Existence of Positive Solutions for Fourth Order Boundary Value Problems

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS.

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August, 2020
Jimma, Ethiopia

Declaration

I, the undersigned declare that, the thesis entitled" Existence of positive solution for fourth order two point boundary value problems" is original and it has not been submitted to any institution else where for the a ward of any degree or like, where other sources of information that have been used, they have been acknowledged
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Acknowledgment

Frist of all I would like to thank the Almighty God who give me health and helped me in different ways to reach this time. Next my special heartfelt thanks to my advisors Dr. Wesen Legese and Mr. Kefyalew Haile for their unreserved support, advise and guidance through the preparation of this thesis and lastly, I would like to thank both Functional Analysis and Differential equation streams and my lovely post graduate students for their constructive comments and provision of some refernces while I was preparing this thesis.

Abstarct

In this thesis, we consider the fourth order two point boundary value problems. It also focused on constructing Green's function for corresponding homogeneous equation by using its property and the main objective of this study was to determine existence of three positive solution by the application of Leggett- William fixed point theorem and we provided examples to demonstrate for the applicability of our main result.

Contents

Chapter 1

Introduction

1.1 Background of the study

The main tools employed are two well known fixed point theorems for operators acting on cones in a Banach space. Fixed point theory has found itself at the center of study of boundary value problems. Many fixed point theorems have provided criteria for the existence of positive solutions or multiple positive solutions of boundary value problems. The use of cone theoretical techniques in the study of solutions to boundary value problems has a rich and diverse history.

Some authors have used fixed point theorems to show the existence of positive solutions to boundary value problems for ordinary differential equation. Major industries like auto mobile, chemical, petroleum, electronics, and a communication as well as emerging technology like biotechnology on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high technological products and more importantly a new inequality will be obtained for an associated Green's function.

Hence we shall exhibit sufficient conditions for the existence of solutions for a family of fourth order boundary value problem. The technique that characterizes Leggett Williams type fixed point theorems on the boundary of sets and role of fixed point index theory and the properties of concave and convex functional in the original work of Leggett Williams the subset can be thought of as the set of all elements of the cone in which $\| x \| \leq b$ and $a(x) = a$. Boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics and so on. Some theories such as the Krasnoselskii fixed point theorem, the Leggett-Williams fixed point theorem,Avery's generalization fixed point theorem and Avery-Henderson fixed point theorems are study about positive solution in different techniques. In this study, we are interes about the fixed point technique.

We list down few of them which are related to our particular problem. Benterki. A. etal. (2018). Existence of Solutions for boundary value problems via fixed point method. Erbe,L.H and Wang,H in (1994).Existence of positive solutions of ODEs.

In Unit two we introduce appropriate definitions and articles. In Unit Three Methodology used to study, In unit four we shall apply the fixed point theorem to boundary value problem to the fourth order and Contraction of Green's function with the boundary condition and obtaining positive solution. In Unit five. Conclusion and future scope .

Motivated by the above papers in this thesis,we establish the existence of positive solution for fourth order three point boundary value problem of

$$u^4(t) + k^2 u''(t) = f(t, u(t)) \tag{1.1}$$

with boundary condition

$$\begin{aligned} \alpha_1 u(0) - \beta_1 u'(0) &= 0 \\ \gamma_1 u(1) + \delta_1 u'(1) &= 0 \\ \alpha_2 u''(0) - \beta_2 u'''(0) &= 0 \\ \gamma_2 u''(1) + \delta_2 u'''(1) &= 0 \end{aligned} \tag{1.2}$$

by applying Leggett-William fixed point theorem.

1.2 Statements of the problem

In this study we focused on establishing existence of positive solution for 4^{th} order BVP by applying Leggett-Williams fixed point theorem

1.3 Objectives of the study

1.3.1 General objective

The main Objective of this thesis was establishing existence of positive solutions for fourth order boundary value problem by applying Leggett-William fixed point theorem

1.3.2 Specific objectives

This study has the following specific objectives:

- To construct the Green's function by following its properties for corresponding homogeneous BVp
- To determine the equivalent integral equation for the given BVPs.
- To prove existence of positive solution for fourth order and to provide illustrative example.

1.4 Significance of the study

The study may have the following importance:

- It provides the technique of constructing green function
- It may familiarize the researcher with scientific communications in applied mathematics.
- It may serve as background information for research that works around this area.

1.5 Delimitation of the Study

The study was focus on investigating existence of positive solutions for the fourth order deferential equation by applying Leggett Williams fixed point theorem

Chapter 2

Review of Related Literatures

2.1 Over view of the study

A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary condition. Boundary value problems arise in several branches of physics as any physical differential equation will have them. problems involving the wave equation, such as the determination of normal modes, are often stated as boundary value problems. Positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in non linear analysis. A boundary value problem for a given differential consists of finding solution of the given differential equation.
The first important and significant of positive solution was proved by Erbe and Wanga in 1994. Positive solutions for the two-point boundary value problems.

$$\begin{aligned} y'' + \alpha(t) f(y(t)) = 0, 0 < t < 1 \\ \alpha y(0) - \beta y'(0) = 0 \\ \gamma y(1) + \delta y'(1) = 0 \end{aligned} \tag{2.1}$$

Prasad etal. in 2016. Multiplicity of positive solutions for second order Sturm-Liouville BVPs

$$\begin{gathered} u'' + k^2 u + f(t,u) = 0, 0 \leq t \leq 1 \\ au(0) - bu'(0) = 0 \; and \; cu(1) + du'(1) = 0 \end{gathered} \tag{2.2}$$

where $k \in (0, \frac{\pi}{2})$ is a constant , by an application of .
Bo Yang in 2005 investigate Positive solution for a fourth order boundary value

problem

$$\begin{aligned} u''''(t) = g(t)f(u(t)), 0 \leq t \leq 1 \\ together\ with\ boundary\ conditions. u(0) = u'(0) = u''(1) = u'''(1) = 0 \end{aligned} \tag{2.3}$$

Tahari and El-Shaheda. in 2008. existence and nonexistence of positive solutions for

$$\begin{aligned} u^4(t) = \lambda \alpha(t) f(u(t)) = 0, 0 \leq t \leq 1 \\ u(0) = u(1) = u''(0) = u'''(0) = 0, \\ \alpha u'(1) + \beta u''(1) = 0, \end{aligned} \tag{2.4}$$

where $\lambda > 0$ is a positive parameter $a : (0,1) \to [0,\infty)$
is continuous and $\int_0^1 dt > 0, f : [0,1] \times [0,\infty) \to [0,\infty)$
by using Kransnoselskiis fixed point theorem in cones.

