SOLUTION OF TWO DIMENSIONAL NONLINEAR SINE-GORDON EQUATION BY USING THE REDUCED DIFFERENTIAL TRANSFORM METHOD

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY IN PARTIAL FULFILMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, the thesis entitled' **solution of two dimensional nonlinear sine-Gordon equation by using the reduced differential transform method "** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged .

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Acronyms

ADM - Adomain decomposition method

DE - Differential equation

DTM - Differential transform method

NLSGE - Nonlinear sine - Gordon equation

ODE - Ordinary differential equation

PDE - Partial differential equation

RDT - Reduced differential transform

RDTM - Reduced differential transform method

SGE - Sine- Gordon equation

VIM - Variational iteration method

Abstract

In this study, the reduced differential transform method (RDTM) was applied to solve two dimensional nonlinear sine-Gordon equation subject to the appropriate initial conditions arising in various physical models. This method provides the solutions in the form of infinite series expansions which converge to their exact solutions with easily computed terms. Using the RDTM the exact solution can be obtained by constructing a recursive formula. Three test modeling problems from mathematical physics, nonlinear Sine-Gordon equations are considered to verify the efficiency, accuracy and convergence of the proposed method. Moreover, Solutions obtained by RDTM are in close conformity with the solutions of earlier studies in the review of the literatures and the results obtained shows that the RDTM technique is highly accurate, efficient, convenient, converge and require less effort in comparison to the other analytical and numerical methods such as DTM. Also, results indicate that the introduced method is promising for solving other type of linear and nonlinear partial differential equations.

Key words: Partial differential equation,Two dimensional nonlinear Sine-Gordon equation, Reduced Differential Transform Method (RDTM), Convergence.

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CHAPTER ONE

Introduction

1.1 Background of the Study

Partial differential equation has big importance in mathematics and other fields of sciences. Especially the nonlinear partial differential equations (PDEs) arise frequently in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in applied mathematics, mathematical physics, and engineering science. One of the most important classes of PDEs occurring in applied mathematics is that associated with name of sine-Gordon. The sine-Gordon (SG) equation is a nonlinear hyperbolic PDE, which was originally considered in the nineteenth century in the course of study of surfaces of constant negative curvature. Then it attracted a lot of attention in the 1970s due to the presence of soliton solutions. In recent years the sine-Gordon equation (SGE) has also been used to describe physical models which possess soliton-like structures in higher dimensions. A typical example is the Josephson junction model which consists of two layers of super conducting material separated by an isolating barrier (Josephson, 1965).

Methods of solving the SGE have been focuses of many recent research works. Su (2019) obtained numerical solution of two dimensional nonlinear sine-Gordon equation using localized method of approximate particular solutions. Baccouch (2019) develop and analyze an energy-conserving local discontinuous Galerkin method for the two dimensional SGE on Cartesian grids. Ji (2018) develop Meshless singular Boundary method for the numerical solution of the time-dependent nonlinear sine-Gordon equation with Neumann boundary condition. In this method, by using a time discrete scheme to approximate the time derivatives, the time-dependent nonlinear problem is transformed into a sequence of time-independent linear boundary value problems. Then, the singular boundary method is used to establish the system of discrete algebraic equations. Duan *et al*. (2018) proposed a numerical model based on lattice Boltzmann method to obtain the numerical solutions of two dimensional generalized sine-Gordon equation and the method was extended to solve the other two dimensional wave equations, such as nonlinear hyperbolic telegraph equation as indicated in (Li *et al*., 2019).

It is well known that most phenomena in our world are essentially nonlinear and are described by nonlinear PDEs. Solutions of PDEs involving non-linear terms are extremely difficult in most of the situations and in some situations they are even not possible to solve completely. In such situations, Mathematicians usually apply some methods which involve lots of compromise and approximations. Other than exact solutions obtained by analytical methods, all these methods are computationally intensive because they are trial and error in nature and need complicated computations. A variety of numerical and analytical approximation methods have been developed to obtain accurate, approximate and analytic solutions for such nonlinear differential equations (DE).

The classical Taylor series method is one of the earliest analytic techniques used to solve many problems, especially ordinary differential equations (ODE). However, since it requires a lot of symbolic calculation for the derivatives of functions, it takes a lot of computational time for higher order derivatives. Later to simplify this complex calculation the updated version of the Taylor series method which is called the Differential Transform Method (DTM) was introduced by Zhou (1986) in Chinese and applied to solve initial value problems for electric circuit analysis. The DTM is one of the numerical methods in solving ordinary and partial differential equations. This method is based on Taylor's series expansion and can be applied to solve both linear and nonlinear ordinary differential equations (ODEs) as well as partial differential equations (PDEs). The DTM is a method used to determine the coefficient's of the Taylor series of the function by solving the induced recursive equation from the given DEs.

In this study, the Reduced Differential Transform Method (RDTM) was applied to solve the two dimensional nonlinear sine-Gordon equation (NLSGE). The RDTM was first envisioned by Keskin (2010) in his PhD thesis and successfully employed to many nonlinear PDEs as indicated in (Keskin and Oturanc, 2010) and (Mohmoud and Gubera, 2016). Keskin and Oturanc (2010) used the RDTM to solve linear and nonlinear wave equations and they showed that the number of iterations used by RDTM is less than the number of iterations used by DTM.

Recently, the RDTM has been widely used by many researchers to solve different nonlinear PDEs such as two dimensional unsteady incompressible Navier-stokes equations (Al-Saif, 2018), nonlinear integral member of kadomtsev-petviashivil hierarchy differential equations (Mohamed and Gepreel, 2017), two and three dimensional second order hyperbolic telegraph equations (Srivastava *et al*., 2017) and one dimensional nonlinear sine-Gordon equation (Ramesh Rao, 2017).

The RDTM recursive equations produce exactly all the Poisson series coefficients of solutions, whereas the DTM recursive equations produce exactly all the Taylor series coefficients of solutions (Gubes and Oturanc, 2016).The solution procedure of the RDTM is simpler than that of traditional DTM and the amount of computation required in RDTM is much less than that in traditional DTM.

In 2016, Gubes and Oturanc used RDTM and DTM to obtain the approximate solutions of coupled Ramani equation. They compare the RDTM solution with exact solution and solution of DTM. In order to test efficiency, convergence and accuracy of the two methods they consider numerical examples and the results shows that both DTM and RDTM are very effective and also provide very accurate solutions. But in comparison of DTM and RDTM, RDTM is used easier than that of DTM, save time of computation and converges faster than the DTM.

On the other hand, in $20l3$, Sarvana and Magesh carried out the comparative study between the RDTM and the Adomian decomposition method (ADM) by handling the Newell–Whitehead–Segel equation. In order to validate and demonstrate the efficiency of the two methods they take two numerical examples and the results have shown that the RDTM is a very simple technique to handle linear and nonlinear Newell–Whitehead–Segel equation than the ADM, and also, it is demonstrated that the RDTM solves such DEs by direct forward calculation without using any complicated polynomials like as the Adomian polynomials to linearize the nonlinear terms. In addition to this, the obtained series solution by the RDTM converges faster than those obtained by the ADM and this shows that RDTM is a powerful technique to handle linear and nonlinear initial value problems.

RDTM was proposed to overcome the demerit of complex calculation of DTM. The main Advantage of this method is the fact that it provides its user with analytical approximation, in many cases an exact solution, in convergent series with elegantly computed terms. Moreover, using RDTM, the solution to initial valued problems can be expressed as an infinite power series expansion, which can be, expressed in a closed form of the exact solution.

However, the solution of two dimensional NLSGE given below is not yet studied by RDTM.

In this study, we are interested in the following two dimensional sine- Gordon equation (Kang *et al*, 2017);

$$\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\phi(x,y)\sin u+h(x,y,t),\ (x,y)\in\Omega,\ t\geq 0, \tag{1.1}$$

subject to the initial conditions

$$u(x,y,0)=\varphi_1(x,y),\ \ x,y\in\Omega, \tag{1.2}$$

$$\frac{\partial}{\partial t}u(x,y,0)=\varphi_2(x,y),\ \ x,y\in\Omega, \tag{1.3}$$

where $\Omega=\{(x,y): a\leq x\leq b,\ c\leq y\leq d\}$, $u(x,y,t)$ represents the wave displacement at a position (x,y) and time "t" which is analytic and sufficiently differentiable function on the domain of interest. The function $h(x,y,t)$ is an analytic function whiles the function $\varphi_1(x,y)$ and $\varphi_2(x,y)$ represents respectively the initial displacement and initial velocity. The nonlinear term $sin(u)$ is the Josephson current across an insulator between two superconductors. Also, the function $\phi(x,y)$ is Josephson current density and the parameters α *and* β are assumed to be non-negative real numbers. When $\beta=0$, the SGE (1.1) reduces to the undamped SG equation in two space variables and when $\beta>0$, to the damped one. This parameter β is known as dissipative term which measures the surface of resistance of superconductors.

Therefore, this study presents the solution of two dimensional NLSGE given by eq (1.1) by using RDTM.

1.2 Statement of the Problem

Recently, the solution of some nonlinear one dimensional SGE was obtained by the reduced differential transform method (keskin *et al*., 2011) and (Ramesh Rao, 2017). Even though SGE can be found in a wide variety of engineering and scientific applications, solving two dimensional NLSGE represented by (1.1) using RDTM is not presumably presented in the existing literature. As a result, this study mainly focuses on the following problems related to two dimensional NLSGE equation given by (1.1).

- Applying RDTM to solve the two dimensional NLSGE Subject to the given initial conditions.
- Verify the applicability of RDTM for two dimensional NLSGE by considering different examples.
- Test the convergence of approximated series solution.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to find the solution of two dimensional NLSGE subject to initial conditions by using RDTM.

1.3.2 Specific Objectives

- To apply RDTM to obtain the solution of two dimensional NLSGE.
- To verify the applicability of RDTM in solving two dimensional NLSGE by providing illustrative examples.
- To show the convergence of approximated series solution.

1.4 Significance of the Study

The outcome of this study is believed to have the following significances

- It develops the researcher skills on mathematical research.
- It may serve as a background information for a researcher who works around this area.

1.5 Delimitation of the Study

The RDTM is used to find solution of many DEs that represents real life problems mathematically. However, this study was delimited to find analytically the solution of inhomogeneous two dimensional NLSGE subject to the appropriate initial conditions by using the RDTM.

CHAPTER TWO

Literature Review

Many real life problems are represented by differential equations (DEs), some of those are Transport equation $(u_x + u_y = 0)$, Laplace equation $(u_{xx} + u_{yy} = 0)$, Heat equation $(u_t - ku_{xx} = 0)$, Wave equation $(u_{tt} - c^2 u_{xx} = 0)$ (Sankara Rao, 2010), SGE in one and two dimensions, etc. DEs are the mathematical expressions of some real life problems arising out of the real world around us such as physical, biological, engineering, financial or sociological fields. DEs can be categorized as Partial differential equation (PDE) and Ordinary differential equation (ODE), where PDE is a differential equation that contains unknown multi-variable functions and their partial derivatives and ODE represents a function of a single variable and their derivatives. A second Order PDEs can also classify as elliptic, parabolic, and hyperbolic PDEs as shown in (Sankara Rao, 2010). There are many problems arising in science and engineering those are modelled using linear or nonlinear PDEs, such as boundary and initial value problems in PDEs occur in fluid mechanics, mathematical physics, astrophysics, biology, materials science, electromagnetism, image processing, computer graphics, etc.

One of these PDEs is SGE. The SGE is a nonlinear hyperbolic partial differential equation which plays an important role in the propagation of magnetic fluxons in Josephson junctions, in differential geometry, relativistic field theory, solid state physics and nonlinear optics etc. For example if the nonlinear term of SGE (1.1) $\sin u(x, y, t)$ is replaced by $u(x, y, t)$ it is the famous Klein-Gordon equation (KGE) (Wazwaz, 2009), which is the name given to equation of motion of quantum scalar (pseudo scalar). So solving this SGE is important as it has various applications in most sciences and engineering. Several mathematical methods have been applied by various researchers in various fields of science and engineering to obtain the approximate/analytical solutions of nonlinear SGEs, which appeared in so many research literatures. We will discuss some of these methods here.

In 2017, Ramesh Rao used RDTM to solve the one dimensional sine Gordon equation:-

$$u_{tt} - c^2 u_{xx} + ksinu = 0,\ x \in \mathbb{R},\ t > 0, \tag{2.1}$$

subject to the initial conditions

$$u(x,0) = f(x), \tag{2.2}$$

$$u_t(x,0) = g(x), \tag{2.3}$$

and compares his solution with the solutions obtained by other methods, such as ADM, Variational iteration method (VIM), etc.The results reveal that the RDTM is very effective and a convenient mathematical tool for finding the exact and semi analytic solution of SGEs as well as other nonlinear DEs.

Keskin *et al*. (2011) implemented the RTDM for solving the SGE (2.1). The potency and the capability of this method for solving SGE was demonstrated by using the modeling problems from mathematical physics. The given nonlinear SGE equation was solved using RDTM and VIM. But when they uses VIM based on correction function iteration formula, they obtain a difficult integral calculation which is not easily computed. Then they apply RDTM and obtain the approximate analytical solution. As a result, they conclude the RDTM is more powerful and effective comparing to VIM and other known methods. Likewise, they used this method to solve the one dimensional KGE subject the initial conditions.

Fayadh and Faraj (2019) also solve SGE (2.1) with initial conditions (2.2) and (2.3) by using combined Laplace transform method and VIM to get the approximate solution of the given SGE.

Kang *et al*. (2017) present an efficient second-order semi-implicit finite difference scheme for solving the two dimensional SGE (1.1) − (1.3), which can admit the discrete energy conservation for the undamped problem. They proposed the efficient linear iteration algorithm for approximating the nonlinear system arising from the implicit treatment of the nonlinear term. Moreover, they prove this iteration algorithm to be a contraction mapping. In turn, based on truncation errors, the convergence analysis of the numerical scheme was also shown. Furthermore, the results of numerical experiments demonstrated the efficiency and the accuracy of the proposed scheme.

Li *et al*. $(20l9)$ develop a mesoscopic lattice Boltzmann Bhatnagar-Gross-Krook (BGK) model to solve two dimensional wave equation with the nonlinear damping and source terms, such as the hyperbolic telegraph equation, damped or undamped sine-Gordon equation, and so on. Using the Chapman-Enskog multiscale expansion, they obtain the macroscopic governing dynamical evolution equation by choosing appropriate local equilibrium distribution functions. In their research work few examples was considered to validate the present mesoscopic model where the exact solution is known. It turned out that the numerical solution is in very good agreement with exact one, which shows that the present mesoscopic model is pretty valid, and can be used to solve more similar nonlinear wave equations with nonlinear damping and source terms, and predict and enrich the internal mechanism of nonlinearity and complexity in nonlinear dynamic phenomenon.

In 2012, Cheng and Liew investigated the analysis of nonlinear two dimensional SGE using the mesh-free reproducing kernel particle Ritz method (kp-Ritz method). They employed the mesh-free kernel particle estimate to approximate the two dimensional displacement field. In their research work, an energy formulation is formulated and a system of non-linear discrete equations is obtained through the application of the Ritz minimization procedure to the energy expressions. To validate the accuracy of the results and stability of the present method, convergence studies were carried out based on influences of support size and number of nodes. The present results were compared with results reported in extant literature and were found to be in good agreement with the literature.

