ANALYSIS OF MAGNETOHYDRODYNAMICS FORCED CONVECTION STAGNATION POINT FLOW THROUGH A PERMEABLE MEDIUM WITH THERMAL RADIATION AND THERMAL SLIP EFFECTS USING OPTIMAL HOMOTOPY ANALYSIS

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES JIMMA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTERS OF SCIENCES IN MATHEMATICS

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Declaration

I here submit the dissertation entitled, **”Analysis of magnetohydrodynamics forced convection stagnation point flow through a permeable medium with thermal radiation and thermal slip effects using optimal homotopy analysis”** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this study, analysis of thermal radiation and thermal slip effects on heat transfer in stagnation point flow over a permeable flat plate is considered. The governing continuity, momentum and energy equations are transformed into nonlinear ordinary differential equations using the similarity transformations and solved analytically by Optimal Homotopy Asymptotic Method. The effect of various physical parameters such as suction/blowing parameter, Eckert number, thermal slip parameter, magnetic parameter, velocity slip parameter, permeability parameter, wall temperature exponent, Prandtl number on velocity and temperature of fluid flow profile. were analyzed and displayed graphically using MATLAB software.

Nomenclature

B_0 applied magnetic field

c_p specific heat at constant pressure (J/kg.K)

D temperature slip factor

f dimensionless stream function

$f_0(\eta)$ initial approximation of f

k coefficient of thermal conductivity (W/mK)

k_1 absorption coefficient

L, N, auxiliary linear operators defined in Eqs.(4.2) & (4.5)

M magnetic parameter

N Navier's constant slip length

P pressure

Pr Prandtl number

qr radiative heat flux (kW/m2)

R radiation parameter

s mass transfer parameter

T temperature of the fluid (K)

T_∞ ambient fluid temperature (K)

u, v velocity components in x- and y-directions (m/s)

u_{slip} velocity slip

u_e free stream velocity (m/s)

v_w uniform surface mass flux

Greek Symbols

α thermal diffusivity

β dimensionless velocity slip parameter

η similarity variable

γ dimensionless thermal slip parameter

λ exponent of the wall temperature

μ coefficient of fluid viscosity

υ kinematic viscosity

Ω permeability parameter

ψ stream function

ρ density (kg/m^3)

σ electrical conductivity

$\sigma*$ Stefan-Boltzmann constant

θ dimensionless temperature

$\theta_0(0)$ initial approximation of θ

Subscripts

w condition at wall

∞ ambient environment

Contents

Chapter 1

Introduction

1.1 Background of the study

A point on the surface of an object in the flow field where the local fluid velocity is zero called stagnation point. The stagnation point flow analysis plays a significant role in the study of numerous natural and industrial phenomena, because of its applications in the exploring of flows over the tips of submarines, tip of ships and aircrafts. It is important in various engineering disciplines like hydrodynamic processes, cooling of nuclear reactors, cooling of electronic devices by fans, etc Sparrow *eta l.*, (1963).

On account of the afore-mentioned applications only, Hiemenz (1911) was the very first researcher to do a pioneering work so as to investigate the viscous fluid motion generated by a two-dimensional stagnation point flow over a flat plate. The flow due to a stretching sheet often occurs in engineering processes. For example, in manufacturing industry, polymer sheets and filaments are manufactured by continuous extrusion of the polymer from a die to a windup roller, which is located at a finite distance away Nield *eta l.*, (2006) .

Crane (1970) was the first to give similarity solution in closed analytical form for two-dimensional flow caused by stretching of a plate. Since the pioneer study of Crane (1970), many authors have showed interest to this type of problem, one of them was Chiam (1994) who combined the stretching plate problem with two-dimensional stagnation point flow. Chiam concluded that the flow near the stretching surface is the same as the inviscid flow far from the surface; hence, no boundary layer is formed. Later, Mahapatra and Gupta (2002) obtained the results that were contrary to the conclusion of Chiam (1994). They claimed that a viscous layer is formed near the stretching surface where the structure of the boundary layer depends on the velocity ratio of the stretching surface to that of the frictionless potential flow in the neighborhood of the stagnation point.

The problem on boundary layer flow and heat transfer in the region of the stagnation point on a stretching surface that had been studied in past few years are mostly concerned with the condition where the sheet is assumed to stretch on its own plane with a velocity proportional to the distance from the stagnation point. This type of problem has been considered in the papers by Mahapatra and Gupta (2001) for magnetohydrodynamics flow, Nazar *eta l.*, (2004) for micro polar fluid, Reza and Gupta (2005) and Lok *eta l.*

The process of suction/injection is of special significance with reference to the practical problems related to the boundary control applications. For example film cooling, fiber coating, and coating of wires. Due to this reason, Schlichting and Bubmann (1943) were the first to analyze the effect of suction on the Hiemenz flow. This problem was further extended by Preston (1948). Ariel (1994) made the analysis of the same problem by way of considering uniform suction. As the magnetohydrodynamics stagnation point flow problems are having their theoretical as well as practical applications in manufacturing processes like boundary layer along material handling conveyers, blood flow problems, extrusion of plastic sheets, cooling of infinite metallic plate in cooling bath, etc., these problems have attracted in the recent past the attention of many a researchers such as Sparrow *eta l.*, (1963); Na (1979), and Ariel (1994).

Rehman *eta l.*, (2017) investigated the stagnation point flow over different geometrical configuration. Because of the numerous engineering applications of permeable media related heat transfer problems in geothermal energy recovery, crude oil extraction, thermal energy storage etc., Ingham and Pop (1998); Vafai (2005); Nield *eta l.*, (2006) and Raptis *eta l.*, (1982) analyzed hydromagnetic free convection flows through porous media. Thereafter, Takhar and Ram (1994) and Yih (1998) investigated the problem under different conditions.

The radiation effects become more important at high absolute temperatures in the context of space technology, comical flight aerodynamics, plasma physics, space craft, aerodynamics, etc. As a consequence of this, the effects of thermal radiation for different geometrical configurations have been investigated by various researchers such as Viskanta and Grosh (1962); Chen *eta l.*, (1984); Elbashbeshy (2000).

The fluid exhibiting wall slips properties are very important on account of their technological applications. For example the polishing of artificial heart valves and internal cavities. So, in order to have a better understanding of the slips phenomena, many researchers like Mooney (1931); Rao and Rajagopal (1999); Khaled and Vafai (2004); Wang (2002); Wang (2006) and Hayat *eta l.*, (2007) examined the effects of slips boundary conditions on fluid flows through different geometries.

Marinca *eta l.*, (2008, 2009) and Marinca and Herisanu (2008) have been the first to propose a type of approximate analytic method which requires no small parameter. This method is known as the optimal homotopy asymptotic method, aiming at solving nonlinear problems without depending on a small parameter and it is used to obtain approximate analytic solution of nonlinear problems of thin film flow of a fourth-grade fluid down a vertical cylinder. In their work, this method was to understand the behavior of nonlinear mechanical vibration of an electrical machine. The same method was also used by Marinca *eta l.*, (2008, 2009) and Marinca and Herisanu to obtain the nonlinear equations solution, arising in the steady state flow of a fourth-grade fluid past a permeable plate and nonlinear equations solution, arising in heat transfer.

The most important feature of the OHAM is the optimal control of the convergence of solutions by means of a particular convergence-control function H(p), which ensures a fast convergence when its components (known as convergent constants) are optimally determined (Marinca and Herisanu, 2008, 2010). The validity of the OHAM is also independent of whether there exist small parameters in the problem being considered.

The aim of the present study is to investigate the effects of thermal radiation Eckart number and partial slips on the flow field under the variable wall temperature condition of the plate by OHAM.

1.2 Statement of the Problem

The study of nonlinear problem is so important in areas of physics and engineering. Because most phenomena of the world are basically nonlinear (Campbell, 1992; Liao, 2003) and described by nonlinear equations. Boundary layer fluid flow problems in different dimensions with heat transfer and magneto hydrodynamic effect have plentiful and inclusive applications in several engineering and industrial sectors. They include glass blowing, melt spinning, heat exchanger design, fiber and wire coating, production of glass fibers, industrialization of rubber and plastic sheets, etc. In addition, the action of thermal radiation is vital to calculating heat transmission in the polymer treating industry Pavlov (1974).

Liao (2003) used HAM to solve nonlinear differential equation. Liao used HAM successfully to investigate a variety of nonlinear problems in science, engineering and finance. Nasreen *eta l.*, (2018) investigate the effect of thermal radiation on steady laminar forced MHD Hiemenz flow past a flat plate in porous medium by the use of HAM.

The present study is aimed to analyze the effects of suction/blowing parameter, Eckert number, thermal slip parameter, magnetic parameter, velocity slip parameter, permeability parameter, wall temperature exponent parameter, Prandtl number on velocity and temperature of fluid flow profile.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of the present study is to analysis the effects of thermal radiation and thermal slips on MHD forced convection stagnation point flow through permeable medium.

1.3.2 Specific Objectives

The specific objectives of the present study are:

- To solve the equation governing the flow problem by using OHAM.
- To identify parameters that affect velocity and temperature of the flow problem.
- To explore the effects of thermal radiation, prandtl number and magnetic field on velocity and temperature profile of the flow.
- Discuss the effects of various parameters on the flow problem in terms of physical meanings.

1.4 Significance of the Study

The out come of this study is expected to have the following significance:

- It may help the researcher to gain a comprehensive understanding on Optimal Homotopy Asymptotic Method for solving nonlinear problems involved in numerous engineering fields.
- It familiarize a researcher with scientific communication in applied mathematics.
- The results of the finding may have an application in the field of fluid mechanics, physics and in different industrial sectors.

1.5 Delimitation of the Study

The study is delimited to the governing partial differential equations of laminar boundary layer and to analyze analytical solution for MHD forced convection stagnation point flow through a permeable medium on stretching sheet with convective boundary conditions.

1.6 Definition of key terms

- **Boundary layer:** Is a fluid character that forms in the flow of fluid through a body of surface.
- **Magnetohydrodynamics**: is the branch of continuum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field.
- **Thermal radiation**: is the procedure in which energy is released in the form of electromagnetic radiation by a surface in all directions. Thermal radiation has numerous uses in the areas of engineering and heat transfer analysis.
- **Stagnation Point Flow** A point on the surface of an object in the flow field where the local fluid velocity is zero is called stagnation point.
- **Optimal homotopy asymptotic method**: is a semi-analytic approximate technique for the treatment of the time-dependent partial differential equation.

Chapter 2

Review Literature

2.1 Magnetohydrodynamics

Magnetohydrodynamics is the branch of continuum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field.

Faraday (1812) did experiments with mercury as a conducting fluid flowing in a glass tube placed in magnetic field and observed that voltage was induced in direction perpendicular to both the direction of flow and magnetic field. He further showed that when an electric field is applied to a conducting fluid in the direction which is perpendicular to magnetic field, a force is exerted on the fluid in the direction perpendicular to both electric field and magnetic field. Since then a lot has been done on MHD and its related fields. Rao *eta l.*, (1990) studied the heat transfer in porous medium in the presence of transverse magnetic field.

MHD is important branch of fluid dynamics. Many technological problems and natural phenomena are susceptible to MHD analysis. Engineers apply MHD principle, in the design of heat exchangers, in creating novel power generating systems, pumps and flow meters, thermal protection, braking, control and re-entry, in space vehicle propulsion (Kumari *eta l.*, 1990).

MHD convection flow problems are also very important in the fields of stellar and planetary magnetosphere, aeronautics, electronics and chemical engineering. Hydromagnetic flow of Newtonian fluid and heat transfer over continuous moving flat surface with uniform suction has been studied by (Prasad *et al.*, 2010). Kumari *eta l.*,(1990) studied the effects of induced magnetic field and heat source/sink on flow and heat transfer characteristic over a stretching surface. Nazar *eta l.*, (2004) investigated the boundary layer over a moving continuous flat plate in an electrically conducting ambient fluid with a step change in applied magnetic field. The Magnetohydrodynamics equations play an important role in many areas of astrophysics, space physics and engineering. Typical applications in those areas require one to capture flow on a range of scales in a way that is as dissipation-free as possible. As a result, there has been considerable interest in bringing accurate and reliable numerical methods to bear on this problem.