Daniel, B.etal. in 2016.Existence of Positive Solution for a Class of fourth order Boundary Value Problems.

$$\begin{aligned} u^4(t) = \lambda h(t, u(t), u''(t)) \\ \alpha_1 u(0) - \gamma_1 u(1) = \beta_1 u'(0) - \delta_1 u'(1) = -\alpha \\ \alpha_2 u''(0) - \gamma_1 u''(1) = \beta_2 u'''(0) - \delta_2 u'''(1) = b. \end{aligned} \tag{2.5}$$

Where $h : [0,1] \times [0,\infty] \times (-\infty,0) \to [0,\infty)$ is nonnegative and continuous
and $a, b, \lambda, \alpha_i, \beta_i, \gamma_i, \delta_i > 0$ for $i = 1,2$.

A. Benterki M. R . (2018). Existence of solutions for Boundary Value problems via fixed point method.

$$\begin{aligned} u'' = f(x,u) - \lambda, x \in (0,1) \\ \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(1) + \delta u'(1) = 0 \end{aligned} \tag{2.6}$$

Cabada . A and Saavedra . L. Existence of solutions for nth -order nonlinear differential boundary value problems by means of new fixed point theorem

$$\begin{aligned} u_1''(t) + \lambda_1 f_1(u_1(t), u_2(t)) = 0, t \in (0,1) \\ u_2''(t) + \lambda_2 f_2(u_1(t), u_2(t)) = 0, t \in (0,1) \\ u_1'(0) = u_1(1) + u_1'(1) = 0 \\ u_2'(0) = u_2(1) + \varepsilon u_2(\eta) = 0, \eta \in (0,1) \end{aligned} \tag{2.7}$$

has a solution for every $\lambda_1, \lambda_2 > 0$ by applying some previously obtained fixed point results on a related system of integral operators.

2.2 Preliminaries

Definition 2.2.1 Let X is a nonempty set and $T : X \to X$ is a mapping. A point $x^* \in X$ is said to be a fixed point of T if $T(x^*) = x^*$.

Definition.2.2.2: When the induced metric is complete, the normed vector space is called a Banach space.

Definition.2.2.3:Let E be a real Banach space. A nonempty closed convex set P is the subset of E is called a cone, if it satisfies the following two conditions:

i. $x \in P, \lambda \geq 0$ implies $\lambda x \in p$

ii. $x \in P, -x \in p$ implies $x = 0$. every cone $P \subset E$ induces an ordering in E given by and $x \leq y$ iff $y - x \in P$

Definition .2.2.4. A normed linear space is a linear space X in which for each vector X there corresponds a real number denoted by $\|X\|$ called the norm of X and has the following properties:

a. $\|x\| \geq 0$, for all $x \in X$ and $\|x\| = 0$, iff $x = 0$

b . $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

c. $\|ax\| = |a|\|x\|$, for all $x \in X$ and for any scalar a.

Definition. 2.2.5. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and $\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$ for all $x, y \in P$ and $t \in [0, 1]$ similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and
$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$ or all $x, y \in P$ and $t \in [0, 1]$

Definition 2.2.6. We consider the second order linear differential equation

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = r(t), t \in J = [0, 1]. \tag{2.8}$$

Where the function $p_0(t), p_1(t), p_2(t)$ *and* $r(t)$ are continuous in $[\alpha, \beta], \alpha, \beta \in R$ and boundary condition of the form

$$\begin{aligned} l_1[y] = a_0 y(0) + a_1 y'(1) + b_0 y(1) + b_1 y'(1) = A \\ l_2[y] = c_0 y(0) + c_1 y'() + d_0 y(1) + d_1 y'(1) = B \end{aligned} \tag{2.9}$$

where a_i, b_i, c_i, d_i $i = 0, 1$ A, B are given constants. the boundary value problem (2.8),(2.9) is called a non homogeneous two point linear boundary value where as the homogeneous deferential equation

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = 0.\ t \in J = [0,1]. \tag{2.10}$$

together with the homogeneous boundary condition

$$l_1[y] = 0, l_2[y] = 0 \tag{2.11}$$

is called a homogeneous two point linear boundary value problems. the function $G(t,s)$ called a Green's function for the corresponding homogeneous boundary value problem $(2.10),(2.11)$ can be explicitly expressed interns of $G(t,s)$. obviously , for the homogeneous problem $(2.10),(2.11)$. the trivial solution always exists Green's function $G(t,s)$ for the boundary value problem $(2.10),(2.11)$ is defined in the square $[\alpha,\beta]\times[\alpha,\beta]$ and possesses the following fundamental properties

i. $G(t,s)$ is continuous in $[0,1]\times[0,1]$

ii.

$$\frac{\partial}{\partial t}G(t,s)$$

is continuous in each of the triangles, $0 \le t \le s \le 1$ and $0 \le s \le t \le 1$: moreover

$$\frac{\partial}{\partial t}G(s^{+^1},s) - \frac{\partial}{\partial t}G(s^{-^1},s) = \frac{1}{p_0(s)}$$

where,

$$\frac{\partial}{\partial t}G(s^{+^1},s) = \lim_{t\to s, t>s}\frac{\partial G(t,s)}{\partial t}$$

and

$$\frac{\partial G(s^{-^1},s)}{\partial t} = \lim_{t\to s, t<s}\frac{\partial G(t,s)}{\partial t}$$

iii. For every fixed $s \in [0,1], z(t) = G(t,s)$ is a solution of the differential equation (2.12) in each of the interval $[0,s]$ and $(s,1]$

iv. For every fixed $s \in [0,1], z(t) = G(t,s)$ satisfies the boundary conditions (2.13). These properties completely characterize Green's function $G(t,s)$

Definition 2.2.7. let X and Y be Banach spaces and $T : X \to Y$ an operator T is said to be completely continuous, if T is continuous and for each bounded sequence $\{x_n\} \subset (X), (Tx_n)$ has a convergent subsequence.

