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. Following a similar procedure, one can readily obtain the expectation value of the solution of Eq. (2.11) to be

$$\langle \hat{b}(t) \rangle = 0. \tag{2.75}$$

Then on account of the linear equation described by Eq. (2.11) and the result given by Eq. (2.75), we realize $\hat{b}(t)$ to be a Gaussian variable with zero mean. Now with the aid of Eqs. (2.74) and (2.75) together with (2.72), we have

$$\langle \hat{c}(t) \rangle = 0. \tag{2.76}$$

These results shows initially, when the deriving coherent light $(\Omega = 0)$, that all the atoms to be in bottom level.

# 3

# Photon Statistics

In this chapter we seek to study the statistical properties of the light produced by dynamics of non-degenerate three-level laser in a closed a cavity and coupled with two-mode vacuum reservoir for single atom via a single-port mirror. Applying the solutions of the equations of evolution of the expectation values for the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global and local photon statistics for light modes $a$ and $b$. In addition, we determine the global photon statistics of the two-mode cavity light.

## 3.1 Single-mode photon statistics

In this section we obtain the global mean and variance of the photon numbers for light modes $a$ and $b$. Moreover, we determine the local mean and variance of the photon numbers for light modes $a$ and $b$.

### 3.1.1 Global mean photon number

Here we seek to calculate the global mean photon numbers of light modes $a$ and $b$, produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled with two-mode vacuum reservoir.

A. Global mean photon number of light mode $a$

We now proceed to obtain the mean photon number of light mode $a$ in the entire frequency interval. The mean photon number of light mode $a$, represented by the operators $\hat{a}$ and $\hat{a}^\dagger$, is defined by

$$\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle. \tag{3.1}$$

We note that the steady-state solution of Eq. (2.10) is

$$\hat{a} = -\frac{2g}{\kappa\sqrt{N}}\hat{\sigma}_a, \tag{3.2}$$

so that introducing Eq. (3.2) and its adjoint into (3.1), we see that

$$\bar{n}_a = \frac{\gamma_c}{\kappa N}\langle\hat{\sigma}_a^\dagger\hat{\sigma}_a\rangle. \tag{3.3}$$

With the help of Eq. (2.46), one can write

$$\hat{\sigma}_a^\dagger\hat{\sigma}_a = \hat{\eta}_a, \tag{3.4}$$

from which follows

$$\langle\hat{\sigma}_a^\dagger\hat{\sigma}_a\rangle = \langle\hat{\eta}_a\rangle = \rho_{aa}. \tag{3.5}$$

On account of Eq. (3.5), Eq. (3.3) can be expressed as

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\rho_{aa}. \tag{3.6}$$

In view of Eq. (2.47), there follows

$$\bar{n}_a = \frac{\gamma_c}{\kappa}N\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{3.7}$$

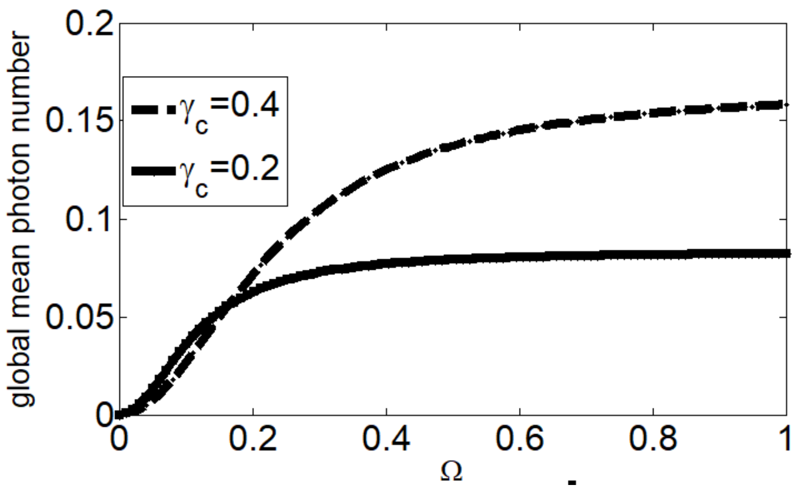


Figure 3.1: Plot of global mean photon number of light mode a

This is the steady-state mean photon number of light mode $a$ produced by dynamics of non-degenerate three-level laser in a closed cavity and coupled with two-mode vacuum reservoir. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.7) reduces to

$$\bar{n}_a = \frac{\gamma_c}{3\kappa}. \tag{3.8}$$

B. Global mean photon number of light mode $b$

Here we seek to determine the mean photon number of light mode $b$ in the entire frequency interval produced by the system under consideration. The mean photon number of light mode $b$, represented by the operators $\hat{b}$ and $\hat{b}^\dagger$, is defined by

$$\bar{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle. \tag{3.9}$$

We note that the steady-state solution of Eq. (2.11) is

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b, \tag{3.10}$$

so that introducing Eq. (3.10) and its adjoint into (3.9), we see that

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_b^\dagger \hat{\sigma}_b \rangle. \tag{3.11}$$

With the help of Eq. (2.47), one can write

$$\hat{\sigma}_b^\dagger \hat{\sigma}_b = \hat{\eta}_b, \tag{3.12}$$

from which follows

$$\langle \hat{\sigma}_b^\dagger \hat{\sigma}_b \rangle = \langle \hat{\eta}_b \rangle = \rho_{bb}. \tag{3.13}$$

On account of Eq. (3.13), Eq. (3.11) can be expressed as

$$\bar{n}_b = \frac{\gamma_c}{\kappa} N \rho_{bb}. \tag{3.14}$$

Now on substituting Eq. (2.48) into (3.14), the mean photon number of light mode $b$ takes, at steady-state, the form

$$\bar{n}_b = \frac{\gamma_c}{\kappa} N \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{3.15}$$

Fig 3.2 describes that the global mean photon number increases as decay constant $(\gamma_c)$ increases. This is the steady-state mean photon number of light mode $b$ produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. We would like to point out that this result is exactly the same as that described by Eq. (3.7). In addition, we note that for $\Omega \gg \gamma_c$,

Eq. (3.15) reduces to

$$\bar{n}_b = \frac{\gamma_c}{3\kappa}. \tag{3.16}$$

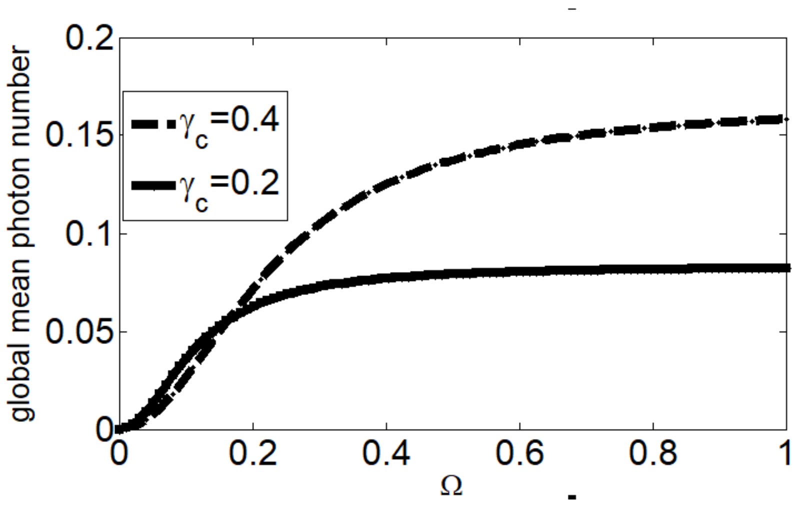


Figure 3.2: Plot of global mean photon number of light mode b for the values of $\kappa$=0.8, $\gamma_c$=0.4

### 3.1.2 Local mean photon number

Here we seek to determine the local mean photon numbers of light modes $a$ and $b$, produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled with two-mode vacuum reservoir.

A. Local mean photon number of light mode $a$

We now proceed to obtain the mean photon number of light mode $a$ in a given frequency interval. To determine the local mean photon number of light mode $a$, we need to consider the power spectrum of light mode $a$. The power spectrum of light mode $a$ with central frequency $\omega_0$ is expressible as [8]

$$P_a(\omega) = \frac{1}{\pi}Re\int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau}\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_{ss}. \tag{3.17}$$

Upon integrating both sides of Eq. (3.17) over $\omega$, we readily get

$$\int_{-\infty}^{\infty} P_a(\omega)d\omega = \bar{n}_a, \tag{3.18}$$

in which $\bar{n}_a$ is the steady-state mean photon number of light mode $a$. From this result, we observe that $P_a(\omega)d\omega$ is the steady-state mean photon number of light mode $a$ in the frequency interval between $\omega$ and $\omega + d\omega$ [8].

We now proceed to determine the two-time correlation function that appears in Eq. (3.17). To this end, we realize that the solution of Eq. (2.10) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\kappa\tau/2} + \frac{g}{\sqrt{1}}e^{-\kappa\tau/2}\int_0^{\tau} e^{\kappa\tau'/2}\hat{\sigma}_a(t+\tau')d\tau'. \tag{3.19}$$

Applying the large time approximation on Eq. (2.32), we have

$$\langle\hat{\sigma}_b^\dagger\rangle = \frac{\Omega}{\gamma_c}\langle\hat{\sigma}_a\rangle. \tag{3.20}$$

Employing this result, Eq. (2.31) takes the form

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\frac{\eta}{2}\langle\hat{\sigma}_a\rangle. \tag{3.21}$$

On the basis of Eq. (3.21), we see that

$$\frac{d}{dt}\hat{\sigma}_a(t) = -\frac{\eta}{2}\hat{\sigma}_a(t) + \hat{F}_a(t), \tag{3.22}$$

in which $\hat{F}_a(t)$ is a noise operator with a vanishing mean and $\eta$ is given by

$$\eta = \left[\frac{\Omega^2 + 2\gamma_c^2}{\gamma_c}\right]. \tag{3.23}$$

The solution of Eq. (3.22) can be put in the form

$$\hat{\sigma}_a(t+\tau) = \hat{\sigma}_a(t)e^{-\eta\tau/2} + e^{-\eta\tau/2}\int_0^\tau e^{\eta\tau'/2}\hat{F}_a(t+\tau')d\tau', \tag{3.24}$$

so that on introducing this into Eq. (3.19), there follows

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\tau/2} + ge^{-\kappa\tau/2}\hat{\sigma}_a(t)\int_0^\tau e^{(\kappa-\eta)\tau'/2}d\tau' \\ &+ ge^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\hat{F}_a(t+\tau'').\end{aligned} \tag{3.25}$$

Thus on carrying out the first integration, we arrive at

$$\begin{aligned}\hat{a}(t+\tau) &= \hat{a}(t)e^{-\kappa\tau/2} + \frac{2g\hat{\sigma}_a(t)}{(\kappa-\eta)}\left[e^{-\eta\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ ge^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\hat{F}_a(t+\tau'').\end{aligned} \tag{3.26}$$

Now multiplying on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{a}^\dagger(t)\hat{\sigma}_a(t)\rangle}{(\kappa-\eta)}\left[e^{-\eta\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ ge^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle.\end{aligned} \tag{3.27}$$

Applying the large-time approximation scheme, one gets from Eq. (2.10)

$$\hat{a}(t) = \frac{2g}{\kappa}\hat{\sigma}_a(t), \tag{3.28}$$

so that in view of this result, we get

$$\hat{\sigma}_a(t) = \frac{\kappa}{2g}\hat{a}(t). \tag{3.29}$$

Thus substitution of Eq. (3.29) into Eq. (3.27) results in

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2}-\frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right]\\ &+ g e^{-\kappa\tau/2}\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\eta)\tau'+\eta\tau'']/2}\langle \hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle. \quad (3.30)\end{aligned}$$

Since a noise operator at a certain time should not affect a light mode operator at an earlier time [8], we note that

$$\langle \hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle = 0. \quad (3.31)$$

It then follows that

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2}-\frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right] \quad (3.32)$$

and at steady-state, we have

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} = \bar{n}_a\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2}-\frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right]. \quad (3.33)$$

Thus on combining Eq. (3.33) with Eq. (3.17), the power spectrum of light mode $a$ with central frequency $\omega_0$ is expressible as

$$P_a(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_a}{\kappa-\eta}\right]Re\left[\kappa\int_0^\infty d\tau e^{-[\eta/2-i(\omega-\omega_0)]\tau}-\eta\int_0^\infty d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}\right], \quad (3.34)$$

so that on carrying out the integration, we readily arrive at

$$P_a(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_a}{\kappa-\eta}\right]Re\left[\frac{\kappa}{[\eta/2-i(\omega-\omega_0)]}-\frac{\eta}{[\kappa/2-i(\omega-\omega_0)]}\right]. \quad (3.35)$$

This can be rewritten as

$$P_a(\omega) = \frac{\kappa\bar{n}_a}{\kappa-\eta}\left[\frac{\eta/2\pi}{[\eta/2]^2+(\omega-\omega_0)^2}\right]-\frac{\eta\bar{n}_a}{\kappa-\eta}\left[\frac{\kappa/2\pi}{[\kappa/2]^2+(\omega-\omega_0)^2}\right]. \quad (3.36)$$

We realize that the mean photon number of light mode $a$ in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [8]

$$\bar{n}_{a\pm\lambda} = \int_{-\lambda}^{\lambda} P_a(\omega')d\omega', \quad (3.37)$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.35) into Eq. (3.37) and carrying out the integration by employing the relation

$$\int_{-\lambda}^{\lambda}\frac{dx}{x^2+a^2} = \frac{2}{a}\tan^{-1}\left(\frac{\lambda}{a}\right), \quad (3.38)$$

The local mean photon number of light mode $a$ produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir is found to be

$$\bar{n}_{a\pm\lambda} = \bar{n}_a z_a(\lambda), \tag{3.39}$$

where $z_a(\lambda)$ is given by

$$z_a(\lambda) = \frac{2\kappa/\pi}{\kappa - \eta}\tan^{-1}\left(\frac{2\lambda}{\eta}\right) - \frac{2\eta/\pi}{\kappa - \eta}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right) \tag{3.40}$$

The local mean photon number increases when $\lambda$ value increases. We see from Eq.