In addition to this, there are also other different methods applied to solve two dimensional NLSGE such as Differential quadrature method (Jiwari *et al*., $20l2$) and Finite difference method (Kurara *et al*., 2016).

But the solution of two dimensional NLSGE (1.1) with initial conditions (1.2) and (1.3) using RDTM has not been discussed so far. Therefore, this study was conducted mainly to solve equation (1.1) with initial condition (1.2) & (1.3) by using RDTM.

CHAPTER THREE

Methodology

3.1 Study Area and Period

This study is conducted to find the solution of two dimensional NLSGE by using the RDTM under the Department of Mathematics, college of Natural Science, Jimma University from August 2018 up to February 2020.

3.2 Study Design

This study designed to be done analytically.

3.3 Source of Information

Secondary data, such as reference books, research papers, journals and internet were used as a source of information for this study.

3.4 Mathematical Procedures

To attain the objective of this study the following procedures are undertaken:

Step 1. Applying properties of reduced differential transform method to NLSGE (1.1) to obtain a recursion system for the unknown function $U_{k+2}(x,y)$, where "k" is non-negative integers.

Step 2. Applying properties of reduced differential transform method to initial conditions (1.2) & (1.3) to obtain the values of unknown functions $U_0(x,y)$ and $U_1(x,y)$.

Step 3. Substituting result(s) obtained from step (2) into step (1) to obtain the values of unknown functions $U_2(x,y), U_3(x,y), U_4(x,y), \ldots$ for $k = 0,1,2,\ldots$

Step 4. Applying definition of inverse reduced differential transform method to determine the solution of two dimensional NLSGE (1.1).

Step 5. Testing convergence of approximated solution using the stated definition 4.3.

Step 6. Mathematica version 7.0 software would be used to sketch the solution curves.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Mathematical Preliminaries

This section presents the basic definitions and properties of reduced differential transform methods that were relevant for this research work. Also, some basic properties of RDTM were proved.

Consider a function of three variables $u(x,y,t)$ and suppose that it can be represented as a product of three single-variable functions, i.e $u(x,y,t)=f(x)h(y)g(t)$. Based on the properties of two-dimensional differential transform, the function $u(x,y,t)$ can be represented as follows (Mohamed and Gepreel, 2017):

$$u(x,y,t)=\left(\sum_{i=0}^{\infty}F(i)x^{i}\right)\left(\sum_{j=0}^{\infty}H(j)y^{j}\right)\left(\sum_{l=0}^{\infty}G(l)t^{l}\right)=\sum_{k=0}^{\infty}U_{k}(x,y)t^{k} \qquad (4.1)$$

where $U_k(x,y)$ is called $t-$dimensional spectrum function of $u(x,y,t)$.

Remark 4.1: The poisson function series generates a multivariate Taylor series expansion of the input expression u, with respect to the variables X, to order n, using the variable weights U.

Remark 4.2: The relationship introduced in eq (4.1) is the Poisson series form of the input expression $u(x,y,t)$, with respect to the space variables x,y and time variable t to order n, using the variable weights $U_k(x,y)$.

4.1.1 Basic ideas of reduced differential transform method (RDTM)

The basic definition and properties of RDTM are introduced as follows (Mohamed and Gepreel, 2017), (Al-Saif, 2018) and (Srivastava *et al*., 2015):

Definition 4. 1:

If a function $u(x,y,t)$ is analytic and differentiated continuously with respect to space variables $x,\ y$ and time variable t in the domain of interest, then let

$$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=t_0} \tag{4.2}$$

Where the t- dimensional spectrum function, $U_k(x,y)$ is the transformed function. In this study, the lowercase $u(x,y,t)$ represent the original function while the uppercase $U_k(x,y)$ stands for the transformed function.

Definition 4. 2:

The inverse reduced differential transform of a sequence $\{U_k(x,y)\}_{k=0}^{\infty}$ is given by

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)(t-t_0)^k \tag{4.3}$$

Combining equations (4.2) & (4.3), yields that

$$u(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=t_0} (t-t_0)^k \tag{4.4}$$

In real applications, the function $u(x,y,t)$ is represented by a finite series of eq(4.3) around $t_0 = 0$ and can be written as $\tilde{u}_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)t^k + R_n(x,y,t)$ where the tail function

$R_n(x,y,t) = \sum_{k=n+1}^{\infty} U_k(x,y)t^k$ is negligibly small.

Furthermore, the reduced differential inverse transform of the set of $\{U_k(x,y)\}_{k=0}^{n}$ gives approximation solution as:

$$\tilde{u}_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)t^k$$

where $'n'$ is the order of the approximate solution. Therefore, the exact solution of the problem is obtained as follows

$$u(x,y,t)=\lim_{n\to\infty}\tilde{u}_n(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k \tag{4.5}$$

$$\Rightarrow u(x,y,t)=U_0(x,y)+U_1(x,y)t^1+U_2(x,y)t^2+U_3(x,y)t^3+U_4(x,y)t^4+U_5(x,y)t^5+\ldots$$

From definition (4.4) above, it can be found that the concept of reduced differential transform method is derived from the power series expansion. For example,

Consider $u(x,y,t)=e^{x+y+t}$, this function can be written as

$$u(x,y,t)=e^{x+y+t}=\underbrace{\left(1+x+\frac{x^2}{2}+\ldots\right)}_{e^x}\underbrace{\left(1+y+\frac{y^2}{2}+\ldots\right)}_{e^y}\underbrace{\left(1+t+\frac{t^2}{2}+\ldots\right)}_{e^t}$$

$$=\left(\sum_{i=0}^{\infty}F(i)x^i\right)\left(\sum_{j=0}^{\infty}H(j)y^j\right)\left(\sum_{l=0}^{\infty}G(l)t^l\right)$$

Otherwise,

$$u(x,y,t)=e^{x+y}\left(e^t\right)=e^{x+y}\underbrace{\left(1+t+\frac{t^2}{2}+\ldots\right)}_{e^t}=e^{x+y}+\left(e^{x+y}\right)t+\left(e^{x+y}\right)\frac{t^2}{2}+\ldots$$

$$=\sum_{k=0}^{\infty}\underbrace{\frac{e^{x+y}}{k!}}_{U_k(x,y)}t^k=\sum_{k=0}^{\infty}U_k(x,y)t^k$$

Theorem 4.1:

If $f(x,y,t)=\frac{\partial^n}{\partial x^n}u(x,y,t)$ then $F_k(x,y)=\frac{\partial^n}{\partial x^n}U_k(x,y)$, $k=0,1,2,\ldots$, $n=0,1,2,\ldots$

Proof:

Suppose $F_k(x,y)$ and $U_k(x,y)$ are the t-dimensional spectrum functions (transformed functions) of $f(x,y,t)$ and $u(x,y,t)$ respectively.

We want to show that $F_k(x,y)=\frac{\partial^n}{\partial x^n}U_k(x,y)$

Applying RDT on both side of $f(x,y,t)=\frac{\partial^n}{\partial x^n}u(x,y,t)$, we have:

$$RDT\left[f(x,y,t)\right]=RDT\left[\frac{\partial^n}{\partial x^n}u(x,y,t)\right] \tag{4.6}$$

Using definition 4.1, we have

$$RDT\left[f(x,y,t)\right]=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}f\left(x,y,t\right)\right]_{t=0}$$

and $$RDT\left[\frac{\partial^n}{\partial x^n}u(x,y,t)\right]=\frac{\partial^n}{\partial x^n}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0} \tag{4.7}$$

Now substituting (4.7) into (4.6), we get

$$\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}f\left(x,y,t\right)\right]_{t=0}=\frac{\partial^n}{\partial x^n}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u\left(x,y,t\right)\right]_{t=0} \tag{4.8}$$

Also definition 4.1, implies

$$\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}f(x,y,t)\right]_{t=0}=F_k(x,y) \text{ and } \frac{\partial^n}{\partial x^n}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}=\frac{\partial^n}{\partial x^n}U_k(x,y) \tag{4.9}$$

Substituting (4.9) into (4.8), yields that

$F_k(x,y)=\dfrac{\partial^n}{\partial x^n}U_k(x,y)$. Hence, proved. □

Theorem 4.2:

If $f(x,y,t)=u(x,y,t)\pm v(x,y,t)$, then $F_k(x,y)=U_k(x,y)\pm V_k(x,y)$, $k=0,1,2,\ldots$

Proof:

Suppose, $F_k(x,y)$, $U_k(x,y)$ and $V_k(x,y)$ are the t-dimensional spectrum functions (transformed functions) of $f(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$ respectively.

We want to show that: $F_k(x,y)=U_k(x,y)\pm V_k(x,y)$

Applying RDT on both sides of $f(x,y,t)=u(x,y,t)\pm v(x,y,t)$ we have:

$$RDT[f(x,y,t)]=RDT[u(x,y,t)]\pm RDT[v(x,y,t)] \tag{4.10}$$

By definition 4.1, we obtain

$$RDT[f(x,y,t)]=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}f(x,y,t)\right]_{t=0},\ RDT[u(x,y,t)]=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}$$

and $$RDT[v(x,y,t)]=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}v(x,y,t)\right]_{t=0},\ k=0,1,2,\ldots \tag{4.11}$$

Now substituting (4.11) in to (4.10), we get

$$\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,y,t)\right]_{t=0} = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y,t)\right]_{t=0} \pm \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} v(x,y,t)\right]_{t=0}$$

(4.12)

Also definition 4.1, implies that

$$F_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,y,t)\right]_{t=0},\ U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y,t)\right]_{t=0} \text{ and}$$

$$V_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} v(x,y,t)\right]_{t=0} \tag{4.13}$$

Substituting (4.13) into (4.12), yields that

$F_k(x,y) = U_k(x,y) \pm V_k(x,y)$, $k = 0, 1, 2, \ldots$ and the proof is completed. □

Theorem 4.3:

Assume that $F_k(x,y), G_k(x,y)$ and $U_k(x,y)$ are the reduced differential transform of the functions $f(x,y,t),\ g(x,y,t)$ and $u(x,y,t)$ respectively, then we have the following;

i) If $f(x,y,t) = \sin u(x,y,t)$, then

$$F_k(x,y) = RDT(f(x,y,t)) = \begin{cases} \sin U_0, & if\ k = 0 \\ \sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right) G_{k_1}(x,y) U_{k-k_1}(x,y), & if\ k \geq 1 \end{cases} \tag{4.14}$$

ii) If $g(x,y,t) = \cos u(x,y,t)$, then

$$G_k(x,y) = RDT(g(x,y,t)) = \begin{cases} \cos U_0, & if\ k = 0 \\ -\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right) F_{k_1}(x,y) U_{k-k_1}(x,y), & if\ k \geq 1 \end{cases} \tag{4.15}$$

Proof:

i) Applying properties of RDTM on both sides of $f(x,y,t) = \sin u(x,y,t)$, we obtain

$$F_0(x,y) = \sin U_0(x,y)$$

Using Leibnitz rule of higher order derivatives of the products, we get

$$f(x,y,t) = \sin u(x,y,t)$$

$$F_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f\left(x,y,t\right)\right]_{t=0} = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\sin u\left(x,y,t\right)\right]_{t=0}$$

$$\frac{\partial^k}{\partial t^k} f(x,y,t)=\frac{\partial^{k-1}}{\partial t^{k-1}}\left(\cos u(x,y,t)\frac{\partial}{\partial t}u\left(x,y,t\right)\right)$$

$$=\sum_{k_1=0}^{k-1}\binom{k-1}{k_1}\frac{\partial^{k_1}g(x,y,t)}{\partial t^{k_1}}\frac{\partial^{k-k_1}u(x,y,t)}{\partial t^{k-k_1}}$$

Therefore,

$$\left[\frac{\partial^k}{\partial t^k} f(x,y,t)\right]_{t=0} = \sum_{k_1=0}^{k-1}\binom{k-1}{k_1}k_1!(k-k_1)!G_{k_1}(x,y)U_{k-k_1}(x,y)$$

$$=\sum_{k_1=0}^{k-1}(k-1)!(k-k_1)G_{k_1}(x,y)U_{k-k_1}\left(x,y\right)$$

and then using definition (4.1), for $k = 1, 2, 3, \ldots$, we get

$$F_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,y,t)\right]_{t=0}$$

$$=\sum_{k_1=0}^{k-1}\frac{1}{k!}(k-1)!(k-k_1)G_{k_1}(x,y)U_{k-k_1}(x,y)$$

$$\Rightarrow F_k(x,y)=\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right)G_{k_1}(x,y)U_{k-k_1}(x,y)\ .\ \square$$

ii) Applying properties of RDTM on both sides of $g(x,y,t)=\cos u(x,y,t),$ we get

$$G_0(x,y)=\cos U_0(x,y)$$

Using Leibnitz rule of higher order derivatives of the products, we get

$$g(x,y,t)=\cos u(x,y,t)$$

$$G_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} g\left(x,y,t\right)\right]_{t=0} = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\cos u\left(x,y,t\right)\right]_{t=0}$$

$$\frac{\partial^k}{\partial t^k} g(x,y,t)=-\frac{\partial^{k-1}}{\partial t^{k-1}}\left(\sin u(x,y,t)\frac{\partial}{\partial t}u(x,y,t)\right)$$

$$=-\sum_{k_1=0}^{k-1}\binom{k-1}{k_1}\frac{\partial^{k_1} f(x,y,t)}{\partial t^{k_1}}\frac{\partial^{k-k_1}u(x,y,t)}{\partial t^{k-k_1}}$$

Therefore,

$$\left[\frac{\partial^k}{\partial t^k} g(x,y,t)\right]_{t=0} = -\sum_{k_1=0}^{k-1}\binom{k-1}{k_1} k_1!(k-k_1)!F_{k_1}(x,y)U_{k-k_1}(x,y)$$

$$= -\sum_{k_1=0}^{k-1}(k-1)!(k-k_1)F_{k_1}(x,y)U_{k-k_1}(x,y)$$

and then using definition 4.1, for $k = 1, 2, 3, \ldots$, we get

$$G_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} g\left(x,y,t\right)\right]_{t=0}$$

$$= \sum_{k_1=0}^{k-1}\frac{1}{k!}(k-1)!(k-k_1)F_{k_1}(x,y)U_{k-k_1}\left(x,y\right)$$

$$\Rightarrow G_k(x,y) = -\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right)F_{k_1}(x,y)U_{k-k_1}(x,y). \ \square$$

Theorem 4.4:

If $w(x,y,t) = \alpha u(x,y,t)$ then $W_k(x,y) = \alpha U_k(x,y)$, ($\alpha$ is constant)

Proof:

$w(x,y,t) = \alpha u(x,y,t),$ by definition 4.1, we get

$$W_k(x,y) = RDT\left(\alpha u(x,y,t)\right) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\left(\alpha u(x,y,t)\right)\right]_{t=0}$$

$$= \alpha\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y,t)\right]_{t=0}$$

$$= \alpha U_k(x,y)$$

$\Rightarrow W_k(x,y) = \alpha U_k(x,y)$, hence, proved. $\square$

Theorem 4.5:

Suppose that $w(x,y,t) = f(x,y)u(x,y,t)$, and then

$$W_k(x,y) = f(x,y)U_k(x,y)$$

Proof:

By definition 4.1 and from the calculus, we have

$$W_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\left(f(x,y)u(x,y,t)\right)\right]_{t=0}$$

$$= f(x,y)\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}$$

$$= f(x,y)U_k(x,y)$$

Therefore, $W_k(x,y) = f(x,y)U_k(x,y)$, which completes the proof. □

Theorem 4. 6: Reduced differential transform of initial condition.