Definition 2.2.8. A Normed linear space is said to be complete, if every Cauchy sequence in X converges to a point in X.
Definition 2.2.9. The function $u(t) \in C[0,1] \times C[0,1]$ is a positive solution of the boundary value problems.

$$u^4(t) + k^2 u''(t) = f(t, u(t)) \tag{2.12}$$

with boundary condition

$$\begin{aligned} \alpha_1 u(0) - \beta_1 u'(0) &= 0 \\ \gamma_1 u(1) + \delta_1 u'(1) &= 0 \\ \alpha_2 u''(0) - \beta_2 u'''(0) &= 0 \\ \gamma_2 u''(1) + \delta_2 u'''(1) &= 0 \end{aligned} \tag{2.13}$$

If u(t) is positive on the given interval and satisfies both the differenctial equation and the boundary conditions.

Chapter 3

Methodology

3.1 Study area and period

This study was conducted at Jimma University under the department of mathematics from September, 2019 G.C. to July, 2020 G.C

3.2 Study design

In order to achive the objective of the study we employed analytical method of the design.

3.3 Source of information

Secondary data such as reference books, research papers, journals and Internet were used as source of information of this study.

3.4 Mathematical Procedure of the Study

In this study we follow the procedures stated below:

- Defining fourth order Leggett-William fixed point theorem.
- Constructing Green's function.
- Determinig the equivalent integral equation for the boundary value problems
- Verfying the main result by providing illustrative examples .

Chapter 4

Main Results and Discussion

4.1 Construction of Green function

In this section, we construct Green's function for the homogeneous problem corresponding to (1.1) and (1.2)
Let G(t,s) be the Green's function for the homogeneous problem

$$(u''(t)+k^2u(t))''=0, 0\leq t\leq 0 \tag{4.1}$$

with the same boundary conditions (1.2)
let $-u''(t)=v(t), -u''(t)\leq 0$.
thus the differential equation (1.1) considering the boundary condition.

$$-(v''(t)+k^2v(t))=0, 0\leq t\leq 1, \tag{4.2}$$

$$\begin{aligned}\alpha_2 v(0)-\beta_2 v'(0)=0\\ \gamma_2 v(1)+\delta_2 v'(1)=0\end{aligned} \tag{4.3}$$

for the equation (4.2) two linearly independent solutions are $v_1(t)=coskt$ and $v_2(t)=sinkt$. Hence , the problem (4.2) and (4.3) have only the trivial solution if and only if

$$\rho=\begin{bmatrix}\alpha_2 v_1(0)-\beta_2 v_1'(0) & \alpha_2 v_2(0)-\beta_2 v_2'(0)\\ \gamma_2 v_1(1)+\delta_2 v_1'(1) & \gamma_2 v_2(1)+\delta_2 v_2'(1)\end{bmatrix}=\begin{bmatrix}\alpha_2 & -\beta_2 k\\ \gamma_2 cosk-\delta_2 k\ sink & \gamma_2 sink+\delta_2 k\ cosk\end{bmatrix}$$

Therefore

$$\rho=(\alpha_2\gamma_2-\beta_2\delta_2 k^2)sink+(\alpha_2\delta_2+\beta_2\gamma_2)k\ cosk\neq 0$$

for $k\in(0,\infty)$
by the property (iii) of the Green's functions there exist four functions, say $\lambda_1(s), \lambda_2(s), \mu_1(s)$ and $\mu_2(s)$ such that

$$G_1(t,s) = \begin{cases} v_1(t)\lambda_1(s) + v_2(t)\lambda_2(s), 0 \le t \le s \le 1, \\ v_1(t)\mu_1(s) + v_2(t)\mu_2(s), 0 \le s \le t \le 1. \end{cases} \tag{4.4}$$

$$G_1(t,s) = \begin{cases} coskt\lambda_1(s) + sinkt\lambda_2(s), 0 \le t \le s \le 1, \\ coskt\mu_1(s) + sinkt\mu_2(s), 0 \le s \le t \le 1. \end{cases} \tag{4.5}$$

Now using properties (i) and (ii), we obtain the following two equations:

$$v_1(s)\lambda_1(s) + v_2(s)\lambda_2(s) = v_1(s)\mu_1(s) + v_2(s)\mu_2(s) \tag{4.6}$$

$$v_1'(s)\mu_1(s) + v_2'(s)\mu_2(s) - v_1'(s)\lambda_1(s) - v_2'(s)\lambda_2(s) = \frac{1}{p_0(t)} \tag{4.7}$$

$cosks\lambda_1(s) + sinks\lambda_2(s) = cosks\mu_1(s) + sinks\mu_2(t)$
$-k\ sinks\mu_1 + k\ cosks\mu_2(s) + k\ sinks\lambda_1(s) - k\ cosks\lambda_2(s) = -1$
Let

$$m_1(s) = \lambda_1(s) - \mu_1(s)$$

and

$$m_2(s) = \lambda_2(s) - \mu_2(s)$$

so that (4.5) and (4.6) can be written as

$$coskt\ m_1(s) + sinkt\ m_2(s) = 0 \tag{4.8}$$

$$sinkt\ m_1(s) - coskt\ m_2(s) = \frac{-1}{k} \tag{4.9}$$

since coskt and sinkt are linearly independent the Wronskian
$W(coskt, sinkt) \neq 0$, for all $t \in (0,1)$.
thus, the relation (4.8) and (4.9) uniquely determine.
$m_1(s) = -\frac{sinks}{k}$ and $m_2(s) = \frac{cosks}{k}$
Now using the relation $\mu_1(s) = \lambda_1(s) + \frac{sinks}{k}$ and $\mu_2(s) = \lambda_2(s) - \frac{cosks}{k}$
Green's function can be written as

$$G_1(t,s) = \begin{cases} coskt\lambda_1(s) + sinkt\lambda_2(s), 0 \le t \le s \le 1 \\ coskt(\lambda_1(s) + \frac{sinks}{k}) + sinkt(\lambda_2(s) - \frac{cosks}{k}), 0 \le s \le t \le 1 \end{cases} \tag{4.10}$$