2.2 Heat Transfer Mechanisms

Heat transfer is the science that predicts energy transfer between material bodies as a consequence of temperature difference. The heat transfer depends not solely on the transfer of heat energy, but also to predict the heat exchange rate, that take place under certain specified conditions. Heat transfer supplements the principles of thermodynamics by providing additional experimental rules that are used to establish energy-transfer rates (Holman *eta l.*, 2010). There are three types of heat transfer mechanisms. These are conduction, convection and radiation heat transfer.

2.2.1 Conduction Heat transfer

When a temperature gradient exists in a body, there is an energy transfer from high temperature region to low temperature region. Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

2.2.2 Convection Heat transfer

Convection heat transfer is the process in which the heat is convected out. The term convection provides with an intuitive notion concerning the heat transfer process. It is the movement of molecules within fluids (liquids, gases). It cannot take place in solids, since neither bulk current flows nor significant diffusion can take place in solids. Convection is one of the major modes of heat transfer and mass transfer Raptis, A. and Tzivanides, G (1983).

In the context of heat and mass transfer, the term ”convection” is used to refer to the sum of advective and diffusive transfer. Convection also includes fluid movement both by bulk motion (advection) and by the motion of individual particles (diffusion). However in some cases, convection is taken to mean only advective phenomena. For instance, in the transport equation, which describes a number of different transport phenomena, terms are separated into convective and diffusive effects Raptis, A. and Tzivanides, G (1983). Convective heat transfer is a mechanism of heat transfer occurring because of bulk motion (observable movement) of fluids. Heat is the entity of interest being advected (carried), and diffused (dispersed). There are two types of convections. These are natural and forced convection.

Natural Convection heat transfer

Natural convection, or free convection, occurs due to temperature differences which affect the density, and thus relative buoyancy, of the fluid. Heavier (more dense) components will fall while lighter (less dense) components rise, leading to bulk fluid movement. Natural convection occurs, only in a gravitational field. It is more likely and/or more rapid with a greater variation in density between the two fluids and a larger distance through the convecting medium. Convection will be less rapid with more rapid diffusion (there by diffusing away the gradient that is causing the convection) and a more viscous (sticky) fluid (Raptis, A. and Tzivanides, G 1983).

Forced Convection heat transfer

In forced convection fluid movement results from external surface forces such as a fan or pump. Forced convection is typically used to increase the rate of heat exchange. Many types of mixing also utilize forced convection to distribute one substance within another. Forced convection also occurs as a by-product to other processes, such as the action of forced convection may produce results more quickly than free convection. For instance, a convection oven works by forced convection, as a fan which rapidly circulates hot air forces heat into food faster than would naturally happen due to simple heating without the fan (Raptis, A., *eta l.*, 2004).

2.2.3 Radiation heat transfer

In contrast to the mechanisms of conduction and convection, where energy transfer through a material medium is involved, heat energy is also transferred through regions where a perfect vacuum exists. The mechanism in this case is electromagnetic radiation. Electromagnetic radiation that is propagated as a result of a temperature difference is known as thermal radiation.

Thermal Radiation

Thermal radiation is electromagnetic radiation from an object that is simply caused by its temperature (Quinn Brewster, M 1992). It rapidly increases in power, and also increases in frequency, with increasing temperature. For example, space craft may have thermal radiators, also called heat radiators to lose excess heat. They tend to be reflective to avoid absorption of solar radiation energy. Examples of thermal radiation are an incandescent light bulb emitting visible-light, infrared radiation emitted by a common household radiator or electric heater, as well as radiation from hot gas in outer space.

Thermal radiation is generated when thermal energy is converted to electromagnetic radiation by the movement of the charges of electrons and protons in the material (Quinn Brewster, M 1992). Sunlight is solar electromagnetic radiation generated by the hot plasma of the Sun, and this thermal radiation heats the Earth by the reverse process of absorption, generating kinetic, thermal energy in electrons and atomic nuclei. The Earth also emits thermal radiation, but at a much lower intensity and different spectral distribution because it is cooler. The balance between heating by incoming solar radiation and cooling by the Earth's outgoing radiation is the primary process that determines Earth's overall temperature.

2.3 Stagnation Point Flow

The planar laminar flow of an incompressible viscous fluid in a steady state close to a stagnation point is also called Hiemenz flow. For this flow in a plane, Hiemenz gave similarity solutions of the governing Navier-Stokes equations. Thereafter, this kind of study was carried forward by researchers like Eckert (1942) and Beard *eta l.*, (1964). Mahapatra and Gupta then re investigate the stagnation point flow towards a stretching sheet by considering different stretching and straining velocities. Many researchers have been working on stagnation point flow by taking consideration on its types of fluid, physical conditions and the effects towards the flow.

Literature study shows that researchers have studied the flow caused by stretching sheet because of its distinctive solution. Sahar investigated the impact of magnetic field flow over a permeable stretching wall in porous medium with heat radiation and suction/injection. Meanwhile, in the presence of radiation and buoyancy effects, Rashidi *eta l.*, conducted a study on free convective heat and mass transfer for MHD flow over a permeable vertical stretch sheet.

Later, the research was proceeded over a stretching porous sheet by Yahaya *eta l.*, (1996). and the issue was solved using the technique of homotopy analysis method. The findings of their study found that when the parameter of buoyancy rises, the velocity of the fluid rises and the heat boundary layer reduces where it was in fact a good agreement with prior studies. As in case of thermal radiation, increasing the thermal radiation parameter produces significant increases in the thermal conditions of the fluid temperature.

2.4 The Optimal Homotopy analysis method

Liao (1992) investigated the homotopy analysis method. The strength of HAM is that it leads to convergent analytic series solutions of strongly nonlinear problems faster than any other existing methods, independent of small or large physical parameters involved in the problem (Liao, 1992). This behavior of HAM makes it a superior technique to the conventional perturbation methods.

Liao (2003), indeed, showed that HAM is the general case and the Adomian decomposition method, expansion method and Lyapunov artificial small parameter method are the special cases of HAM. Moreover, He's homotopy perturbation method (2000) is also a special case of the HAM (cf. Liao (2005). In 1998 He proposed the Homotopy Perturbation Method which is valid in general for nonlinear differential equations.

In 2015 Marinca and Herisanu proposed Optimal Homotopy Asymptotic Method. Instead of an infinite series, they need only a few terms, mostly two terms. The procedure is successful to obtain the analytical (in classical sense) approximate solutions of currently important problems in practice and its effective and reliable (Marinca and Herisanu, 2015). OHAM is a powerful method for solving nonlinear problems without depending on small or large parameters, which shows its validity and potential for the solution of nonlinear problems in science and engineering applications (Marinca and Herisanu, 2015).

Chapter 3

Methodology

3.1 Study Area and Period

The study was conducted at Jimma University under the department of mathematics from September, 2019 G.C. to August, 2020 G.C.

3.2 Study Design

In the study documentary review study design was analytical.

3.3 Source of Information

The sources of information used to conduct this study were:

1. Related reference book

2. Published articles.

3. Journals and etc.

3.4 Mathematical Procedure of the Study

The study has been conducted through the following general procedures :

1. Formulate the equation governing the flow problem.

2. Change the partial differential equation governing the flow problem in step 1 in to equivalent ordinary differential equation by using suitable similarity transformation.

3. Apply OHAM to the equation obtained in step 2 to get an expression determining the velocity and temperature of the flow.

4. Solve the equation obtaind in step 3 by using respective boundary conditions.

5. Analyze effects of different parameters embedded in the governing problem on velocity and temperature of the flow.

6. Visualize the effect of different parameters on velocity and temperature profile graphically by using mathlab.

Chapter 4

Result and discussion

4.1 Basic Principles of Optimal Homotopy Asymptotic Method

The basic principles of OHAM as expanded by Marinca, Herisanu and other researchers were as follows:
1. Consider the following differential equations

$$A\,[V] + a(x) = 0,\ \mathrm{x}\ \in \Omega \tag{4.1}$$

Where Ω is problem domain, A(v) = L(v) + N(v), where L, N are linear & nonlinear operators, v(x) is an unknown function, a(x) is a known function.
2. Construct an optimal homotopy equation as

$$(1-p)\,[L(\phi(x,p)) + a(x)] - H(p)\,[A(\phi(x,p)) + a(x)] = 0, \tag{4.2}$$

Where $0 \leq p \leq 1$ is an embedding parameter H(p) = $\sum_{k=1}^{m} p^k c_k$ is an auxiliary function on which the convergence of the solution greatly dependent. The auxiliary function H(p) also adjusts the convergence domain and controls the convergence region.
3. Expand $\phi(x,p,c_j)$ in Taylor's series about p, one has an approximate solution.

$$\phi(x,p,c_j) = v_0(x) + \sum_{k=1}^{\infty} v(x,c_j)p^k, j = 1,2,3,... \tag{4.3}$$

Many researchers have observed that the convergence of the series Eq. (4.3) depends upon c_j, (j = 1, 2, 3, ..., m)

$$\hat{v} = v_0(x) + \sum_{k=1}^{m} v(x,c_j) \tag{4.4}$$

4. Substituting Eq.(4.4) in to Eq.(4.1) we have the following residual:

$$R(x,c_j) = L(\hat{v}(x,c_j)) + a(x) + N(\hat{v}(x,c_j)), j = (1,2,3,...,m) \tag{4.5}$$

If R(x,c_j) = 0, then $\hat{v}$ will be the exact solution.
5. Finally, substituting those constants in Eq. (4.4) and one can get the approximate solution.

4.2 Mathematical formulation

Following the Raptis and Takher (1987) model for the permeability medium and by introducing the boundary layer approximation the governing continuity, momentum and energy equation of the flow can be written as follows.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4.6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-1}{\rho}\frac{\partial p}{\partial x}+\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\upsilon}{K}u-\frac{\sigma B_0^2}{\rho}\mu, \tag{4.7}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho c_p}\frac{\partial q_r}{\partial y}+\frac{\upsilon}{\rho c_p}(\frac{\partial u}{\partial y})^2, \tag{4.8}$$

Where the fluid is placed in a two dimensional environment where the x-axis is taken parallel to the surface while y-axis extends upwards normal to the surface and u and v are velocity components in x-and y-directions, respectively; P is the pressure; ρ is the fluid density; $\upsilon=\frac{\mu}{\rho}$ is the kinematic viscosity where μ is the coefficient of fluid viscosity; K is porosity parameter, σ is the electrical conductivity; B_0 is the applied magnetic field along y-direction; T is the temperature of the fluid and the porous medium which are in local thermal equilibrium; $\alpha=\frac{k}{\rho c_p}$ is the equivalent thermal diffusivity, where k is coefficient of thermal conductivity; c_p is the specific heat at constant pressure, and q_r is the radiative heat flux. The boundary conditions are defined as follows:

$$\begin{cases} y=0; v=v_w, u=u_w+u_{slip}=N\upsilon\frac{\partial u}{\partial y}, \\ T=T_w+T_{silp}=T_\infty+AX^\lambda+D\frac{\partial T}{\partial y}, \\ y\to\infty; u=u_e=ax, T=T_\infty, \end{cases} \tag{4.9}$$

Where v_w is the uniform surface mass flux positive for blowing and negative for suction; u_{slip} is velocity slip, which is proportional to the local wall shear stress and is given by $N\upsilon\frac{\partial u}{\partial y}$, where N is Naviers constant slip length; λ is the exponent of the wall temperature; D is temperature slip factor, and u_e = ax is the free stream velocity where a is a positive number.