If $\frac{\partial^n}{\partial t^n}u(x,y,0) = \varphi(x,y),$ then $U_n(x,y) = \frac{\varphi(x,y)}{n!}$

Proof:

By definition 4.1, we have

$$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}$$

and so, when $k = n,$ we have

$$U_n(x,y) = \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}u(x,y,t)\right]_{t=0}$$

From the initial condition, $\frac{\partial^n}{\partial t^n}u(x,y,0) = \left[\frac{\partial^n}{\partial t^n}u(x,y,t)\right]_{t=0} = \varphi(x,y)$

Therefore,

$U_n(x,y) = \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}u(x,y,t)\right]_{t=0} = \frac{1}{n!}\varphi(x,y) = \frac{\varphi(x,y)}{n!}$ .this completes the proof. □

Theorem 4. 7:

If $u(x,y,t) = \sin(\alpha x + \beta y + \omega t),$ then $U_k(x,y) = \frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2!} + \alpha x + \beta y\right)$

Proof:

$$u(x,y,t) = \sin(\alpha x + \beta y + \omega t)$$

$$\frac{\partial}{\partial t}u(x,y,t) = \omega\cos(\alpha x + \beta y + \omega t) = \omega\sin\left(\frac{\pi}{2} + \alpha x + \beta y + \omega t\right)$$

$$\frac{\partial^2}{\partial t^2}u(x,y,t) = \frac{\partial}{\partial t}\left[\omega\cos(\alpha x + \beta y + \omega t)\right] = -\omega^2\sin\left(\alpha x + \beta y + \omega t\right)$$

$$= \omega^2\sin\left(\frac{2\pi}{2} + \alpha x + \beta y + \omega t\right)$$

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$$\frac{\partial^k}{\partial t^k}u(x,y,t)=\omega^k\sin\left(\frac{k\pi}{2}+\alpha x+\beta y+\omega t\right)$$

Hence, by definition 4.1

$$U_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\sin\left(\alpha x+\beta y+\omega t\right)\right]_{t=0}$$

$$=\frac{1}{k!}\left[\omega^k\sin\left(\frac{k\pi}{2!}+\alpha x+\beta y+\omega t\right)\right]_{t=0}$$

$$=\frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2!}+\alpha x+\beta y\right)$$

Therefore, $U_k(x,y)=\frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2!}+\alpha x+\beta y\right)$ and the proof of theorem 4.7 is completed. □

Theorem 4.8: If $u(x,y,t)=e^{\lambda t}$, then $U_k(x,y)=\frac{\lambda^k}{k!}$, where λ is constant.

Proof:

$$u(x,y,t)=e^{\lambda t}$$

$$\frac{\partial}{\partial t}u(x,y,t)=\lambda e^{\lambda t}$$

$$\frac{\partial^2}{\partial t^2}u(x,y,t)=\frac{\partial}{\partial t}\left[\lambda e^{\lambda t}\right]=\lambda^2e^{\lambda t}$$

$$\frac{\partial^3}{\partial t^3}u(x,y,t)=\frac{\partial}{\partial t}\left[\lambda^2e^{\lambda t}\right]=\lambda^3e^{\lambda t}$$

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$$\frac{\partial^k}{\partial t^k}u(x,y,t)=\lambda^ke^{\lambda t}$$

So, by definition 4.1, we have

$$U_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}e^{\lambda t}\right]_{t=0}=\frac{1}{k!}\left[\lambda^k e^{\lambda t}\right]_{t=0}$$

$$=\frac{1}{k!}\lambda^k\left[e^{\lambda t}\right]_{t=0}=\frac{\lambda^k}{k!}e^0=\frac{\lambda^k}{k!}.1=\frac{\lambda^k}{k!}$$

Therefore, $U_k(x,y)=\frac{\lambda^k}{k!}$. so the proof is completed. □

Theorem 4. 9:

If $w(x,y,t)=\frac{\partial^r}{\partial t^r}u(x,y,t)$ then $W_k(x,y)=\frac{(k+r)!}{k!}U_{k+r}(x,y)$

Proof:

By definition 4.1, we obtain

$$W_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\left(\frac{\partial^r}{\partial t^r}u(x,y,t)\right)\right]_{t=0}$$

From the calculus, we have that

$$W_k(x,y)=\frac{1}{k!}\left[\frac{\partial^{k+r}}{\partial t^{k+r}}u(x,y,t)\right]_{t=0}$$

Since the reduced differential transform of function $u(x,y,t)$ is

$$U_{k+r}(x,y)=\frac{1}{(k+r)!}\left[\frac{\partial^{k+r}}{\partial t^{k+r}}u(x,y,t)\right]_{t=0}$$

$$\Rightarrow\left[\frac{\partial^{k+r}}{\partial t^{k+r}}u(x,y,t)\right]_{t=0}=(k+r)!U_{k+r}(x,y)$$

Hence, we obtain $W_k(x,y)=\frac{1}{k!}\left[\frac{\partial^{k+r}}{\partial t^{k+r}}u(x,y,t)\right]_{t=0}=\frac{1}{k!}(k+r)!U_{k+r}(x,y)$

$\Rightarrow W_k(x,y)=\frac{(k+r)!}{k!}U_{k+r}(x,y)$, this can be written as

$W_k(x,y)=(k+1)(k+2)...(k+r)U_{k+r}(x,y)$, which completes the proof of theorem 4.9. □

Table 1: Basic properties of reduced differential transform method (Al-Saif, 2018) and (Neog, 2015)

$Original\ function$	$Transformed\ function$
$u(x,y,t)$	$U_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}$
$w(x,y,t)=u(x,y,t)\pm v(x,y,t)$	$W_k(x,y)=U_k(x,y)\pm V_k(x,y)$
$w(x,y,t)=\alpha u(x,y,t)$	$W_k(x,y)=\alpha U_k(x,y)$, where α $is\ constant.$
$w(x,y,t)=x^m y^n t^p$	$W_k(x,y)=x^m y^n \delta(k-p)$, where $\delta(k-p)=\begin{cases}1, if\ \ k=p\\0, if\ k\neq p\end{cases}$
$w(x,y,t)=x^m y^n t^p u(x,y,t)$	$W_k(x,y)=x^m y^n U_{(k-p)}(x,y)$
$w(x,y,t)=u(x,y,t)v(x,y,t)$	$W_k(x,y)=\sum_{r=0}^{k}U_k(x,y)V_{k-r}(x,y)=\sum_{r=0}^{k}V_k(x,y)U_{k-r}(x,y)$
$w(x,y,t)=\frac{\partial^r}{\partial t^r}u(x,y,t)$	$W_k(x,y)=\frac{(k+r)!}{k!}U_{k+r}(x,y)=(k+1)(k+2)\ldots(k+r)U_{k+r}(x,y)$
$w(x,y,t)=\frac{\partial^n}{\partial x^n}u(x,y,t)$	$W_k(x,y)=\frac{\partial^n}{\partial x^n}U_k(x,y)$
$w(x,y,t)=e^{\lambda t}$	$W_k(x,y)=\frac{\lambda^k}{k!}$, where λ is constant.
$w(x,y,t)=\frac{\partial^n}{\partial y^n}u(x,y,t)$	$W_k(x,y)=\frac{\partial^n}{\partial y^n}U_k(x,y)$
$w(x,y,t)=\sin(\alpha x+\beta y+\omega t)$	$W_k(x,y)=\frac{\omega^k}{k!}\sin\left(\frac{\pi k}{k!}+\alpha x+\beta y\right)$, where α, β and ω are constants.
$w(x,y,t)=\cos(\alpha x+\beta y+\omega t)$	$W_k(x,y)=\frac{\omega^k}{k!}\cos\left(\frac{\pi k}{k!}+\alpha x+\beta y\right)$, where α, β and ω are constants.

4. 2 Main result

The aim of this study is to obtain the solution of two dimensional NLSGE by using the RDTM. This is done by extending the work of Ramesh Rao (2017) and Keskin *et al*. (2010) that was used to solve one dimensional NLSGE by using the RDTM. So, the definitions and theorems mentioned in the preceding section and some properties of reduced differential transform method listed in the table would be applied here.

Now consider the two dimensional nonlinear sine- Gordon equation(Kang, Fend, Cheng, and Guo, 2017);

$$\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\phi(x,y)\sin u+h(x,y,t),\left(x,y\right)\in\Omega,\ t\geq 0, \tag{4.16}$$

subject to the initial conditions

$$u(x,y,0)=\varphi_1(x,y),\quad x,y\in\Omega\ , \tag{4.17}$$

$$\frac{\partial}{\partial t}u(x,y,0)=\varphi_2(x,y),\ x,y\in\Omega\ . \tag{4.18}$$

To find the solution of NLSGE (4.16) we apply RDTM, since the reduced differential transform method is one of the best modern methods used to solve such nonlinear DEs.

- The steps/procedures to be carried out in the process of finding the solution of two dimensional NLSGE by RDTM are described below.

Applying the properties of RDTM to both sides of equation (4.16), we construct the following iteration formula:

$$(k+2)(k+1)U_{k+2}(x,y)+\beta(k+1)U_{k+1}(x,y)=\alpha\left(\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{\partial^2}{\partial y^2}U_k(x,y)\right)$$
$$-\phi(x,y)F_k(x,y)+H_k(x,y)$$

$$\Rightarrow U_{k+2}(x,y)=\frac{-1}{(k+2)(k+1)}\left[\begin{array}{l}\beta(k+1)U_{k+1}(x,y)-\alpha\left(\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{\partial^2}{\partial y^2}U_k(x,y)\right)\\+\phi(x,y)F_k(x,y)-H_k(x,y)\end{array}\right], \tag{4.19}$$

where $F_k(x,y)$ is the reduced differential transform of nonlinear term $\sin u(x,y,t)$ and $H_k(x,y)$ is the reduced differential transform of inhomogeneous term $h(x,y,t)$. The reduced differential transform of $H_k(x,y)$ is computed using definition 4.1 and theorems 4.5, 4.7 and 4.8. The reduced differential transform of nonlinear term $F_k(x,y)$ is computed using theorem 4.3.

According to theorem 4.3 the first few nonlinear terms are given as follows:

$$F_0(x,y) = \sin U_0$$

$$F_1(x,y) = G_0(x,y)U_1(x,y)$$

$$F_2(x,y) = G_0(x,y)U_2(x,y) - \frac{1}{2}F_0(x,y)U_1^2(x,y)$$

$$F_3(x,y) = G_0(x,y)U_3(x,y) - F_0(x,y)U_1(x,y)U_2(x,y) - \frac{1}{6}G_0(x,y)U_1^3(x,y)$$

$$F_4(x,y) = G_0(x,y)U_4(x,y) - F_0(x,y)U_1(x,y)U_3(x,y) - \frac{1}{2}F_0(x,y)U_2^2(x,y) - \frac{1}{2}G_0(x,y)U_1^2(x,y)U_2(x,y) + \frac{1}{24}F_0(x,y)U_1^4(x,y)$$

Applying the reduced differential transform to initial conditions (4.17) and (4.18) by using theorem 4.6, we get;

$$U_0(x,y) = \varphi_1(x,y) \tag{4.20}$$

$$U_1(x,y) = \varphi_2(x,y) \tag{4.21}$$

Substituting (4.21) and (4.20) into (4.19) and straight forward iterative calculations, we get the following $U_k(x,y)$ values $U_2(x,y), U_3(x,y), U_4(x,y), \ldots$

Then, the inverse reduced differential transformation of the set of values $\{U_k(x,y)\}_{k=0}^{n}$ gives approximation solution as,

$$\tilde{u}_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)t^k$$

where "n" is order of approximation solution.

Therefore, the exact solution of problem (4.16) is given by

$$u(x,y,t) = \lim_{n\to\infty}\tilde{u}_n(x,y,t) = U_0(x,y) + U_1(x,y)t + U_2(x,y)t^2 + U_3(x,y)t^3 + U_4(x,y)t^4 + U_5(x,y)t^5 + \cdots$$

4.3 **Convergence analysis**

In this section the convergence analysis of the approximated analytical solutions which are computed from the application of RDTM (Al-Saif, 2018) are presented.

Let us consider the SGE (4.16) in the following functional equation form:

$$u(x,y,t)=\mathcal{F}\left(u(x,y,t)\right)$$

where $\mathcal{F}$ is a general non-linear operator involving both linear and non-linear terms.

According to RDTM the two dimensional NLSGE given in eq (4.16) has a solution of the form

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k=\sum_{k=0}^{\infty}\beta_k$$

It is noted that the solutions by RDTM is equivalent to the determining the sequences

$$S_0=U_0(x,y)=\beta_0,$$

$$S_1=U_0(x,y)+U_1(x,y)t=\beta_0+\beta_1,,$$

$$S_2=U_0(x,y)+U_1(x,y)t+U_2(x,y)t^2=\beta_0+\beta_1+\beta_2,$$

$$\vdots$$

$$S_n=\sum_{k=0}^{n}U_k(x,y)t^k=\sum_{k=0}^{n}\beta_k\ ,$$

by using iterative scheme

$$S_{n+1}=\mathcal{F}(S_n),$$

associated with the functional equation

$$S=\mathcal{F}(S).$$

Hence, the solution obtained by RDTM, $u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k=\sum_{k=0}^{\infty}\beta_k$ is equivalent to

$$u(x,y,t)=U_0(x,y)+U_1(x,y)t+U_2(x,y)t^2+U_3(x,y)t^3+U_4(x,y)t^4+\ldots=\left\{S_n\right\}_{n=0}^{\infty}$$

The sufficient condition for convergence of the series solution $\left\{S_n\right\}_{n=0}^{\infty}$ is given in the following theorem;

Theorem 10: Let $\mathcal{F}$ be an operator from a from Hilbert space $\mathcal{H}$ in to $\mathcal{H}$. Then the series solution $\{S_n\}_{n=0}^{\infty}$ converges whenever there is "α" such that $0<\alpha<1$, and $\|\beta_{k+1}\| \le \alpha \|\beta_k\|$.