Finally,using the property (iv) with boundary condition we get.

$$\begin{cases} \alpha_2[cosk(0)\lambda_1(s)+sink(0)\lambda_2(s)]-\beta_2[cosk'(0)\lambda_1(s)+sink'(0)\lambda_2(s)]=0 \\ \gamma_2[cosk(1)(\lambda_1(s)+\frac{sinks}{k})+sink(1)(\lambda_2(s)-\frac{cosks}{k})]+ \\ \delta_2[cosk'(1)(\lambda_1(s)+\frac{sinks}{k})+sink'(1)(\lambda_2(s)-\frac{cosks}{k})=0 \end{cases}$$

$$\begin{cases} \alpha_2\lambda_1(s)-\beta_2k\lambda_2(s)=0 \\ (\gamma_2kcosk-\delta_2k^2sink)\lambda_1(s)+(\gamma_2ksink+\delta_2k^2cosk)\lambda_2(s)=\gamma_2sink\ cosks \\ -\gamma_2cosk\ sinks+\delta_2ksink\ sinks+\delta_2kcosk\ cosks \end{cases} \tag{4.11}$$

$$\rho=\begin{bmatrix} \alpha_2 & -\beta_2k \\ (\gamma_2kcosk-\delta_2k^2sink) & \gamma_2k\ sink+\delta_2k^2cosk \end{bmatrix}$$

$$\rho=(\alpha_2\gamma_2-\beta_2\delta_2k^2)ksink+(\alpha_2\delta_2+\beta_2\gamma_2)k^2cosk\neq 0$$

for $k\in(0,\infty)$

from (4.11) we determine $\lambda_1(s)$ and $\lambda_2(s)$ *as*

$\lambda_1(s)=\frac{1}{\rho}\begin{bmatrix} 0 & -\beta_2k \\ \gamma_2sink\ cosks-\gamma_2cosk\ sinks+\delta_2k\ sink\ sinks+\delta_2k\ cosk\ cosks & (\gamma_2k\ sink+\delta_2k^2cosk) \end{bmatrix}$

$\lambda_1(s)=\frac{\beta_2\gamma_2k\ sink\ cosks-\beta_2\gamma_2k\ cosk\ sinks+\beta_2\delta_2k^2sink\ sinks+\beta_2\delta_2k^2cosk\ cosks}{\rho}$

$\lambda_2(s)=\frac{1}{\rho}\begin{bmatrix} \alpha_2 & 0 \\ (\gamma_2kcosk-\delta_2k^2sink) & \gamma_2sink\ cosks-\gamma_2cosk\ sinks+\delta_2ksink\ sinks+\delta_2k\ cosk\ cosks \end{bmatrix}$

$=\frac{\alpha_2\gamma_2sink\ cosks-\alpha_2\gamma_2\ cosk\ sinks+\alpha_2\delta_2k\ sink\ sinks+\alpha_2\delta_2k\ cosk\ cosks}{\rho}$

$\mu_1(s)=\lambda_1(s)+\frac{sinks}{k}$

$\mu_1(s)=\frac{\beta_2\gamma_2ksink\ cosks-\beta_2\gamma_2k\ cosk\ sinks+\beta_2\delta_2k^2sink\ sinks+\beta_2\delta_2k^2cosk\ cosks}{\rho}+\frac{sinks}{k}$

$=\frac{\beta_2\gamma_2k\ sink\ cosks+\beta_2\delta_2k^2cosk\ cosks+\alpha_2\gamma_2sink\ sinks+\alpha_2\delta_2kcosk\ sinks}{\rho}$

and

$\mu_2(s)=\lambda_2(s)-m_2(s)=\lambda_2(s)-\frac{cosks}{k}$

$\mu_2(s)=\frac{\alpha_2\gamma_2sink\ cosks-\alpha_2\gamma_2\ cosk\ sinks+\alpha_2\delta_2ksink\ sinks+\alpha_2\delta_2kcosk\ cosks}{\rho}-\frac{cosks}{k}$

$\mu_2(s)=\frac{\alpha_2\delta_2ksink\ sinks-\alpha_2\gamma_2cosk\ sinkc+\beta_2\delta_2k^2sink\ cosks-\beta_2\delta_2kcosk\ cosks}{\rho}$

then by substituting in to equation (4.10) which gives

$$G_1(t,s)=\begin{cases} coskt[\frac{\beta_2\gamma_2 ksink\ cosks-\beta_2\gamma_2 k\ cosk\ sinks+\beta_2\delta_2 k^2 sink\ sinks+\beta_2\delta_2 k^2 cosk\ cosks}{\rho}] \\ +sinkt[\frac{\alpha_2\gamma_2 sink\ cosks-\alpha_2\gamma_2\ cosk\ sinks+\alpha_2\delta_2 ksink\ sinks+\alpha_2\delta_2 kcosk\ cosks}{\rho}], 0\le t\le s\le 1 \\ coskt\{\frac{\beta_2\gamma_2 ksink\ cosks+\beta_2\delta_2 k^2 cosk\ cosks+\alpha_2\gamma_2 sink\ sinks+\alpha_2\delta_2 kcosk\ sinks}{\rho}\} \\ +sinkt([\frac{\alpha_2\delta_2 ksink\ sinks-\alpha_2\gamma_2 cosk\ sinkc+\beta_2\delta_2 k^2 sink\ cosks-\beta_2\delta_2 kcosk\ cosks}{\rho}]), 0\le s\le t\le 1 \end{cases} \tag{4.12}$$