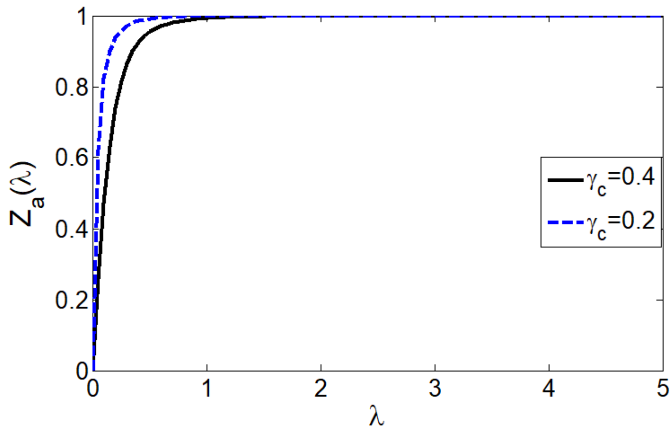


Figure 3.3: Plot of $z_a(\lambda)$ that $\bar{n}_{a\pm\lambda}$ increases with $\lambda$

(3.39) along with the plot of $z_a(\lambda)$ that $\bar{n}_{a\pm\lambda}$ increases with $\lambda$ until it reaches the maximum value of the local mean photon number. From the plots in Fig. (3.39), we find the values indicated below:

| $\gamma_c$ | $z_a(0.5)$ | $z_a(1)$ | $z_a(1.5)$ | $z_a(2)$ |
|---|---|---|---|---|
| 0.2 | 0.6457 | 0.8113 | 0.8886 | 0.9274 |
| 0.4 | 0.6259 | 0.7884 | 0.866 | 0.9062 |

Table 3.1: Values of $z_b(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from these results $z_a(\lambda)$ in the value of stimulated emission $(\gamma_c = 0.4)$ is less than in the value of stimulated emission $(\gamma_c = 0.2)$. Moreover, using the above results of $z_a(\lambda)$ and on account of Eq. (3.61), we have

We therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

| $\gamma_c$ | $\bar{n}_{a\pm 0.5}$ | $\bar{n}_{a\pm 1}$ | $\bar{n}_{a\pm 2}$ |
|---|---|---|---|
| 0.2 | 0.81 | 0.97 | 1.09 |
| 0.4 | 0.79 | 0.95 | 1.07 |

Table 3.2: Values of $\bar{n}_{b\pm\lambda}$ for $\gamma_c = 0.4$, $\kappa = 0.6$, and $\Omega = 2$.

## B. Local mean photon number of light mode $b$

We now proceed to obtain the mean photon number of a light mode $b$ in a given frequency interval produced by the system under consideration. To determine the local mean photon number of light mode $b$, we need to consider the power spectrum of light mode $b$. The power spectrum of light mode $b$ with central frequency $\omega_0$ is expressible as

$$P_b(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle_{ss}. \tag{3.41}$$

Upon integrating both sides of Eq. (3.41) over $\omega$, we readily get

$$\int_{-\infty}^{\infty} P_b(\omega) d\omega = \bar{n}_b, \tag{3.42}$$

in which $\bar{n}_b$ is the steady-state mean photon number of light mode $b$. From this result, we observe that $P_b(\omega)d\omega$ is the steady-state mean photon number of light mode $b$ in the frequency interval between $\omega$ and $\omega + d\omega$. We now proceed to calculate the two-time correlation function that appears in Eq. (3.42). To this end, we realize that the solution of Eq. (2.11) can be written as

$$\hat{b}(t+\tau) = \hat{b}(t)e^{-\kappa\tau/2} + ge^{-\kappa\tau/2} \int_0^{\tau} e^{\kappa\tau'/2} \hat{\sigma}_b(t+\tau')d\tau'. \tag{3.43}$$

Applying the large time approximation on Eq. (2.31), we have

$$\langle \hat{\sigma}_a \rangle = \frac{\Omega}{\gamma_c} \langle \hat{\sigma}_b^\dagger \rangle. \tag{3.44}$$

Employing this result, Eq. (2.32) takes the form

$$\frac{d}{dt} \langle \hat{\sigma}_b^\dagger \rangle = -\frac{1}{2} \mu \langle \hat{\sigma}_b^\dagger \rangle. \tag{3.45}$$

On the basis of Eq. (3.45), we see that

$$\frac{d}{dt} \hat{\sigma}_b(t) = -\frac{\mu}{2} \hat{\sigma}_b(t) + \hat{F}_b(t), \tag{3.46}$$

in which $\hat{F}_b(t)$ is a noise operator with a vanishing mean and $\mu$ is given by

$$\mu = \left[\frac{\Omega^2 + 2\gamma_c^2}{2\gamma_c}\right]. \tag{3.47}$$

The solution of equation (3.46) can be put in the form

$$\hat{\sigma}_b(t+\tau') = \hat{\sigma}_b(t)e^{-\mu\tau'/2} + e^{-\mu\tau'/2}\int_0^{\tau'} e^{-\mu\tau''/2}\hat{F}_b(t+\tau'')d\tau'', \tag{3.48}$$

so that on introducing this into Eq. (3.43), we have

$$\begin{aligned}\hat{b}(t+\tau) &= \hat{b}(t)e^{-\kappa\tau/2} + ge^{-\kappa\tau/2}\hat{\sigma}_b(t)\int_0^{\mu} e^{(\kappa-\mu)\tau'/2}d\tau' \\ &+ ge^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\hat{F}_b(t+\tau''). \end{aligned}\tag{3.49}$$

Thus on carrying out the first integration, we arrive at

$$\begin{aligned}\hat{b}(t+\tau) &= \hat{b}(t)e^{-\kappa\tau/2} + \frac{2g\hat{\sigma}_b(t)}{(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ ge^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\hat{F}_b(t+\tau''). \end{aligned}\tag{3.50}$$

Now multiplying both sides on the left by $\hat{b}^\dagger(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle e^{-\kappa\tau/2} + \frac{2g\langle\hat{b}^\dagger(t)\hat{\sigma}_b(t)\rangle}{(\kappa-\mu)}\left[e^{-\mu\tau/2} - e^{-\kappa\tau/2}\right] \\ &+ ge^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle. \end{aligned}\tag{3.51}$$

Applying the large-time approximation scheme, one gets from Eq. (2.11)

$$\hat{b}(t) = \frac{2g}{\kappa}\hat{\sigma}_b(t). \tag{3.52}$$

In view of Eq. (3.52), we see that

$$\hat{\sigma}_b(t) = \frac{\kappa}{2g}\hat{b}(t). \tag{3.53}$$

With this substituted into Eq. (3.49), there follows

$$\begin{aligned}\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right] \\ &+ ge^{-\kappa\tau/2}\int_0^{\tau} d\tau'\int_0^{\tau'} d\tau'' e^{[(\kappa-\mu)\tau'+\mu\tau'']/2}\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle \end{aligned}\tag{3.54}$$

and taking into account the fact that

$$\langle\hat{b}^\dagger(t)\hat{F}_b(t+\tau'')\rangle = 0, \tag{3.55}$$

we arrive at

$$\langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle = \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.56}$$

Therefore, at steady-state, Eq. (3.56) takes the form

$$\langle \hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle_{ss} = \bar{n}_b\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \tag{3.57}$$

Thus on combining Eq. (3.57) with Eq. (3.41), the power spectrum of light mode $b$ with central frequency $\omega_0$ can be put in the form

$$P_b(\omega) = \frac{1}{\pi}\left[\frac{\bar{n}_b}{\kappa-\mu}\right]Re\left[\kappa\int_0^\infty d\tau e^{-[\mu/2-i(\omega-\omega_0)]\tau} - \mu\int_0^\infty d\tau e^{-[\kappa/2-i(\omega-\omega_0)]\tau}, \tag{3.58}$$

so that on carrying out the integration, we readily arrive at

$$P_b(\omega) = \frac{\kappa\bar{n}_b}{\kappa-\mu}\left[\frac{\mu/2\pi}{[\mu/2]^2+(\omega-\omega_0)^2}\right] - \frac{\mu\bar{n}_b}{\kappa-\mu}\left[\frac{\kappa/2\pi}{[\kappa/2]^2+(\omega-\omega_0)^2}\right]. \tag{3.59}$$

We realize that the mean photon number of light mode $b$ in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{b\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \tag{3.60}$$

in which $\omega' = \omega - \omega_0$. Therefore, upon substituting Eq. (3.59) into Eq. (3.60), and performing the integration by using the relation given by Eq. (3.38), we readily get

$$\bar{n}_{b\pm\lambda} = \bar{n}_b z_b(\lambda), \tag{3.61}$$

where $z_b(\lambda)$ is given by

$$z_b(\lambda) = \frac{2\kappa/\pi}{\kappa-\mu}\tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{2\mu/\pi}{\kappa-\mu}\tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.62}$$

We see from Eq. (3.61) along with the plot of $z_b(\lambda)$ that $\bar{n}_{b\pm\lambda}$ increases with $\lambda$ until it reaches the maximum value of the global mean photon number. From the plots in Fig.3.4, we find the values indicated below:

| $\gamma_c$ | $z_b(0.5)$ | $z_b(1)$ | $z_b(1.5)$ | $z_b(2)$ |
|---|---|---|---|---|
| 0.4 | 0.95 | 0.9986 | 0.9996 | 0.9998 |
| 0.2 | 0.99 | 0.9986 | 0.9996 | 0.9998 |

Table 3.3: Values of $z_b(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

We see from these results $z_b(\lambda)$ in the value of stimulated emission $(\gamma_c = 0.2)$ is greater than in the value of stimulated emission $(\gamma_c = 0.4)$. Moreover, using the above results of $z_b(\lambda)$ and on account of Eq. (3.61), we have

From the plots in Fig. 3.4, we therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval.

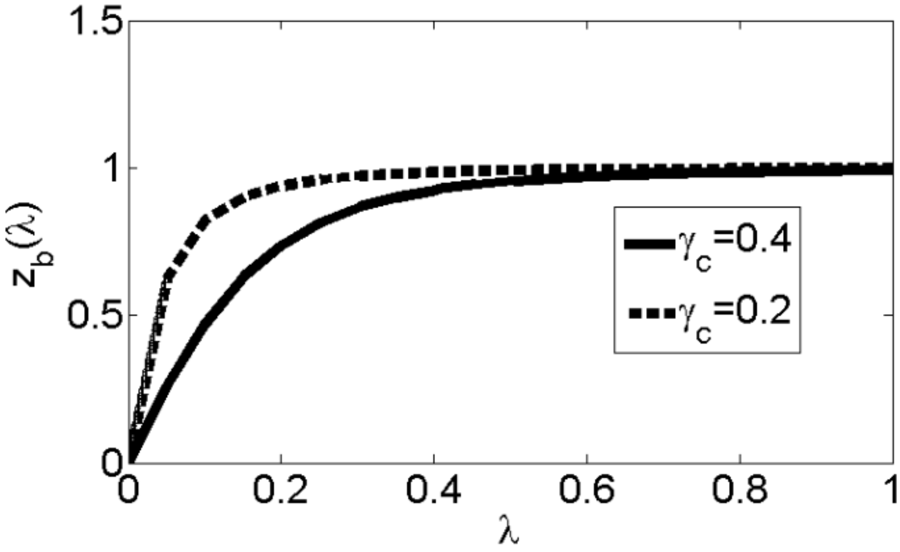


Figure 3.4: Plot of local mean photon number for light mode b for the values of $\kappa$=0.8,$\gamma_c$=0.4

| $\gamma_c$ | $\bar{n}_{b\pm0.5}$ | $\bar{n}_{b\pm1}$ | $\bar{n}_{b\pm2}$ |
|---|---|---|---|
| 0.4 | 1.114 | 1.163 | 1.164 |
| 0.2 | 1.155 | 1.163 | 1.164 |

Table 3.4: Values of $\bar{n}_{b\pm\lambda}$ for $\gamma_c = 0.4$, $\kappa = 0.6$, and $\Omega = 2$.

### 3.1.3 Global photon-number variance

Here we seek to obtain the global photon number variance of light modes $a$ and $b$, produced by the dynamics of non-degenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir.

#### A. Global photon-number variance of light mode $a$

We now proceed to calculate the photon number variance of light mode $a$ in the entire frequency interval. The photon number variance of light mode $a$ is expressible as

$$(\Delta n)_a^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \tag{3.63}$$

Applying the fact that $\hat{a}$ is a Gaussian variable with zero mean, we arrive at

$$(\Delta n)_a^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.64}$$

In view of Eq. (3.2), we see that

$$\langle \hat{a}^2 \rangle = 0, \tag{3.65}$$

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa}\rho_{bb}. \tag{3.66}$$

Thus on account of Eqs. (3.6), (3.65) and (3.66), the photon number variance (3.62) turns out to be

$$(\Delta n)_a^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \rho_{aa}\rho_{bb}. \tag{3.67}$$

With the aid of Eqs. (2.47) and (2.48), the photon number variance of light mode $a$ takes, at steady-state, the form

$$(\Delta n)_a^2 = \left[\frac{\gamma_c}{\kappa}\right]^2 \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]^2 . \tag{3.68}$$

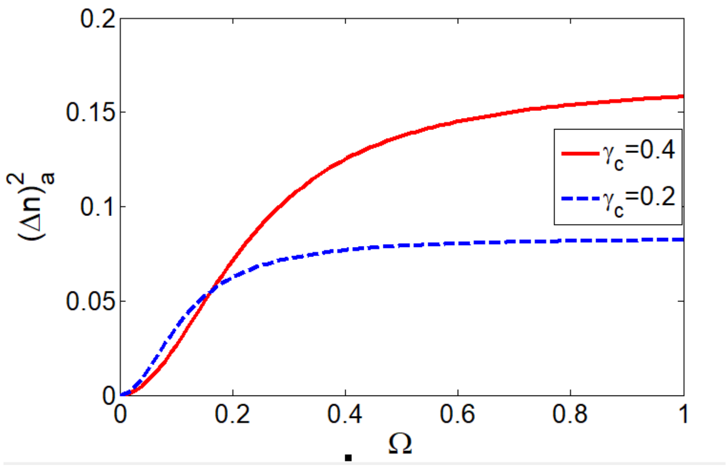


Figure 3.5: Plot of global photon number variance of light mode a versus $\Omega$ for the values of $\kappa$=0.8, $\gamma_c$=0.4

The fig 3.5 shows that the global photon number variance increases from $\Omega$=0 to $\Omega$ =2, This is the global photon number variance of light mode $a$, produced by the dynamics of non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. Moreover, in view of Eq. (3.7), we have

$$(\Delta n)_a^2 = \bar{n}_a^2, \tag{3.69}$$

Which represents the normally-ordered variance of the photon number for chaotic light.

we compare the global mean photon number and photon number variance of light mode a. Global mean photon number of light mode a is very larger than that of global photon number variance having the same value of parameters. The figure 3.6