Proof:

Firstly, we show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$. For this reason, we suppose that

$$\|S_{n+1}-S_n\| = \|\beta_{n+1}\| \le \alpha \|\beta_n\| + \alpha^2 \|\beta_{n-1}\| \le \ldots \le \alpha^{n+1} \|\beta_0\|$$

But for every $n, m \in \mathbb{N}$, $n \ge m$, by triangle inequality and formula for the sum of geometric sequence we find that

$$\begin{aligned}
\|S_n - S_m\| &= \|S_n - S_{n-1} + S_{n-1} - S_{n-2} + \ldots + S_{m+1} - S_m\| \\
&\le \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \ldots + \|S_{m+1} - S_m\| \\
&\le \alpha^n \|\beta_0\| + \alpha^{n-1} \|\beta_0\| + \alpha^{n-2} \|\beta_0\| + \ldots + \alpha^{m+1} \|\beta_0\| \\
&= \alpha^{m+1} \left(\alpha^{n-m-1} + \alpha^{n-m-2} + \alpha^{n-m-3} + \ldots + 1\right) \|\beta_0\| \\
&= \alpha^{m+1} \left(\frac{1-\alpha^{n-m-1}}{1-\alpha}\right) \|\beta_0\|
\end{aligned}$$

Since $0<\alpha<1$, in the numerator we have $1-\alpha^{n-m-1}<1$. Consequently,

$$\|S_n - S_m\| \le \frac{\alpha^{m+1}}{1-\alpha} \|\beta_0\|$$

On the right $0<\alpha<1$ and $\|\beta_0\| < \infty$, we then have

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0$$

Thus, we conclude that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$, hence, the series solution $\{S_n\}_{n=0}^{\infty}$ converges to some $\{S\} \in \mathcal{H}$. □

Theorem 11:

Let $\mathcal{F}$ be a non-linear operator satisfies Lipschitz condition from Hilbert space $\mathcal{H}$ in to $\mathcal{H}$ and $u(x,y,t)$ be exact solution of the given SGE. If the series solution $\{S_n\}_{n=0}^{\infty}$ converges, then it converged to $u(x,y,t)$.

Proof:

To prove that $\{S_n\}_{n=0}^{\infty}$ converges to $u(x,y,t)$.

$$u(x,y,t)=\mathcal{F}\left(u(x,y,t)\right), \quad \text{since} \quad S=\mathcal{F}(S)$$

$$u(x,y,t)=\mathcal{F}\left(u(x,y,t)\right)=\mathcal{F}\left(\sum_{k=0}^{\infty}\beta_k\right)=\mathcal{F}\left(\lim_{n\to\infty}\sum_{k=0}^{n}\beta_k\right)$$

$$=\lim_{n\to\infty}\mathcal{F}\left(\sum_{k=0}^{n}\beta_k\right)=\lim_{n\to\infty}\mathcal{F}\left(S_n\right)=\lim_{n\to\infty}S_{n+1}=S$$

Since $\{S_n\}_{n=0}^{\infty}$ converges to S and $S=u(x,y,t)$, so the series solution $\{S_n\}_{n=0}^{\infty}$ converges to the exact solution $u(x,y,t)$.

To prove that the SGE given in (4.16) has a unique solution;

Let $u_1(x,y,t)$ & $u_2(x,y,t)$ be the solution of the given SGE, then we have

$$\left\|\mathcal{F}(u_1)-\mathcal{F}(u_2)\right\|\le\alpha\left\|u_1-u_2\right\|,\ 0<\alpha<1$$

Since, $u_1=\mathcal{F}(u_1)$ and $u_2=\mathcal{F}(u_2)$ we get

$$\left\|u_1-u_2\right\|=\left\|\mathcal{F}(u_1)-\mathcal{F}(u_2)\right\|\le\alpha\left\|u_1-u_2\right\|$$

Which implies $\left\|u_1-u_2\right\|=0$ since $0<\alpha<1$ and this gives us

$u_1(x,y,t)=u_2(x,y,t)$.

Therefore, from Banach fixed theorem, there is unique solution of the problem (4.16).

and the theorem is proved. □

Definition 4.3 For $k\in\mathbb{N}\bigcup\{0\}$, we define

$$\alpha_k=\begin{cases}\dfrac{\left\|\beta_{k+1}\right\|}{\left\|\beta_k\right\|}=\dfrac{\left\|U_{k+1}(x,y)t^{k+1}\right\|}{\left\|U_k(x,y)t^k\right\|}, & if\ \left\|\beta_k\right\|=\left\|U_k(x,y)t^k\right\|\neq 0,\\ 0, & if\ \left\|\beta_k\right\|=\left\|U_k(x,y)t^k\right\|=0.\end{cases}$$

Then we can say that the series approximate solution $\{S_n\}_{n=0}^{\infty}$ converges to the exact solution $u(x,y,t)$ when $0\le\alpha_k<1$ for $k=0,1,2,\ldots$.

4.4. Supportive examples

In this section we consider three examples that verify the efficiency and accuracy of the proposed method.

EXAMPLE 4. 1: (Kang *et al*, 2017)

Consider the two dimensional sine- Gordon equation (4.16) on the domain

$\Omega = \left[\frac{-1}{2}, \frac{1}{2}\right] \times \left[\frac{-1}{2}, \frac{1}{2}\right]$ with $\beta = 0,\ \phi(x, y) = 1,\ \alpha = \frac{1}{2\pi^2}$ and

$h(x, y, t) = \sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right);$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2\pi^2}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - \sin u + \sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right), \tag{4.22}$$

with initial conditions

$$u(x, y, 0) = \cos(\pi x)\cos(\pi y), \tag{4.23}$$

$$\frac{\partial}{\partial t} u(x, y, 0) = 0. \tag{4.24}$$

Solution: Applying properties of RDTM to both sides of equation (4.22), we construct the following recursive formula:

$$(k+2)(k+1)U_{k+2}(x, y) = \frac{1}{2\pi^2}\left[\frac{\partial^2}{\partial x^2}U_k(x, y) + \frac{\partial^2}{\partial y^2}U_k(x, y)\right] - F_k(x, y) + H_k(x, y)\ , \tag{4.25}$$

where $F_k(x, y)$ is the reduced differential transform of nonlinear term $sinu(x, y, t)$ and $H_k(x, y)$ is the reduced differential transform of inhomogeneous term $\sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right)$.

Again applying properties of RDTM to initial conditions (4.23) and (4.24), we get

$$U_0(x, y) = \cos(\pi x)\cos(\pi y) \tag{4.26}$$

$$U_1(x, y) = 0 \tag{4.27}$$

Now taking the values of k, (k =0, 1, 2, …) and applying theorem 4.3 and definition 4.1 in equation (4.25), we obtain the following successive values of $U_k(x,y)$.

For $k=0$ eq (4.25) has the form

$$(0+2)(0+1)U_{0+2}(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_0(x,y)}{\partial x^2}+\frac{\partial^2 U_0(x,y)}{\partial y^2}\right]-F_0(x,y)+H_0(x,y)$$

$$2U_2(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_0(x,y)}{\partial x^2}+\frac{\partial^2 U_0(x,y)}{\partial y^2}\right]-F_0(x,y)+H_0(x,y) \tag{4.28}$$

$$\frac{\partial^2 U_0(x,y)}{\partial x^2}=\frac{\partial^2}{\partial x^2}\left(\cos(\pi x)\cos(\pi y)\right)=-\pi^2\cos(\pi x)\cos(\pi y)$$

$$=\frac{\partial^2}{\partial y^2}\left(\cos(\pi x)\cos(\pi y)\right)=\frac{\partial^2 U_0(x,y)}{\partial y^2} \tag{4.29}$$

Using theorem 4.3,

$$F_0(x,y)=\sin U_0(x,y)=\sin\left(\cos(\pi x)\cos(\pi y)\right) \tag{4.30}$$

By definition 4.1

$$H_0(x,y)=\frac{1}{0!}\left[\sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right)\right]_{t=0}=\sin\left(\cos(\pi x)\cos(\pi y)\right) \tag{4.31}$$

Substituting (4.31), (4.30) and (4.29) in to eq (4.28), we obtain

$$2U_2(x,y)=\frac{1}{2\pi^2}\left[-\pi^2\cos(\pi x)\cos(\pi y)-\pi^2\cos(\pi x)\cos(\pi y)\right]$$

$$-\sin\left(\cos(\pi x)\cos(\pi y)\right)+\sin\left(\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow U_2(x,y)=-\frac{1}{2}\cos(\pi x)\cos(\pi y)$$

For $k=1$ eq (4.25) has the form

$$(1+2)(1+1)U_{1+2}(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_1(x,y)}{\partial x^2}+\frac{\partial^2 U_1(x,y)}{\partial y^2}\right]-F_1(x,y)+H_1(x,y)$$

$$\Rightarrow 6U_3(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_1(x,y)}{\partial x^2}+\frac{\partial^2 U_1(x,y)}{\partial y^2}\right]-F_1(x,y)+H_1(x,y) \qquad (4.32)$$

But $\frac{\partial^2 U_1(x,y)}{\partial x^2}=\frac{\partial^2}{\partial x^2}(0)=0=\frac{\partial^2 U_1(x,y)}{\partial y^2}$ (4.33)

Using theorem 4.3, $F_1(x,y)=G_0(x,y)U_1(x,y)=\cos\big(U_0(x,y)\big)U_1(x,y)$

$$=\cos\big(\cos(\pi x)\cos(\pi y)\big)\big(0\big)=0 \qquad (4.34)$$

By definition 4.1

$$H_1(x,y)=\frac{1}{1!}\left[\frac{\partial}{\partial t}\sin\big(\cos(\pi x)\cos(\pi y)\cos(t)\big)\right]_{t=0}$$

$$=\left[-\cos\big(\cos(\pi x)\cos(\pi y)\cos(t)\big)\big(\cos(\pi x)\cos(\pi y)\sin(t)\big)\right]_{t=0}=0$$

$$\Rightarrow H_1(x,y)=0 \qquad (4.35)$$

Substituting (4.35), (4.34) and (4.33) in to eq (4.32), we obtain

$$6U_3(x,y)=\frac{1}{2\pi^2}(0)-0+0$$

$$\Rightarrow U_3(x,y)=0$$

For $k=2$ eq (4.25) has the form

$$(2+2)(2+1)U_{2+2}(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_2(x,y)}{\partial x^2}+\frac{\partial^2 U_2(x,y)}{\partial y^2}\right]-F_2(x,y)+H_2(x,y)$$

$$\Rightarrow 12U_4(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_2(x,y)}{\partial x^2}+\frac{\partial^2 U_2(x,y)}{\partial y^2}\right]-F_2(x,y)+H_2(x,y) \qquad (4.36)$$

But $\frac{\partial^2 U_2(x,y)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{-1}{2}\cos(\pi x)\cos(\pi y)\right) = \frac{\pi^2}{2}\left(\cos(\pi x)\cos(\pi y)\right)$

$$= \frac{\partial^2}{\partial y^2}\left(\frac{-1}{2}\cos(\pi x)\cos(\pi y)\right) = \frac{\partial^2 U_2(x,y)}{\partial y^2} \qquad (4.37)$$

Using theorem 4.3

$$F_2(x,y) = G_0(x,y)U_2(x,y) - \frac{1}{2}F_0(x,y)U_1^2(x,y)$$

$$= \cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\frac{-1}{2}\cos(\pi x)\cos(\pi y)\right) - \frac{1}{2}\sin\left(\cos(\pi x)\cos(\pi y)\right)(0)$$

$$\Rightarrow F_2(x,y) = -\frac{1}{2}\cos\left(\cos(\pi x)\cos(\pi y)\right)\cos(\pi x)\cos(\pi y) \qquad (4.38)$$

By definition 4.1

$$H_2(x,y) = \frac{1}{2!}\left[\frac{\partial^2}{\partial t^2}\sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right)\right]_{t=0}$$

$$= -\frac{1}{2}\left[\cos\left(\cos(\pi x)\cos(\pi y)\right)\cos(\pi x)\cos(\pi y)\right] \qquad (4.39)$$

Substituting (4.39), (4.38) and (4.37) in to eq (4.36), we obtain

$$12U_4(x,y) = \frac{1}{2\pi^2}\left[\frac{\pi^2}{2}\left(\cos(\pi x)\cos(\pi y)\right) + \frac{\pi^2}{2}\left(\cos(\pi x)\cos(\pi y)\right)\right]$$

$$+\frac{1}{2}\cos\left(\cos(\pi x)\cos(\pi y)\right)\cos(\pi x)\cos(\pi y)$$

$$-\frac{1}{2}\left[\cos\left(\cos(\pi x)\cos(\pi y)\right)\cos(\pi x)\cos(\pi y)\right]$$

$$\Rightarrow U_4(x,y) = \frac{1}{24}\left[\cos(\pi x)\cos(\pi y)\right]$$

For $k = 3$ eq (4.25) takes the form

$$(3+2)(3+1)U_{3+2}(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_3(x,y)}{\partial x^2}+\frac{\partial^2 U_3(x,y)}{\partial y^2}\right]-F_3(x,y)+H_3(x,y)$$

$$20U_5(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_3(x,y)}{\partial x^2}+\frac{\partial^2 U_3(x,y)}{\partial y^2}\right]-F_3(x,y)+H_3(x,y) \quad (4.40)$$

But $\frac{\partial^2 U_3(x,y)}{\partial x^2}=\frac{\partial^2}{\partial x^2}(0)=0=\frac{\partial^2 U_3(x,y)}{\partial y^2}$ (4.41)

Using theorem 4.3

$$F_3(x,y)=G_0(x,y)U_3(x,y)-F_0(x,y)U_1(x,y)U_2(x,y)-\frac{1}{6}G_0(x,y)U_1^3(x,y)$$

$$\Rightarrow F_3(x,y)=0-0-0=0 \quad (4.42)$$

By definition 4.1

$$H_3(x,y)=\frac{1}{3!}\left[\frac{\partial^3}{\partial t^3}\sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right)\right]_{t=0}=0 \quad (4.43)$$

Substitutng (4.43), (4.42) and (4.41) in to eq (4.40), we obtain

$$20U_5(x,y)=\frac{1}{2\pi^2}(0)-0+0$$

$$\Rightarrow U_5(x,y)=0$$

For $k=4$ eq (4.25) takes the form

$$(4+2)(4+1)U_{4+2}(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_4(x,y)}{\partial x^2}+\frac{\partial^2 U_4(x,y)}{\partial y^2}\right]-F_4(x,y)+H_4(x,y)$$

$$\Rightarrow 30U_6(x,y)=\frac{1}{2\pi^2}\left[\frac{\partial^2 U_4(x,y)}{\partial x^2}+\frac{\partial^2 U_4(x,y)}{\partial y^2}\right]-F_4(x,y)+H_4(x,y) \quad (4.44)$$

$$\frac{\partial^2 U_4(x,y)}{\partial x^2}=\frac{\partial^2}{\partial x^2}\left(\frac{1}{24}\cos(\pi x)\cos(\pi y)\right)=-\frac{\pi^2}{24}\left(\cos(\pi x)\cos(\pi y)\right)$$

$$= \frac{\partial^2}{\partial y^2}\left(\frac{1}{24}\cos(\pi x)\cos(\pi y)\right) = \frac{\partial^2 U_4(x,y)}{\partial y^2} \quad (4.45)$$

Using theorem 4.3

$$F_4(x,y) = G_0(x,y)U_4(x,y) - F_0(x,y)U_1(x,y)U_3(x,y) - \frac{1}{2}F_0(x,y)U_2^2(x,y)$$

$$-\frac{1}{2}G_0(x,y)U_1^2(x,y)U_2(x,y) + \frac{1}{24}F_0(x,y)U_1^4(x,y)$$

$$\Rightarrow F_4(x,y) = \frac{1}{24}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right)$$

$$-\frac{1}{8}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right)^2 \quad (4.46)$$