$$G_1(t,s)=\begin{cases} \frac{1}{\rho}(\alpha_2 sinkt+\beta_2 kcoskt)(\gamma_2 sink(1-s)+\delta_2 kcosk(1-s)), 0\le t\le s\le 1 \\ \frac{1}{\rho}(\alpha_2 sinks+\beta_2 kcosks)(\gamma_2 sink(1-t)+\delta_2 kcosk(1-t)), 0\le s\le t\le 1 \end{cases} \tag{4.13}$$

$$-u''(t)=v(t)=\int_0^1 G(t,s)f(s,u(s))ds \tag{4.14}$$

we consider

$$-u''(t)=0, 0\le t\le 1 \tag{4.15}$$

with the boundary condition

$$\begin{aligned} \alpha_1 u(0)-\beta_1 u'(0)=0 \\ \gamma_1 u(1)+\delta_2 u'(1)=0 \end{aligned} \tag{4.16}$$

the two linearly independent solution of (4.15) is $u_1(t)=1$ and $u_2(t)=t$.
hence the problem (4.15) has trivial solution if and only if

$$\Delta=\begin{bmatrix} \alpha_1 u_1(0)-\beta_1 u_1'(0) & \alpha_1 u_2(0)-\beta_1 u_2'(0) \\ \gamma_1 u_1(1)+\delta_1 u_1'(1) & \gamma_1 u_2(1)+\delta_1 u_2'(0) \end{bmatrix}=\begin{bmatrix} \alpha_1 & -\beta_1 \\ \gamma_1 & \gamma_1+\delta_1 \end{bmatrix}=(\alpha_1)(\gamma_1+\delta_1)+\beta_1\gamma_1\neq 0$$

from the property (iii) there exist four functions, say, $\lambda_1(s),\lambda_2(s),\mu_1(s)$ and $\mu_2(s)$ such that

$$G_2(t,s)=\begin{cases} \lambda_1(s)+t\lambda_2(s), 0\le s\le t\le 1 \\ \mu_1(s)+t\mu_2(s), 0\le t\le t\le 1 \end{cases} \tag{4.17}$$

Now using the property (i) and (ii) , we obtain the following two equations.

$$\lambda_1(s)+t\lambda_2(s)=\mu_1(s)+t\mu_2(s) \tag{4.18}$$

$$\mu_2(s)-\lambda_2(s)=-1 \tag{4.19}$$

let $m_1(s)=\mu_1(s)-\lambda_1(s)$ and $m_2(s)=\mu_2(s)-\lambda_2(s)$ thus $m_1(s)=s$ and $m_2(s)=-1$
thus $\mu_1(s)=\lambda_1(s)+m_1(s)$
$\mu_1(s)=\lambda_1(s)+s$
and $\mu_2(s)=\lambda_2(s)+m_2(s)$
$\mu_2(s)=\lambda_2(s)-1.$
Green's function can be written as

$$G_2(t,s)=\begin{cases}\lambda_1(s)+t\lambda_2(s), 0\le s\le t\le 1\\ \lambda_1(s)+s+t\lambda_2(s)-t, 0\le t\le s\le 1\end{cases} \tag{4.20}$$

by using the boundary condition

$$G_2(t,s)=\begin{cases}\alpha_1[u_1(0)\lambda_1(s)+u_2(0)\lambda_2(s)]-\beta_1[u_1'\lambda_1(s)+u_2'(0)\lambda_2(s)]=0\\ \gamma_1[u_1(1)\mu_1(s)+u_2(1)\mu_2(s)]+\delta_1[u_1'(1)\mu_1(s)+u_2'(1)\mu_2(s)]=0\end{cases}$$

$$G_2(t,s)=\begin{cases}\alpha_1\lambda_1(s)-\beta_1\lambda_2(s)=0\\ \gamma_1\lambda_1(s)+(\gamma_1+\delta_1)\lambda_2(s)=\delta_1-\gamma_1 s+\gamma_1\end{cases} \tag{4.21}$$

by taking

$$\Delta=(\alpha_1)(\gamma_1+\delta_1)+\beta_1\gamma_1\neq 0$$

$$\lambda_1(s)=\frac{\begin{bmatrix}0 & -\beta_1\\ (\delta_1+\gamma_1-\gamma_1 s) & \gamma_1+\delta_1\end{bmatrix}}{\Delta}=\frac{\beta_1(\delta_1+\gamma_1-\gamma_1 s)}{\Delta}$$

and

$\lambda_2(s)=\frac{\begin{bmatrix}\alpha_1 & 0\\ \gamma_1 & (\delta_1+\gamma_1-\gamma_1 s)\end{bmatrix}}{\Delta}=\frac{\alpha_1[\delta_1+\gamma_1-\gamma_1 s]}{\Delta}$

substituting the values if $\lambda_1(s)$ and $\lambda_2(s)$ in to (4.20)

$$G_2(t,s)=\begin{cases}\lambda_1(s)+t\lambda_2(s), 0\le s\le t\le 1\\ \lambda_1(s)+s+t\lambda_2(s)-t, 0\le t\le s\le 1\end{cases}$$

$$=\begin{cases}\frac{1}{\Delta}\beta_1(\delta_1+\gamma_1-\gamma_1 s)+t\frac{1}{\Delta}\alpha_1(\delta_1+\gamma_1-\gamma_1 s)\\ \frac{1}{\Delta}[\gamma_1(\beta_1+\alpha_1 s)+\delta_1(\beta_1+\alpha_1 s)-\frac{1}{\Delta}t[\gamma_1(\beta_1+\alpha_1 s)]]\end{cases}$$

$$G_2(t,s)=\begin{cases}\frac{1}{\Delta}(\delta_1+\gamma_1-\gamma_1 s)(\beta_1+\alpha_1 t), 0\le s\le t\le 1\\ \frac{1}{\Delta}(\beta_1+\alpha_1 s)(\delta_1+\gamma_1-\gamma_1 t), 0\le t\le s\le 1\end{cases} \tag{4.22}$$

$$u(t)=\int_0^1 G_2(t,s)(\int_0^1 G_1(s,\tau)f(\tau,u(\tau))d\tau)ds \tag{4.23}$$

is the solution of the fourth order boundary value problem.