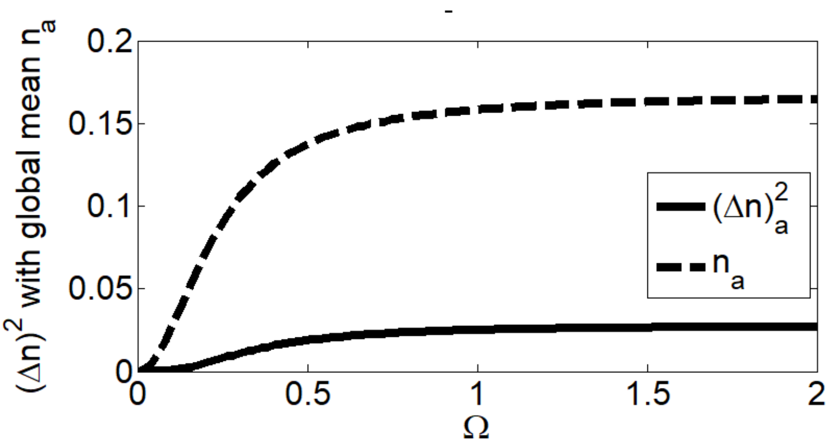


Figure 3.6: Plot of global mean photon number and photon number variance of light mode a for the values of $\kappa$=0.8, $\gamma_c$=0.4

shows that the global mean photon number is much more greater than that of global photon number variance of the light mode (i.e light mode a).
In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.68) reduces to

$$(\Delta n)^2_a = \left[\frac{\gamma_c}{3\kappa}\right]^2, \tag{3.70}$$

so that with the aid of Eq. (3.7), we see that

$$(\Delta n)^2_a = \bar{n}^2_a. \tag{3.71}$$

## B. Global photon-number variance of light mode $b$

Here we seek to obtain the photon number variance of light mode $b$ in the entire frequency interval. The photon number variance of light mode $b$ is defined as

$$(\Delta n)^2_b = \langle \hat{b}^\dagger \hat{b} \hat{b}^\dagger \hat{b} \rangle - \langle \hat{b}^\dagger \hat{b} \rangle^2 \tag{3.72}$$

and using the fact that $\hat{b}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2_b = \langle \hat{b}^\dagger \hat{b} \rangle \langle \hat{b} \hat{b}^\dagger \rangle + \langle \hat{b}^{\dagger 2} \rangle \langle \hat{b}^2 \rangle. \tag{3.73}$$

In view of Eq. (3.10), we have

$$\langle \hat{b}^2 \rangle = 0, \tag{3.74}$$

$$\langle \hat{b} \hat{b}^\dagger \rangle = \frac{\gamma_c}{\kappa}\rho_{cc}. \tag{3.75}$$

Thus on account of Eqs. (3.12), (3.74) and (3.75), the photon number variance (3.79) turns out to be

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \rho_{cc}\rho_{bb}, \tag{3.76}$$

from which follows

$$(\Delta n)_b^2 = \bar{n}_b\left[\frac{\gamma_c}{\kappa} - 2\bar{n}_b\right]. \tag{3.77}$$

With the aid of Eq. (3.15), the photon number variance of light mode $b$ takes, at steady-state, the form

$$(\Delta n)_b^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\frac{\Omega^2(\gamma_c^2+\Omega^2)}{(\gamma_c^2+3\Omega^2)^2}\right]. \tag{3.78}$$

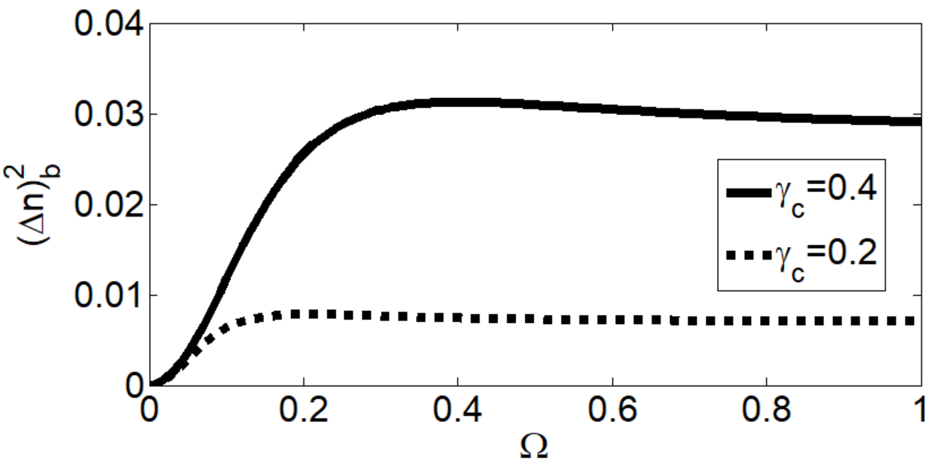


Figure 3.7: Plot of global photon number variance of light mode b for the values of $\kappa$=0.8, and $\gamma_c$=0.4

The fig. 3.7 describes that $(\Delta n)_b^2$ increases with increasing in $\Omega$ value. This is the steady-state global photon number variance of of light mode $b$ produced by the dynamics of non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.78) reduces to

$$(\Delta n)_b^2 = \left[\frac{\gamma_c}{3\kappa}\right]^2 \tag{3.79}$$

and in view of Eq. (3.16), there follows

$$(\Delta n)_b^2 = \bar{n}_b^2, \tag{3.80}$$

which represents the normally-ordered variance of the photon number for chaotic light. We readily observe from the plots in Fig. (7) that the photon number variance of light mode $b$ is $(\Delta n)_b^2 = 0.031$ and occurs when the three-level laser is operating at $\Omega = 0.40$.

### 3.1.4 Local photon-number variance

Here we seek to study the local photon number variance of light modes $a$ and $b$, produced by the coherently driven non degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir.

#### A. Local photon-number variance of light mode $a$

We now proceed to obtain the photon number variance of light mode $a$ in a given frequency interval. To determine the local photon number variance of light mode $a$, we need to consider the spectrum of photon number fluctuations of light mode $a$. The spectrum of photon number fluctuations of light mode $a$ with central frequency $\omega_0$ is expressible as [4]

$$S_a(\omega) = \frac{1}{\pi}\int_0^\infty d\tau e^{i(\omega-\omega_0)\tau}\langle \hat{n}_a(t), \hat{n}_a(t+\tau)\rangle_{ss}, \tag{3.81}$$

where

$$\hat{n}_a(t) = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.82}$$

and

$$\hat{n}_a(t+\tau) = \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau). \tag{3.83}$$

Upon integrating both sides of Eq. (3.81) over $\omega$, we find

$$\int_{-\infty}^\infty S_a(\omega)d\omega = (\Delta n)_a^2, \tag{3.84}$$

in which $(\Delta n)_a^2$ is the steady-state photon number variance of light mode $a$. From this result, we realize that $S_a(\omega)d\omega$ is the photon number variance of light mode $a$ in the frequency interval between $\omega$ and $\omega + d\omega$ [4].

We now proceed to evaluate the two-time correlation function that appears in Eq. (3.81). Applying the notation [9]

$$\langle \hat{A}, \hat{B}\rangle = \langle \hat{A}\hat{B}\rangle - \langle \hat{A}\rangle\langle \hat{B}\rangle, \tag{3.85}$$

we see that

$$\langle \hat{n}_a(t), \hat{n}_a(t+\tau)\rangle = \langle \hat{n}_a(t)\hat{n}_a(t+\tau)\rangle - \langle \hat{n}_a(t)\rangle\langle \hat{n}_a(t+\tau)\rangle. \tag{3.86}$$

On account of Eqs. (3.82) and (3.83) and using the fact that $a$ is a Gaussian variable with zero mean given by Eq. (2.74), we have

$$\begin{aligned}\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\langle\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle \\ &+ \langle\hat{a}(t)\hat{a}(t+\tau)\rangle\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle \\ &+ \langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle. \end{aligned} \tag{3.87}$$

Thus substitution of Eq. (3.87) into Eq. (3.86) results in

$$\begin{aligned}\langle\hat{n}_a(t),\hat{n}_a(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle\langle\hat{a}(t)\hat{a}(t+\tau)\rangle \\ &+ \langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle. \end{aligned} \tag{3.88}$$

With the help of Eq. (3.26), one can readily establish that

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}^{\dagger 2}(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right], \tag{3.89}$$

$$\langle\hat{a}(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^{2}(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right], \tag{3.90}$$

$$\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}(t)\hat{a}^\dagger(t)\rangle\left[\frac{\kappa}{\kappa-\eta}e^{-\eta\tau/2} - \frac{\eta}{\kappa-\eta}e^{-\kappa\tau/2}\right]. \tag{3.91}$$

Now employing Eqs. (3.32), (3.89), (3.90), and (3.91), we obtain

$$\begin{aligned}\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle &= \left[\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle + \langle\hat{a}^2(t)\rangle\langle\hat{a}^{\dagger 2}(t)\rangle\right] \\ &\times \left[\left(\frac{\eta}{\kappa-\eta}\right)^2 e^{-\kappa\tau} + \left(\frac{\kappa}{\kappa-\eta}\right)^2 e^{-\eta\tau} - \frac{2\kappa\eta}{(\kappa-\eta)^2}e^{-(\kappa+\eta)\tau/2}\right]. \end{aligned} \tag{3.92}$$

This can be rewritten as

$$\langle\hat{n}_a(t)\hat{n}_a(t+\tau)\rangle_{ss} = \frac{(\Delta n)_a^2}{(\kappa-\eta)^2}\left[\eta^2 e^{-\kappa\tau} + \kappa^2 e^{-\eta\tau} - 2\kappa\eta e^{-(\kappa+\eta)\tau/2}\right], \tag{3.93}$$

in which $(\Delta n)_a^2$ is the steady-state photon number variance of light mode $a$ given by Eq. (3.68). Therefore, in view of Eq. (3.93), the spectrum of photon number fluctuations can be put in the form

$$\begin{aligned}S_a(\omega) &= \frac{(\Delta n)_a^2}{\pi(\kappa-\eta)^2}Re\left[\eta^2\int_0^\infty d\tau e^{-[\kappa-i(\omega-\omega_0)]\tau}\right. \\ &+ \left.\kappa^2\int_0^\infty d\tau e^{-[\eta-i(\omega-\omega_0)]\tau} - 2k\eta\int_0^\infty d\tau e^{-[\frac{(\kappa+\eta)}{2}-i(\omega-\omega_0)]\tau}\right]. \end{aligned} \tag{3.94}$$

Thus on carrying out the integration, the spectrum of photon number fluctuations of light mode $a$ turns out to be

$$S_a(\omega) = \frac{(\Delta n)_a^2}{(\kappa-\eta)^2}\left[\frac{\eta^2\kappa/\pi}{\kappa^2+(\omega-\omega_0)^2} + \frac{\kappa^2\eta/\pi}{\eta^2+(\omega-\omega_0)^2} - \frac{2\kappa\eta(\kappa+\eta)/2\pi}{(\frac{\kappa+\eta}{2})^2+(\omega-\omega_0)^2}\right]. \tag{3.95}$$

Now we realize that the photon number variance in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [4]

$$(\Delta n)^2_{a\pm\lambda} = \int_{-\lambda}^{\lambda} S_a(\omega')d\omega', \tag{3.96}$$

in which $\omega' = \omega - \omega_0$. Therefore, substituting Eq. (3.93) into Eq. (3.94) leads to

$$(\Delta n)^2_{a\pm\lambda} = \frac{(\Delta n)^2_a}{\pi(\kappa-\eta)^2}\Bigg[\int_{-\lambda}^{\lambda} \frac{\eta^2\kappa d\omega'}{\kappa^2+\omega'^2} - \int_{-\lambda}^{\lambda} \frac{2\kappa\eta(\kappa+\eta)d\omega'}{(\frac{\kappa+\eta}{2})^2+\omega'^2} + \int_{-\lambda}^{\lambda} \frac{\eta\kappa^2 d\omega'}{\eta^2+\omega'^2}\Bigg]. \tag{3.97}$$

Employing the relation given by Eq. (3.38), the local photon number variance of light mode $a$ produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled with a two-mode vacuum reservoir is found to be

$$(\Delta n)^2_{a\pm\lambda} = (\Delta n)^2_a z_a'(\lambda), \tag{3.98}$$

where $z_a'(\lambda)$ is given by

$$z_a'(\lambda) = \frac{2\eta^2/\pi}{(\eta-\kappa)^2}tan^{-1}\left(\frac{\lambda}{\kappa}\right) + \frac{2\kappa^2/\pi}{(\kappa-\eta)^2}tan^{-1}\left(\frac{\lambda}{\eta}\right) - \frac{4\kappa\eta/\pi}{(\kappa-\eta)^2}tan^{-1}\left(\frac{2\lambda}{\kappa+\eta}\right). \tag{3.99}$$

We see from Eq. (3.98) along with the plot $z_a'(\lambda)$ that $(\Delta n)^2_{a\pm\lambda}$ increases with $\lambda$ until it

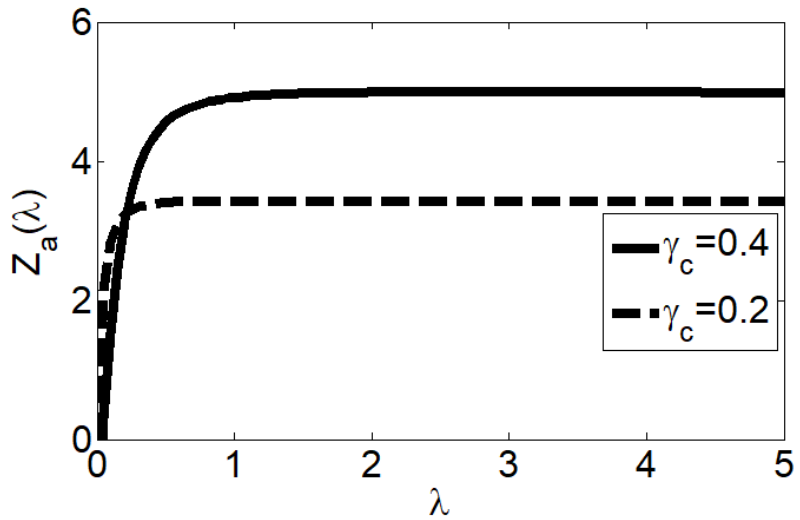


Figure 3.8: Plot of Local photon number variance for light mode a for the values of $\kappa$=0.8 and $\gamma_c$=0.4 and 0.2

reaches the maximum value of the global photon number variance. From the plots in Fig.3.8 we find the values indicated below:

We see from these results that $z_a'(\lambda)$ the stimulated emission decay constant $(\gamma_c = 0.4)$ is less than the stimulated emission decay constant $(\gamma_c = 0.2)$. Moreover, using the above results of $z_a'(\lambda)$ and on account of Eq. (3.99), we have We therefore observe

| $\gamma_c$ | $z_a^{'}(0.5)$ | $z_a^{'}(1)$ | $z_a^{'}(2)$ |
|---|---|---|---|
| $0.2$ | $3.403$ | $3.418$ | $3.413$ |
| $0.4$ | $4.567$ | $4.901$ | $4.98$ |

Table 3.5: Values of $z_a^{'}(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

| $\gamma_c$ | $(\Delta n)^2_{a\pm 0.5}$ | $(\Delta n)^2_{a\pm 1}$ | $(\Delta n)^2_{a\pm 2}$ |
|---|---|---|---|
| $0.2$ | $3.43$ | $3.445$ | $3.440$ |
| $0.4$ | $4.594$ | $4.928$ | $5.007$ |

Table 3.6: Values of $(\Delta n)^2_{a\pm\lambda}$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

that a large part of the total variance of photon number is confined in a relatively small frequency interval.