By definition 4.1

$$H_4(x,y) = \frac{1}{4!}\left[\frac{\partial^4}{\partial t^4}\sin\left(\cos(\pi x)\cos(\pi y)\cos(t)\right)\right]_{t=0}$$

$$= \frac{1}{24}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right)$$

$$-\frac{1}{8}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right)^2 \quad (4.47)$$

Substituting $(4.47), (4.46)$ and (4.45) into eq (4.44), we obtain

$$30U_6(x,y) = \frac{1}{2\pi^2}\left[-\frac{\pi^2}{24}\left(\cos(\pi x)\cos(\pi y)\right) - \frac{\pi^2}{24}\left(\cos(\pi x)\cos(\pi y)\right)\right]$$

$$-\frac{1}{24}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right) + \frac{1}{8}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right)^2$$

$$= \frac{1}{24}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right) - \frac{1}{8}\cos\left(\cos(\pi x)\cos(\pi y)\right)\left(\cos(\pi x)\cos(\pi y)\right)^2$$

$$\Rightarrow U_6(x,y) = \frac{1}{720}\left[\cos(\pi x)\cos(\pi y)\right] = \frac{1}{6!}\left[\cos(\pi x)\cos(\pi y)\right]$$

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$$U_k(x,y)=\begin{cases}\frac{1}{k!}\left[\cos(\pi x)\cos(\pi y)\right], & \text{for "}k\text{" is even,}\\ 0, & \text{for "}k\text{" is odd.}\end{cases}$$

Now using definition of inverse differential Transform, we get

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k$$

$$=U_0(x,y)t^0+U_1(x,y)t^1+U_2(x,y)t^2+U_3(x,y)t^3+U_4(x,y)t^4+U_5(x,y)t^5+U_6(x,y)t^6+...$$

$$=\cos(\pi x)\cos(\pi y)-\frac{1}{2}\left[\cos(\pi x)\cos(\pi y)\right]t^2+\frac{1}{24}\left[\cos(\pi x)\cos(\pi y)\right]t^4$$

$$-\frac{1}{720}\left[\cos(\pi x)\cos(\pi y)\right]t^6+...$$

$$=\cos(\pi x)\cos(\pi y)\left(1-\frac{t^2}{2}+\frac{t^4}{24}-\frac{t^6}{720}+...\right)$$

$$=\cos(\pi x)\cos(\pi y)\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+...\right)$$

$$=\cos(\pi x)\cos(\pi y)\cos(t)$$

Hence, the exact solution is

$$u(x,y,t)=\cos(\pi x)\cos(\pi y)\cos(t).$$

This is exactly the same as the result obtained by Kang *et al*. (2017).

➤ To test the convergence of the approximated solution, we calculated "α_k" as:

Using definition 4.3, $\alpha_k = \begin{cases} \dfrac{\|\beta_{k+1}\|}{\|\beta_k\|}, \; if \;\; \|\beta_k\| \neq 0, \\ 0\,, \quad if \quad \|\beta_k\| = 0. \end{cases}$

For $k=0$,

$$\alpha_0 = \frac{\|\beta_1\|}{\|\beta_0\|} = \frac{\|U_1(x,y)t\|}{\|U_0(x,y)\|} = \frac{\|0\|}{\|\cos(\pi x)\cos(\pi y)\|} = 0 < 1$$

For $k=1$,

By definition 4.3 $\alpha_1 = 0 < 1$ since $\|\beta_1\| = 0$.

For $k=2$,

$$\alpha_2 = \frac{\|\beta_3\|}{\|\beta_2\|} = \frac{\|U_3(x,y)t^3\|}{\|U_2(x,y)t^2\|} = \frac{0}{\left\|\frac{-1}{2}\cos(\pi x)\cos(\pi y)\right\|} = 0 < 1$$

For $k=3$,

By definition 4.3 $\alpha_3 = 0 < 1$ since $\|\beta_3\| = 0$.

For $k=4$,

$$\alpha_4 = \frac{\|\beta_5\|}{\|\beta_4\|} = \frac{\|U_5(x,y)t^5\|}{\|U_4(x,y)t^4\|} = \frac{0}{\frac{1}{24}\left[\cos(\pi x)\cos(\pi y)\right]} = 0 < 1$$

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$$\alpha_k = \begin{cases} \dfrac{\|\beta_{k+1}\|}{\|\beta_k\|} = \dfrac{\|U_{k+1}(x,y)t^{k+1}\|}{\|U_k(x,y)t^k\|} = \dfrac{\|0\|}{\|U_k(x,y)t^k\|} = \dfrac{\|0\|}{\left\|\dfrac{1}{(k+1)!}\cos(\pi x)\cos(\pi y)\right\|} = 0 < 1, & for\ k\ is\ even, \\ 0 = 0 < 1, \quad by\ defitinion\ 4.3, & for\ k\ is\ odd. \end{cases}$$

Hence, for $x, y \in \Omega$ *and* $t \geq 0$, we obtain

$\alpha_0 = 0 < 1,\ \alpha_1 = 0 < 1,\ \alpha_2 = 0 < 1,\ \alpha_3 = 0 < 1,\ \alpha_4 = 0 < 1, \ldots, \alpha_k = 0 < 1.$ Therefore, using the stated definition 4.3 the solution of eq (4.22) by RDTM is converges to the exact solution.

The solution curves of two dimensional nonlinear Sine-Gordon equation given in Example 4.1 above for different order of truncation "n" is depicted in Figure 1 below

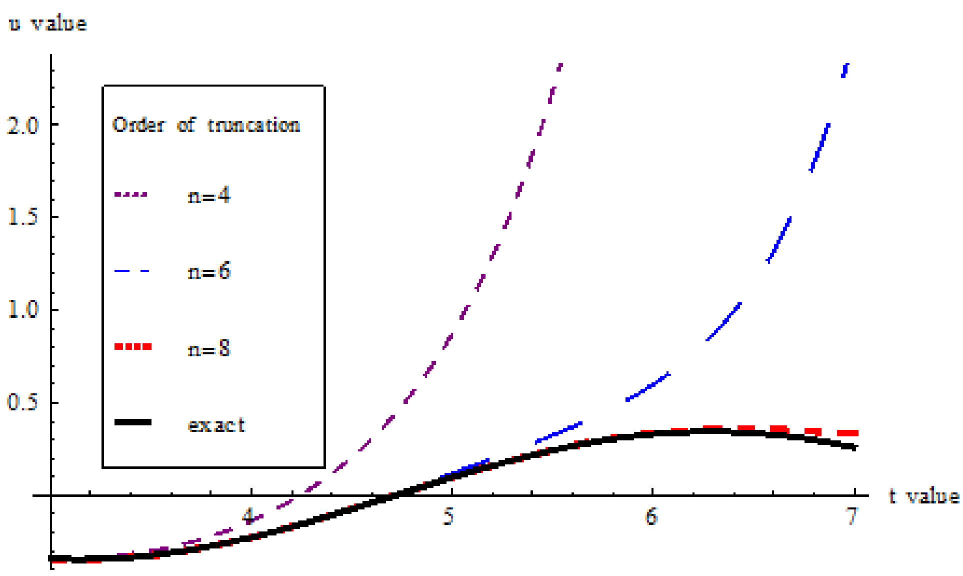

Figure 1. Solution curves of example 4.1 for $x = y = 0.3$ and $t \in [3,7]$

EXAMPLE 4. 2: [Li *et al*., 2019]

Consider the following two dimensional sine- Gordon equations on the domain $\Omega=[0,2]\times[0,2]$;

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-2\sin u+2\sin\left[e^{-t}\left(1-\cos(\pi x)\right)\left(1-\cos\left(\pi y\right)\right)\right]$$

$$-\pi^2 e^{-t}\left[\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right], \qquad (4.48)$$

with initial conditions

$$u(x,y,0)=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right), \qquad (4.49)$$

$$\frac{\partial}{\partial t}u(x,y,0)=-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right). \qquad (4.50)$$

Solution: Applying properties of RDTM to both sides of equation (4.48), we construct the following recursive formula:

$$(k+2)(k+1)U_{k+2}(x,y)+(k+1)U_{k+1}(x,y)=\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{\partial^2}{\partial y^2}U_k(x,y)-2F_k(x,y)+H_k(x,y), \quad (4.51)$$

where $F_k(x,y)$ is the reduced differential transform of nonlinear term $sinu(x,y,t)$ and $H_k(x,y)$ is the reduced differential transform of inhomogeneous term $\left[2\sin e^{-t}\left((1-\cos(\pi x))(1-\cos(\pi y))\right)\right]-\pi e^{-t}\left[\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right]$.

Let us define $H_k(x,y)$ as follows:

$H_k(x,y)=D_k(x,y)-Z_k(x,y)$, where $D_k(x,y)$ is the reduced differential transform of $2\sin e^{-t}\left((1-\cos(\pi x))(1-\cos(\pi y)\right)$ and $Z_k(x,y)$ is the reduced differential transform of $\pi e^{-t}\left[\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right]$.

Again applying properties of RDTM to initial conditions (4.49) and (4.50), we get

$$U_0(x,y)=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \tag{4.52}$$

$$U_1(x,y)=-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \tag{4.53}$$

Now taking the values of k, ($k = 0,1,2,...$) and applying theorem 4.3, definition 4.1, theorem 4.5 and theorem 4.8 in equation (4.51), we obtain the following successive values of $U_k(x,y)$.

For $k=0$ eq (4.51) becomes

$$(0+2)(0+1)U_{0+2}(x,y)+(0+1)U_{0+1}(x,y)=\frac{\partial^2}{\partial x^2}U_0(x,y)+\frac{\partial^2}{\partial y^2}U_0(x,y)-2F_0(x,y)+H_0(x,y)$$

$$\Rightarrow 2U_2(x,y)+U_1(x,y)=\frac{\partial^2}{\partial x^2}U_0(x,y)+\frac{\partial^2}{\partial y^2}U_0(x,y)-2F_0(x,y)+H_0(x,y) \tag{4.54}$$

But $\frac{\partial^2}{\partial x^2}U_0(x,y)=\frac{\partial^2}{\partial x^2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)=\pi^2\cos(\pi x)\left(1-\cos(\pi y)\right)$ (4.55)

$$\frac{\partial^2}{\partial y^2}U_0(x,y)=\frac{\partial^2}{\partial y^2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)=\pi^2\cos(\pi y)\left(1-\cos(\pi x)\right) \tag{4.56}$$

Using theorem 4.3

$$F_0(x,y)=\sin U_0(x,y)=\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right) \tag{4.57}$$

By definition 4.1

$$D_0(x,y)=\frac{1}{0!}\left[2\sin e^{-t}\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\right]_{t=0}$$

$$=2\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)$$

By theorems 4.5 and 4.8, we have

$$Z_0(x,y)=\frac{(-1)^0}{0!}\left[\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)\right]$$

$$=\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

Therefore, $H_0(x,y)=D_0(x,y)-Z_0(x,y)$

$$\Rightarrow H_0(x,y)=2\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)$$

$$-\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right) \qquad (4.58)$$

Now, substituting (4.58), (4.57), (4.56), (4.55) and (4.53) in to eq (4.54), we obtain

$$2U_2(x,y)-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)=\pi^2\cos(\pi x)\left(1-\cos(\pi y)\right)+\pi^2\cos(\pi y)\left(1-\cos(\pi x)\right)$$

$$-2\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)$$

$$+2\sin\left(\left(1-\cos(\pi x)\right)(1-\cos(\pi y)\right)-\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow 2U_2(x,y)=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)+\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$-\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow U_2(x,y)=\frac{1}{2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \qquad (4.59)$$

For $k=1$ eq (4.51) becomes

$$(1+2)(1+1)U_{1+2}(x,y)+(1+1)U_{1+1}(x,y)=\frac{\partial^2}{\partial x^2}U_1(x,y)+\frac{\partial^2}{\partial y^2}U_1(x,y)-2F_1(x,y)+H_1(x,y)$$

$$\Rightarrow 6U_3(x,y)+2U_2(x,y)=\frac{\partial^2}{\partial x^2}U_1(x,y)+\frac{\partial^2}{\partial y^2}U_1(x,y)-2F_1(x,y)+H_1(x,y) \qquad (4.60)$$

But $\frac{\partial^2}{\partial x^2}U_1(x,y)=\frac{\partial^2}{\partial x^2}\left[-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right]=-\pi^2\cos(\pi x)\left(1-\cos(\pi y)\right)$ (4.61)

$$\frac{\partial^2}{\partial y^2}U_1(x,y)=\frac{\partial^2}{\partial y^2}\left[-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right]=-\pi^2\cos(\pi y)\left(1-\cos(\pi x)\right) \quad (4.62)$$

Using theorem 4.3

$$F_1(x,y)=G_0(x,y)U_1(x,y)=\cos U_0(x,y)U_1(x,y)$$

$$=-\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \quad (4.63)$$

By definition 4.1

$$D_1(x,y)=\frac{1}{1!}\left[\frac{\partial}{\partial t}\left(2\sin e^{-t}\left((1-\cos(\pi x))(1-\cos(\pi y))\right)\right)\right]_{t=0}$$

$$=-2\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)$$

By theorems 4.5 and 4.8, we have

$$Z_1(x,y)=\frac{(-1)^1}{1!}\left[\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)\right]$$

$$=-\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

Therefore, $H_1(x,y)=D_1(x,y)-Z_1(x,y)$

$$\Rightarrow H_1(x,y)=-2\cos\left(\left(1-\cos(\pi x)\right)1\left(-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)$$

$$+\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right) \quad (4.64)$$

Substituting (4.64), (4.63), (4.62) , (4.61)and (4.59) in to eq (4.60), we obtain

$$6U_3(x,y)+2\left(\frac{1}{2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)=-\pi^2\cos(\pi x)\left(1-\cos(\pi y)\right)-\pi^2\cos(\pi y)\left(1-\cos(\pi x)\right)$$
$$+2\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)$$

$$-2\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)$$

$$+\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow 6U_3(x,y)=-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)-\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$+\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow U_3(x,y)=-\frac{1}{6}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \tag{4.65}$$

For $k=2$ eq (4.51) becomes

$$(2+2)(2+1)U_{2+2}(x,y)+(2+1)U_{2+1}(x,y)=\frac{\partial^2}{\partial x^2}U_2(x,y)+\frac{\partial^2}{\partial y^2}U_2(x,y)-2F_2(x,y)+H_2(x,y)$$

$$\Rightarrow 12U_4(x,y)+3U_3(x,y)=\frac{\partial^2}{\partial x^2}U_2(x,y)+\frac{\partial^2}{\partial y^2}U_2(x,y)-2F_2(x,y)+H_2(x,y) \tag{4.66}$$

But $\frac{\partial^2}{\partial x^2}U_2(x,y)=\frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right]=\frac{1}{2}\pi^2\cos(\pi x)\left(1-\cos(\pi y)\right)$ (4.67)

$$\frac{\partial^2}{\partial y^2}U_2(x,y)=\frac{\partial^2}{\partial y^2}\left[\frac{1}{2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right]=\frac{1}{2}\pi^2\cos(\pi y)\left(1-\cos(\pi x)\right) \tag{4.68}$$