Lemma 4.1.1: The Green's function $G_1(t,s)$ satisfies the following inequalities

$i).G_1(t,s)>0$, for all $,s\in(0,1)$

$ii).G_1(t,s)\le MG_1(s,s), for\ all\ t,s\in[0,1]\times[0,1]$

$iii).G_1(t,s)\ge\frac{1}{M}G_1(s,s)\ for\ all\ t,s\in[0,1]\times[0,1]$

$Where\ M=max\{\frac{\alpha_2+\beta_2 k}{\beta_2 kcosk},\frac{\gamma_2+\delta_2 k}{\delta_2 kcosk}\}$

Proof: i). Since $k^2\le\frac{\alpha_2\gamma_2}{\beta_2\delta_2}$ the Green's function $G_1(t,s)$ is positive for all $t,s\in(0,1)$

ii). Let $t\le s\ then$

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{(\alpha_2 sinkt+\beta_2 kcoskt)}{(\alpha_2 sinks+\beta_2 kcosks)}\le\frac{\alpha_2+\beta_2 k}{\beta_2 kcosk}.$$

Let $s\le t$

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{\gamma_2 sink(1-t)+\delta_2 kcosk(1-t)}{\gamma_2 sinks(1-s)+\delta_2 kcosk(1-s)}\le\frac{\gamma_2+\delta_2 k}{\delta_2 kcosk}$$

Therefore $G_1(t,s)\le MG_1(s,s)$ is bounded for all $(t,s)\in[0,1]\times[0,1]$

where

$$M=max\{\frac{\alpha_2+\beta_2 k}{\beta_2 kcosk},\frac{\gamma_2+\delta_2 k}{\delta_2 kcosk}\}.$$

iii). Let $t\le s$ then

$$\frac{G_1(t,s)}{G_1(s,s)}=\frac{(\alpha_2 sinkt+\beta_2 kcoskt)}{(\alpha_2 sinks+\beta_2 kcosks)}\le\frac{\alpha_2+\beta_2 k}{\beta_2 kcosk}.$$

Let $s \leq t$

$$\frac{G_1(t,s)}{G_1(s,s)} = \frac{\gamma_2 sink(1-t) + \delta_2 kcosk(1-t)}{\gamma_2 sinks(1-s) + \delta_2 kcosk(1-s)} \leq \frac{\gamma_2 + \delta_2 k}{\delta_2 kcosk}.$$

Therefore $G_1(t,s) \geq \frac{1}{M} G_1(s,s)$ is bounded for all $(t,s) \in [0,1] \times [0,1]$
where $M = max\{\frac{\alpha_2+\beta_2 k}{\beta_2 kcosk}, \frac{\gamma_2+\delta_2 k}{\delta_2 kcosk}\}$.

Lemma 4.1.2:The Green's function $G_2(t,s)$ satisfies the following inequalities
i).$G_2(t,s) > 0$, for all $t,s \in (0,1)$
$ii).G_2(t,s) \leq G_2(s,s), for\ all\ 0 \leq t,s \leq 1$
$iii).G_2(t,s) \geq NG_2(s,s) for\ all\ t,s \in [0,1] \times [0,1]$
Where $N = min\{\frac{\gamma_1+4\delta_1}{4(\gamma_1+\delta_1)}, \frac{\alpha_1+4\beta_1}{4(\alpha_1+\beta_1)}\} \leq 1$.
Proof: i).The Green's function $G_2(t,s)$ is positive for all $t,s \in (0,1)$
ii). Let $s \leq t$, then

$$\frac{G_2(t,s)}{G_2(s,s)} = \frac{(\delta_1 + \gamma_1 - \gamma_1 t)(\beta_1 + \alpha_1 s)}{(\delta_1 \gamma_1 - \gamma_1 s)(\beta_1 + \alpha_1 s)} = \frac{(\delta_1 + \gamma_1 - \gamma_1 t)}{(\delta_1 + \gamma_1 - \gamma_1 s)} \leq 1.$$

Let $t \leq s$, then

$$\frac{G_2(t,s)}{G_2(s,s)} = \frac{(\delta_1 \gamma_1 - \gamma_1 s)(\beta_1 + \alpha_1 t)}{(\delta_1 + \gamma_1 - \gamma_1 s)(\beta_1 + \alpha_1 s)} = \frac{(\beta_1 + \alpha_1 t)}{(\beta_1 + \alpha_1 s)} \leq 1.$$

So that $G_2(t,s) \leq G_2(s,s)$ is bounded
furthermore, for $\frac{1}{4} \leq t \leq \frac{3}{4}$.
iii). Let $s \leq t, then$
$\frac{G_2(t,s)}{G_2(s,s)} = \frac{(\delta_1+\gamma_1-\gamma_1 t)(\beta_1+\alpha_1 s)}{(\delta_1+\gamma_1-\gamma_1 s)(\beta_1+\alpha_1 s)} = \frac{(\delta_1+\gamma_1-\gamma_1 t)}{(\delta_1+\gamma_1-\gamma_1 s)} \geq \frac{(\delta_1+\gamma_1-\gamma_1(\frac{3}{4}))}{(\delta_1+\gamma_1-\gamma_1(\frac{1}{4}))} = \frac{4\delta_1+\gamma_1}{4\delta_1+3\gamma_1}$
$\frac{G_2(t,s)}{G_2(s,s)} \geq \frac{4\delta_1+\gamma_1}{4\delta_1+3\gamma_1}$.
Let $t \leq s$, then

$$\frac{G_2(t,s)}{G_2(s,s)} = \frac{(\delta_1 + \gamma_1 - \gamma_1 s)(\beta_1 + \alpha_1 t)}{(\delta_1 + \gamma_1 - \gamma_1 s)(\beta_1 + \alpha_1 s)} = \frac{(\beta_1 + \alpha_1 t)}{(\beta_1 + \alpha_1 s)} \geq \frac{(\beta_1 + \alpha_1(\frac{1}{4}))}{(\beta_1 + \alpha_1(\frac{3}{4}))} = \frac{4\beta_1 + \alpha_1}{4\beta_1 + 3\alpha_1}$$

$$\frac{G_2(t,s)}{G_2(s,s)} \geq \frac{4\beta_1 + \alpha_1}{4\beta_1 + 3\alpha_1}$$

$$\frac{G_2(t,s)}{G_2(s,s)} \geq N, \frac{1}{4} \leq t \leq \frac{3}{4}, N = N = min\{\frac{\gamma_1 + 4\delta_1}{4(\gamma_1 + \delta_1)}, \frac{\alpha_1 + 4\beta_1}{4(\alpha_1 + \beta_1)}\} \leq 1.$$