The governing partial differential equations representing the flow problem are reduced to nonlinear higher order ordinary differential equations by using similarity transformations.
Using the free stream velocity $u = u_e = ax$ in equation (4.7) we have
$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\partial u_e}{\partial x}$
$\implies u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\partial u_e}{\partial x} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u_e}{\partial y^2} - \frac{\upsilon}{K}u_e - \frac{\sigma B_0^2 u_e}{\rho}$
$\implies u_e\frac{\partial u_e}{\partial x} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u_e}{\partial y^2} - \frac{\upsilon}{K}u_e - \frac{\sigma B_0^2 u_e}{\rho}$

$$\implies \frac{-1}{\rho}\frac{\partial p}{\partial x} = u_e\frac{\partial u_e}{\partial x} + \frac{\upsilon}{K}u_e + \frac{\sigma B_0^2 u_e}{\rho}. \tag{4.10}$$

plugging Eq. (4.10) in to Eq. (4.7) we have:
$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\partial u_e}{\partial x} + \frac{\upsilon}{K}u_e + \frac{\sigma B_0^2 u_e}{\rho} + \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\upsilon}{K}u - \frac{\upsilon}{K}u_e - \frac{\sigma B_0^2 u}{\rho}$,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u_e}{\partial y^2} + u_e\frac{\partial u_e}{\partial x} - \frac{\upsilon}{K}(u - u_e) - \frac{\sigma B_0^2}{\rho}(u - u_e). \tag{4.11}$$

Equation (4.11) illustrated that the free stream velocity affects the flow of a fluid, and therefore convection of heat will be affected considerably. This shows that the flow is of forced convection type. It is assumed that the viscous dissipation is neglected, the physical properties of the fluid are constant and the boussinesq and boundary layer approximation may be adopted for steady laminar flow. The fluid is considered to be gray, absorbing-emitting radiation but non-scatter medium. The radiative heat flux is described by Rosseland approximation.

Next, we can express the radiative heat flux in terms of temperature using Rosseland approximation for radiation Brewster(1992) as follows.

$$q_r = \frac{-4\sigma'}{3k_1}\frac{\partial T^4}{\partial y}, \tag{4.12}$$

Where σ' depicts the Stefan-Boltzmann constant and k_1 the mean absorption coefficient. Suppose the temperature differences within the flow are sufficiently small, so that T^4 can be expressed as linear function after using Taylor series to expand T^4 about the free stream temperature T_∞ and neglecting higher order terms. This result is the following approximation:

$$T^4 \equiv 4T_\infty^3 T - 3T_\infty^4. \tag{4.13}$$

Plug Eq. (4.12) & Eq. (4.13) in to Eq(4.8) gives
$q_r = \frac{-4\sigma'}{3k_1}\frac{\partial}{\partial y}[4T_\infty^3 T - 3T_\infty^4] = \frac{-16\sigma'}{3k_1}(\frac{\partial T}{\partial y})T_\infty^3 = \frac{-16\sigma' T_\infty^3}{3k_1}\frac{\partial T}{\partial y}$

$$\Longrightarrow \frac{\partial q_r}{\partial y} = \frac{-16\sigma' T_\infty^3}{3k_1}\frac{\partial T}{\partial y}. \tag{4.14}$$

Plugging Eq. (4.14) in to Eq. (4.8) we get:

$$u\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{16\sigma' T_\infty^3}{3k_1\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\upsilon}{\rho c_p}(\frac{\partial u}{\partial y})^2, \tag{4.15}$$

Where $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity, from this equation it is clearly seen that the influence of radiation is to enhance the thermal diffusivity.

Now, let us Introduce stream function ψ defined as

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}. \tag{4.16}$$

In view of relations in Eq. (4.16), Eq. (4.6) is satisfied automatically.

We then Introduce a similarity variable η and dimensionless stream function $f(\eta)$ and temperature defined as follows;

$$\eta = y\sqrt{\frac{a}{\alpha}}, \psi = \sqrt{a\alpha}xf(\eta), \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{4.17}$$

From Eq. (4.16) and Eq. (4.17) we get

$$\begin{cases} u = \frac{\partial \psi}{\partial y} = axf'(\eta), \\ v = -\frac{\partial \psi}{\partial x} = -\sqrt{a\alpha}f(\eta), \end{cases} \tag{4.18}$$

Where $f'(\eta) = \frac{df}{d\eta}$, T_w is a constant temperature of the wall, $\theta(\eta)$ is non-dimensional form of the temperature.

Using Eq. (4.7)-Eq. (4.18) we get

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} =& v\frac{\partial^2 u}{\partial y^2} + u_e\frac{\partial u_e}{\partial x} - \frac{\upsilon}{k}(u - u_e) - \frac{\sigma B_0^2}{\rho}(u - u_e), \\ &\implies (axf')(af') - \sqrt{a\alpha}f(axf'')(\sqrt{\frac{a}{\alpha}}) = v(\frac{a^2xf'''}{\alpha}) + \frac{\upsilon}{k}(axf' - ax) - \frac{\sigma B_0^2}{\rho}(axf' - ax), \\ &\Leftrightarrow a^2x(f')^2 - a^2xf''f = \frac{v}{\alpha}a^2xf''' + a^2x - \frac{\upsilon}{k}ax(f' - 1) - \frac{\sigma B_0^2}{\rho}ax(f' - 1), \\ &\Leftrightarrow f'^2 - f''f = \frac{v}{\alpha}f''' + 1 - \frac{\upsilon}{ak}(1 - f') + \frac{\sigma B_0^2}{\rho a}(f' - 1), \\ &\Leftrightarrow \frac{v}{\alpha}f''' + 1 - f'^2 + ff''' + \frac{\upsilon}{ka}(f' - 1) - \frac{\sigma B_0^2}{\rho a}(f' - 1) = 0, \\ &\Leftrightarrow Prf''' + (1 - f'^2) + ff'' + (\Omega + M^2)(1 - f') = 0, \end{aligned} \tag{4.19}$$

Where Pr = $\frac{v}{\alpha}$ is the Prdtal number, Ω = $\frac{v}{ka}$ is the permeability parameter, M = $\sqrt{\frac{\sigma B_0^2}{\rho a}}$ is the magnetic parameter.

From Eqs. (4.9) and Eq. (4.19) we get the following transformed velocity boundary condition;
at y = 0, v = v_w = $-\sqrt{a\alpha}\ f(0)$,
$\implies f(0) = \frac{-v_w}{\sqrt{a\alpha}}$ = s and u = $Nv\frac{\partial u}{\partial y} = axf'(0)$,
$\implies f'(0) = \frac{Nv}{ax}\ \frac{\partial u}{\partial x} = \frac{Nv}{ax}(axf''\sqrt{\frac{a}{\alpha}}) = Nvf''\sqrt{\frac{a}{\alpha}} = \beta f''(0)$,

Where s = -$\frac{v_w}{\sqrt{a\alpha}}$, $\beta = Nv\sqrt{\frac{a}{\alpha}}$,
As y $\to \infty$; $u = u_e = ax$ and $u = axf'$.
$\implies f_0'(\eta)$ = 1 as $\eta \to \infty$.

Using Eqs. (4.17) and (4.9) we get the transformed energy equation of the flows

$$\begin{cases} \frac{\partial T}{\partial y} = \theta'(T_w - T_\infty)\sqrt{\frac{a}{\alpha}}, \text{ and } \frac{\partial^2 T}{\partial y^2} = \theta'' \frac{a}{\alpha}(T_w - T_\infty), \\ T = T_\infty + Ax^\lambda + D\frac{\partial T}{\partial y} = T_w + T_{slip}, \\ T - T_\infty = Ax^\lambda + T - T_w, \\ T_w - T_\infty = Ax^\lambda, \end{cases} \tag{4.20}$$

From $T - T_\infty = Ax^\lambda + T - T_w$, we get

$$T - T_\infty = \theta(Ax^\lambda) \implies \frac{\partial T}{\partial x} = A\lambda\theta x^{\lambda-1}, \tag{4.21}$$

Thus, equation. (4.15) and (4.20) gives

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \alpha\frac{\partial^2 T}{\partial y^2} + \left(\frac{4\sigma' T_\infty^3}{k_1 k}\frac{4}{3}\alpha\right)\frac{\partial^2 T}{\partial y^2} + \frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 \\ &= \alpha\frac{\partial^2 T}{\partial y^2}\left(1 + \frac{4}{3}R\right) + \frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 \\ &\implies (axf')(A\lambda\theta x^{\lambda-1}) - (\sqrt{a}\alpha f)\theta'(T_w - T_\infty)\sqrt{\frac{a}{\alpha}} \\ &= \left(1 + \frac{4}{3}R\right)\frac{\theta''}{\alpha}a(T_w - T_\infty) + \frac{\upsilon}{c_p\alpha}u_e^2 a f''^2 \\ &\implies \frac{Ax^\lambda}{T_w - T_\infty}\theta f'\lambda - \theta' f = \left(1 + \frac{4}{3}R\right)\theta'' + pr\frac{u_e}{(T_w - T_\infty)}f''^2 \\ &\implies \left(1 + \frac{4}{3}R\right)\theta'' + f\theta' - \lambda f'\theta + prE_c f''^2 = 0. \end{aligned} \tag{4.22}$$

at $y = 0, T = T_w + T_{slip} = T_\infty + Ax^\lambda + D\frac{\partial T}{\partial y}$,
as $y \to \infty, T = T_\infty$ and $\theta = \frac{T - T_\infty}{T_w - T_\infty}$,
at $y = 0, \theta = \frac{Ax^\lambda + D\frac{\partial T}{\partial y}}{T_w - T_\infty} = \frac{Ax^\lambda + D\theta'(T_w - T_\infty)\sqrt{\frac{a}{\alpha}}}{T_w - T_\infty}$,

$\implies \frac{Ax^{\lambda}}{T_w - T_\infty} + D\theta' \sqrt{\frac{a}{\alpha}} = \theta,$
$\implies \theta(0) = 1 + \gamma\theta'(0)$ as $y \to \infty, T = T_\infty \implies \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} = 0.$

Hence the transformed equation representing the flow problem with respect to their boundary conditions are given as Eq. (4.23) and Eq. (4.24) below

$$\begin{cases} Prf''' + (1 - f'^2) + ff'' + (\Omega + M^2)(1 - f') = 0, \\ (1 + \frac{4}{3}R)\theta'' + f\theta' - \lambda f'\theta + PrE_c f''^2 = 0, \end{cases} \tag{4.23}$$

With boundary conditions

$$f(0) = s, f'(0) = \beta f''(0) \text{ and } f' = f(\infty) = 1, \theta(0) = 1 + \gamma\theta'(0), \theta(\infty) = 0, \tag{4.24}$$

Where, primes denote differentiation with respect to η, $\gamma = D\sqrt{\frac{a}{\alpha}}$, is the dimensionless thermal slip parameter, R = $\frac{4\sigma * T_\infty^3}{k_1 k}$ is the radiation parameter, $\alpha = \frac{k}{\rho c_p}$ is the equivalent thermal diffusivity, $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity(monument diffusivity), $E_c = \frac{u_e^2}{c_p T_w - T_\infty}$ is the Eckert number, $Pr = \frac{\upsilon}{\alpha}$ is the Pradtl number, k is the thermal conductivity, c_p the constant pressure specific heat capacity, μ is the dynamic viscosity, s = $-\frac{v_w}{\sqrt{a\alpha}}$, is the mass transfer parameter where $s > 0$ for suction, $s < 0$ for blowing and $s = 0$ reflects the impermeable surface, $\beta = Nv\sqrt{\frac{a}{\alpha}}$ is the dimensionless velocity slip parameter.