Using theorem 4.3

$$F_2(x,y)=G_0(x,y)U_2(x,y)-\frac{1}{2}F_0(x,y)U_1^2(x,y)$$

$$F_2(x,y)=\frac{1}{2}\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)$$

$$-\frac{1}{2}\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^2 \tag{4.69}$$

By definition 4.1

$$D_2(x,y)=\frac{1}{2!}\left[\frac{\partial^2}{\partial t^2}\left[2\sin e^{-t}\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\right]\right]_{t=0}$$

$$=\cos\left((1-\cos(\pi x))(1-\cos(\pi y))\right)(1-\cos(\pi x))(1-\cos(\pi y))$$

$$-\sin\left((1-\cos(\pi x))(1-\cos(\pi y))\right)\left((1-\cos(\pi x))(1-\cos(\pi y))\right)^2$$

By theorems 4.5 and 4.8, we have

$$Z_2(x,y)=\frac{(-1)^2}{2!}\left[\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)\right]$$
$$=\frac{1}{2}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

Therefore, $H_2(x,y)=D_2(x,y)-Z_2(x,y)$

$$\Rightarrow H_2(x,y)=\cos\left((1-\cos(\pi x))(1-\cos(\pi y))\right)(1-\cos(\pi x))(1-\cos(\pi y))$$

$$-\sin\left((1-\cos(\pi x))(1-\cos(\pi y))\right)\left((1-\cos(\pi x))(1-\cos(\pi y))\right)^2$$

$$-\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right) \qquad (4.70)$$

Substituting (4.70), (4.69), (4.68), (4.67) and (4.65) in to eq (4.66), we obtain

$$12U_4(x,y)+3\left(-\frac{1}{6}(1-\cos(\pi x))(1-\cos(\pi y))\right)=\frac{1}{2}\pi^2\cos(\pi x)(1-\cos(\pi y))+\frac{1}{2}\pi^2\cos(\pi y)(1-\cos(\pi x))$$
$$-\cos\left((1-\cos(\pi x))(1-\cos(\pi y))\right)(1-\cos(\pi x))(1-\cos(\pi y))$$

$$+\sin\left((1-\cos(\pi x))(1-\cos(\pi y))\right)\left((1-\cos(\pi x))(1-\cos(\pi y))\right)^2$$

$$+\cos\left((1-\cos(\pi x))(1-\cos(\pi y))\right)(1-\cos(\pi x))(1-\cos(\pi y))$$

$$-\sin\left((1-\cos(\pi x))(1-\cos(\pi y))\right)\left((1-\cos(\pi x))(1-\cos(\pi y))\right)^2$$
$$-\frac{1}{2}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow 12U_4(x,y) = \frac{1}{2}\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big) + \frac{1}{2}\pi^2\left(\cos(\pi x)+\cos(\pi y) - 2\cos(\pi x)\cos(\pi y)\right)$$

$$-\frac{1}{2}\pi^2\left(\cos(\pi x)+\cos(\pi y) - 2\cos(\pi x)\cos(\pi y)\right)$$

$$= \frac{1}{24}\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)$$

$$\Rightarrow U_4(x,y) = \frac{1}{24}\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big) \tag{4.71}$$

For $k=3$ eq (4.51) becomes

$$\Rightarrow 20U_5(x,y) + 4U_4(x,y) = \frac{\partial^2}{\partial x^2}U_3(x,y) + \frac{\partial^2}{\partial y^2}U_3(x,y) - 2F_3(x,y) + H_3(x,y) \tag{4.72}$$

But $\frac{\partial^2}{\partial x^2}U_3(x,y) = \frac{\partial^2}{\partial x^2}\left[-\frac{1}{6}\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\right] = -\frac{1}{6}\pi^2\cos(\pi x)\big(1-\cos(\pi y)\big)$ (4.73)

$$\frac{\partial^2}{\partial y^2}U_3(x,y) = \frac{\partial^2}{\partial y^2}\left[-\frac{1}{6}\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\right] = -\frac{1}{6}\pi^2\cos(\pi y)\big(1-\cos(\pi x)\big) \tag{4.74}$$

Using theorem 4.3

$$F_3(x,y) = G_0(x,y)U_3(x,y) - F_0(x,y)U_1(x,y)U_2(x,y) - \frac{1}{6}G_0(x,y)U_1^3(x,y)$$

$$\Rightarrow F_3(x,y) = \frac{1}{6}\begin{bmatrix} 3\sin\Big(\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\Big)\Big(\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\Big)^2 \\ -\cos\Big(\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\Big)\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big) \\ +\cos\Big(\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\Big)\Big(\big(1-\cos(\pi x)\big)\big(1-\cos(\pi y)\big)\Big)^3 \end{bmatrix} \tag{4.75}$$

By definition 4.1

$$D_3(x,y)=\frac{1}{3!}\left[\frac{\partial^3}{\partial t^3}\left[2\sin e^{-t}\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\right]\right]_{t=0}$$

$$=\frac{1}{3}\begin{bmatrix}3\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^2\\ -\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\\ +\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^3\end{bmatrix}$$

By theorems 4.5 and 4.8, we have

$$Z_3(x,y)=\frac{(-1)^3}{3!}\left[\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)\right]$$

$$=-\frac{1}{6}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

Therefore, $H_3(x,y)=D_3(x,y)-Z_3(x,y)$

$$\Rightarrow H_3(x,y)=\frac{1}{3}\begin{bmatrix}3\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^2\\ -\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\\ +\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^3\end{bmatrix}+\frac{1}{6}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right) \tag{4.76}$$

Substituting (4.76), (4.75), (4.74), (4.73) and (4.71) in to eq (4.72), we obtain

$$20U_5(x,y)+4\left(\frac{1}{24}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)=-\frac{1}{6}\pi^2\cos(\pi x)\left(1-\cos(\pi y)\right)$$

$$-\frac{1}{6}\pi^2\cos(\pi y)\left(1-\cos(\pi x)\right)$$

$$-\frac{1}{3}\begin{bmatrix}3\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^2\\ -\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\\ +\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^3\end{bmatrix}$$

$$+\frac{1}{3}\begin{bmatrix} 3\sin\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^2 \\ -\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \\ +\cos\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)\left(\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\right)^3 \end{bmatrix}$$

$$+\frac{1}{6}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$\Rightarrow 20U_5(x,y)=-\frac{1}{6}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)-\frac{1}{6}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$+\frac{1}{6}\pi^2\left(\cos(\pi x)+\cos(\pi y)-2\cos(\pi x)\cos(\pi y)\right)$$

$$=-\frac{1}{6}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)$$

$$\Rightarrow U_5(x,y)=-\frac{1}{120}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right) \tag{4.77}$$

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$$U_k(x,y)=\begin{cases} \frac{1}{k!}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right), & for\ "k"\ is\ even, \\ -\frac{1}{k!}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right), & for\ "k"\ is\ odd. \end{cases}.$$

Now using the definition of inverse differential transform method, we get

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k$$

$$=U_0(x,y)t^0+U_1(x,y)t^1+U_2(x,y)t^2+U_3(x,y)t^3+U_4(x,y)t^4+U_5(x,y)t^5+U_6(x,y)t^6+\ldots$$

$$=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)-\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)t^1$$

$$+\frac{1}{2}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)t^2-\frac{1}{6}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)t^3$$

$$+\frac{1}{24}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)t^4-\frac{1}{120}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)t^5+\ldots$$

$$=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\left[1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+\ldots\right]$$

$$=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\left[1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5+\ldots\right]$$

$$=\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right)\left(e^{-t}\right)$$

$$\Rightarrow u(x,y,t)=e^{-t}\left(1-\cos(\pi x)\right)\left(1-\cos(\pi y)\right).$$

This result shows an excellent agreement with the one obtained by Li *et al*. (2019) and Liu *et al*. (2015).

- To test the convergence of the approximated solution, we calculated "α_k" as:

First let us take $x=y=1$ & $t=0.5$, then using definition 4.3, we get,

$$\alpha_k=\begin{cases}\dfrac{\|\beta_{k+1}\|}{\|\beta_k\|},\ if\ \ \|\beta_k\|\neq 0,\\ 0\ ,\quad if \quad \|\beta_k\|=0.\end{cases}$$

For $k=0$,

$$\alpha_0 = \frac{\|\beta_1\|}{\|\beta_0\|} = \frac{\|U_1(x,y)t\|}{\|U_0(x,y)\|} = \frac{\|(1-\cos(\pi x))(1-\cos(\pi y))t\|}{\|-(1-\cos(\pi x))(1-\cos(\pi y))\|} = \|0.5\| = 0.5 < 1$$

For $k = 1$,

$$\alpha_1 = \frac{\|\beta_2\|}{\|\beta_1\|} = \frac{\|U_2(x,y)t^2\|}{\|U_1(x,y)t\|} = \frac{\left\|\frac{1}{2}(1-\cos(\pi x))(1-\cos(\pi y))t^2\right\|}{\|-(1-\cos(\pi x))(1-\cos(\pi y))t\|} = \|0.25\| = 0.25 < 1$$

For $k = 2$,

$$\alpha_2 = \frac{\|\beta_3\|}{\|\beta_2\|} = \frac{\|U_3(x,y)t^3\|}{\|U_2(x,y)t^2\|} = \frac{\left\|-\frac{1}{6}(1-\cos(\pi x))(1-\cos(\pi y))t^3\right\|}{\left\|\frac{1}{2}(1-\cos(\pi x))(1-\cos(\pi y))t^2\right\|}$$

$$= \|-0.1666666667\| = 0.1666666667 < 1$$

For $k = 3$,

$$\alpha_3 = \frac{\|\beta_4\|}{\|\beta_3\|} = \frac{\|U_4(x,y)t^4\|}{\|U_3(x,y)t^3\|} = \frac{\left\|\frac{1}{24}(1-\cos(\pi x))(1-\cos(\pi y))t^4\right\|}{\left\|-\frac{1}{6}(1-\cos(\pi x))(1-\cos(\pi y))t^3\right\|} = \|-0.125\| = 0.125 < 1$$

For $k = 4$,

$$\alpha_4 = \frac{\|\beta_5\|}{\|\beta_4\|} = \frac{\|U_5(x,y)t^5\|}{\|U_4(x,y)t^4\|} = \frac{\left\|-\frac{1}{120}(1-\cos(\pi x))(1-\cos(\pi y))t\right\|}{\left\|\frac{1}{24}(1-\cos(\pi x))(1-\cos(\pi y))\right\|} = \|-0.1\| = 0.1 < 1$$

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For ‘k’ is even,

$$\alpha_k = \frac{\|\beta_{k+1}\|}{\|\beta_k\|} = \frac{\|U_{k+1}(x,y)t^{k+1}\|}{\|U_k(x,y)t^k\|} = \frac{\left\|-\frac{1}{(k+1)!}(1-\cos(\pi x))(1-\cos(\pi y))t\right\|}{\left\|\frac{1}{(k)!}(1-\cos(\pi x))(1-\cos(\pi y))\right\|} = \left\|-\frac{1}{2(k+1)}\right\| = \frac{1}{2(k+1)} < 1$$

For ‘k’ is odd,

$$\alpha_k = \frac{\|\beta_{k+1}\|}{\|\beta_k\|} = \frac{\|U_{k+1}(x,y)t^{k+1}\|}{\|U_k(x,y)t^k\|} = \frac{\left\|\frac{1}{(k+1)!}(1-\cos(\pi x))(1-\cos(\pi y))t\right\|}{\left\|-\frac{1}{(k)!}(1-\cos(\pi x))(1-\cos(\pi y))\right\|} = \left\|-\frac{1}{2(k+1)}\right\| = \frac{1}{2(k+1)} < 1.$$

Hence, for $x, y \in \Omega = [0,2]^2$, $t \geq 0$ and taking $x = y = 1$ & $t = 0.5$, we obtain, $\alpha_0 = 0.5 < 1$, $\alpha_1 = 0.25 < 1$, $\alpha_2 = 0.1666666667 < 1$, $\alpha_3 = 0.125 < 1$, $\alpha_4 = 0.1 < 1$, ..., $\alpha_k = \frac{1}{2(k+1)} < 1.$

Therefore, using the stated definition 4.3 the solution of eq (4.48) by RDTM is converges to the exact solution.

The solution curves of two dimensional nonlinear Sine-Gordon equation given in Example 4.2 above for different order of truncation "n" is depicted in Figure 2 below

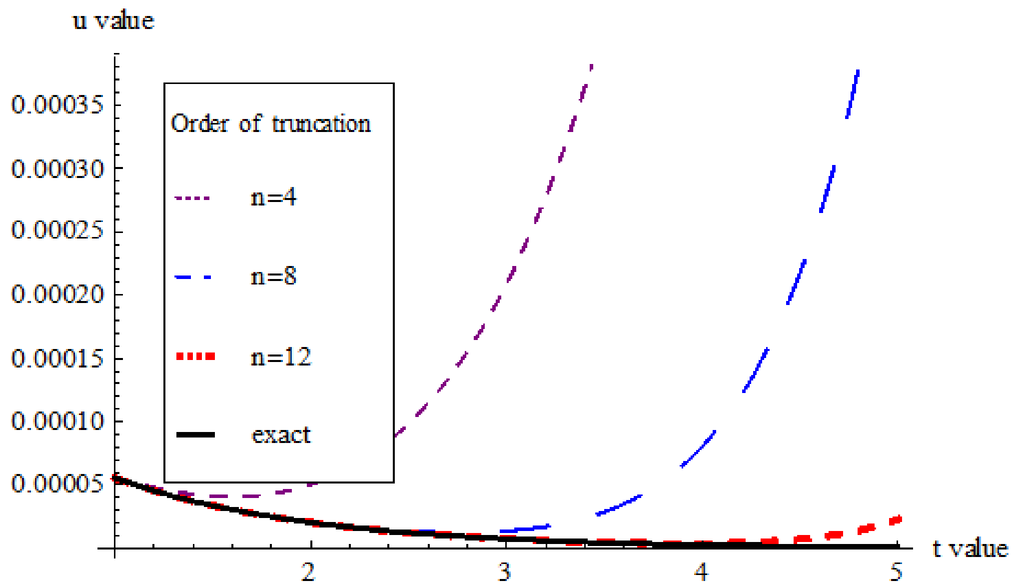

Figure 2. Solution curves of example 4.2 for $x = y = 0.05$ and $t \in [1,5]$

EXAMPLE 4. 3: [Baccouch, 2019]

Consider the following two dimensional inhomogeneous sine- Gordon equations,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \sin u + \sin\left(\sin(x+y+t)\right) + \sin(x+y+t)\,,\ (x,y) \in \Omega = \left[0, 2\pi\right]^2 \ \&\, t \geq 0, \quad (4.78)$$

with initial conditions

$$u(x,y,0) = \sin(x+y) \quad , \quad x, y \in \Omega, \quad (4.79)$$

$$\frac{\partial u(x,y,0)}{\partial t} = \cos(x+y)\,,\ x, y \in \Omega. \quad (4.80)$$

Solution: Applying properties of RDTM to both sides of equation (4.78), we construct the following recursive formula:

$$(k+2)(k+1)U_{k+2}(x,y) = \frac{\partial^2}{\partial x^2}U_k(x,y) + \frac{\partial^2}{\partial y^2}U_k(x,y) - F_k(x,y) + H_k(x,y)\ , \quad (4.81)$$

where $F_k(x,y)$ is the reduced differential transform of nonlinear term $sinu(x,y,t)$ and $H_k(x,y)$ is the reduced differential transform of inhomogeneous term $\left[\sin\left(\sin(x+y+t)\right) + \sin(x+y+t)\right]$.