Hence

$$G_2(t,s) \geq NG_2(s,s)$$

for all

$$\frac{1}{4} \leq t,s \leq \frac{3}{4}.$$

The proof is complete

Theorem 4.1.1: (Leggett-William). Suppose $T : \bar{k_c} \to \bar{k_c}$ is completely continuous and suppose there exists a concave positive functional α on k such that $\alpha(u) \leq \|u\|$ for $u \in \bar{k_c}$. Suppose there exist constant $0 < a < b < d \leq c$ such that

(B1). $\{u \in k(a,b,d) : \alpha(u) > b\} \neq \emptyset$ and $(Tu) > b$ if $u \in k(a,b,d)$;

(B2).$\|Tu\| < u$ if $u \in k_a$ and

(B3). $\alpha(Tu) > b$ for $u \in k(a,b,c)$ with $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2, and u_3 such that $\|u_1\| < a, b < \alpha(u_2)$ and $\|u_3\| > a$ with $\alpha(u_3) < b$.

In this thesis we consider the fourth-order boundary value problem

$u^4(t) + k^2u''(t) = f(t,u(t))$, $0 \leq t \leq 1$

with boundary condition (1.2).

The following condition will be assumed throughout

(A1): $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function

$(A2) : 0 \leq \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau,u(\tau)) d\tau ds$

$(A3) : \rho = (\alpha_2\gamma_2 - \beta_2\delta_2k^2)ksink + (\alpha_2\delta_2 + \beta_2\gamma_2)k^2cosk > 0, k \in (0,\infty)$ and $\Delta = (\alpha_1)(\gamma_1 + \delta_1) + \beta_1\gamma_1 > 0, for\ \alpha_i, \gamma_i, \delta_i, \beta_i \geq 0, for\ i = 1,2.$

4.2 Results

Define $\mathbb{B} = (C[0,1], \|\cdot\|)$ where $\|u\| = \max_{t\in[0,1]} |u(t)|$. Then $\mathbb{B}$ is a Banach space. define the cone $\mathscr{K} \subset \mathbb{B}$ by

$$\mathscr{K} = \{u \in \mathbb{B} : u(t) \geq 0, t \in [0,1]\}$$

$$\min_{t\in[0,1]} u(t) \geq m\|u\|$$

where $m = \frac{N}{M}$
and the operator$T : \mathscr{K} \to \mathscr{K}$ by

$$Tu(t) = \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds \tag{4.24}$$

Lemma 4.2.1: If $(A1)-(A3)$ are holds, then the operator $T : \mathbb{B} \to \mathscr{K}$ is bounded.
proof: Since $G_2(t,s)$ and$G_1(t,s)$ are positive , then $Tu(t) \geq 0$ for all $\forall t \in [0,1]$

$$Tu(t) = \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds$$
$$\leq \int_0^1 G_2(s,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds$$

Therefore $Tu(t)$ is bounded.
Lemma 4.2.2: If $(A1)-(A3)$ are holds, then the operator $T : \mathscr{K} \to \mathscr{K}$ is completely continuous.
proof. From the continuity of f, we know that $Tu \in \mathscr{K}$ for each $u \in \mathbb{B}$

$$Tu(t) = \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds$$
$$\leq \int_0^1 G_2(s,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds.$$

Note that by non-negativity of $G_2(s,t), G_1(s,t)$ and f

$$\|Tu\| \leq \int_0^1 G_2(s,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds \tag{4.25}$$

now for $\frac{1}{4} \leq t \leq \frac{3}{4}$

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} Tu(t) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds$$
$$\geq \int_0^1 NG_2(s,s) \int_0^1 \frac{1}{M} G_1(\tau,\tau) f(\tau, u(\tau)) d\tau ds$$
$$\geq \frac{N}{M} \int_0^1 G_2(s,s) \int_0^1 G_1(\tau,\tau) f(\tau, u(\tau)) d\tau ds$$
$$\|Tu\| \geq m\|Tu\|, Tu \in \mathscr{K}. \tag{4.26}$$

Therefore $T : \mathscr{K} \to \mathscr{K}$ is self map. since $G_2(t,s), G_1(t,s)$ and $f(t,u)$ are continuous, that $T : \mathscr{K} \to \mathscr{K}$ is completely continuous.

The proof is completed

Theorem 4.2.2. Let $\beta \in (0,1), 0 < M \leq \frac{\|u\|}{c}$ and $\mathbb{B} \geq [\beta m \int_\beta^1 G_2(t,s) \int_\beta^1 G_1(\tau,\tau) d\tau ds]^{-1}$
Let a, b and c be such that $0 < a < b < c$. Assume that the following hypothesis are satisfied

(H3). $f(t,u) < Ma$ for all $t \in [0,1]$ and $u \in [0,a]$

$(H4).f(u) > Nb$ for all $t \in [0,1]$ and $u \in [b,c]$

$(H5).f(u) \leq Mc$ for all $t \in [0,1]$ and $u \in [0,c]$.

Then the boundary value problem has three positive solutions $u_1, u_2, u_3 \in \mathscr{K}$ satisfying

$\|u_1\| < a,\ b < \alpha(u_2),\ a < \|u_3\|$ with $a(u_3) < b$.

Proof. Define a nonnegative functional on β by $\alpha(u) = \min_{\beta \leq t \leq 1} |u(t)|$, we show that the condition of Theorem 4.1.1 are satisfied.