## B. Local photon-number variance for light mode $b$

We now proceed to obtain the photon number variance of light mode $b$ in a given frequency interval produced by the system under consideration. To determine the local photon number variance of light mode $b$, we need to consider the spectrum of photon number fluctuations of light mode $b$. We define the spectrum of photon number fluctuations of light mode $b$ with central frequency $\omega_0$ by

$$S_b(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{n}_b(t), \hat{n}_b(t+\tau) \rangle_{ss}, \tag{3.100}$$

where

$$\hat{n}_b(t) = \hat{b}^\dagger(t)\hat{b}(t), \tag{3.101}$$

$$\hat{n}_b(t+\tau) = \hat{b}^\dagger(t+\tau)\hat{b}(t+\tau). \tag{3.102}$$

Upon integrating both sides of Eq. (3.100) over $\omega$, we easily find

$$\int_{-\infty}^\infty S_b(\omega) d\omega = (\Delta n)^2_b, \tag{3.103}$$

in which $(\Delta n)^2_b$ is the steady-state photon number variance of the light mode $b$. We can then assert that $S_b(\omega)d\omega$ is the steady-state photon number variance of light mode $b$ in the frequency interval between $\omega$ and $\omega + d\omega$.

We now proceed to evaluate the two-time correlation function that appears in Eq. (3.103). Applying the relation given by Eq. (3.85), we see that

$$\langle \hat{n}_b(t), \hat{n}_b(t+\tau) \rangle = \langle \hat{n}_b(t)\hat{n}_b(t+\tau) \rangle - \langle \hat{n}_b(t) \rangle \langle \hat{n}_b(t+\tau) \rangle. \tag{3.104}$$

On account of Eqs. (3.101) and (3.102), we have

$$\begin{aligned}\langle\hat{n}_b(t)\hat{n}_b(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\langle\hat{b}^\dagger(t+\tau)\hat{b}(t+\tau)\rangle\\ &+ \langle\hat{b}(t)\hat{b}(t+\tau)\rangle\langle\hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle\\ &+ \langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle\langle\hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle. \qquad (3.105)\end{aligned}$$

Thus substitution of Eq. (3.104) into Eq. (3.105) results in

$$\begin{aligned}\langle\hat{n}_b(t),\hat{n}_b(t+\tau)\rangle &= \langle\hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle\langle\hat{b}(t)\hat{b}(t+\tau)\rangle\\ &+ \langle\hat{b}^\dagger(t)\hat{b}(t+\tau)\rangle\langle\hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle. \qquad (3.106)\end{aligned}$$

With the help of Eq. (3.50), one can readily obtain the following equations

$$\langle\hat{b}^\dagger(t)\hat{b}^\dagger(t+\tau)\rangle = \langle\hat{b}^{\dagger 2}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \qquad (3.107)$$

$$\langle\hat{b}(t)\hat{b}(t+\tau)\rangle = \langle\hat{b}^{2}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right], \qquad (3.108)$$

$$\langle\hat{b}(t)\hat{b}^\dagger(t+\tau)\rangle = \langle\hat{b}(t)\hat{b}^\dagger(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\mu\tau/2} - \frac{\mu}{\kappa-\mu}e^{-\kappa\tau/2}\right]. \qquad (3.109)$$

Hence on account of Eqs. (3.56), (3.107), (3.108), and (3.109), Eq. (3.106) can be put in the form

$$\begin{aligned}\langle\hat{n}_b(t)\hat{n}_b(t+\tau)\rangle &= \left[\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle\langle\hat{b}(t)\hat{b}^\dagger(t)\rangle + \langle\hat{b}^2(t)\rangle\langle\hat{b}^{\dagger 2}(t)\rangle\right]\\ &\times \left[\left(\frac{\mu}{\kappa-\mu}\right)^2 e^{-\kappa\tau} + \left(\frac{\kappa}{\kappa-\mu}\right)^2 e^{-\mu\tau} - \frac{2\kappa\mu}{(\kappa-\mu)^2}e^{-(\kappa+\mu)\tau/2}\right]. \qquad (3.110)\end{aligned}$$

This can be rewritten as

$$\langle\hat{n}_b(t)\hat{n}_b(t+\tau)\rangle_{ss} = \frac{(\Delta n)_b^2}{(\kappa-\mu)^2}\left[\mu^2 e^{-\kappa\tau} + \kappa^2 e^{-\mu\tau} - 2\kappa\mu e^{-(\kappa+\mu)\tau/2}\right], \qquad (3.111)$$

in which $(\Delta n)_b^2$ is the steady-state photon number variance of light mode $b$ given by Eq. (3.78). With the help of Eq. (3.111), the spectrum of photon number fluctuations can be put in the form

$$\begin{aligned}S_b(\omega) &= \frac{(\Delta n)_b^2}{\pi(\kappa-\mu)^2}Re\left[\mu^2\int_0^\infty d\tau e^{-[\kappa-i(\omega-\omega_0)]\tau}\right.\\ &+ \kappa^2\int_0^\infty d\tau e^{-[\mu-i(\omega-\omega_0)]\tau}\\ &- \left.2\kappa\mu\int_0^\infty d\tau e^{-[\frac{(\kappa+\mu)}{2}-i(\omega-\omega_0)]\tau}\right] \qquad (3.112)\end{aligned}$$

and carrying out the integration, we obtain

$$S_b(\omega) = \frac{(\Delta n)_b^2}{(\kappa-\mu)^2}\left[\frac{\mu^2\kappa/\pi}{\kappa^2+(\omega-\omega_0)^2} - \frac{2\kappa\mu(\kappa+\mu)/2\pi}{(\frac{\kappa+\mu}{2})^2+(\omega-\omega_0)^2} + \frac{\kappa^2\mu/\pi}{\mu^2+(\omega-\omega_0)^2}\right]. \qquad (3.113)$$

Now we realize that the photon number variance in the frequency interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$(\Delta n)^2_{b\pm\lambda} = \int_{-\lambda}^{\lambda} S_b(\omega')d\omega', \tag{3.114}$$

in which $\omega' = \omega - \omega_0$. Therefore, substitution of Eq. (3.113) into Eq. (3.114) leads to

$$(\Delta n)^2_{b\pm\lambda} = \frac{(\Delta n)^2_b}{\pi(\kappa-\mu)^2}\Bigg[\int_{-\lambda}^{\lambda}\frac{\mu^2\kappa d\omega'}{\kappa^2+\omega'^2} - \int_{-\lambda}^{\lambda}\frac{2\kappa\mu(\kappa+\mu/2)d\omega'}{(\frac{\kappa+\mu}{2})^2+\omega'^2} + \int_{-\lambda}^{\lambda}\frac{\kappa^2\mu d\omega'}{\mu^2+\omega'^2}\Bigg]. \tag{3.115}$$

Employing the relation given by Eq. (3.38), the local photon number variance of light mode $b$ produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir is found to be

$$(\Delta n)^2_{b\pm\lambda} = (\Delta n)^2_b z'_b(\lambda), \tag{3.116}$$

where $z'_b(\lambda)$ is given by

$$z'_b(\lambda) = \frac{2\mu^2/\pi}{(\mu-\kappa)^2}tan^{-1}\left(\frac{\lambda}{\kappa}\right) + \frac{2\kappa^2/\pi}{(\kappa-\mu)^2}tan^{-1}\left(\frac{\lambda}{\mu}\right) - \frac{4\kappa\mu/\pi}{(\kappa-\mu)^2}tan^{-1}\left(\frac{2\lambda}{\kappa+\mu}\right). \tag{3.117}$$

We see from Eq. (3.116) along with the plot $z'_b(\lambda)$ that $(\Delta n)^2_{b\pm\lambda}$ increases with $\lambda$ until

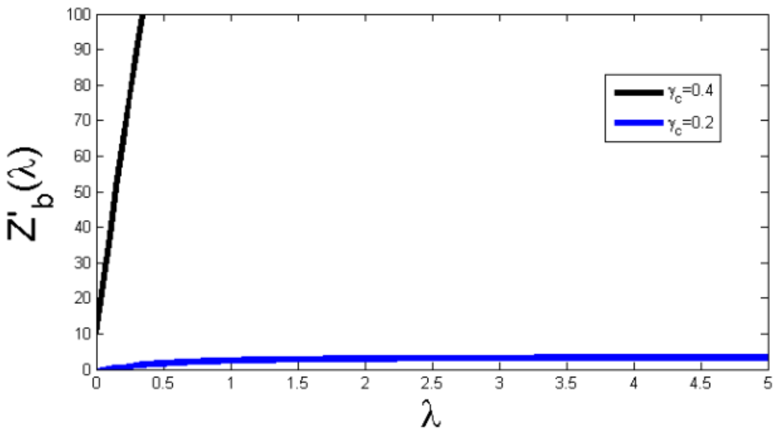


Figure 3.9: Plot of Local photon number variance of light mode b for the values of $\kappa$=0.8, and $\gamma_c$=0.4 and 0.2

it reaches the maximum value of the local photon number variance. From the plots in Fig. 3.9, we find the values indicated below:
We see from these results, $z'_b(\lambda)$ in the stimulated emission decay constant $(\gamma_c = 0.4)$ is greater than in the stimulated emission decay constant $(\gamma_c = 0.2)$. Moreover, using the above results of $z'_b(\lambda)$ and on account of Eq. (3.116), we have We therefore observe that a large part of the total variance of photon number is confined in a relatively small frequency interval.

| $\gamma_c$ | $z_b^{'}(0.5)$ | $z_b^{'}(1)$ | $z_b^{'}(2)$ |
|---|---|---|---|
| 0.2 | 1.5 | 2.4 | 2.9 |
| 0.4 | - | - | – |

Table 3.7: Values of $z_b^{'}(\lambda)$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

| $\gamma_c$ | $(\Delta n)^2_{b\pm 0.5}$ | $(\Delta n)^2_{b\pm 1}$ | $(\Delta n)^2_{b\pm 2}$ |
|---|---|---|---|
| 0.2 | 1.58 | 2.48 | 2.98 |
| 0.4 | – | – | – |

Table 3.8: Values of $(\Delta n)^2_{b\pm\lambda}$ for $\gamma_c = 0.4$, $\kappa = 0.8$, and $\Omega = 2$.

## 3.2 Two-mode photon statistics

In this section, applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we seek to obtain the mean and variance of the photon numbers for the two-mode light beam.

### 3.2.1 Two-mode mean photon number

Here we seek to calculate the steady-state mean photon number of the two-mode cavity light beam. The mean photon number of the two-mode light beam, represented by the operators $\hat{c}$ and $\hat{c}^\dagger$, is defined by

$$\bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle. \tag{3.118}$$

The steady-state solution of Eq. (2.71) is found to be

$$\hat{c} = \frac{2g}{\kappa}\sigma. \tag{3.119}$$

Hence at steady state the mean photon number goes over into

$$\bar{n} = \frac{\gamma_c}{\kappa}\left[\rho_{aa} + \rho_{bb}\right]. \tag{3.120}$$

We see from Eq. (3.120) that the mean photon number of the two-mode light beam is the sum of the mean photon numbers of the separate single-mode light beams given by Eqs. (3.6) and (3.14). Therefore, on account of Eqs. (2.47) and (2.48), Eq. (3.120) turns out to be

$$\bar{n} = \frac{\gamma_c}{\kappa}\left[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]. \tag{3.121}$$

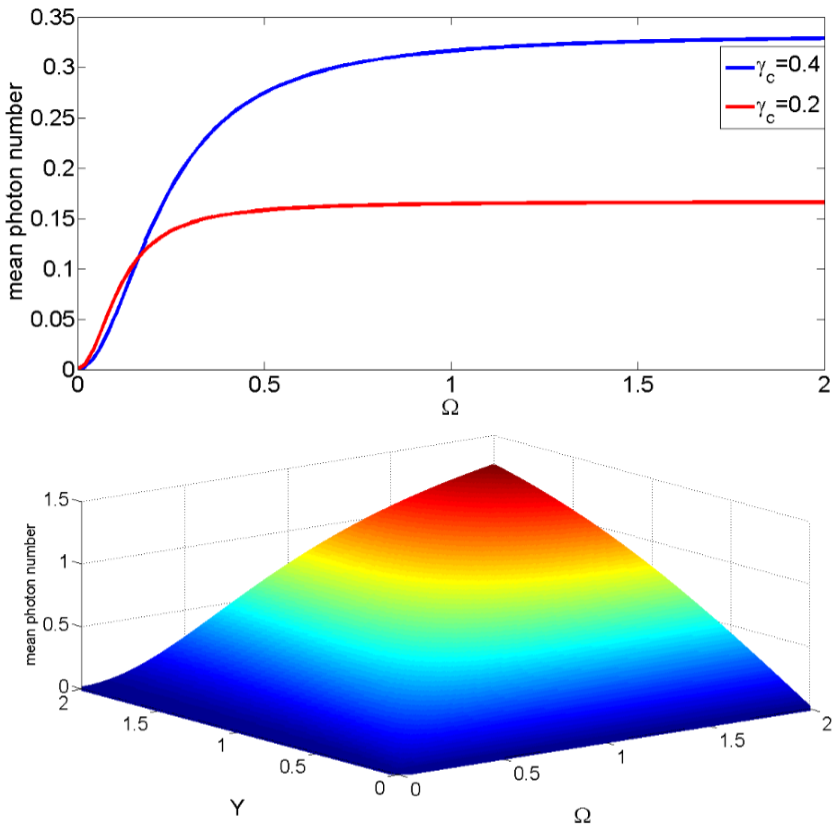


Figure 3.10: plot of mean photon number versus Ω in 2D and 3D

Figure 3.10 describes the mean photon number increases when the value of decay constant increases. This is the steady-state mean photon number of a two-mode cavity light produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. The result described by Eq. (3.121) is exactly the same as the one obtained by Fesseha [8]. Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.121) reduces to

$$\bar{n} = \frac{2\gamma_c}{3\kappa}. \tag{3.122}$$

We observe from the plots in Fig. 3.10 that the mean photon number of the two-mode light beam is greater when than when $\gamma c$=0.4, $\kappa$ =0.8 and $\Omega$=2.