4.3 Solution of the problem

Applying OHAM to the nonlinear ordinary differential Eq. (4.23) we have:

$$\begin{cases} Prf'''(1 - p) = H_1(p)[Prf''' + (1 - f'^2) + ff'' + (\Omega + M^2)(1 - f')], \\ (1 + \frac{4}{3}R)\theta''(1 - p) = H_1(p)[(1 + \frac{4}{3}R)\theta'' + f\theta' - \lambda f'\theta + PrE_c f''^2], \end{cases} \tag{4.25}$$

Where the primes denote differentiation of the function f with respect to η We consider f, θ, H_1 and H_2 as follows, (Fazle *et .al* ., 2013).

$$
\begin{cases}
f(\eta) = f_0 + f_1 p + f_2 p^2 + f_3 p^3, \\
\theta(\eta) = \theta_0 + \theta_1 p + \theta_2 p^2 + \theta_3 p^3, \\
H_1(p) = c_1 p + c_2 p^2 + c_3 p^3, \\
H_2(p) = c_4 p + c_5 p^2 + c_6 p^3, p \in [0,1],
\end{cases}
\tag{4.26}
$$

Using Eq. (4.26) in Eq. (4.25) we obtain:

$$
\begin{aligned}
Prf'''(1-p) =& Pr(1-p)[f_0''' + f_1''' p + f_2''' p^2 + f_3''' p^3] \\
=& Pr[f''' + (f_1''' - f_0''')p + (f_2''' - f_1''')p^2 + (f_3''' - f_2''')p^3 - f_3''' p^4].
\end{aligned}
$$

$$
\begin{aligned}
PrH_1(p)f''' =& Pr[c_1 p + c_2 p^2 + c_3 p^3](f_0''' + f_1''' p + f_2''' p^2 + f_3''' p^3) \\
=& Pr[c_1 f_0''' p + (c_1 f_1''' + c_2 f_0''')p^2 + (c_1 f_2''' + c_2 f_1''')p^3 + ...].
\end{aligned}
$$

$$
\begin{aligned}
-H_1(p)f'^2 =& -(c_1 p + c_2 p^2 + c_3 p^3)[f_0^2 + 2f_0' f_1' p + (2f_0' f_2')p^2 + (2f_0' f_3' + 2f_1' f_2')p^3 + ...] \\
=& -[c_1 f^2 p + (2f_0' f_1' c_1 + c_2 f'^2)p^2 + [(c_1(2f_0' f_2' + f_1^2)) + 2c_2 f_0' f_1' + c_3 f_0'^2]p^3 \\
&+ c_1(2f_0' f_3' + 2f_1' f_2')p^4 + ...].
\end{aligned}
$$

$$
\begin{aligned}
H_1(p)ff'' =& (f_0 + f_1 p + f_2 p^2 + f_3 p^3)(f_0'' + f_1'' p + f_2'' p^2 + f_3'' p^3)(c_1 p + c_2 p^2 + c_3 p^3) \\
=& (f_0 f_0'' + (f_0 f_1'' + f_1 f_1'')p + (f_0 f_2'' + f_1 f_1'' + f_2 f_2'')P^2 \\
&+ (f_0 f_3'' + f_1 f_2'' + f_2 f_1'' + f_3 f_0'')p^3 + ...)H_1(p) \\
=& c_1 f_0 f_0'' p + [c_1(f_0 f_1'' + f_1 f_0'') + c_2 f_0 f_0'']p^2 + [c_1(f_0 f_2'' + f_1 f_1'' + f_2 f_0'') \\
&+ c_2(f_0 f_1'' + f_1 f_0'') + c_3 f_0 f_0'']p^3 + ...
\end{aligned}
$$

$$
\begin{aligned}
H_1(\Omega + M^2)(1-f') =& (\Omega + M^2)(1 - f_0' - f_1' p - f_2' p^2 - f_3' p^3)(c_1 p + c_2 p^2 + c_3 p^3) \\
=& (\Omega + M^2)[c_1(1-f_0')p + [c_2(1-f_0') - c_1 f_1']p^2 + [c_3(1-f_0') - c_2 f_1' - c_1 f_2']p^3].
\end{aligned}
$$

Collecting similar terms based on power of p, we obtain:

The Zeroth order problem:

$f_0'''(\eta) = 0,$

With boundary condition,

$f_0(0) = \mathrm{S}$, $f_0'(0) = \beta f_0''(0)$, $f_0'(\eta) = 1$, as $\eta \to \infty$.

The First order problem:

$Pr(f_1''' - f_0''') = Prc_1 f_0''' + c_1 - c_1 f_0'^2 + c_1 f_0 f_0'' + c_1(\Omega + M^2)(1 - f_0'),$

$$\Longrightarrow\ Prf_1''' = Pr(c_1+1)f_0''' + c_1 - c_1 f_0'^2 + c_1 f_0 f_0'' + C_1(\Omega+M^2)(1-f_0')),$$
$$\Longrightarrow\ f_1'''(\eta,c_1) = \tfrac{C_1}{Pr}(1-f_0'^2+f_0f_0''+(\Omega+M^2)(1-f_0')),$$

With boundary condition,
$f_1(0) = 0 = f_1'(0),\ f_1'(\eta) = 0,$ as $\eta \to \infty$.

The Second order problem:

$$\begin{aligned}
Pr(f_2'''-f_1''') =& Pr(c_1 f_1''' + c_2 f_0''') + c_2 - (2f_0'f_1'c_1 + c_2 f_0'^2) + c_1(f_0 f_1'' + f_1 f_0'') + c_2 f_0 f_0'' \\
&+ (\Omega+M^2)(c_2 - c_2 f_0' - c_1 f_1'),\\
\Longrightarrow\ f_2'''(\eta,c_1,c_2) =& (c_1+1)f_1''' + \frac{c_2}{Pr}(1-f_0'^2+f_0f''+(\Omega+M^2)(1-f_0'))\\
&+\frac{c_1}{Pr}(f_0f_2''+f_1f_0''-2f_0'f_1'-f_1).
\end{aligned}$$

With boundary condition,
$f_2(0) = f_2'(0) = 0,\ f_2'(\eta) = 0$ as $\eta \to \infty$

The Third order problem:

$$\begin{aligned}
Pr(f_3'''-f_2''') =& Pr(c_1 f_2''' + c_2 f_1''' + c_3 f_0''') + c_3 - c_1(2f_0'f_2' + f_1'^2) - 2c_2 f_0'f_1' - c_3 f_0'^2\\
&+ c_2(f_0f_1''+f_1f_0'') + c_3 f_0 f_0'' + (\Omega+M^2)[c_3(1-f_0') - c_2 f_1' - c_1 f_2']\\
\Longrightarrow\ f'''(\eta,c_1,c_2,c_3) =& (1+c_1)f_2''' + c_2 f_1'' + \frac{c_3}{Pr}(1-f_0'^2+f_0F_0'')(\Omega+M^2)(1-f_0')\\
&+\frac{c_3}{Pr}(f_0f_1''+f_1f_0''-2f_0'f_1'-(\Omega+M^2)f_1')\\
&-\frac{c_3}{Pr}(2f_0'f_2'+f_1'^2+(\Omega+M^2)f_2')
\end{aligned}$$

With boundary condition,
$f_3(0) = f_3'(0) = 0,\ f_3'(\eta) = 0$ as $\eta \to \infty$.

Similarly applying OHAM onto the thermal equation of the problem we have:
$(1-p)(1+\frac{4}{3}R)\theta'' = H_2(p)[(1+\frac{4}{3}R)\theta'' + f\theta' - \lambda f'\theta + PrE_c f''^2]$

$$
\begin{aligned}
(1-p)(1+\frac{4}{3}R)\theta'' =&(1-p)(1+\frac{4}{3}R)(\theta_0''+\theta_1''p+\theta_2''p^2+\theta_3''p^3)\\
=&(1+\frac{4}{3}R)[\theta_0''+(\theta_1''-\theta_0'')p+(\theta_2''-\theta_1'')p^2+(\theta_3''-\theta_2'')p^3-\theta_3''p^4]
\end{aligned}
$$

$$
\begin{aligned}
H_2(p)(1+\frac{4}{3}R)\theta'' =&(c_4p+c_5p^2+c_6p^3)(1+\frac{4}{3}R)[(\theta_0''+\theta_1''p+\theta_2''p^2+\theta_3''p^3)]\\
=&\ (1+\frac{4}{3}R)[c_4\theta_0''p+(c_4\theta_1''+c_5\theta_0'')p^2+(c_4\theta_2''+c_5\theta_1''+c_6\theta_0'')p^3+...]
\end{aligned}
$$

$$
\begin{aligned}
f\theta' =&(f_0+f_1p+f_2p^2+f_3p^3)(\theta_0'+\theta_1'p+\theta_2'p^2+\theta_3'p^3)\\
=&\ f_0\theta_0'+(f_1\theta_0'+f_0\theta_1')p+(f_0\theta_2'+f_1\theta_1'+f_2\theta_0')p^2+(f_0\theta_3'+f_1\theta_2'+f_2\theta_1'+f_3\theta_0')p^3+...
\end{aligned}
$$

$$
\begin{aligned}
H_2(p)f\theta' =&(c_4p+c_5p^2+c_6p^3)(f\theta')\\
=&\ c_4f_0\theta'p+[c_4(f_1\theta_0'+f_0\theta_1')\\
&+c_5f_0\theta_0']p^2+[c_4(f_0\theta_2'+f_1\theta_1'+f_2\theta_0')+c_5(f_1\theta_0'+f_0\theta_1')]p^3+...
\end{aligned}
$$

$$
\begin{aligned}
-\lambda f'\theta =&-\lambda(f_0'+f_1p+f_2'p^2+f_3'p^3)(\theta_0+\theta_1p+\theta_2p^2+\theta_3p^3)\\
=&-\lambda(f_0'\theta_0+(f_1'\theta_0+f_0'\theta_1)p+(f_0'\theta_2+f_1'\theta_1+f_2'\theta)p^2\\
&+[(f_0\theta_2'+f_1\theta_1'+f_2\theta_0')+c_5(f_1\theta_0'+f_0\theta_1')p^3+...]
\end{aligned}
$$

$$
\begin{aligned}
H_2(p)(-\lambda f'\theta) =&-\lambda(c_4p+c_5p^2+c_6p^3)(f'\theta)\\
=&-\lambda[c_4f_0'\theta_0p+[c_4(f_0'\theta_0+(f_0'\theta_1)+c_5f_0'\theta_0)]p^2+(c_4(f_0'\theta_2+f_1'\theta_1+f_2'\theta_0)\\
&+c_5(f_1'\theta_0+f_0'\theta_1)+c_6f_0'\theta_0)p^3+...]
\end{aligned}
$$

$$
\begin{aligned}
f''^2 =&(f_0''+f_1''p+f_2''p^2+f_3''p^3)(f_0''+f_1''p+f_1''p^2+f_3''p^3)\\
=&\ f_0''^2+2f_0''f_1''p+(2f_0''f_2''+f_1''^2)p^2+(2f_0''f_3''+2f_1''f_2'')p^3+...
\end{aligned}
$$

$$
\begin{aligned}
PrE_cH_2(p)f''^2 =&PrE_c(c_4p+c_5p^2+c_6p_3)f''^2\\
=&\ PrE_c[c_4f_0''^2p+[2c_4f_0''f_1''+c_5f_0''^2]p^2+[c_4(2f_0''f_2''+f_1''^2)+2c_5f_0''f_1'']p^3+...]
\end{aligned}
$$

Collecting similar terms based on the power of p we get the Zeroth, first, second and the third order problem with their respective boundary condition.