Let us define $H_k(x,y)$ as follows:

$H_k(x,y) = D_k(x,y) + Z_k(x,y)$, where $D_k(x,y)$ is the reduced differential transform of $\sin\left(\sin(x+y+t)\right)$ and $Z_k(x,y)$ is the reduced differential transform of $\sin(x+y+t)$.

Again applying properties of RDTM to initial conditions (4.79) and (4.780), we get

$$U_0(x,y) = \sin(x+y) \quad (4.82)$$

$$U_1(x,y) = \cos(x+y) \quad (4.83)$$

Now taking the values of k, (k =0,1,2,…) and applying theorem 4.3, definition 4.1 and theorem 4.7 in equation (4.81) we obtain the following successive values of $U_k(x,y)$.

For $k=0$ eq (4.81) takes the form

$$(0+2)(0+1)U_{0+2}(x,y)=\frac{\partial^2}{\partial x^2}U_0(x,y)+\frac{\partial^2}{\partial y^2}U_0(x,y)-F_0(x,y)+H_0(x,y)$$

$$\Rightarrow 2U_2(x,y)=\frac{\partial^2}{\partial x^2}U_0(x,y)+\frac{\partial^2}{\partial y^2}U_0(x,y)-F_0(x,y)+H_0(x,y) \tag{4.84}$$

But $\frac{\partial^2}{\partial x^2}U_0(x,y)=\frac{\partial^2}{\partial x^2}\left(\sin(x+y)\right)=-\sin(x+y)=\frac{\partial^2}{\partial y^2}\left(\sin(x+y)\right)=\frac{\partial^2}{\partial y^2}U_0(x,y)$ (4.85)

Using theorem 4.3

$$F_0(x,y)=\sin\left(U_0(x,y)\right)=\sin\left(\sin(x+y)\right) \tag{4.86}$$

By definition 4.1

$$D_0(x,y)=\frac{1}{0!}\left[\sin\left(\sin(x+y+t)\right)\right]_{t=0}=\sin\left(\sin(x+y)\right)$$

By theorem 4.7

$$Z_0(x,y)=\frac{1^0}{0!}\left[\sin\left(\frac{\pi(0)}{2!}+x+y\right)\right]=\sin(x+y)$$

Therefore, $H_0(x,y)=D_0(x,y)+Z_0(x,y)=\sin\left(\sin(x+y)\right)+\sin(x+y)$ (4.87)

Now, substituting (4.87), (4.86) and (4.85) in to eq (4.84), we obtain

$$2U_2(x,y)=-\sin(x+y)-\sin(x+y)-\sin\left(\sin(x+y)\right)+\sin\left(\sin(x+y)\right)+\sin(x+y)$$

$$=-\sin(x+y)$$

$$\Rightarrow U_2(x,y)=-\frac{1}{2}\sin(x+y)$$

For $k=1$ eq (4.81) takes the form

$$(1+2)(1+1)U_{1+2}(x,y)=\frac{\partial^2}{\partial x^2}U_1(x,y)+\frac{\partial^2}{\partial y^2}U_1(x,y)-F_1(x,y)+H_1(x,y)$$

$$\Rightarrow 6U_3(x,y)=\frac{\partial^2}{\partial x^2}U_1(x,y)+\frac{\partial^2}{\partial y^2}U_1(x,y)-F_1(x,y)+H_1(x,y) \qquad (4.88)$$

But $\frac{\partial^2}{\partial x^2}U_1(x,y)=\frac{\partial^2}{\partial x^2}\left(\cos(x+y)\right)=-\cos(x+y)=\frac{\partial^2}{\partial y^2}\left(\cos(x+y)\right)=\frac{\partial^2}{\partial y^2}U_1(x,y)$ (4.89)

Using theorem 4.3

$$F_1(x,y)=G_0(x,y)U_1(x,y)=\cos\left(U_0(x,y)\right)U_1(x,y)$$

$$\Rightarrow F_1(x,y)=\cos\left(\sin(x+y)\right)\left(\cos(x+y)\right) \qquad (4.90)$$

By definition 4.1

$$D_1(x,y)=\frac{1}{1!}\left[\frac{\partial}{\partial t}\sin\left(\sin(x+y+t)\right)\right]_{t=0}=\cos\left(\sin(x+y)\right)\cos(x+y)$$

By theorem 4.7

$$Z_1(x,y)=\frac{1^1}{1!}\left[\sin\left(\frac{\pi(1)}{2!}+x+y\right)\right]=\cos(x+y)$$

Therefore, $H_1(x,y)=D_1(x,y)+Z_1(x,y)=\cos\left(\sin(x+y)\right)\cos(x+y)+\cos(x+y)$ (4.91)

Now, substituting (4.91), (4.90) and (4.89) in to eq (4.88), we obtain

$$6U_3(x,y)=-\cos(x+y)-\cos(x+y)-\cos\left(\sin(x+y)\right)\left(\cos(x+y)\right)$$

$$+\cos\left(\sin(x+y)\right)\cos(x+y)+\cos(x+y)$$

$$=-\cos(x+y)$$

$$\Rightarrow U_3(x,y) = -\frac{1}{6}\cos(x+y)$$

For $k=2$ eq (4.81) takes the form

$$(2+2)(2+1)U_{2+2}(x,y) = \frac{\partial^2}{\partial x^2}U_2(x,y) + \frac{\partial^2}{\partial y^2}U_2(x,y) - F_2(x,y) + H_2(x,y)$$

$$\Rightarrow 12U_4(x,y) = \frac{\partial^2}{\partial x^2}U_2(x,y) + \frac{\partial^2}{\partial y^2}U_2(x,y) - F_2(x,y) + H_2(x,y) \qquad (4.92)$$

But $\frac{\partial^2}{\partial x^2}U_2(x,y) = \frac{\partial^2}{\partial x^2}\left(-\frac{1}{2}\sin(x+y)\right) = \frac{1}{2}\sin(x+y) = \frac{\partial^2}{\partial y^2}\left(-\frac{1}{2}\sin(x+y)\right) = \frac{\partial^2}{\partial y^2}U_2(x,y)$ (4.93)

Using theorem 4.3

$$F_2(x,y) = G_0(x,y)U_2(x,y) - \frac{1}{2}F_0(x,y)U_1^2(x,y)$$

Therefore,

$$F_2(x,y) = -\frac{1}{2}\cos\left(\sin(x+y)\right)\sin(x+y) - \frac{1}{2}\sin\left(\sin(x+y)\right)\left(\cos(x+y)\right)^2$$

$$= -\frac{1}{2}\left[\sin\left(\sin(x+y)\right)\cos^2(x+y) + \cos\left(\sin(x+y)\right)\sin(x+y)\right] \qquad (4.94)$$

By definition 4.1

$$D_2(x,y) = \frac{1}{2!}\left[\frac{\partial^2}{\partial t^2}\sin\left(\sin(x+y+t)\right)\right]_{t=0} = \frac{1}{2!}\left[\frac{\partial}{\partial t}\cos\left(\sin(x+y+t)\right)\cos(x+y+t)\right]_{t=0}$$

Hence, $D_2(x,y) = -\frac{1}{2}\left[\sin\left(\sin(x+y)\right)\cos^2(x+y) + \cos\left(\sin(x+y)\right)\sin(x+y)\right]$

By theorem 4.7

$$Z_2(x,y) = \frac{1^2}{2!}\left[\sin\left(\frac{\pi(2)}{2!} + x + y\right)\right] = -\frac{1}{2}\sin\left(x+y\right)$$

Therefore, $H_2(x,y) = D_2(x,y) + Z_2(x,y)$

$$\Rightarrow H_2(x,y) = -\frac{1}{2}\left[\sin\left(\sin(x+y)\right)\cos^2(x+y) + \cos\left(\sin(x+y)\right)\sin(x+y)\right] - \frac{1}{2}\sin(x+y) \quad (4.95)$$

Substituting (4.95), (4.94) and (4.93) in to eq (4.92), we obtain

$$12U_4(x,y) = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x+y)$$

$$+\frac{1}{2}\left[\sin\left(\sin(x+y)\right)\cos^2(x+y) + \cos\left(\sin(x+y)\right)\sin(x+y)\right]$$

$$-\frac{1}{2}\left[\sin\left(\sin(x+y)\right)\cos^2(x+y) + \cos\left(\sin(x+y)\right)\sin(x+y)\right] - \frac{1}{2}\sin(x+y)$$

$$= \frac{1}{2}\sin(x+y)$$

$$\Rightarrow U_4(x,y) = \frac{1}{24}\sin(x+y)$$

For $k = 3$ eq (4.81) takes the form

$$20U_5(x,y) = \frac{\partial^2}{\partial x^2}U_3(x,y) + \frac{\partial^2}{\partial y^2}U_3(x,y) - F_3(x,y) + H_3(x,y) \quad (4.96)$$

But $\frac{\partial^2}{\partial x^2}U_3(x,y) = \frac{\partial^2}{\partial x^2}\left(-\frac{1}{6}\cos(x+y)\right) = \frac{1}{6}\cos(x+y) = \frac{\partial^2}{\partial y^2}U_3(x,y)$ (4.97)

Using theorem 4.3

$$F_3(x,y) = G_0(x,y)U_3(x,y) - F_0(x,y)U_1(x,y)U_2(x,y) - \frac{1}{6}G_0(x,y)U_1^3(x,y)$$

Therefore,

$$F_3(x,y) = -\frac{1}{6}\cos\left(\sin(x+y)\right)\cos(x+y) + \frac{1}{2}\sin\left(\sin(x+y)\right)\cos(x+y)\sin(x+y)$$
$$-\frac{1}{6}\cos\left(\sin(x+y)\right)\cos^3(x+y)$$

$$\Rightarrow F_3(x,y) = -\frac{1}{6}\begin{bmatrix}\cos\left(\sin(x+y)\right)\cos(x+y) - 3\sin\left(\sin(x+y)\right)\cos(x+y)\sin(x+y) \\ +\cos\left(\sin(x+y)\right)\cos^3(x+y)\end{bmatrix} \quad (4.98)$$

By definition 4.1

$$D_3(x,y) = \frac{1}{3!}\left[\frac{\partial^3}{\partial t^3}\sin\left(\sin(x+y+t)\right)\right]_{t=0}$$

Hence,

$$D_3(x,y) = -\frac{1}{6}\begin{bmatrix}\cos\left(\sin(x+y)\right)\cos(x+y) - 3\sin\left(\sin(x+y)\right)\cos(x+y)\sin(x+y) \\ +\cos\left(\sin(x+y)\right)\cos^3(x+y)\end{bmatrix}$$

By theorem 4.7

$$Z_3(x,y) = \frac{1^3}{3!}\left[\sin\left(\frac{\pi(3)}{2!} + x + y\right)\right] = -\frac{1}{6}\cos\left(x+y\right)$$

Therefore,

$$H_3(x,y) = D_3(x,y) + Z_3(x,y)$$

$$= -\frac{1}{6}\begin{bmatrix}\cos\left(\sin(x+y)\right)\cos(x+y) - 3\sin\left(\sin(x+y)\right)\cos(x+y)\sin(x+y) \\ +\cos\left(\sin(x+y)\right)\cos^3(x+y)\end{bmatrix} - \frac{1}{6}\cos\left(x+y\right) \quad (4.99)$$

Substituting (4.99), (4.98) and (4.97) in to eq (4.96), we obtain

$$20U_5(x,y)=\frac{1}{6}\cos(x+y)+\frac{1}{6}\cos(x+y)$$

$$+\frac{1}{6}\begin{bmatrix}\cos\left(\sin(x+y)\right)\cos(x+y)-3\sin\left(\sin(x+y)\right)\cos(x+y)\sin(x+y)\\+\cos\left(\sin(x+y)\right)\cos^3(x+y)\end{bmatrix}$$

$$-\frac{1}{6}\begin{bmatrix}\cos\left(\sin(x+y)\right)\cos(x+y)-3\sin\left(\sin(x+y)\right)\cos(x+y)\sin(x+y)\\+\cos\left(\sin(x+y)\right)\cos^3(x+y)\end{bmatrix}-\frac{1}{6}\cos\left(x+y\right)$$

$$=\frac{1}{6}\cos(x+y)$$

$$\Rightarrow 20U_5(x,y)=\frac{1}{6}\cos(x+y)$$

Therefore, $U_5(x,y)=\dfrac{1}{120}\cos(x+y)$

For $k=4$ eq (4.81) takes the form

$$30U_6(x,y)=\frac{\partial^2}{\partial x^2}U_4(x,y)+\frac{\partial^2}{\partial y^2}U_4(x,y)-F_4(x,y)+H_4(x,y) \tag{4.100}$$

But $\dfrac{\partial^2}{\partial x^2}U_4(x,y)=\dfrac{\partial^2}{\partial x^2}\left(\dfrac{1}{24}\sin(x+y)\right)=-\dfrac{1}{24}\sin(x+y)=\dfrac{\partial^2}{\partial y^2}U_4(x,y)$ (4.101)

Using theorem 4.3

$$F_4(x,y)=G_0(x,y)U_4(x,y)-F_0(x,y)U_1(x,y)U_3(x,y)-\frac{1}{2}F_0(x,y)U_2^2(x,y)$$

$$-\frac{1}{2}G_0(x,y)U_1^2(x,y)U_2(x,y)+\frac{1}{24}F_0(x,y)U_1^4(x,y)$$

$$\Rightarrow F_4(x,y)=\frac{1}{24}\begin{bmatrix}4\sin\left(\sin(x+y)\right)\cos^2(x+y)+\cos\left(\sin(x+y)\right)\sin(x+y)\\+6\cos\left(\sin(x+y)\right)\cos^2(x+y)\sin(x+y)\\-3\sin\left(\sin(x+y)\right)\sin^2(x+y)+\sin\left(\sin(x+y)\right)\cos^4(x+y)\end{bmatrix} \tag{4.102}$$

By definition 4.1

$$D_4(x,y)=\frac{1}{4!}\left[\frac{\partial^4}{\partial t^4}\sin\left(\sin(x+y+t)\right)\right]_{t=0}$$

Hence,

$$D_4(x,y)=\frac{1}{24}\begin{bmatrix}4\sin\left(\sin(x+y)\right)\cos^2(x+y)+\cos\left(\sin(x+y)\right)\sin(x+y)\\ +6\cos\left(\sin(x+y)\right)\cos^2(x+y)\sin(x+y)\\ -3\sin\left(\sin(x+y)\right)\sin^2(x+y)+\sin\left(\sin(x+y)\right)\cos^4(x+y)\end{bmatrix}$$