### 3.2.2 Two-mode photon-number variance

Here we proceed to study the steady-state photon number variance of the two-mode light beam, produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. The photon number variance for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{3.123}$$

Since $\hat{c}$ is Gaussian variable with zero mean, the variance of the photon number can be written as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c} \hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle \langle \hat{c}^2 \rangle. \tag{3.124}$$

With the aid of Eq. (3.119), one can easily establish that

$$\langle \hat{c} \hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa} \big[ \rho_{bb} + \rho_{cc} \big], \tag{3.125}$$

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \rho_{ac}. \tag{3.126}$$

Since $\langle \hat{\sigma}_c \rangle$ is real, then $\langle \hat{c}^2 \rangle = \langle \hat{c}^{\dagger 2} \rangle$. Therefore, with the aid of Eqs. (3.120), (3.125) and (3.126), the variance of the photon number for the two-mode cavity light turns out to be

$$(\Delta n)^2 = \left( \frac{\gamma_c}{\kappa} \right)^2 \big[ \big( \rho_{aa} + \rho_{bb} \big) \big( \rho_{bb} + \rho_{cc} \big) + \rho_{ac}^2 \big] . \tag{3.127}$$

We observe from Eq. (3.127) that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beams given by Eqs. (3.69) and (3.76). Furthermore, upon substituting of Eqs. (2.46)-(2.49) into Eq. (3.127), the steady-state variance of the photon number goes over into

$$(\Delta n)^2 = \left( \frac{\gamma_c}{\kappa} \right)^2 \left[ \frac{4\Omega^4 + 3\Omega^2 \gamma_c^2}{(\gamma_c^2 + 3\Omega^2)^2} \right] . \tag{3.128}$$

Fig.3.11 shows that the photon number variance of two mode light versus $\Omega$.

This is the steady-state photon number variance of the two-mode light beam, produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. Furthermore, we note that for $\Omega \gg \gamma_c$, Eq. (3.128) reduces to

$$(\Delta n)^2 = \left[ \frac{2\gamma_c}{3\kappa} \right]^2 \tag{3.129}$$

and in view of Eq. (3.124), we have

$$(\Delta n)^2 = \bar{n}^2, \tag{3.130}$$

which represents the normally-ordered variance of the photon number for chaotic light.

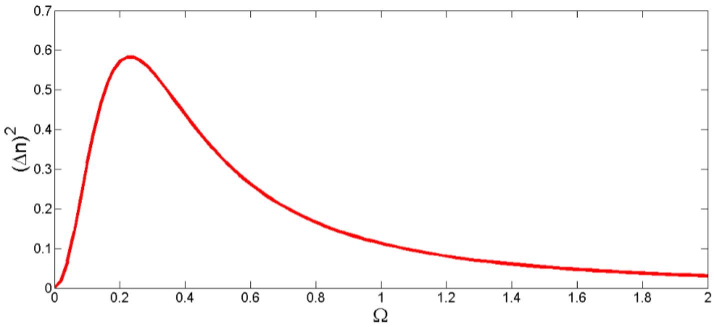


Figure 3.11: Plot of photon number variance of two mode light for the values of $\kappa$=0.8, $\gamma_c$=0.4 and 0.2

## 3.3 The normalized Photon number Correlation

In order to determine whether the photon numbers of mode $a$ and mode $b$ are correlated or not, we must examine the normalized photon numbers correlation. Thus the photon numbers correlation for light mode $a$ and light mode $b$ can be defined as

$$g^{(2)}_{(a,b)}(t) = \frac{\langle \hat{n}_a \hat{n}_b \rangle}{\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle}, \tag{3.131}$$

in which

$$\langle \hat{n}_a \rangle = \langle \hat{a}^\dagger \hat{a} \rangle, \tag{3.132}$$

$$\langle \hat{n}_b \rangle = \langle \hat{b}^\dagger \hat{b} \rangle, \tag{3.133}$$

$$\langle \hat{n}_a \hat{n}_b \rangle = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle \langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle \langle \hat{a} \hat{b}^\dagger \rangle. \tag{3.134}$$

It then follows that

$$g^{(2)}_{(a,b)}(t) = 1 + \frac{\langle \hat{a}^\dagger \hat{b}^\dagger \rangle \langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle \langle \hat{a} \hat{b}^\dagger \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}. \tag{3.135}$$

In view of Eqs. (3.2) and (3.10) along with their conjugates, we have

$$\langle \hat{n}_a \rangle = \frac{\gamma_c}{\kappa} \rho_{aa}, \tag{3.136}$$

$$\langle \hat{n}_b \rangle = \frac{\gamma_c}{\kappa} \rho_{bb}, \tag{3.137}$$

$$\langle \hat{n}_a \hat{n}_b \rangle = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\rho_{aa} \rho_{bb} + \rho_{ac}^2\right]. \tag{3.138}$$

Since $\hat{a}$ and $\hat{b}$ are Gaussian variables of zero mean, one can verify that

$$g^{(2)}_{(a,b)}(t) = 1 + \frac{\langle \hat{a}^\dagger \hat{b}^\dagger \rangle \langle \hat{a} \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle \langle \hat{a} \hat{b}^\dagger \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle}. \tag{3.139}$$

We realize that the operators in Eqs. (3.136)-(3.138) are in the normal order. Therefore, Eq. (3.139) can be expressed as

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\rho^2_{ac}}{\rho_{aa}\rho_{bb}}. \tag{3.140}$$

It then follows that

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\gamma^2_c}{\Omega^2}. \tag{3.141}$$

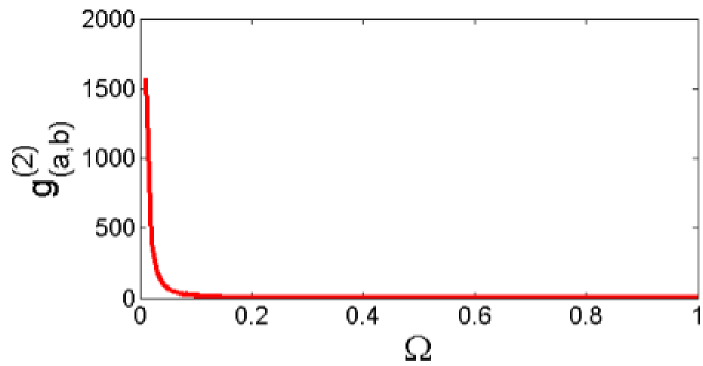


Figure 3.12: Plot of second order photon number correlation for the values of $\kappa$=0.8, $\gamma_c$=0.4 and 0.2

Fig. 3.12 indicates that the photon numbers correlation is different from one. Thus the photon numbers of mode $a$ and mode $b$ of a pair of a two-mode laser light beams are correlated. It can be that from this result the second-order correlation function of the two-mode light does not depend on the number of atoms. Moreover, we can see from Eq. (3.140) that the expectation value of the product of number operators, $\langle \hat{n}_a \hat{n}_b \rangle$ is different from $\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle$. This implies that there is an intermode correlation. Thus this intermode correlation must be due to the atomic coherence induced by the atoms in coherent coupling of the top and bottom levels. One can see from this figure that $g^{(2)}_{(a,b)}(0)$ decreses when $\Omega$ increases. It can be observed from the same figure that the second-order correlation function vanishes for $\Omega < 0.05$.

Now it is essential to calculate the second-order correlation function for the individual mode to have an insight for the previous result. To this end, the second order correlation function for mode $a$ is given by

$$g^{(2)}_{(a,a)}(0) = \frac{\langle : \hat{n}_a \hat{n}_a : \rangle}{\langle \hat{n}_a \rangle^2}, \tag{3.142}$$

where :: represent normal ordering and $\hat{n}_a = \hat{a}^\dagger \hat{a}$ is the photon number operator for mode $a$. Since $\hat{a}$ is a Gaussian variable with vanishing mean, one can easily verify that

$$g^{(2)}_{(a,a)}(0) = 2. \tag{3.143}$$

Similarly, the second-order correlation function for mode $b$ is found to be

$$g^{(2)}_{(b,b)}(0) = 2. \tag{3.144}$$

From the expressions 3.143 and (3.144), we note that the second-order correlation function for light in a vacuum state. So, the cavity modes $a$ and $b$ are separately in a vacuum state.

Furthermore, in order to quantify the correlation between the two modes, we introduce the linear correlation coefficient in terms of a covariance as [15]

$$J_{(\hat{n}_a,\hat{n}_b)} = \frac{cov(\hat{n}_a, \hat{n}_b)}{\sqrt{\Delta\hat{n}_a^2}\sqrt{\Delta\hat{n}_b^2}}, \tag{3.145}$$

where $\Delta\hat{n}_a^2$ and $\Delta\hat{n}_b^2$ are the variances of the photon number for modes $a$ and $b$, respectively. So, the covariance of the photon numbers is defined by

$$cov(\hat{n}_a, \hat{n}_b) = \langle\hat{n}_a\hat{n}_b\rangle - \langle\hat{n}_a\rangle\langle\hat{n}_b\rangle. \tag{3.146}$$

One can easily verify, using the fact that $\hat{a}$ and $\hat{b}$ are Gaussian variables, in the steady state that

$$cov(\hat{n}_a, \hat{n}_b) = \langle\hat{b}\hat{a}\rangle_{ss}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}. \tag{3.147}$$

Since the cavity modes are separately in a chaotic state the variances of the photon numbers obey the relation for a chaotic state,

$$\Delta\hat{n}_a^2 = \langle\hat{n}_a\rangle + \langle\hat{n}_a\rangle^2, \tag{3.148}$$

$$\Delta\hat{n}_b^2 = \langle\hat{n}_b\rangle + \langle\hat{n}_b\rangle^2. \tag{3.149}$$

On account of this fact and (3.147), the correlation function can be rewritten as

$$J_{(\hat{n}_a,\hat{n}_b)} = \frac{\langle\hat{b}\hat{a}\rangle_{ss}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}}{\sqrt{\langle\hat{n}_a\rangle_{ss} + \langle\hat{n}_a\rangle_{ss}^2}\sqrt{\langle\hat{n}_b\rangle_{ss} + \langle\hat{n}_b\rangle_{ss}^2}}. \tag{3.150}$$

It then follows that

$$J_{(\hat{n}_a,\hat{n}_b)} = \left(\frac{\gamma_c}{\kappa}\right)\left[\frac{\Omega^2\gamma_c^2}{\Omega^2\gamma_c^2 + 3\Omega^4 + \frac{\gamma_c}{\kappa}\Omega^4}\right]. \tag{3.151}$$

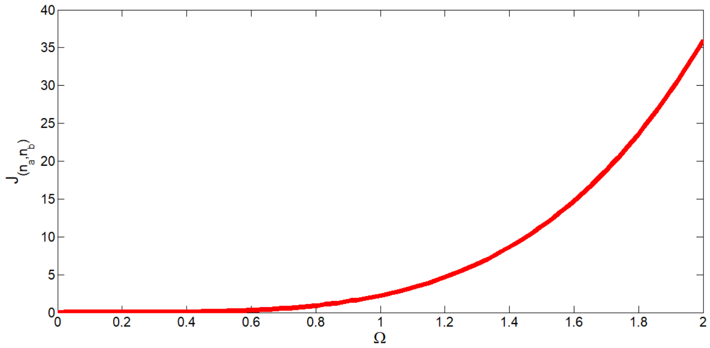


Figure 3.13: photon number correlation for the values of $\kappa$=0.8 and $\gamma_c$=0.4 and 0.2

In Figure 3.13, the linear correlation coefficient versus the amplitude of the driving coherent light, $\Omega$ is plotted. It is also found from this figure that for $\Omega$ very close to $0$ the inter-mode correlation would be significantly large, since the mean photon numbers of the light in modes $b$ is very close to zero when initially almost single atom is exists in the lower level. Moreover, similar to the second-order correlation function, the plots of Figure 3.13 show that the linear correlation coefficient vanishes when $\Omega < 0.05$.

## 3.4 Intensity difference Fluctuations

On the other hand, the variance of the intensity difference can be defined as

$$\Delta I_D^2 = \langle \hat{I}_D^2 \rangle - \langle \hat{I}_D \rangle^2, \tag{3.152}$$

where the difference of intensity is

$$\hat{I}_D = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}. \tag{3.153}$$

Hence making use of Eq. (3.153), it is possible to express

$$\langle I_D^2 \rangle = \langle \hat{a}^\dagger \hat{a} \rangle [1 + 2\langle \hat{a}^\dagger \hat{a} \rangle] + \langle \hat{b}^\dagger \hat{b} \rangle [1 + 2\langle \hat{b}^\dagger \hat{b} \rangle] - 2\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle - 2\langle \hat{a}\hat{b} \rangle^2, \tag{3.154}$$

$$\langle I_D \rangle^2 = \langle \hat{a}^\dagger \hat{a} \rangle^2 + \langle \hat{b}^\dagger \hat{b} \rangle^2 - 2\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle, \tag{3.155}$$

as a result the variance of the intensity difference can finally take the form

$$\Delta I_D^2 = \langle \hat{a}^\dagger \hat{a} \rangle [1 + \langle \hat{a}^\dagger \hat{a} \rangle] + \langle \hat{b}^\dagger \hat{b} \rangle [1 + \langle \hat{b}^\dagger \hat{b} \rangle] - 2\langle \hat{a}\hat{b} \rangle^2. \tag{3.156}$$

In view of Eqs. (3.2) and (3.11) along with their conjugates, we have

$$\Delta I_D^2 = \frac{\gamma_c}{\kappa} \big[ \rho_{aa} + \rho_{bb} + 2\rho_{aa}\rho_{bb} - 2\rho_{ac}^2 \big]. \tag{3.157}$$

Since $\rho_{aa} = \rho_{bb}$, we see that

$$\Delta I_D^2 = \frac{2\gamma_c}{\kappa}\left[\rho_{aa} + \rho_{aa}^2 - \rho_{ac}^2\right]. \tag{3.158}$$

On account of Eqs. (2.46) and (2.47), Eq. (3.159) can be rewritten as

$$\Delta I_D^2 = \frac{2\gamma_c}{\kappa}\left[\frac{\Omega^4 - 2\Omega^2\gamma_c^2}{\left[\gamma_c^2 + 3\Omega^2\right]^2}\right]. \tag{3.159}$$

This is the steady-state variance of the intensity difference produced by the dynamics of non-degenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. In addition, we note that for $\Omega \gg \gamma_c$, Eq. (3.159) reduces to

$$\Delta I_D^2 = \frac{2\gamma_c}{\kappa}\left[\Omega^2 + \frac{1}{9}\right]. \tag{3.160}$$

From the fig. 3.14 we understand that the variance of intensity increases when $\Omega$

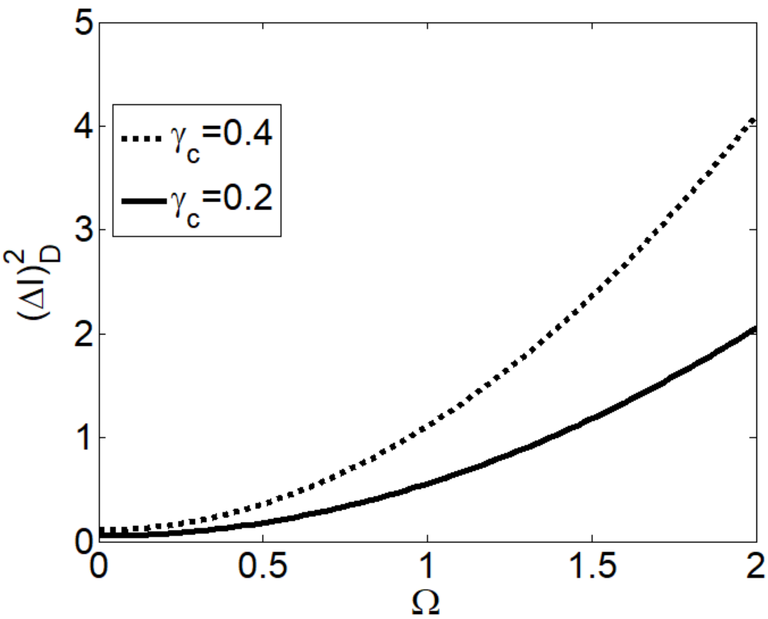


Figure 3.14: Plot of intensity difference fluctuation for the values of $\kappa$=0.8 and $\gamma_c$=0.4 and 0.2

increases. From these equation we understand that the variance of the intensity difference increases as $\Omega$ increases. On the other hand, we determined the Plots of variance of the intensity difference ($\Delta I_D^2$) of the two-mode cavity light at steady-state versus $\Omega$ for $\gamma_c = 0.4$, $\kappa = 0.8$. From these plots we see that variance of the intensity difference decreases as the initially seeded coherent light increases. And also the variance of the intensity difference increase with increasing the value of $\Omega$.