The Zeroth order problem:

$(1+\frac{4}{3}R)\theta'' = 0 \Longrightarrow\ \theta''(\eta) = 0$

With boundary conditions;

$\theta_0(0) = 1+\gamma\theta_0'(0), \theta_0(\infty) = 0$

First order problem:

$(1+\frac{4}{3}R)(\theta_1''-\theta_0'') = (1+\frac{4}{3}R)c_4\theta_0''+c_4 f_0\theta_0''-c_4\lambda\theta_0 f_0'+PrE_c c_4 f_0''^2$

$\theta''(\eta,c_4) = (1+c_4)\theta_0''+\frac{3c_4}{3+4R}[f_0\theta_0'-\lambda f_0'\theta_0+PrE_c f_0''^2]$

With boundary conditions;

$\theta_0(0)=0, \theta_0(\infty)=0$

The Second order problem:

$$\begin{aligned}(1+\frac{4}{3}R)(\theta_2''-\theta_1'') =& (1+\frac{4}{3}R)(c_4\theta_1''+c_5\theta_0'')+C_4(f_1\theta_0'+f_0\theta_1')\\ &+c_5 f_0\theta_0'-\lambda c_4(f_1'\theta_0+f_0'\theta_1)-\lambda c_5 f_0'\theta_0+PrE_c(2c_4 f_0''f_1''+c_5 f_0''^2)\end{aligned}$$

$$\begin{aligned}\implies \theta_2''(\eta,c_4,c_5) =& (1+c_4)\theta_1''+c_5\theta_0''+\frac{3c_4}{3+4R}[f_1\theta_0'+f_0\theta_1'-\lambda(f_1'\theta_0+f_0\theta_1)\\ &+2PrE_c f_0''f_1''+\frac{3c_5}{3+4R}[f_0\theta_0'-\lambda f_0'\theta_0+PrE_c f_0''^2]]\end{aligned}$$

With boundary conditions;

$\theta_2(0)=0, \theta_2(\infty)=0$

The third order problem:

$$\begin{aligned}(1+\frac{4}{3}R)(\theta_3''-\theta_2'') =& (1+\frac{4}{3}R)(c_4\theta_2''+c_5\theta_1''+C_6\theta_0'')\\ &+C_4(f_0\theta_2'+f_1\theta_1'+f_2\theta_0')+c_5(f_1\theta_0'+f_0\theta_1')\\ &-c_4\lambda(f_0\theta_2'+f_1'\theta_1+f_2'\theta_0)c_5\lambda(f_1\theta_0'+f_0'\theta_1)-c_6\lambda f_0'\theta_0\\ &+PrE_c c_4(2f_0''f_2''+f_1''^2)+2PrE_c c_5 f_0''f_1''\end{aligned}$$

$$\begin{aligned}\implies \theta_2''(\eta,c_4,c_5,c_6) =& (1+c_4)\theta_2''+c_5\theta_1''+c_6\theta_0''\\ &+\frac{3c_4}{3+4R}[f_0\theta_2'+f_1\theta_1'+f_2\theta_0'-\lambda(f_0'\theta_2+f_1'\theta_1+f_2'\theta_0)+PrE_c(2f_0''f_2''+f_1''^2)]\\ &+\frac{3c_4}{3+4R}[f_1\theta_0'+f_0\theta_1'-\lambda(f_1'\theta_0+f_0'\theta_1)+2PrE_c f_0''f_1'']+\frac{-3c_6\lambda}{3+4R}f_0'\theta_0'\end{aligned}$$

With boundary condition;
$\theta_3(0) = 0, \theta_3(\infty) = 0$

Thus, Zeroth order problems:

$$\begin{cases} f_0'''(\eta) = 0 \\ \theta_0''(\eta) = 0 \end{cases} \tag{4.27}$$

With boundary conditions;

$$\begin{cases} f_0(0) = s, f_0'(0) = \eta f_0''(0), f_0'(\eta) = 1 \text{ as } \eta \to \infty \\ \theta_0(0) = 1 + \gamma\theta_0'(0), \theta_0(\eta) = 0 \text{ as } \eta \to \infty \end{cases} \tag{4.28}$$

Since this problem has semi infinity boundary conditions, we need to transform the coordinate to solve problem with boundary conditions analytically.

coordinate transformation

Let $\varepsilon = \frac{\eta}{\eta_\infty} \implies \eta = \varepsilon\eta_\infty \implies \frac{d\eta}{d\varepsilon} = \eta_\infty$

$f_0'(\varepsilon) = f_0'(\eta)\eta_\infty, f_0''(\varepsilon) = f_0''(\eta)\eta_\infty^2$ and $f_0'''(\varepsilon) = f_0'''(\eta)\eta_\infty^3,$

Thus, $f_0'''(\eta) \implies f_0'''(\varepsilon) = 0$

With boundary conditions; $f_0(0)$ = s , $f_0'(0) = \frac{\beta f_0''(0)}{\eta_\infty}$ and $f_0'(1) = \eta_\infty$ at $\varepsilon \to 1$ and
$\theta_0'(\varepsilon) = \theta_0'(\eta)\eta_\infty \implies \theta_0'(\eta) = \frac{\theta_0'(\varepsilon)}{\eta_\infty}$ (by coordinate transformation).
$\therefore \theta_0''(\varepsilon) = 0,$

With boundary condition;
$\theta_0(0) = 1 + \frac{\gamma\theta_0'(0)}{\eta_\infty}, \theta_0(1) = 0.$

Thus, the transformed zeroth order problem becomes

$$\begin{cases} f_0'''(\varepsilon) = 0, \\ \theta_0''(\varepsilon) = 0, \end{cases} \tag{4.29}$$

With boundary conditions;

$$\begin{cases} f_0(0) = s, f_0'(0) = \beta f_0''(0), \\ f_0'(\varepsilon) = \eta_\infty \text{ as } \varepsilon \to 1 \\ \theta_0(0) = 1 + \frac{\gamma\theta_0'(0)}{\eta_\infty}, \theta_0(1) = 0. \end{cases} \tag{4.30}$$

Eq. (4.29) is solved by successive integration with respect to ε and using boundary conditions Eq. (4.30) we get.
$f_0(\varepsilon) = \frac{(\eta_\infty^2 - \beta f_0''(0))\varepsilon^2}{2\eta_\infty} + \frac{\beta f_0''(0)\varepsilon}{\eta_\infty} + s$
$\implies f_0(\eta) = \frac{(\eta_\infty^2 - \beta f_0''(0))}{2\eta_\infty}(\frac{\eta}{\eta_\infty})^2 + \frac{\beta f_0''(0)\eta_\infty^2}{\eta_\infty}\frac{\eta}{\eta_\infty}n + s$

$$\Leftrightarrow f_0(\eta) = \frac{1}{2\eta_\infty}(1 - \beta f_0''(0))\eta^2 + \beta f_0''(0)\eta + s \tag{4.31}$$

Similarly, applying successive integration to the second Eq. (4.29) and using respective boundary conditions we get
$\theta_0(\varepsilon) = (\frac{\eta_\infty + \gamma\theta_0'(0)}{\eta_\infty})(1 - \varepsilon)$

$$\implies \theta_0(\eta) = (1 + \gamma\theta_0'(0))(1 - \frac{\eta}{\eta_\infty}). \tag{4.32}$$

First order problem:

$$\begin{cases} f_1'''(\eta, c_1) = \frac{C_1}{Pr}(1 - f_0'^2 + f_0 f_0'' + (\Omega + M^2)(1 - f_0')), \\ \theta_1''(\eta, c_4) = \frac{3c_4}{3+4R}[f_0\theta_0' - \lambda f_0'\theta_0 + PrE_c f_0''^2], \end{cases} \tag{4.33}$$

With boundary condition:

$$f_1(0) = f_1'(0) = 0, \theta_1(0) = 0, f_1(\eta) = \theta_1(\eta) = 0, \text{ as } \eta \to \infty. \tag{4.34}$$

Plugging Eq.(4.31), Eq.(4.33) and their derivatives and using coordinate transformation we get

$$
\begin{cases}
f_1''(\varepsilon,c_1) = & \frac{C_1\eta_\infty}{Pr}\left[\eta_\infty^2+(\Omega+M^2)(\eta_\infty^2-\beta f_0''(0))+s\eta_\infty-\frac{\beta f_0''(0)}{\eta_\infty}(s+\beta f_0''(0))\right] \\
& +\frac{C_1\eta_\infty}{Pr}\left[(\frac{\beta f_0''(0)}{\eta_\infty})^2-\beta f_0''(0)+(\Omega+M^2)(\beta f_0''(0)-\eta_\infty^2)\right]\varepsilon \\
& +\frac{C_1\eta_\infty}{Pr}\left[\frac{\eta_\infty^2}{2}-\eta_\infty^2+\beta f_0''(0)-\frac{1}{2}(\frac{\beta f_0''(0)}{\eta_\infty})^2\right]\varepsilon^2 \\
\theta_1''(\varepsilon,c_4) = & \frac{-3c_4}{3+4R}\left[\frac{\eta_\infty+\gamma\theta_0'(0)}{\eta_\infty}(s+\beta f_0''(0)+(\beta f_0''(0)+\lambda(\eta_\infty^2-\beta f_0''(0)))\right]\varepsilon \\
& +\frac{-3c_4}{3+4R}\left[\frac{\eta_\infty+\gamma\theta_0'(0)}{\eta_\infty}(\frac{\eta_\infty^2}{2}-\frac{\beta f_0''(0)}{2}+\lambda\beta f_0''(0)-\lambda\eta_\infty^2)\right]\varepsilon^2 \\
& -\frac{-3c_4}{3+4R}\left[\frac{PrE_c}{\eta_\infty^2}(\eta_\infty^2-2\beta f_0''(0)+(\frac{\beta f_0''(0)}{\eta_\infty})^2)\right]
\end{cases}
\tag{4.35}
$$

With boundary condition;

$$
f_1(0)=f_1'(0)=0, \theta_1(0)=0, f_1(\varepsilon)=\theta_1(\varepsilon)=0 \text{ as } \varepsilon\to\infty. \tag{4.36}
$$

Next, solve Eq.(4.35) by successive integration with respect to ε with the respective boundary condition in Eq.(4.36) and using the coordinate transformation we get the following solutions for the first order problem:

$$
\begin{cases}
f_1(\eta,c_1) = & \frac{c_1\eta_\infty}{2Pr}\left[(1+\Omega+M^2)(1-\beta f_0''(0))(\frac{s}{\eta_\infty}-\beta f_0''(0))(\frac{s}{\eta_\infty}-\beta f_0''(0))(\frac{\eta^3}{3\eta_\infty}-\frac{\eta^2}{2})\right] \\
& +\frac{c_1\eta_\infty}{2Pr}\left[\frac{1}{6}\left[\beta f_0''(0)(\beta f_0''(0)-1)+(\Omega+M^2)(\beta f_0''(0)-\frac{1}{\eta_\infty})\right]\right](\frac{\eta^4}{2\eta^2}-\eta^2) \\
& +\frac{c_1\eta_\infty}{2Pr}\left[\frac{1}{6}[\beta f_0''(0)(1-\frac{\beta f_0''(0)}{2})-\frac{1}{2}](\frac{\eta^5}{5\eta_\infty^3}-\frac{\eta^2}{2})\right] \\
\theta_1(\eta,c_4) = & \frac{-3c_4(1+\gamma\theta_0'(0))}{6+8R}[(s+\beta f_0''(0))(\eta^2-\eta_\infty\eta)+\frac{1}{3}(\beta f_0''(0) \\
& +\lambda(1-2\beta f_0''(0)))(\frac{\eta^3}{\eta}-\eta_\infty\eta)+\frac{1}{12}((1-2\lambda)(1-\beta f_0''(0)))(\frac{\eta^4}{\eta_\infty^2}-\eta_\infty\eta)] \\
& +\frac{3c_4PrE_c}{6+8R}[1+\beta f_0''(0)(\beta f_0''(0)-2)](\frac{\eta^2}{\eta_\infty^2}-\frac{\eta}{\eta_\infty})
\end{cases}
\tag{4.37}
$$