Let $u \in \mathscr{K}_c$. then $\|u\| \leq c$ and by (H5) and define $\|u\| = \max_{0 \leq t \leq 1} \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) d\tau ds$

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds$$

$$\leq M \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) d\tau dsc$$

$$\leq \|u\|$$

$$\leq c.$$

Hence $T : k_c \to k_c$ by Lemma 4.2.2. T is completely continuous.

Using an analogous argument, it follows from condition (H3) that if $u \in k_a$ then $\|Tu\| < a$ condition (B2) of theorem 4.1.1 hold.

Let d be a fixed constant such that $b < d \leq c$ then $\alpha(d) = d > b$ and $\|d\| = d$ as such $\mathscr{K}(\alpha, b, d) \neq \emptyset$.

Let $u \in \mathscr{K}(\alpha, b, d)$ then $\|u\| \leq d \leq c$ and $\min_{\beta \leq t \leq 1} u(s) \geq b$ by assumption (H4)

$$(Tu) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \int_0^1 G_2(t,s) \int_0^1 G_1(s,\tau) f(\tau, u(\tau)) d\tau ds$$

$$\geq \frac{N}{M} \int_0^1 G_2(s,s) \int_0^1 G_1(\tau,\tau) f(\tau, u(\tau)) d\tau ds$$

$$> m \int_0^1 G_2(s,s) (\int_0^1 G_1(\tau,\tau) f(\tau, u(\tau)) d\tau ds) Nb$$

$$> b.$$

Thus for all $u \in k(a,b,d), \alpha(Tu) > b$, condition (B1)of theorem 4.1.1 hold.
Finlay, if $u \in k(\alpha,b,c)$ with $\|Tu\| > d$ then $\|u\| \leq c$ and $\lim_{\beta \leq t \leq 1} u(s) \geq b$ and from assumption (H4) we can show $\alpha(Tu) > b$ condition (B3) of theorem 4.1,1 holds as consequence of theorem 4.1.1 T has three fixed points u_1, u_2, u_3 such that $\|u_1\| \leq a,\ b < \alpha(u_2),\ a < \|u_3\|$ with $\alpha(u_3) < b$ these fixed points are solutions of the boundary condition 1.2.

4.3 Example

Let us consider an example to see validity of our main result for the fourth order
Consider the following fourth- order differential equation

$$u^4(t) + \frac{1}{8}u''(t) = f(t,u(t)), 0 \leq t \leq 1$$

subject to the boundary condition

$$\begin{aligned} u(0) - 3u'(0) &= 0 \\ 2u(1) + u'(1) &= 0 \\ 4u''(0) - u'''(0) &= 0 \\ 5u''(1) + u'''(1) &= 0. \end{aligned}$$

Where

$$f(t,u) = \frac{4(u+1)^5}{193(u^2+73)}.$$

The Green's function for the homogeneous problem

$$G_1(t,s) = \begin{cases} \frac{(4sin\frac{\sqrt{2}}{4}\ t+\frac{\sqrt{2}}{4}\ cos\frac{\sqrt{2}}{4}\ t)(5sin\frac{\sqrt{2}}{4}\ (1-s)+\frac{\sqrt{2}}{4}\ cos\frac{\sqrt{2}}{4}\ (1-s))}{3.5213}, 0 \leq t \leq s \leq 1 \\ \frac{(4sin\frac{\sqrt{2}}{4}\ s+\frac{\sqrt{2}}{4}\ cos\frac{\sqrt{2}}{4}\ s)(5sin\frac{\sqrt{2}}{4}\ (1-t)+\frac{\sqrt{2}}{4}\ cos\frac{\sqrt{2}}{4}\ (1-t))}{3.5213}, 0 \leq s \leq t \leq 1 \end{cases}$$

$$v(t) = \int_0^1 G_1(t,s) f(s,u(s)) ds$$

we consider $-u'' = v(t) = \int_0^1 G_1(t,s) f(s,u(s)) ds$ with boundary condition

$$-u''(t) = 0, \quad 0 \leq t \leq 1$$

$$G_2(t,s) = \begin{cases} \frac{1}{9}(3-2s)(3+t),\ 0 \le s \le t \le 1 \\ \frac{1}{9}(3+s)(3-2t),\ 0 \le t \le s \le 1 \end{cases}$$

$$u(t) = \int_0^1 G_1(t,s) \int_0^1 G_2(s,\tau) f(\tau, u(\tau)) d\tau ds$$

by algebraic expressions, we get $M = 15.1424, N = 0.5, m = 0.03302$
clearly f is continuous and increasing on $[0,\infty)$. If we choose $a = 0.001, b = 6.25$, and $c = 22$ then, $0 < a < b < c$ and f satisfies the following condition
(i) $f(t,u) < 0.01514$ for $t \in [0,1]$ and $u \in [0, 0.001]$
(ii) $f(t,u) > 3.125$ for $t \in [0,1]$ and $u \in [6.25, 22]$
(iii) $f(t,u) \le 333.1333$ for $t \in [0,1]$ and $u \in [0,22]$
all the conditions of Theorem 3.2.1 are satisfied, with the boundary value problems. Then three positive solutions , $u_1, u_2, u_3 \in \mathscr{K}$ satisfying

$$\|u_1\| < 0.001, \quad 6.25 < \alpha(u_2)$$

$$0.001 < \|u_3\|, \ with \ \ \alpha(u_3) < 6.25$$

these example satisfies the final result.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis existence of positive solution for fourth order two boundary value problem of Leggett-William fixed point theorem is verified.
We changed fourth order differential equation to second order differential equation, in order to use Green's function to determine non-trivial solution for homogeneous boundary value problem. Different Lemmas are stated and provide for Green's function , then three positive solution has been obtained for the BVPs. We formulate equivalent integral equation for the BVP (1)-(1.2) supported by example.

5.2 Future scope

In our day to day life different deferential equation arise. Specially positive solution have many applications in different disciplines. Therefore finding the solution of such differential equation is important to solve the problem. Any interested researchers may conduct the research on existence of positive solution more than fourth order with different coefficient by application of Leggett-William fixed point theorem.

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