# 4

# Quadrature Squeezing

In this chapter we seek to study the quadrature variance and the quadrature squeezing of the light produced by the dynamics of non degenerate three-level laser in a closed cavity and coupled with a two-mode vacuum reservoir for single atom via a single-port mirror. Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global quadrature variances for light modes a and b. In addition, we determine the global quadrature squeezing of the two-mode cavity light.

## 4.1 Single-mode quadrature variance

The squeezing properties of a cavity mode a are described by two quadrature operators defined by

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger \tag{4.1}$$

And

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{4.2}$$

The operators $\hat{a}_-$ and $\hat{a}_+$ represents physical quantities called the plus and minus quadrature. Using Eqs. (4.1) and (4.2), one can write

$$[\hat{a}_-, \hat{a}_+] = [i(\hat{a}^\dagger - \hat{a}), \hat{a} + \hat{a}^\dagger], \tag{4.3}$$

with the aid of the identity

$$[\hat{A} + \hat{B}; \hat{C} + \hat{D}] = [\hat{A}; \hat{C}] + [\hat{A}; \hat{D}] + [\hat{B}; \hat{C}] + [\hat{B}; \hat{D}] \tag{4.4}$$

$$[\hat{a}_-, \hat{a}_+] = i(\langle|\hat{a}^\dagger, \hat{a}|\rangle + \langle[|\hat{a}^\dagger, \hat{a}^\dagger|]\rangle - \langle[\hat{a}, \hat{a}]\rangle - \langle[\hat{a}, \hat{a}^\dagger]\rangle), \tag{4.5}$$

so that in view of equation.(3.1) one can express this commutation relation in the form

$$[\hat{a}_-, \hat{a}_+] = i\frac{\gamma_c}{\kappa}(\langle[\hat{\sigma}_a^\dagger, \hat{\sigma}_a]\rangle + \langle[\hat{\sigma}_a^\dagger, \hat{\sigma}_a^\dagger]\rangle - \langle[\hat{\sigma}_a, \hat{\sigma}_a]\rangle - \langle[\hat{\sigma}_a, \langle\hat{\sigma}_a^\dagger]\rangle), \tag{4.6}$$

using the commutation relation

$$[\hat{\sigma}a, \hat{\sigma}_a^\dagger] = [|b\rangle\langle b|, |a\rangle\langle a|] \tag{4.7}$$

And

$$[\hat{\sigma}_a, \hat{\sigma}_a] = 0, \tag{4.8}$$

along with equation 4.6, we obtain

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}[\rho_{aa} - \rho_{bb}] \tag{4.9}$$

And taking in to account 2.43 one easily gets

$$[\hat{a}_-, \hat{a}_+] = 0 \tag{4.10}$$

Furthermore, we recall that if

$$[\hat{A}, \hat{B}] = i\hat{C} \tag{4.11}$$

Then

$$\Delta\hat{A}\Delta\hat{B} \geqslant \frac{1}{2}|\langle\hat{C}\rangle| \tag{4.12}$$

Thus on account of equation $(4.10)$ and $(4.12)$, the uncertainty relation for cavity mode a can be expressed as

$$\Delta\hat{a}_-\Delta\hat{a}_+ \geqslant 0 \tag{4.13}$$

The quadrature variance for mode a is defined by

$$(\Delta a_\pm)^2 = \langle\hat{a}_\pm^2\rangle - \langle\hat{a}\rangle^2 \tag{4.14}$$

In view of (4.1) and (4.2), the quadrature variance can be put in the form

$$(\Delta a_\pm)^2 = \pm(\langle\hat{a}^{\dagger 2}\rangle) + \langle\hat{a}\rangle^2 \pm \langle\hat{a}^\dagger\hat{a}\rangle \pm \langle\hat{a}\hat{a}^\dagger\rangle \pm 2\langle\hat{a}^{\dagger 2}\rangle\hat{a}\rangle) \tag{4.15}$$

we now proceed to determine the various expectation values involved in equation (4.15) then using eq. (3.2), we find

$$\langle\hat{a}\rangle = -2\frac{g}{k}\langle\hat{\sigma}_a\rangle \tag{4.16}$$

using the steady state solution of (2.58), we have

$$\langle\hat{\sigma}_a\rangle = 0 \tag{4.17}$$

And

$$\langle\hat{a}\rangle = 0 \tag{4.18}$$

we observe on the basis of equation (2.9) and (4.18) that $\hat{a}$ is a Gaussian variable with zero mean. in addition one can also write

$$\langle \hat{a}^{\dagger} \rangle \langle \hat{a} \rangle = \langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle = 0 \tag{4.19}$$

With the help of (3.23), we also find

$$\langle \hat{a}\hat{a}^{\dagger} \rangle = \frac{\gamma_c}{k} \langle \hat{\sigma_a} \hat{\sigma_a}^{\dagger} \rangle \tag{4.20}$$

hence on account of the fact that

$$\hat{\sigma_a} \hat{\sigma_a}^{\dagger} = |b\rangle\langle b| \tag{4.21}$$

we easily get

$$\langle \hat{a}\hat{a}^{\dagger} \rangle = \frac{\gamma_c}{k} \rho_{bb} \tag{4.22}$$

using equation (3.11) one can also write

$$\langle \hat{a}^2 \rangle = \frac{\gamma_c}{k} \langle \hat{\sigma_b} \hat{\sigma_b} \rangle \tag{4.23}$$

Now with the help of identity

$$\hat{\sigma_a} \hat{\sigma_a} = 0 \tag{4.24}$$

We have

$$\langle \hat{a}^2 \rangle = 0 \tag{4.25}$$

Applying equation (4.18), (4.19) and (4.25), we arrive at

$$(\Delta a_{\pm})^2 = \bar{n}_a + \frac{\gamma_c}{\kappa} \rho_{bb} \tag{4.26}$$

so that in view (3.14), the quadrature variance can be rewritten as

$$(\Delta a_{\pm})^2 = \bar{n}_a + \bar{n}_b \tag{4.27}$$

Finally with the help of eq.(3.11), the quadrature variance can be put in the form

$$(\Delta a_{+})^2 = (\Delta a_{-})^2 = 2\bar{n}_a \tag{4.28}$$

where $\bar{n}_a$ is given by equation (3.9).We then see that cavity mode a is in chaotic state.the squeezing property of cavity mode b are described by two quadrature operators defined by

$$\hat{b}_{+} = \hat{b} + \hat{b}^{\dagger} \tag{4.29}$$

And

$$\hat{b}_{-} = i(\hat{b} - \hat{b}^{\dagger}) \tag{4.30}$$

these operators are Hermitian and satisfy the common relation

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}(\rho_{bb} - \rho_{cc}) \tag{4.31}$$

it then follows that

$$\Delta b_+ \Delta b_- \geqslant \frac{\gamma_c}{\kappa}(\rho_{bb} - \rho_{aa}) \tag{4.32}$$

the quadrature variance for mode b is defined by

$$(\Delta b_\pm)^2 = \langle (b_\pm^2) \rangle - \langle (b_\pm)^2 \rangle \tag{4.33}$$

with the help of (4.29), the quadrature variance can be put in the form

$$(\Delta b_\pm)^2 = \pm(\langle \hat{b}^{\dagger 2} \rangle + \langle \hat{b}^2 \rangle \pm \langle \hat{b}\hat{b}^\dagger \rangle) \mp (\langle \hat{b}^\dagger \rangle^2 + \langle \hat{b} \rangle^2 \pm 2\langle \hat{b}^\dagger \rangle \langle \hat{b} \rangle) \tag{4.34}$$

Then, we find the variance expectation values involved in equation (3.34). using eq. (3.8), we have

$$\langle \hat{b} \rangle = -2\frac{g}{\kappa}\langle \sigma_b \rangle \tag{4.35}$$

now introducing eq.(2.54) in to eq. (4.35), we get

$$\langle \hat{b} \rangle = 0 \tag{4.36}$$

On the basis of equation (2.10) and (4.36), we also note that $\hat{b}$ is a Gaussian variable with zero mean. then we see that

$$\langle \hat{b}^\dagger \rangle \langle \hat{b} \rangle = 0 \tag{4.37}$$

Applying eq. (3.8),one obtains

$$\langle \hat{b}\hat{b}^\dagger \rangle = \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_b \hat{\sigma}_b^\dagger \rangle \tag{4.38}$$

with the aid of the identity

$$\hat{\sigma}_b \hat{\sigma}_b^\dagger = |c\rangle\langle c| \tag{4.39}$$

We have

$$\langle \hat{b}\hat{b}^\dagger \rangle = \frac{\gamma_c}{\kappa}\rho_{cc} \tag{4.40}$$

on account of (2.30), one can also find

$$\langle \hat{b}^2 \rangle = 0 \tag{4.41}$$

thus in view of equations (4.36),(4.37), (4.40) and (4.41) the quadrature variance can be written as

$$(\Delta b_+^2) = (\Delta b_-)^2 = \bar{n}_b + \frac{\gamma_c}{\kappa}\rho_{cc} \tag{4.42}$$

In which

$$\rho_{cc} = \frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2} \tag{4.43}$$

And we note that for $\gamma_c \ll \Omega$, this result can be put as

$$\rho_{cc} = \frac{1}{3} \tag{4.44}$$

on account of this, eq. (4.42) takes the form

$$(\Delta b_+)^2 = (\Delta b_-)^2 = 2(\bar{n}_b) \tag{4.45}$$

where

$$\bar{n}_b = \frac{\gamma_c}{3\kappa} \tag{4.46}$$

We note that cavity mode b is in a chaotic state for $\gamma_c << \Omega$, In addition, for $\gamma_c >> \Omega$ we have

$$(\Delta b_+)^2 = (\Delta b_-)^2 = \frac{\gamma_c}{\kappa} \tag{4.47}$$

And

$$\Delta b_+ \Delta b_- \geqslant \frac{\gamma_c}{\kappa} \tag{4.48}$$

Hence on the basis of this result,we conclude that cavity mode b is in a coherent state

## 4.2 Two-mode quadrature variance

In the previous section, we have considered the squeezing properties of modes a and b. We now extend our analysis to the superposed cavity modes. The squeezing properties of the superposed cavity modes is described by the quadrature operators defined by

$$\hat{C}_+ = \hat{a}_+ + \hat{b}_+ \tag{4.49}$$

And

$$\hat{C}_- = \hat{a}_- + \hat{b}_- \tag{4.50}$$

Where $\hat{a}_\pm$ and $\hat{b}_\pm$ are defined by 4.1,4.2, 4.29 and 4.30.the operators $\hat{c}_+$ and $\hat{c}_-$ are Hermitian. using eq. 4.49 and 4.50, one can write

$$[\hat{C}_-, \hat{C}_+] = \langle [\hat{a}_- + \hat{b}_-, \hat{a}_+ + \hat{b}_+] \rangle \tag{4.51}$$

In view of 4.4 this can be rewritten as

$$[\hat{C}_-, \hat{C}_+] = \langle [\hat{a}_-, \hat{a}_+] \rangle + \langle [\hat{a}_-, \hat{b}_+] \rangle + \langle [\hat{b}_-, \hat{a}_+] \rangle + \langle [\hat{b}_-, \hat{b}_+] \rangle \tag{4.52}$$

Thus with the aid of 4.10 and 4.31 we arrive at

$$[\hat{C}_-, \hat{C}_+] = 2i\frac{\gamma_c}{\kappa}(\rho_{bb} - \rho_{cc}) \tag{4.53}$$

it then follows that

$$\Delta c_+ \Delta c_- \geqslant \frac{\gamma_c}{\kappa} \frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2} \tag{4.54}$$

We note that for $\gamma_c >> \Omega$. eq.$4.54$ reduce to

$$\Delta c_+ \Delta c_- \geqslant \frac{\gamma_c}{\kappa} \tag{4.55}$$

The variance of quadrature operators is defined by

$$(\Delta C_\pm)^2 = \langle \hat{c}_\pm^2 \rangle - \langle \hat{c}_\pm^2 \rangle^2 \tag{4.56}$$

Now with the help of eq. 4.50 one can write eq. 4.56 as

$$(\Delta C_\pm)^2 = \langle \hat{a}_\pm^2 \rangle + \langle \hat{b}_\pm^2 \rangle + \langle \hat{a}_\pm \hat{b}_\pm \rangle - \langle \hat{a}_\pm \rangle^2 - \langle \hat{b}_\pm \rangle^2 - 2\langle \hat{b}_\pm \rangle \langle \hat{a}_\pm \rangle \tag{4.57}$$

then we determine the various expectation values involved in eq. 4.57. using eq. 4.1, 4.2, 4.29 and 4.30, we find

$$\langle \hat{b}_\pm \hat{a}_\pm \rangle = \pm \langle \hat{b}^\dagger \hat{a}^\dagger \rangle + \langle \hat{b} \hat{a}^\dagger \rangle + \langle \hat{b}^\dagger \hat{a} \rangle \pm \langle \hat{b} \hat{a} \rangle \tag{4.58}$$