Second order problem:

$$\begin{cases} f_2'''(\eta,c_1,c_2) = & (c_1+c_2)f_1''' + \frac{c_2}{Pr}(1-f_0'^2+f_0f'' \\ & +(\Omega+M^2)(1-f_0')) + \frac{c_2}{Pr}(f_0f_2''+f_1f_0''-2f_0'f_1'-f_1) \\ \theta_2''(\eta,c_3,c_4) = & (1+c_3)\theta_1'' + \frac{3c_4}{3+4R}[f_1\theta_0'+f_0\theta_1'-\lambda(f_1'\theta_0+f_0\theta_1) \\ & +2PrE_cf_0''f_1'' + \frac{3c_4}{3+4R}[f_0\theta_0'-\lambda f_0\theta_0+PrE_cf_0''^2]] \end{cases} \tag{4.38}$$

With boundary condition;

$$f_2(0) = f_2'(0) = 0, f_2'(\eta) = 0, \theta_2(0) = \theta_2(1) = 0. \tag{4.39}$$

Eq. (4.36) is solved by successive integration with respect to η with the boundary condition in Eq. (4.37) is employed and its solution is:

$$
\begin{aligned}
f_2(\eta,c_1,c_2) =& \frac{\eta_\infty}{Pr}[\left[1+(\Omega+M^2)(1-\beta f_0''(0))+\frac{s}{\eta_\infty}-\beta f_0''(0)(\frac{s}{\eta_\infty}+\beta f_0''(0))\right]((c_1+c_1^2)(\frac{\eta^3}{6}-\frac{\eta^2}{4})\\
&+\frac{c_1^2}{Pr}(\frac{\eta^4}{24\eta_\infty}-\frac{\eta^3}{12}+\frac{\eta^2\eta_\infty}{24})+\frac{c_1^2}{Pr}(1-\beta f_0''(0))(\frac{-\eta^6}{144\eta_\infty^2}+\frac{\eta^5}{80\eta_\infty}-\frac{\eta_\infty^2\eta^2}{96})\\
&-\frac{c_1^2}{Pr}(2\beta f_0''(0)+1)(\frac{\eta^5}{120\eta_\infty}-\frac{\eta^4}{48\eta_\infty}+\frac{\eta_\infty^2\eta^2}{48}))]\\
&+\frac{\eta_\infty}{Pr}[[\beta f_0''(0)(\beta f_0''(0)-1)+(\Omega+M^2)(\beta f_0''(0)-\frac{1}{\eta_\infty})]((c_1+c_1^2)(\frac{\eta^4}{24\eta_\infty^2}-\frac{\eta^2}{12})\\
&+\frac{c_1^2}{Pr}(\frac{\eta^5}{12\eta_\infty^2}-\frac{\eta_\infty^3\eta^2}{36}+\frac{\eta_\infty^2\eta^2}{48})+(\frac{1-\beta f_0''(0)}{Pr})c_1^2(\frac{-\eta^7}{720\eta_\infty^3}+\frac{\eta^5}{240\eta_\infty}-\frac{13\eta^2\eta_\infty^2}{1440})\\
&-\frac{c_1^2}{Pr}(2\beta f_0''(0)+1)((\frac{\eta^6}{720\eta_\infty}-\frac{\eta^4}{144\eta_\infty}+\frac{7\eta_\infty^2\eta^2}{720}))]\\
&+\frac{\eta_\infty}{Pr}[[\beta f_0''(0)(1-\frac{\beta f_0''(0)}{2})-\frac{1}{2}][(c_1+c_1^2+c_2)(\frac{\eta^5}{60\eta_\infty^3}-\frac{\eta^2}{24})+\frac{c_1^2}{Pr}(\frac{\eta^6}{360\eta_\infty^3}-\frac{\eta^3}{72}\\
&+\frac{\eta_\infty\eta_\infty^2}{80}+(1-\beta f_0''(0))(\frac{\eta^5}{120\eta_\infty}-\frac{\eta^4}{48\eta_\infty}+\frac{\eta_\infty^2\eta^2}{48}))]]\\
&+\frac{\eta_\infty}{Pr}[[\beta f_0''(0)(1-\frac{\beta f_0''(0)}{2})-\frac{1}{2}][(c_1+c_1^2+c_2)(\frac{\eta^5}{60\eta_\infty^3}-\frac{\eta^2}{24})\\
&+\frac{c_1^2}{pr}(\frac{\eta^6}{360\eta_\infty^3}-\frac{\eta^3}{72}+\frac{\eta_\infty\eta^2}{80}+(1-\beta f_0''(0))(\frac{\eta^8}{20160\eta_\infty^4}-\frac{\eta^5}{1440\eta_\infty}+\frac{31\eta_\infty^2\eta^2}{20160}))]]\\
&+\frac{\eta_\infty}{Pr}[\frac{c_1^2}{Pr}(\beta f_0''(0)(\beta f_0''(0)-2))[(\beta f_0''(0)+\frac{1}{2})(\frac{\eta^7}{252\eta_\infty^3}-\frac{\eta^4}{288}+\frac{\eta_\infty^2\eta^2}{180})\\
&+(1-\beta f_0''(0))(\frac{\eta^8}{4032\eta_\infty}-\frac{\eta^5}{720\eta_\infty}+\frac{5\eta_\infty^2\eta^2}{2016})]]\\
&+\frac{\eta_\infty}{Pr}[[(1+\Omega+M^2)(\frac{\eta^3}{6\eta_\infty}-\frac{\eta^2}{4})+[\eta_\infty\beta f_0''(0)(\beta f_0''(0)-1)\\
&+(\Omega+M^2)(\beta f_0''(0)-1)](\frac{\eta^4}{24\eta_\infty^2}-\frac{\eta^2}{12})\\
&\quad+[s-\beta f_0''(0)(s+\eta_\infty\beta f_0''(0))-\beta f_0''(0)\eta_\infty(\Omega+M^2)](\frac{\eta^3}{6\eta_\infty^2}-\frac{\eta^2}{4\eta_\infty})]]
\end{aligned}
$$

$$
\begin{aligned}
\theta_2(\eta,c_4,c_5) =& \frac{-3c_4(\eta_\infty+\gamma\theta_0'(0))}{\eta_\infty(3+4R)}[(\frac{3c_4(s+\beta f_0''\eta_\infty^2)}{6+8R})[\frac{(1+c_4)(3+4R)}{3c_4}(\frac{\eta^2}{\eta_\infty^2}-1)\\
&+\frac{1}{2}(\eta_\infty-\beta f_0''(0))(\frac{\eta^5}{10\eta_\infty^3}-\frac{\eta^4}{12\eta_\infty^2}+\frac{\eta_\infty^2}{60})+s(\frac{\eta^3}{3\eta_\infty^2}-\frac{\eta^2}{2\eta_\infty}+\frac{\eta_\infty}{6})\\
&+\lambda(1-\beta f_0''(0))(\frac{\eta^4}{12\eta_\infty^2}-\frac{\eta^5}{20\eta_\infty^3}-\frac{\eta_\infty^2}{30})+\beta f_0''(0)[\frac{\eta^4}{6\eta_\infty^2}-\frac{\eta^3}{6\eta_\infty}+\lambda[\frac{\eta_\infty^3}{6}-\frac{\eta^4}{12\eta_\infty^2}-\frac{\eta_\infty^2}{12}]]]]\\
&+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[\frac{c_4(\beta f_0''(0))+\lambda(1-2\beta f_0''(0))}{6+8R}[\frac{(1+c_4)(3+4R)}{3c_4}(\frac{\eta^3}{3\eta_\infty}-\frac{\eta_\infty^2}{3})\\
&+\frac{1}{2}(\eta_\infty-\beta f_0''(0))(\frac{\eta^6}{30\eta_\infty^2}-\frac{\eta^4}{36}-\frac{\eta_\infty^4}{180})+\beta f_0''(0)(\frac{\eta^5}{20\eta_\infty}-\frac{\eta_\infty\eta^3}{18}+\frac{\eta_\infty^4}{180})\\
&+s(\frac{\eta^4}{12\eta_\infty^3}-\frac{\eta_\infty\eta^2}{6}+\frac{\eta_\infty^3}{12})+\lambda(1-\beta f_0''(0))(\frac{\eta^4}{36}-\frac{\eta^6}{90\eta_\infty^2}-\frac{\eta^4}{60})\\
&+\lambda[\frac{\eta_\infty\eta_3}{18}-\frac{\eta^5}{60\eta_\infty}-\frac{7\eta_\infty^4}{180}]]\\
&+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[\frac{3c_4(1-\beta f_0''(0)+2\lambda(\beta f_0''(0)-1))}{3+4R}[\frac{(1+c_4)(3+4R)}{3c_4}\\
&(\frac{\eta^4}{6\eta_\infty^2}-\frac{\eta_\infty^2}{6})+\frac{1}{36}(\eta_\infty-\beta f_0''(0))(\frac{\eta^6}{30\eta_\infty^2}-\frac{\eta^4}{36}-\frac{\eta_\infty^4}{180})+\beta f_0''(0)(\frac{2\eta^7}{7\eta_\infty^3}-\frac{\eta^4}{4}+\frac{\eta_\infty^4}{28})+\\
&\frac{\beta f_0''(0)}{6}(\frac{2\eta^6}{15\eta_\infty^2}-\frac{\eta_\infty\eta^3}{6})+\frac{s}{6}(\frac{\eta^5}{5\eta_\infty^2}-\frac{\eta_\infty\eta^2}{2})\\
&+\frac{\lambda}{18}(1-\beta f_0''(0))(\frac{\eta^4}{4}-\frac{\eta^7}{14\eta_\infty^3}-\frac{5\eta^4}{28})+\frac{\lambda\beta f_0''(0)}{6}(\frac{\eta_\infty\eta^3}{6}-\frac{\eta^6}{30\eta_\infty^2}-\frac{2\eta_\infty^4}{15})]]\\
&+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[[1+(\Omega+M^2)(1-\beta f_0''(0))+\frac{s}{\eta_\infty}\\
&-\frac{\beta f_0''(0)}{\eta_\infty}(s+\eta_\infty\beta f_0''(0))]\frac{c_1}{pr}(\frac{\eta^3}{120\eta_\infty}-\frac{\eta^4}{48}-\frac{\eta_\infty^4}{80}+\lambda(\frac{\eta^4}{12}-\frac{\eta_\infty\eta^3}{12}-\frac{\eta^5}{40\eta_\infty}+\frac{\eta_\infty^4}{80}))]\\
&+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[\frac{c_1}{pr}[(\beta f_0''(0))^2-\beta f_0''(0)+(\Omega+M^2)(\beta f_0''(0)-\frac{1}{\eta_\infty})]\\
&[\frac{\eta^6}{720\eta_\infty}-\frac{\eta^4}{144}+\frac{\eta_\infty^4}{180}+\lambda(\frac{\eta^5}{120\eta_\infty}-\frac{\eta_\infty\eta^3}{36}+\frac{\eta^4}{72}-\frac{\eta^6}{180\eta_\infty^2}+\frac{\eta_\infty^4}{90})]]\\
&+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[\frac{c_1}{Pr}(\beta f_0''(0)-\frac{1}{2}(\beta f_0''(0))^2-\frac{1}{2})(\frac{\eta^7}{2520\eta_\infty^3}-\frac{\eta^4}{288}\\
&+\lambda(\frac{\eta^6}{360\eta_\infty^2}-\frac{\eta_\infty\eta\eta^3}{72}+\frac{\eta^4}{144}-\frac{\eta^7}{504\eta_\infty^2}))]\\
&+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[3c_4PrE_c[(\frac{1-2\beta f_0''(0)+(\beta f_0''(0))^2}{6+8R})[\frac{(1+c_1)(3+4R)}{3c_4}(\frac{\eta^2}{\eta_\infty^2}-1)]]]
\end{aligned}
$$