By theorem 4.7

$$Z_4(x,y)=\frac{1^3}{4!}\left[\sin\left(\frac{\pi(4)}{2!}+x+y\right)\right]=\frac{1}{24}\sin\left(x+y\right)$$

Therefore,

$$H_4(x,y)=D_4(x,y)+Z_4(x,y)$$

$$=\frac{1}{24}\begin{bmatrix}4\sin\left(\sin(x+y)\right)\cos^2(x+y)+\cos\left(\sin(x+y)\right)\sin(x+y)\\ +6\cos\left(\sin(x+y)\right)\cos^2(x+y)\sin(x+y)\\ -3\sin\left(\sin(x+y)\right)\sin^2(x+y)+\sin\left(\sin(x+y)\right)\cos^4(x+y)\end{bmatrix}+\frac{1}{24}\sin\left(x+y\right) \quad (4.103)$$

Substituting (4.103), (4.102) and (4.101) in to eq (4.100), we obtain

$$30U_6(x,y)=-\frac{1}{24}\sin(x+y)-\frac{1}{24}\sin(x+y)$$

$$-\frac{1}{24}\begin{bmatrix}4\sin\left(\sin(x+y)\right)\cos^2(x+y)+\cos\left(\sin(x+y)\right)\sin(x+y)\\ +6\cos\left(\sin(x+y)\right)\cos^2(x+y)\sin(x+y)\\ -3\sin\left(\sin(x+y)\right)\sin^2(x+y)+\sin\left(\sin(x+y)\right)\cos^4(x+y)\end{bmatrix}$$

$$+\frac{1}{24}\begin{bmatrix}4\sin\left(\sin(x+y)\right)\cos^2(x+y)+\cos\left(\sin(x+y)\right)\sin(x+y)\\+6\cos\left(\sin(x+y)\right)\cos^2(x+y)\sin(x+y)\\-3\sin\left(\sin(x+y)\right)\sin^2(x+y)+\sin\left(\sin(x+y)\right)\cos^4(x+y)\end{bmatrix}+\frac{1}{24}\sin\left(x+y\right)$$

$$\Rightarrow 30U_6(x,y)=-\frac{1}{24}\sin(x+y)$$

Therefore, $U_6(x,y)=-\frac{1}{720}\sin(x+y)$

$$U_7(x,y)=-\frac{1}{5040}\cos(x+y)$$

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$$U_k(x,y)=\begin{cases}\pm\frac{1}{k!}\sin(x+y), & for\ \ "k"\ is\ even,\\ \pm\frac{1}{k!}\cos(x+y), & for\ \ "k"\ is\ odd.\end{cases}$$

Now using definition of inverse differentia transform method, we get

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k$$

$$=U_0(x,y)t^0+U_1(x,y)t^1+U_2(x,y)t^2+U_3(x,y)t^3+U_4(x,y)t^4+U_5(x,y)t^5+U_6(x,y)t^6+\ldots$$

$$=\sin(x+y)+\cos(x+y)t-\frac{1}{2}\sin(x+y)t^2-\frac{1}{6}\cos(x+y)t^3+\frac{1}{24}\sin(x+y)t^4$$

$$+\frac{1}{120}\cos(x+y)t^5-\frac{1}{720}\sin(x+y)t^6-\frac{1}{5040}\cos(x+y)t^7+\ldots$$

$$=\left(\sin(x+y)-\frac{1}{2}\sin(x+y)t^2+\frac{1}{24}\sin(x+y)t^4-\frac{1}{720}\sin(x+y)t^6+\ldots\right)$$

$$+\left(\cos(x+y)t-\frac{1}{6}\cos(x+y)t^3+\frac{1}{120}\cos(x+y)t^5-\frac{1}{5040}\cos(x+y)t^7+\ldots\right)$$

$$=\sin(x+y)\left(1-\frac{t^2}{2}+\frac{t^4}{24}-\frac{t^6}{720}+\ldots\right)+\cos(x+y)\left(t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}+\ldots\right)$$

$$=\sin(x+y)\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\ldots\right)+\cos(x+y)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\ldots\right)$$

$$=\sin(x+y)\cos t+\cos(x+y)\sin t=\sin\left(x+y+t\right)$$

Therefore, $u\left(x,y,t\right)=\sin\left(x+y+t\right)$

Which is exactly the same as the result obtained by Baccouch (2019).

- To test the convergence of the approximated solution, we calculated "α_k" as:

First let us take $x = y = \dfrac{\pi}{8}$ & $t = 0.25$, then using definition 4.3, we get

$$\alpha_k = \begin{cases} \dfrac{\|\beta_{k+1}\|}{\|\beta_k\|}, \ if \ \|\beta_k\| \neq 0, \\ 0 \ , \quad if \quad \|\beta_k\| = 0. \end{cases}$$

For $k = 0,$

$$\alpha_0 = \frac{\|\beta_1\|}{\|\beta_0\|} = \frac{\|U_1(x,y)t\|}{\|U_0(x,y)\|} = \frac{\|\cos(x+y)t\|}{\|\sin(x+y)\|} = \frac{\left\|\cos\left(\dfrac{\pi}{4}\right)(0.25)\right\|}{\left\|\sin\left(\dfrac{\pi}{4}\right)\right\|}$$

$$= \frac{\left\|\left(\dfrac{\sqrt{2}}{2}\right)(0.25)\right\|}{\left\|\dfrac{\sqrt{2}}{2}\right\|} = \|0.25\| = 0.25 < 1$$

For $k = 1,$

$$\alpha_1 = \frac{\|\beta_2\|}{\|\beta_1\|} = \frac{\|U_2(x,y)t^2\|}{\|U_1(x,y)t\|} = \frac{\left\|-\dfrac{1}{2}\sin(x+y)t\right\|}{\|\cos(x+y)\|} = \frac{\left\|-\dfrac{1}{2}\sin\left(\dfrac{\pi}{4}\right)(0.25)\right\|}{\left\|\cos\left(\dfrac{\pi}{4}\right)\right\|}$$

$$= \frac{\left\|\left(\dfrac{\sqrt{2}}{2}\right)(0.125)\right\|}{\left\|\dfrac{\sqrt{2}}{2}\right\|} = \|0.125\| = 0.125 < 1$$

For $k = 2$,

$$\alpha_2 = \frac{\|\beta_3\|}{\|\beta_2\|} = \frac{\|U_3(x,y)t^3\|}{\|U_2(x,y)t^2\|} = \frac{\left\|-\frac{1}{6}\cos(x+y)t\right\|}{\left\|-\frac{1}{2}\sin(x+y)\right\|} = \frac{\left\|-\frac{1}{6}\cos\left(\frac{\pi}{4}\right)(0.25)\right\|}{\left\|-\frac{1}{2}\sin\left(\frac{\pi}{4}\right)\right\|}$$

$$= \frac{\left\|-\frac{1}{6}\left(\frac{\sqrt{2}}{2}\right)(0.25)\right\|}{\left\|-\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\right\|} = \frac{\|0.04166667\|}{\|0.5\|} = 0.083334 < 1$$

For $k = 3$,

$$\alpha_3 = \frac{\|\beta_4\|}{\|\beta_3\|} = \frac{\|U_4(x,y)t^4\|}{\|U_3(x,y)t^3\|} = \frac{\left\|\frac{1}{24}\sin(x+y)t\right\|}{\left\|-\frac{1}{6}\cos(x+y)\right\|} = \frac{\left\|\frac{1}{24}\sin\left(\frac{\pi}{4}\right)(0.25)\right\|}{\left\|-\frac{1}{6}\cos\left(\frac{\pi}{4}\right)\right\|}$$

$$= \frac{\left\|\frac{1}{24}\left(\frac{\sqrt{2}}{2}\right)(0.25)\right\|}{\left\|-\frac{1}{6}\left(\frac{\sqrt{2}}{2}\right)\right\|} = \|0.0625\| = 0.0625 < 1$$

For $k = 4$,

$$\alpha_4 = \frac{\|\beta_5\|}{\|\beta_4\|} = \frac{\|U_5(x,y)t^5\|}{\|U_4(x,y)t^4\|} = \frac{\left\|\frac{1}{120}\cos(x+y)t\right\|}{\left\|\frac{1}{24}\sin(x+y)\right\|} = \frac{\left\|\frac{1}{120}\sin\left(\frac{\pi}{4}\right)(0.25)\right\|}{\left\|\frac{1}{24}\cos\left(\frac{\pi}{4}\right)\right\|}$$

$$= \frac{\left\|\frac{1}{120}\left(\frac{\sqrt{2}}{2}\right)(0.25)\right\|}{\left\|\frac{1}{24}\left(\frac{\sqrt{2}}{2}\right)\right\|} = \|0.05\| = 0.05 < 1$$

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For 'k' is even,

$$\alpha_k = \frac{\|\beta_{k+1}\|}{\|\beta_k\|} = \frac{\|U_{k+1}(x,y)t^{k+1}\|}{\|U_k(x,y)t^k\|} = \frac{\left\|\pm\frac{1}{(k+1)!}\cos(x+y)t\right\|}{\left\|\pm\frac{1}{(k)!}\sin(x+y)\right\|}$$

$$= \frac{\left\|\cos\left(\frac{\pi}{4}\right)(0.25)\right\|}{\left\|\frac{1}{(k+1)}\sin\left(\frac{\pi}{4}\right)\right\|} = \left\|\frac{1}{4(k+1)}\right\| = \frac{1}{4(k+1)} < 1$$

For 'k' is odd,

$$\alpha_k = \frac{\|\beta_{k+1}\|}{\|\beta_k\|} = \frac{\|U_{k+1}(x,y)t^{k+1}\|}{\|U_k(x,y)t^k\|} = \frac{\left\|\pm\frac{1}{(k+1)!}\sin(x+y)t\right\|}{\left\|\pm\frac{1}{(k)!}\cos(x+y)\right\|}$$

$$= \frac{\left\|\sin\left(\frac{\pi}{4}\right)(0.25)\right\|}{\left\|\frac{1}{(k+1)}\cos\left(\frac{\pi}{4}\right)\right\|} = \left\|\frac{1}{4(k+1)}\right\| = \frac{1}{4(k+1)} < 1$$

Hence, for $(x,y)\in\Omega=[0,2\pi]^2$ & $t\geq 0$ and taking $x=y=\frac{\pi}{8}$ & $t=0.25,$ we obtain

$$\alpha_0 = 0.25<1,\ \alpha_1 = 0.125<1,\ \alpha_2 = 0.083334<1,\ \alpha_3 = 0.0625<1,\ \alpha_4 = 0.05<1,\ldots,\alpha_k = \frac{1}{4(k+1)}<1$$

Therefore, using the stated definition 4.3 the solution of eq (4.78) by RDTM is converges to the exact solution.

The solution curves of two dimensional nonlinear Sine-Gordon equation given in Example 4.3 above for different order of truncation "n" is depicted in Figure 3 below

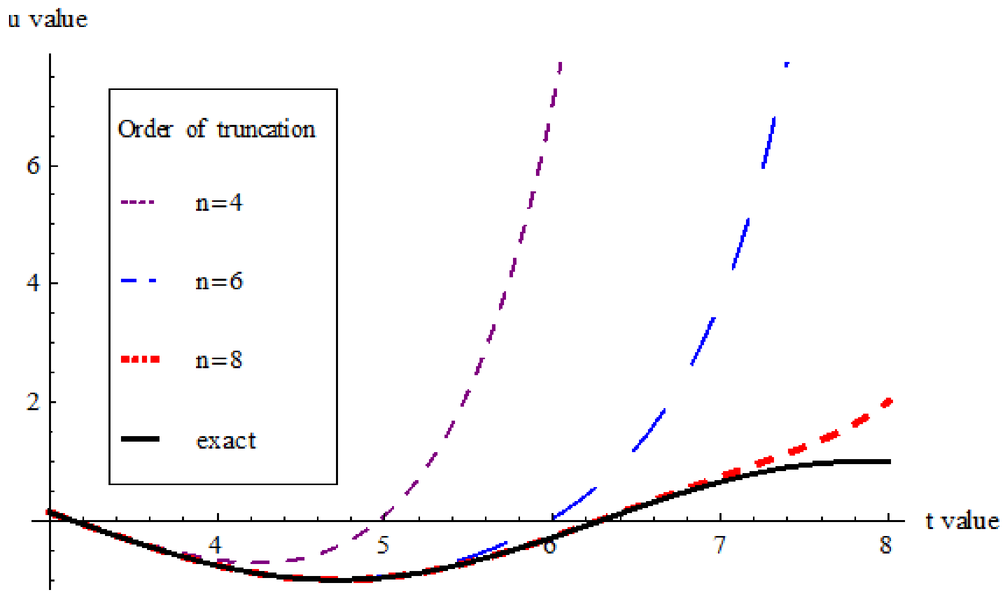

Figure 3. Solution curves of example 4.3 for $x = y = 2\pi$ and $t \in [3, 8]$

4.5 Discussion

In this manuscript, the reduced differential transform method (RDTM) was presented to solve two dimensional nonlinear sine- Gordon equations.

- The results of both exact solution and approximated solution are graphically plotted by using Mathematica software version 7.0 and compared for different order of truncation "n" as shown in Figure 1, Figure 2 and Figure 3.
- The solution curves of both exact and approximated solution shows that the approximated series solution are closer and closer to the exact solution as the order of truncation "n" is increases and both exact and approximated solutions are almost the same for very large order of truncation "n".
- In figure 1, the approximated solution are drawn for $x = y = 0.3$ and $t \in [3,7]$ and for $n = 4,\ n = 6\ and\ n = 8$. From those curve we can observe that the curves of the approximated solution for $n = 6$ is closer to the curve of exact solution than the curve of the approximated solution for $n = 4$ and the curve of the approximated solution for $n = 8$ is more closer and closer to the curve of exact solution than the curve of the approximated solution when $n = 4$ and $n = 6$.
- Also, Figure 2 and Figure 3 indicated that if the term of approximated series solution gets larger and larger the approximated solution fit with the exact solution.

CHAPTER FIVE

CONCLUSION AND FUTURE SCOPE

5.1 Conclusion

In this study, the reduced differential transform method (RDTM) is employed to obtain the solution of two dimensional nonlinear sine-Gordon equation subject to the appropriate initial conditions. The proposed method is applied in a direct way without using linearization, discretization or any other restrictive conditions and gives the solution in the form of convergent power series with elegantly computed components. The main advantage of the RDTM is that it requires less amount of computation compared to another method such as DTM and the approximate series solution is converges to the exact solution. Using this method the solution procedure is simpler and effective than other existing methods like DTM. Moreover, the results obtained by RDTM from tasted examples were in excellent agreement with the exact solutions and results of other methods available in the literature. Also, the results reveal that the RDTM is a powerful, highly accurate, reliable, very effective and an easily implementable mathematical method for solving linear and nonlinear PDEs arising in various domains of science and engineering.

5.2 Future scope

The reduced differential transform method may be used to solve some higher order linear and nonlinear partial differential equations and also for three dimensional linear and nonlinear sine-Gordon equation. Thus, we recommend that the proposed method can be applied to other some linear and nonlinear multidimensional partial differential equations those models problems emerging in various domains of science and engineering.

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