So that in view of (3.2) and (3.10), this can be rewritten as

$$\langle \hat{b}_\pm \hat{a}_\pm \rangle = \pm \frac{\gamma_c}{\kappa} \langle \hat{\sigma}_b^\dagger \hat{\sigma}_a^\dagger \rangle + \frac{\gamma_c}{\kappa} \langle \hat{\sigma}_b \hat{\sigma}_a^\dagger \rangle + \frac{\gamma_c}{\kappa} \langle \hat{\sigma}_b^\dagger \hat{\sigma}_a \rangle \pm \frac{\gamma_c}{\kappa} \langle \hat{\sigma}_b \hat{\sigma}_a \rangle \tag{4.59}$$

And on account of the fact that

$$\hat{\sigma}_b^\dagger \hat{\sigma}_a^\dagger = 0 \tag{4.60}$$

$$\hat{\sigma}_b \hat{\sigma}_a^\dagger = 0 \tag{4.61}$$

$$\hat{\sigma}_b^\dagger \hat{\sigma}_a = 0 \tag{4.62}$$

$$\hat{\sigma}_b \hat{\sigma}_a = 0 \tag{4.63}$$

along with eq.(4.12), we have

$$\langle \hat{b}_\pm \hat{a}_\pm \rangle = \pm \frac{\gamma_c}{\kappa} \rho_{ac} \tag{4.64}$$

In which

$$\rho_{ac} = \frac{\gamma_c \Omega}{\gamma_c^2 + 3\Omega_2} \tag{4.65}$$

Similarly

$$\langle \hat{a}_\pm \hat{b}_\pm \rangle = \pm \frac{\gamma_c}{\kappa} \rho_{ca} \tag{4.66}$$

So that on account of the fact that $\rho_{ac} = \rho_{ca}$ this can be rewritten as

$$\langle \hat{a}_\pm \hat{b}_\pm \rangle = \pm \frac{\gamma_c}{\kappa} \rho_{ca} \tag{4.67}$$

Now with the aid of equation (4.18) and (4.19) in to eq. (4.67) there follows

$$\langle \hat{a}_{\pm} \rangle^2 = 0 \tag{4.68}$$

on the other hand,with the aid of (4.29) and (4.30), we get

$$\langle \hat{b}_{\pm} \rangle^2 = \pm \langle \hat{b}^{\dagger} \rangle^2 + 2\langle \hat{b}^{\dagger} \rangle \langle \hat{b}^{\dagger} \rangle \tag{4.69}$$

hence on account of (4.36) and (4.37),we have

$$\langle \hat{b}_{\pm} \rangle^2 = 0 \tag{4.70}$$

Then, we can also see that

$$\langle \hat{a}_{\pm} \rangle \langle \hat{b}_{\pm} \rangle = \langle \hat{b}_{\pm} \rangle \langle \hat{a}_{\pm} \rangle = 0 \tag{4.71}$$

using eq.(4.1) and (4.2), we find

$$\langle \hat{a}_{\pm}^2 \rangle = \pm \langle \hat{a}^{\dagger 2} \rangle \pm \langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{a} \hat{a}^{\dagger} \rangle \tag{4.72}$$

We recall that

$$\langle \hat{a}^2 \rangle = 0 \tag{4.73}$$

$$\langle \hat{a} \hat{a}^{\dagger} \rangle = \frac{\gamma_c}{\kappa} \rho_{bb} \tag{4.74}$$

and

$$\langle \hat{a}^{\dagger} \hat{a} \rangle = \frac{\gamma_c}{\kappa} \rho_{aa} \tag{4.75}$$

so that in view of this result eq.(4.26),takes the form

$$\langle \hat{a}_{\pm}^2 \rangle = \frac{\gamma_c}{\kappa} \rho_{aa} + \frac{\gamma_c}{\kappa} \rho_{bb} \tag{4.76}$$

Applying eq.(3.6) and (3.14), one get

$$\langle \hat{a}_{\pm}^2 \rangle = \bar{n}_a + \bar{n}_b \tag{4.77}$$

since $\bar{n}_a = \bar{n}_b$ ,then equation (4.77) reduces to

$$\langle \hat{a}_{\pm}^2 \rangle = 2\bar{n}_a \tag{4.78}$$

on the other hand, with the aid of (4.29) and (4.30),we arrive at

$$\langle \hat{b}_{\pm}^2 \rangle = \pm \langle \hat{b}^{\dagger 2} \rangle \pm \langle \hat{b}^2 \rangle + \langle \hat{b}^{\dagger} \hat{b} \rangle + \langle \hat{b} \hat{b}^{\dagger} \rangle \tag{4.79}$$

Hence in view of (3.7), (4.36), and (4.40),(4.79) takes the form

$$\langle \hat{b}_{\pm}^2 \rangle = \bar{n}_b + \frac{\gamma_c}{\kappa} \rho_{cc} \tag{4.80}$$

And for $\gamma_c << \Omega$,this result can be rewritten as

$$\langle\hat{b}_{\pm}^2\rangle = 2\bar{n}_b \tag{4.81}$$

With $\bar{n}_b$ is given by eq,(4.46). Finally applying eq.(4.65),(4.68),(4.70),(4.72),(4.73),(4.78) and (4.80) in to eq.(4.58) the quadrature variance for the superposed cavity modes can be expressed as

$$(\Delta C_{\pm})^2 = 2\bar{n}_a + \bar{n}_b + \frac{\gamma_c}{\kappa}\rho_{cc} \pm 2\frac{\gamma_c}{\kappa}\rho_{ac} \tag{4.82}$$

so that employing eq.(4.8) in to eq. (4.82). We readily obtained the quadrature variance for the superposed cavity modes

$$(\Delta C_{\pm})^2 = 3\bar{n}_a + \frac{\gamma_c}{\kappa}\rho_{cc} \pm 2\frac{\gamma_c}{\kappa}\rho_{ac} \tag{4.83}$$

And taking in to account of eq. (4.7) we get

$$(\Delta C_{\pm})^2 = 3\frac{\gamma_c}{\kappa}\rho_{aa} + \frac{\gamma_c}{\kappa} \pm 2\frac{\gamma_c}{\kappa}\rho_{ac} \tag{4.84}$$

therefore, the quadrature variance of the superposed cavity modes can be put into the form

$$(\Delta C_{+})^2 = 3\frac{\gamma_c}{\kappa}\rho_{aa} + \frac{\gamma_c}{\kappa}\rho_{cc} + 2\frac{\gamma_c}{\kappa}\rho_{ac} \tag{4.85}$$

And

$$(\Delta C_{-})^2 = 3\frac{\gamma_c}{\kappa}\rho_{aa} + \frac{\gamma_c}{\kappa}\rho_{cc} - 2\frac{\gamma_c}{\kappa}\rho_{ac} \tag{4.86}$$

Now substitution of equations (2.47),(4.43) and (4.66) in to equations (4.85) and (4.86), yields

$$(\Delta C_{+})^2 = \frac{\gamma_c}{\kappa}(\frac{4\Omega^2 + \gamma_c^2 + 2\gamma_c\Omega}{\gamma_c^2 + 3\Omega^2}) \tag{4.87}$$

And

$$(\Delta C_{-})^2 = \frac{\gamma_c}{\kappa}(\frac{4\Omega^2 + \gamma_c^2 - 2\gamma_c\Omega}{\gamma_c^2 + 3\Omega^2}) \tag{4.88}$$

Then dividing both the numerator and denominator by $\gamma_c^2$, one can easily obtains

$$(\Delta C_{+})^2 = \frac{\gamma_c}{\kappa}(\frac{1 + 4\eta^2 + 2\eta}{1 + 3\eta^2}) \tag{4.89}$$

$$(\Delta C_{-})^2 = \frac{\gamma_c}{\kappa}(\frac{1 + 4\eta^2 - 2\eta}{1 + 3\eta^2}) \tag{4.90}$$

Where

$$\eta = \frac{\Omega}{\gamma_c} \tag{4.91}$$

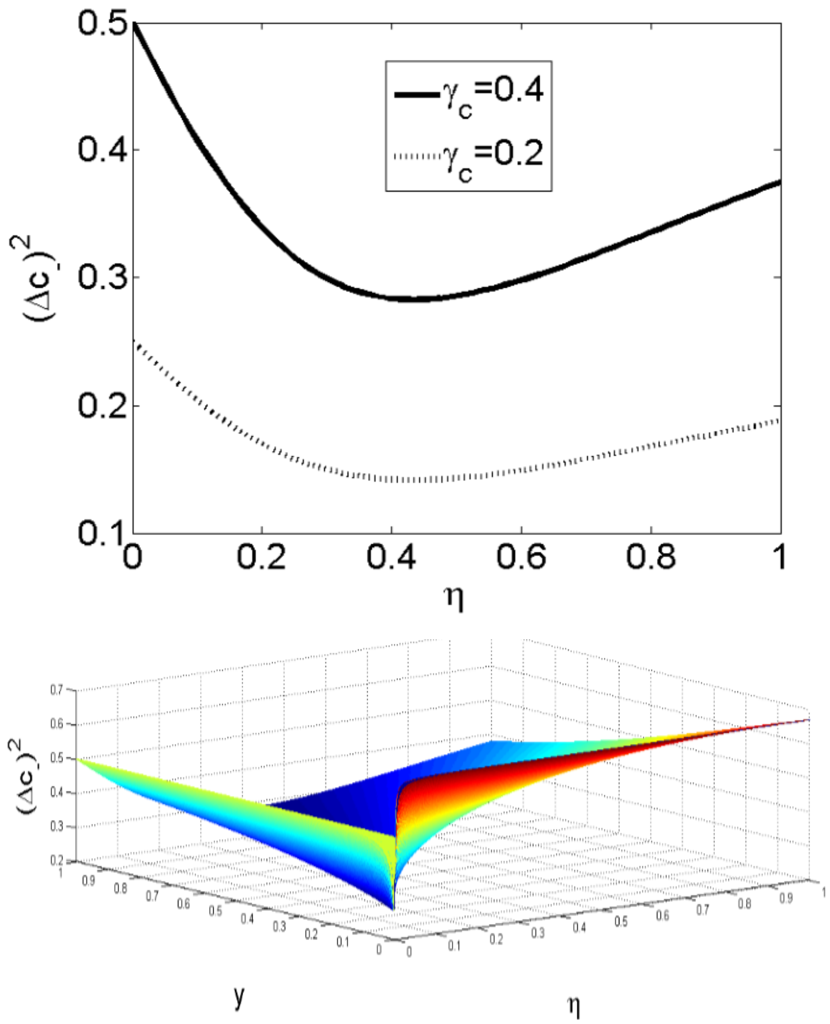


Figure 4.1: Plot of minus quadrature in 2D and 3D for the values of $\kappa$=0.8 and $\gamma_c$=0.4, 0.2

The above graph shows that minus quadrature has the minimum value at $\eta$ =$0.4343$ which is $0.2829$ for $\eta \ll 1$ equation (4.89) and (4.90) reduce to

$$(\Delta C_+)^2 = (\Delta C_-)^2 = \frac{\gamma_c}{\kappa} \tag{4.92}$$

On the basis of this result and equation (4.56) the superposed cavity modes are in a coherent state. the two cavity modes are said to be in a squeezed state if either $\Delta c_+ < \frac{\gamma_c}{\kappa}$ or $\Delta c_- < \frac{\gamma_c}{\kappa}$ such that the uncertainty relation $\Delta C_+ \Delta C_- \geqslant \frac{\gamma_c}{\kappa}$ is not violated. Fig 4.1 clearly indicates that the superposed cavity modes are in squeezed state for all values of $\eta$ between 0 and 1 and the squeezing occurs in the minus quadrature. We next proceed to obtain the quadrature squeezing of the superposed cavity modes relative to the quadrature variance of the superposed coherent light. We define the quadrature squeezing of the superposed modes by

$$S = \frac{(\frac{\gamma_c}{\kappa} - \Delta C_-{}^2)}{\frac{\gamma_c}{\kappa}} \tag{4.93}$$

Now employing eq. 4.90, one can express eq.4.93 in the form

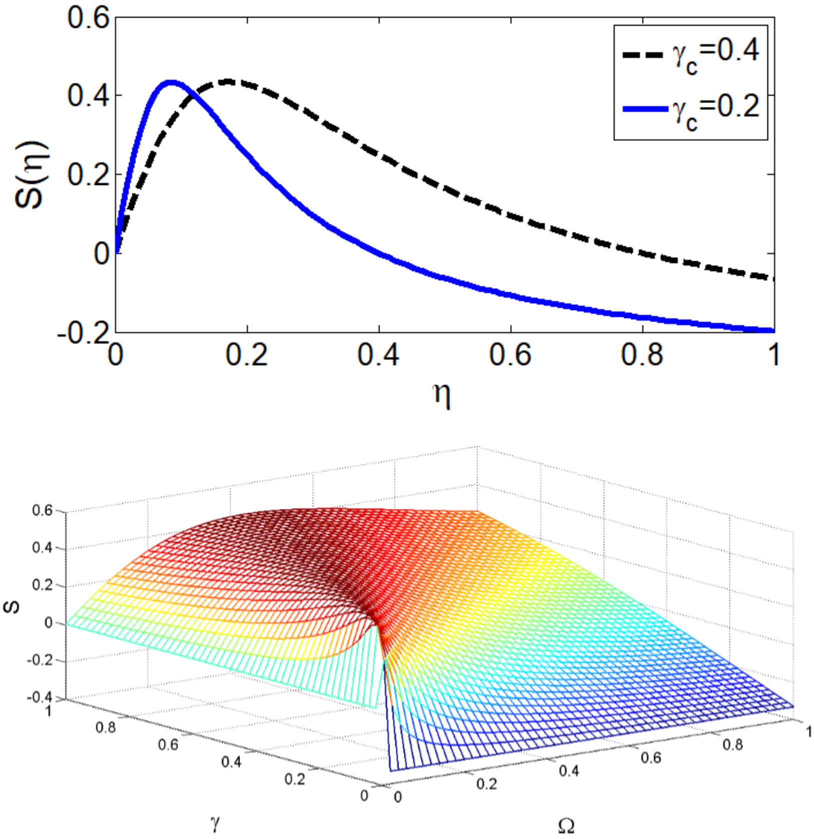


Figure 4.2: plot of quadrature squeezing in 2D and 3D for the values of $\kappa$=0.8 and $\gamma_c$=0.4, 0.2

$$S = 1 - (\frac{1 + 4\eta^2 + 2\eta}{1 + 3\eta^2}) \tag{4.94}$$

Then

$$S = \frac{2\eta - \eta^2}{1 + 3\eta^2} \tag{4.95}$$

We clearly see from Figure 4.2: Plot of the quadrature squeezing $s(\eta)$ versus $\eta$ .The maximum quadrature squeezing is found to be $43.43\%$ when $\gamma_c = 0.4$ and $\gamma_c = 0.2$ at $\eta$=$0.2829$ and $\eta$=$0.1414$ respectively below the vacuum-state level. Hence we observe that the degree of squeezing for the superposed cavity modes increases in the interval between 0 and 0.2829 and it decreases in the intervals between 0.2829 and 1.