$$+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[(\frac{\eta_\infty}{2}-\frac{\beta f_0''(0)}{2})+(\frac{\eta^5}{10\eta_\infty^4}-\frac{\eta^4}{12\eta_\infty^3}-\frac{\eta_\infty}{60})+\beta f_0''(0)(\frac{\eta^4}{6\eta_\infty^2}-\frac{\eta^3}{6\eta_\infty})]$$
$$+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[s(\frac{\eta^3}{3\eta_\infty^2}-\frac{\eta^3}{2\eta_\infty}+\frac{\eta_\infty}{6})+\lambda(1-\beta f_0''(0))(\frac{\eta^4}{12\eta_\infty^2}-\frac{\eta^5}{20\eta_\infty^3}-\frac{\eta_\infty^2}{30})]$$
$$+\frac{-3c_4(\eta_\infty)+\gamma\theta_0'(0)}{\eta_\infty(3+4R)}[\lambda\beta f_0''(0)(\frac{\eta^3}{6\eta_\infty}-\frac{\eta^4}{12\eta_\infty^2}-\frac{\eta_\infty^2}{12})]$$
$$-(\frac{3c_5}{3+4R})(1+\gamma\theta_0'(0))[(\frac{1}{2}-\frac{\beta f_0''(0)}{2}+\lambda(\beta f_0''(0)-1))(\frac{\eta^4}{12\eta_\infty^2}-\frac{\eta_\infty^2}{12})]$$
$$+(\frac{3c_5}{3+4R})(1+\gamma\theta_0'(0))[(\beta f_0''(0)+\lambda(1-2\beta f_0''(0)))(\frac{\eta^3}{6\eta_\infty}-\frac{\eta_\infty^2}{6})+(\frac{s+\lambda\beta f_0''(0)\eta_\infty}{2})(\frac{\eta^2}{\eta_\infty}-\eta_\infty)]$$
$$+\frac{3c_5 prE_c}{3+4R}((\beta f_0''(0))^2-2\beta f_0''(0)+1)(\frac{\eta^2}{2\eta_\infty}-\frac{\eta_\infty}{2}\eta_\infty)$$

Using OHAM for p = 1, we obtain the three terms solution:

$$\begin{cases} f(\eta,c_1,c_2)=f_0(\eta)+f_1(\eta,c_1)+f_2(\eta,c_1,c_2) \\ \theta(\eta,c_4,c_5)=\theta_0(\eta)+\theta_1(\eta,c_4)+\theta_2(\eta,c_4,c_5) \end{cases} \tag{4.40}$$

We use the method of least squares to obtain the four unknown convergent constants in Eq.(4.40).

For example, in case of Pr = 2, R = 1, $M=1$, $\Omega=1$, β = 0.2, s = 1, γ = 0.2, $E_c=0.1$ and λ = 1 the values of constants are $c_1=-0.01379646$, $c_2=4.181133$, $c_4=-0.04417926$ and $c_5=-905.104899$

By substituting those constants into Eq. (4.40), we obtain the approximate solution for the given problem on Eqs. (4.23).

4.4 Discussion

To obtain the solution of the differential equation(4.23) with the boundary conditions(4.24) a procedure based on OHAM is employed. The effects of various parameters on the flow (velocity) and temperature profile have been determined for different values of Prandtl number, Permeability parameter, Magnetic parameter, suction/blowing, Velocity slip parameter, Radiation parameter, Eckert number, exponent of wall temperature and thermal slip parameter.

The effects of various parameters on the non-dimensional velocity profile within the boundary layer are depicted in Figures $1-6$. The Figures $7-14$ illustrate the behaviors of temperature profile for different values of embedded parameters in the boundary layer.

Figure 4.1 illustrates the effects of magnetic parameter on the velocity profile. It shows an increasing trend in presence of slips corresponding to the increasing values of magnetic parameter. This occurs because of the fact that the magnetic force increases the fluid motion in the boundary-layer due to the presence of the term($u_e - u$) in the momentum equation.

Figure 4.2 describes the effects of permeability parameter on the velocity profiles in the presence of slip parameters. The figure shows that the velocity profile increase for increasing values of permeability parameter.

Figure 4.3 makes the graphical representation of the effects of velocity slip parameter on momentum profiles is an increase in slip parameter, results into a decrease in the momentum. The fluid velocity remains unaffected of the variations in the values of radiation parameter, wall temperature exponent and thermal slip . This is simply because of the flow problem being uncoupled from the thermal problem.

Figure 4.4 it is obvious that the velocity profiles exhibit an increasing trend with respect to the increasing values of suction/blowing parameter.

Figure 4.5 Velocity profiles for different values of Eckert number. It illustrates that E_c has no effect on the monument of fluid velocity.

Figure 4.6 illustrates the effects of Prandtl number on the temperature profiles in the presence of slip parameters. The figure shows that temperature profiles is increasing with respect to the increasing values of prandtl number.

Figure 4.7 shows that temperature profiles is decreasing with respect to the increasing values of suction/blowing parameter. This proves the profundity of the effect of suction/blowing parameter on the boundary-layer thickness. The suction reduces the thermal boundary-layer thickness while blowing thickens it. As a result, the process of suction can be used effectively for fast cooling. As the thermal boundary thickness increases with strong blowing, the heated fluid moves farther from the wall and forms an insulating layer of nearly the same temperature as that of the wall. This results into a decrease in the heat transfer rate from the wall, and hence leads to slower cooling.

Figure 4.8 shows that the thermal boundary-layer thicknesses show the same trend with the increasing values of the permeability parameter. In case of suction, the effect of permeability parameter on temperature profile is almost negligible.

Figure 4.9 makes the graphical representation of the effects of velocity slip on thermal boundary-layer thicknesses.Thus, an increase in slip parameter, results into a decrease thermal boundary-layer thicknesses.

Figure 4.10 it is concluded that the thermal boundary layer thickness and the temperature distribution increase with the increasing values of the thermal radiation parameter. This is due to the fact that the divergence of the radiative heat flux ($\frac{\partial qr}{\partial y}$) increases along with the decreasing values of the Rosseland radiative absorptivity (k_1). This, in turn, shows an increase in the rate of radiative heat transfer to the fluid. This causes the fluid temperature to increase. In view of this fact, the effect of radiation becomes more significant as $R \to \infty$, and the radiation effect is negligible as $R \to 0$

From the Figure 4.11 it is concluded that the thermal boundary layer thickness decreasing and the temperature distribution increase with the increasing values of the Eckert number parameter.

In the figure 4.12 the effect of wall temperature exponent on thermal boundary layer has been investigated, and it has been found that the temperature profile decreasing trend as the wall temperature exponent increases. In this case, the thermal boundary layer becomes thin. Further, from the figure the increase of exponent of wall temperature parameter results into a increase in thermal boundary layer thickness.

Figure 4.13 makes the graphical representation of the effects of thermal slip parameter on thermal boundary-layer thicknesses.Thus, an increase in thermal slip parameter, results into a decrease thermal boundary-layer thicknesses both section and blowing.

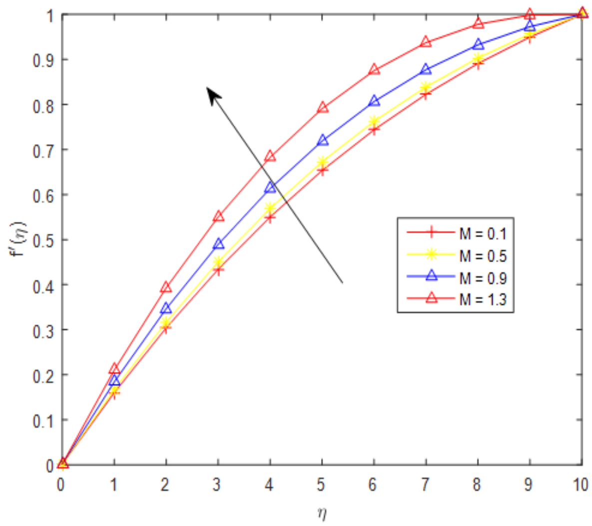

Figure 4.1: Velocity profiles for different values of M when $pr = 0.72$, $\Omega = 1$, $\beta = 0.2$, $R = 1$, $s = 1$, $\gamma = 0.2$, $E_c = 0.1$ and $\lambda = 1$.

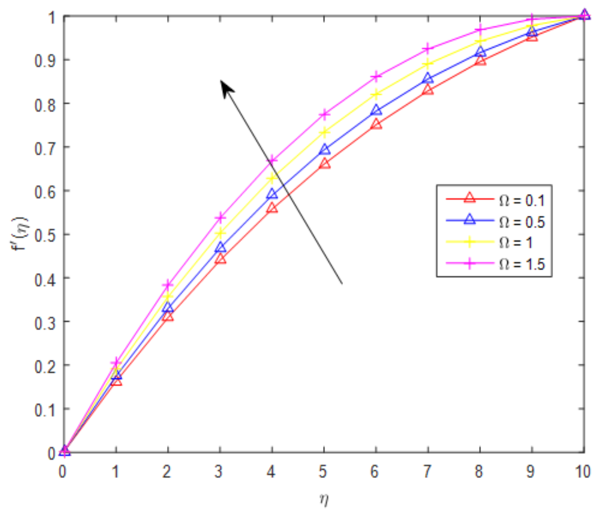

Figure 4.2: Velocity profiles for different values of Ω when Pr = 2, M = 1, β = 0.2, R = 1, s = 1, γ = 0.2, E_c = 0.1 and λ = 1.

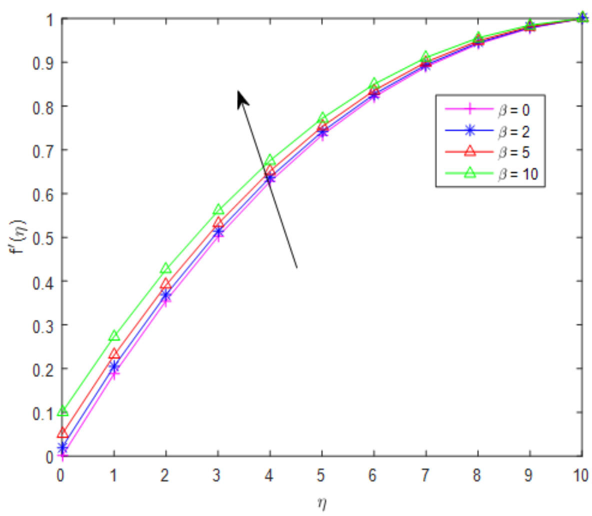

Figure 4.3: Velocity profiles for different values of β when Pr = 2, M = 1, R = 1, s = 1, γ = 0.2, Ω = 1, E_c = 0.1 and λ = 1.

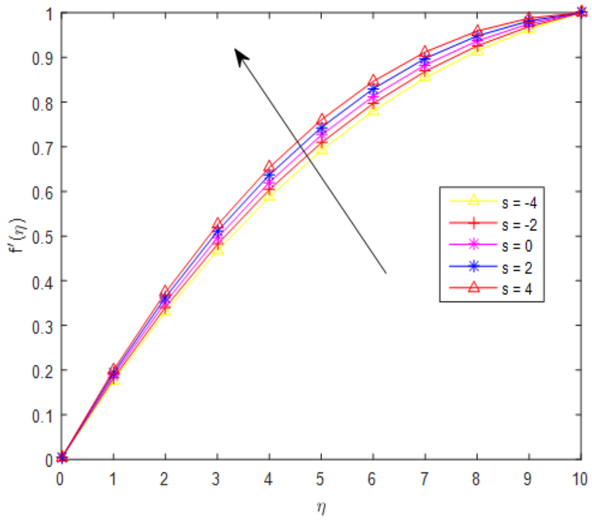

Figure 4.4: Velocity profiles for different values of s when Pr = 2, M = 1, R = 1, β = 0.3, γ = 0.2, Ω = 1, E_c = 0.1 and λ = 1.