# 5

# Entanglement properties of the two-mode light

Quantum entanglement is the term given to the phenomena, whereby particles can be generated or interact in ways such that the quantum state of each particle cannot be described independently. In such cases, the system of particles is said to be entangled, and itis not proper to consider any of the individual particles in isolation from the others, but only as a single entangled state. Moreover, the entanglement is one of the most counter-intuitive aspects of the quantum world and an enigmatic powerful property. The generation and manipulation of the entanglement have attracted a great interest owing to their wide applications in quantum teleportation [28], quantum dense coding [29], quantum computation [30], quantum error correction [31], and quantum cryptography [32].

In this chapter we seek to study the photon entanglement as well as atom entanglement of a two-mode laser light beams produced by the dynamics of driven non-degenerate three-level lasers with in a closed cavities and coupled to the two-mode vacuum reservoirs via single-port mirrors. Applying the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the entanglement of the two-mode light beams.

## 5.1 Photon Entanglement

Here, we prefer to analyze the entanglement of photon-states in the laser cavity. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, polarization, etc. performed on entangled particles are found to be appropriately correlated. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have non-classical and non-local properties lead to a better understanding of the basic quantum principles. It is in this spirit that this section is devoted to the analysis of the entanglement of the two-mode photon states. In other words, it is a well-known fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product density operators of the constituents,

$$\hat{\rho} \neq \sum_k p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}, \tag{5.1}$$

in which $p_k \gg 0$ and $\sum_k p_k = 1$ to verify the normalization of the combined density states. On the other hand, a maximally entangled CV state can be expressed as a coeigen state of a pair of EPR-type operators[33] such as $\hat{x}_a - \hat{x}_b$ and $\hat{P}_a - \hat{P}_b$.The total variance of these two operators reduces to zero for maximally entangled CV states. According to the inseparable criteria given by Duan et al [34], cavity photon-states of a system are entangled, if the sum of the variance of a pair of EPR-like operators,

$$\hat{s} = \hat{x}_a - \hat{x}_b, \tag{5.2}$$

$$\hat{t} = \hat{p}_a + \hat{p}_b, \tag{5.3}$$

where

$$\hat{x}_a = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right), \tag{5.4}$$

$$\hat{x}_b = \frac{1}{\sqrt{2}}\left(\hat{b} + \hat{b}^\dagger\right), \tag{5.5}$$

$$\hat{p}_a = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right), \tag{5.6}$$

$$\hat{p}_b = \frac{i}{\sqrt{2}}\left(\hat{b}^\dagger - \hat{b}\right), \tag{5.7}$$

are quadrature operators for modes $a$ and $b$, satisfy

$$\Delta s^2 + \Delta t^2 < 2 \tag{5.8}$$

and recalling the cavity mode operators $\hat{\sigma_a}$ and $\hat{\sigma_b}$ are Gaussian variables with zero mean, we readily get

$$\begin{aligned}\Delta s^2 + \Delta t^2 = &\left[\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}\hat{b}^\dagger\rangle\right] \\ &-\left[\langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{b}\hat{a}\rangle + \langle\hat{b}^\dagger\hat{a}^\dagger\rangle\right].\end{aligned} \tag{5.9}$$

Thus with the aid of Eqs. (3.2) and (3.11), we see that

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa}\left[N + \langle\hat{\rho}_b\rangle - 2\langle\hat{\sigma}_c\rangle\right]. \tag{5.10}$$

It then follows that

$$\Delta s^2 + \Delta t^2 = 2\Delta c_-^2. \tag{5.11}$$

where $\Delta c_-^2$ is given by (4.90). One can readily see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It is noted that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing as given by (4.95). It also follows that like the mean photon number and quadrature variance the degree of entanglement depends on the number of atom. With the help of the criterion (5.8) that a significant entanglement between the states of the light generated in the cavity. This is due to the strong correlation between the radiation emitted when the atoms decay from the upper energy level to the lower via the intermediate level.

On account of Eqs. (4.86) and (4.88), the photon entanglement of the two-mode cavity light takes, at steady-state, the form

$$\Delta s^2 + \Delta t^2 = \left(\frac{2\gamma_c}{\kappa}N\right)\left[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right]. \tag{5.12}$$

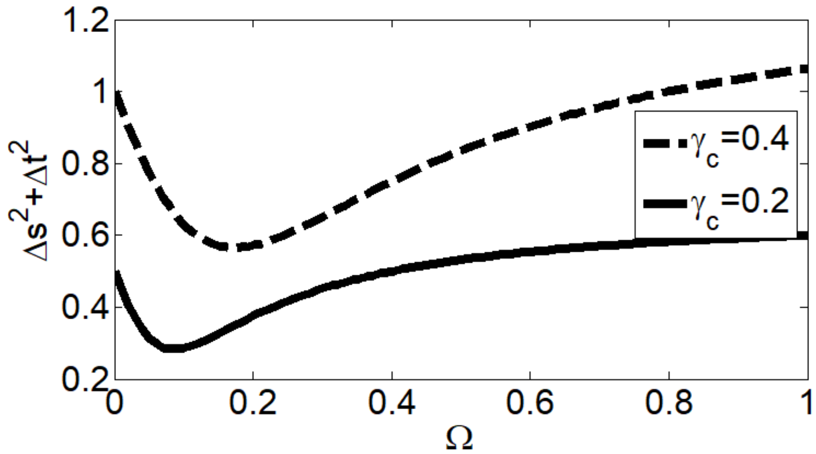


Figure 5.1: Plot of photon entanglement for the values of $\kappa$=0.8 and $\gamma_c$=0.4, 0.2

Fig. 5.1 Describes the steady-state photon entanglement of a two-mode cavity light produced by the dynamics of non-degenerate three-level laser with in a closed cavity and coupled to a two-mode vacuum reservoir.

In addition, we note that for $\Omega \gg \gamma_c$, Eq. (5.12) reduces to

$$\Delta s^2 + \Delta t^2 = \frac{8\gamma_c}{3\kappa}N. \tag{5.13}$$

This can be rewritten as

$$\Delta s^2 + \Delta t^2 = 4\bar{n}, \tag{5.14}$$

where $\bar{n}$ is given by Eq. (3.121). From the plots of Fig. (5.1) along with Eq. (5.12), we have

| $\gamma_c$ | $\Delta s^2 + \Delta t^2$ | $\Omega$ |
|---|---|---|
| $0.2$ | $0.2832$ | $0.09091$ |
| $0.2$ | $0.598$ | $1$ |
| $0.4$ | $0.5658$ | $0.1717$ |
| $0.4$ | $1.061$ | $1$ |

Table 5.1: Values of $\Delta s^2 + \Delta t^2$ for $\gamma_c = 0.4$,and $\gamma_c = 0.2$ $\kappa = 0.8$, and $N = 1$.

When we see the plots on Fig. (5.1) that as the stimulated emission decay constant increases the photon entanglement also increases. Similarly, as we observe from the data on Table 5.1, the photon entanglement is increases with increasing of the stimulated emission decay constant. From these plots and values of $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma_c = 0.2$, and $N = 1$, we determined the maximum photon entanglement is $72\%$ and it occurs at $\Omega = 0.1717$, $\Omega = 1$, and for $\gamma_c = 0.4$.

## 5.2 Cavity Atomic-States Entanglement

The quantum entanglement between the two cavity modes $a$ and $b$ proposed by Duan-Giedke-Cirac-Zoller (DGCZ) [35], which is a sufficient condition for entangled quantum states. According to DGCZ, a quantum state of a system is said to be entangled if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality

$$\Delta u^2 + \Delta v^2 < 2. \tag{5.15}$$

On the other hand, cavity atomic-states of a system are entangled, if the sum of the variance of a pair of EPR-like operators,

$$\hat{u} = \hat{x}'_a - \hat{x}'_b, \tag{5.16}$$

$$\hat{v} = \hat{p}'_a + \hat{p}'_b, \tag{5.17}$$

where

$$\hat{x}'_a = \frac{1}{\sqrt{2}}\left(\hat{\sigma}_a + \hat{\sigma}_a^\dagger\right), \tag{5.18}$$

$$\hat{x}'_b = \frac{1}{\sqrt{2}}\left(\hat{\sigma}_b + \hat{\sigma}_b^\dagger\right), \tag{5.19}$$

$$\hat{p}'_a = \frac{i}{\sqrt{2}}\left(\hat{\sigma}_a^\dagger - \hat{\sigma}_a\right), \tag{5.20}$$

$$\hat{p}'_b = \frac{i}{\sqrt{2}}\left(\hat{\sigma}_b^\dagger - \hat{\sigma}_b\right), \tag{5.21}$$

Since $\hat{\sigma}_a$ and $\hat{\sigma}_b$ are Gaussian variables with zero means, so one can easily verify that

$$\Delta u^2 + \Delta v^2 = \left[\langle\hat{\sigma}_a^\dagger\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a\hat{\sigma}_a^\dagger\rangle + \langle\hat{\sigma}_b^\dagger\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b\hat{\sigma}_b^\dagger\rangle - \langle\hat{\sigma}_b^\dagger\hat{\sigma}_a^\dagger\rangle - \langle\hat{\sigma}_a\hat{\sigma}_b\rangle\right]. \tag{5.22}$$

Now with the aid of (2.70) and (2.71), Eq. (5.22) takes the form

$$\Delta u^2 + \Delta v^2 = N\big[N + \langle\hat{\rho}_a\rangle - 2\langle\hat{\sigma}_c\rangle\big]. \tag{5.23}$$

On account of Eqs. (4.86) and (4.88), the cavity atomic-states entanglement of the two-mode cavity light takes, at steady-state, the form

$$\Delta u^2 + \Delta v^2 = \left[\frac{\gamma_c^2 + 4\Omega^2 - 2\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right]N^2. \tag{5.24}$$

This is the steady-state the cavity atomic-states entanglement of a two-mode cavity light produced by the dynamics of non-degenerate three-level laser with in a closed cavity and coupled to a two-mode vacuum reservoir.

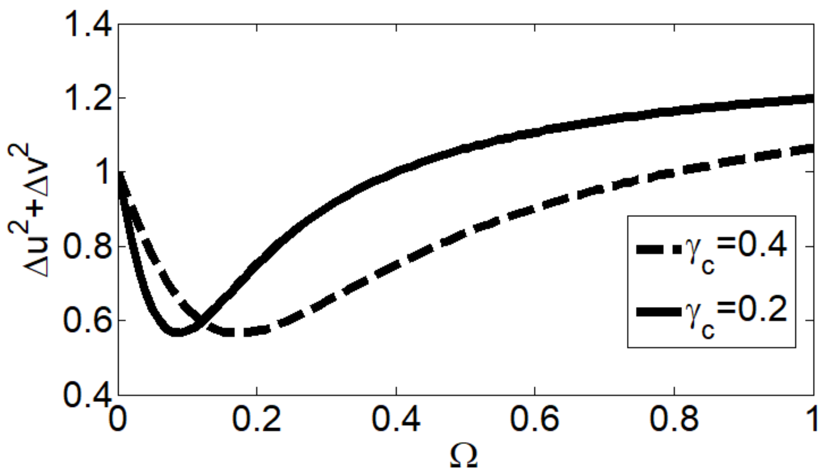


Figure 5.2: Plot of atom entanglement

From this plot and values of $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma_c = 0.2$, and $N = 1$, we determined the maximum atom entanglement is $72\%$ and it occurs at $\Omega = 0.1818$, $\Omega = 1$, and for $\gamma_c = 0.2$. From the plots of Fig. (5.2) along with Eq. (5.24), we have

| $\gamma_c$ | $\Delta u^2 + \Delta v^2$ | $\Omega$ |
|---|---|---|
| $0.2$ | $0.5672$ | $0.0808$ |
| $0.2$ | $1.197$ | $1$ |
| $0.4$ | $0.5658$ | $0.1818$ |
| $0.4$ | $1.063$ | $1$ |

Table 5.2: Values of $\Delta u^2 + \Delta v^2$ for $\gamma_c = 0.4$, $\gamma_c = 0.2$ $\kappa = 0.8$, and $N = 1$.

When we see the stimulated emission decay constant increases the atom entanglement decreases. Similarly, as we observe from the data on Table 5.2, the atom entanglement increased with decreasing of the stimulated emission decay constant($\gamma_c$).

We note that for $\Omega \gg \gamma_c$, Eq. (5.24) reduces to

$$\Delta u^2 + \Delta v^2 = \frac{4}{3}N^2. \tag{5.25}$$

Furthermore, when $\Omega = 0$, Eq. (5.24) also turns out to be

$$\Delta u^2 + \Delta v^2 = N^2. \tag{5.26}$$

On the basis of the criteria (5.8) and (5.15), we clearly see that the two states of the generated light are strongly entangled at steady-state. Moreover, the system is closed vacuum reservoir the generated light leads to an increase in the degree of entanglement.

# 6

# Conclusion

In this thesis we have studied the squeezing and statistical properties of the light produced by the dynamics of non-degenerate three-level laser in a closed cavity and coupled with two-mode vacuum reservoir for single atom via a single-port mirror. We have carried out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators,we have found that mean and variance of photon number as well as quadrature squeezing.

We have found that the global mean photon number of light mode a and b are equal. We have seen that the mean and variance of the photon numbers of light modes a and b in the interval between $\omega = \omega_0 - \lambda$ and $\omega = \omega_0 + \lambda$ increase with $\lambda$ until it reach the maximum values of the global mean and variance of the photon numbers of light modes a and b. Our results show that a large part of the total mean and variance of the photon numbers are confined in a relatively small frequency interval. Moreover, we have shown that the mean photon number of the two-mode light beam is the sum of mean photon numbers of the separate single-mode light beam. However, we have observed that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beam. And also the light generated by the three-level laser is in a squeezed state and the squeezing occurs in the minus quadrature. It is found to be $43.43\%$ when $\gamma_c = 0.4$ and $\gamma_c = 0.2$ at $\eta$=$0.2829$ and $\eta$=$0.1414$ respectively below the vacuum-state level. Unlike the mean photon number and the quadrature variance, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode light beam is independent of the number of photons. It is found that the photon entanglement of a two-mode light cavity increases with increasing in stimulated emission.

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