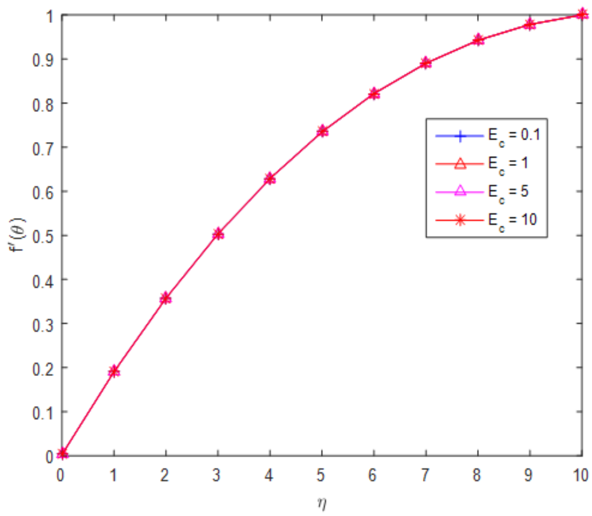

Figure 4.5: Velocity profiles for different values of E_c when Pr = 2, M = 1, R = 1, β = 0.3, γ = 0.2, Ω = 1, s = 1 and λ = 1.

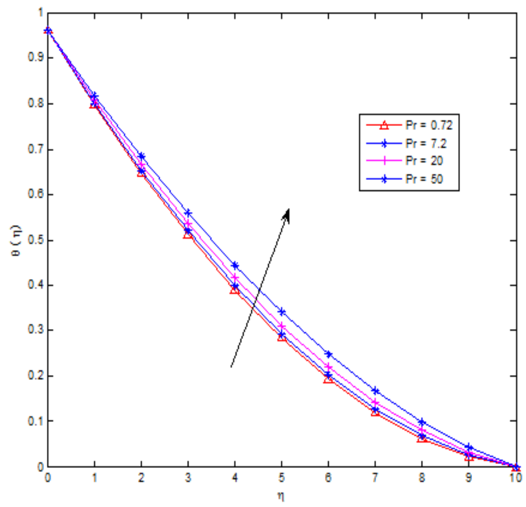

Figure 4.6: Temperature profiles for different values of Pr when M = 1, Ω = 2, β = 0.2, R = 1, s = 1, γ = 0.2, E_c = 0.1 and λ =1.

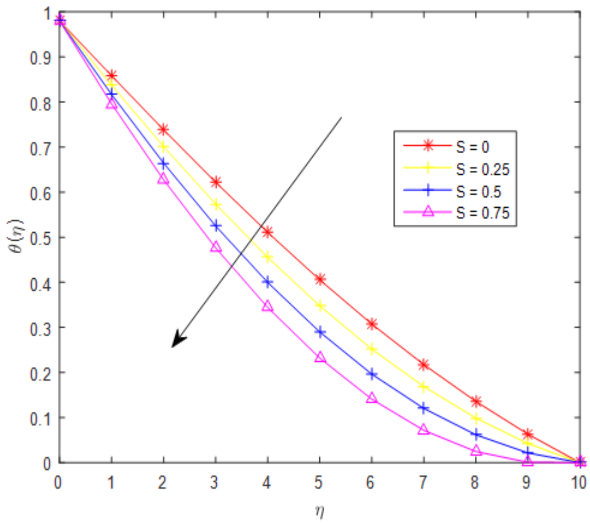

Figure 4.7: Temperature profiles for different values of s when Pr = 2, M = 1, R = 1, β = 0.2, γ = 0.2, Ω = 1, E_c = 0.1 and λ = 1.

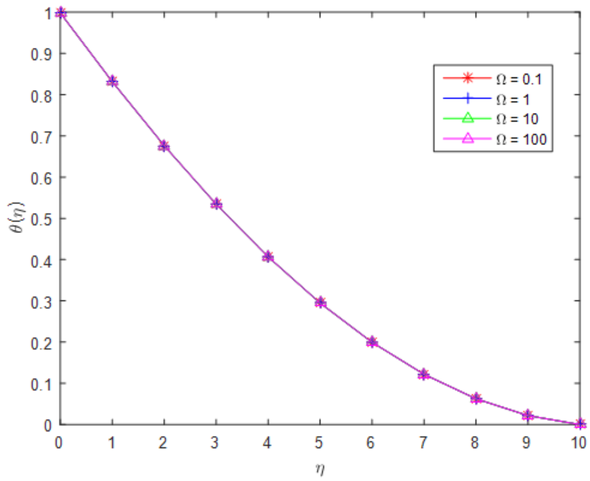

Figure 4.8: Temperature profiles for different values of Ω when $Pr = 2$, $M = 1$, $\beta = 0.2$, $R = 1$, $s = 0.5$, $\gamma = 0.2$, $E_c = 0.1$ and $\lambda = 1$.

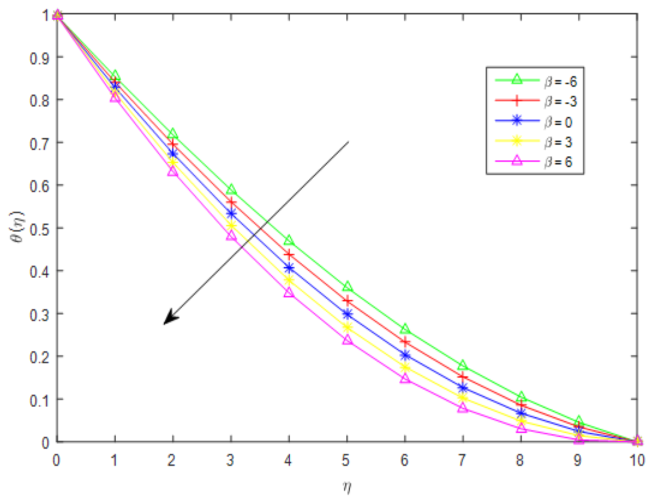

Figure 4.9: Temperature profiles for different values of β when $Pr = 2$, $M = 2$, $R = 2$, $s = 1$, $\gamma = 0.5$, $\Omega = 2$, $E_c = 1$ and $\lambda = 1$.

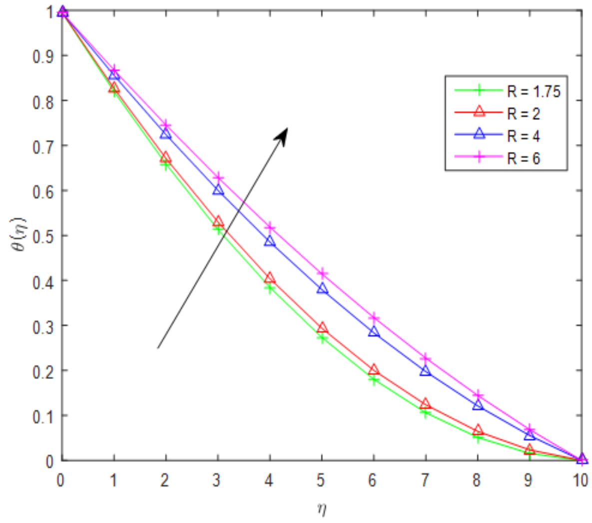

Figure 4.10: Temperature profiles for different values of R when M = 2, Pr = 0.72, Ω = 2, β = 0.2, s = 1, γ = 0.5, E_c = 1 and λ = 1.

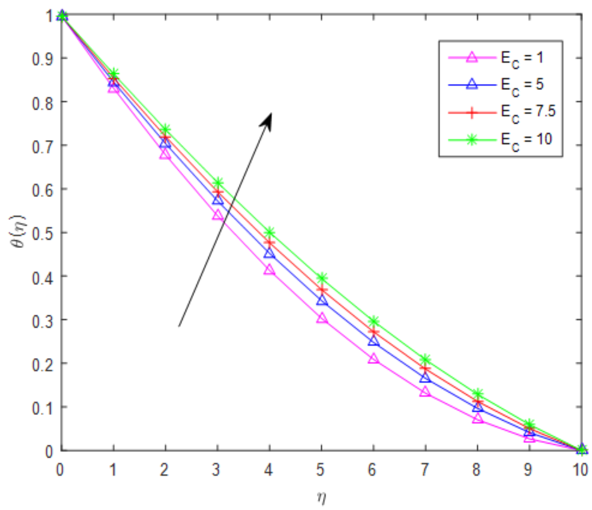

Figure 4.11: Temperature profiles for different values of Ec when M = 2, Pr = 7, Ω = 2, β = 0.2, R = 2, s = 1, γ = 0.4 and λ = 1.

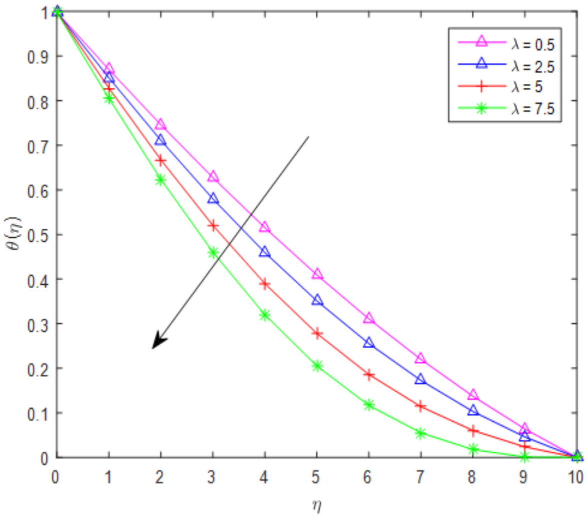

Figure 4.12: Temperature profiles for different values of λ when M = 2, Pr = 7, Ω = 2, β = 0.2, R = 2, s = 1, γ = 0.4 and E_c = 10.

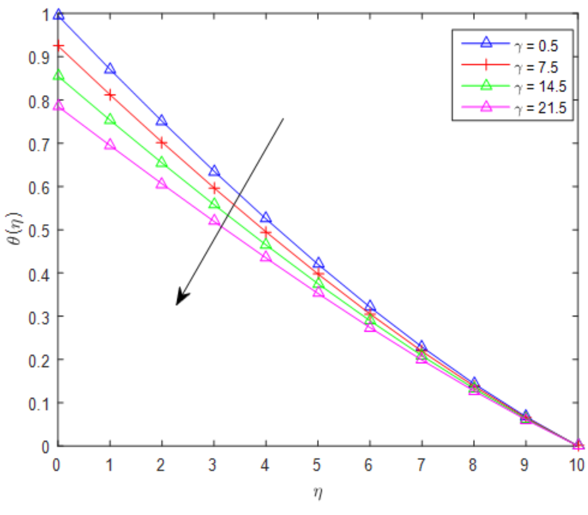

Figure 4.13: Temperature profiles for different values of γ when M = 2, Pr = 7, Ω = 2, β = 0.2, R = 2, s = 1, λ = 0.4 and E_c = 10.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

The present study is carried out to analyze the two dimensional steady MHD laminar forced convection stagnation point flow and heat transfer over a flat plate in a permeable medium with varying wall temperature by considering the radiation effect and partial slip conditions. The analysis has been done with the help of math lab based on the OHAM. The effects of the governing parameters Pr, Ω, M, s, β, E_c, R, λ and γ on the velocity and temperature profiles are examined in details. The following significant conclusions are drawn from the analysis:

1. An increase in Eckert number increase the temperature profile.

2. An increase in thermal slip parameter, results into a decrease thermal boundary-layer thickness both section and blowing.

3. The velocity profile is an increasing function of the parameters Ω, M, s and β.

4. An increase in permeability parameter does not change the temperature profile.

5. The thermal boundary layer thickness exhibit increasing trend along with Pr, E_c and R.

6. An increase in parameters β, s, λ and γ reduces the temperature profile.

7. An increase in Eckert number has no significant change in velocity profile.

5.2 Recommendation

In this study, OHAM is a powerful method for solving nonlinear problems without depending on small or large parameters, which shows its validity and potential for the solution of nonlinear problems in science and engineering applications.

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