Existence of Positive Solutions for Second Order Undamped Integral Boundary Value Problems

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

By: Deginet Worku

Advisor: Wesen Legesse(Ph.D.)
Co-Advisor: Girma Kebede(M.Sc.)

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Jimma, Ethiopia

Declaration

I, the undersigned declared that, the thesis entitled existence of positive solutions for second-order undamped integral boundary value problems is original and it has not been submitted to any institution elsewhere for the award of any degree or like, where other sources of information that have been used, they have been acknowledge.
Name: Deginet Worku
Signiture ....
Date ....
The work has been done under the supervision and approval of:

Dr. Wesen Legesse (Advisor)

Signiture...................................

Date..

Mr. Girma Kebede (Co-Advisor)

Signiture...................................

Date..

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Abstract

This thesis is concerned with existence of positive solutions for second order undamped integral boundary value problem . It also presents the construction of Green's function for corresponding non-trivial homogeneous equation by using its properties. Under the suitable conditions, we established the existence of at least $2n$ *positive solution by applying Avery-Henderson fixed point theorem. We provided example to demonstrate for the applicability of our main result.*

Contents

1 Introduction

1.1 Background of the study

Boundary value problems associated with linear as well as non-linear ordinary differential equations or finite difference equations have created a great deal of interest and play an important role in many fields of applied mathematics such as engineering design and manufacturing.

Major industries like automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications as well as emerging technologies like biotechnology and nanotechnology rely on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high-technological products. In these applied setting, positive solutions are meaningful.

In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, a class of boundary value problems with integral boundary conditions arise naturally in thermal conduction problems by Canon(1963), semiconductor problems by Ionkin(1977), hydrodynamic problems by Chegis(1984).

The existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years. At the same time, we notice that a class of BVPs with integral boundary conditions appeared in heat conduction, chemical engineering, underground water flow, thermoelasticity and plasma physics.

Boundary value problem with integral conditions constitutes a very important class of problems. These BVP include two, three, multi point and non-local BVP as a special case. The existence and multiplicity of positive solutions for such problems have received a great deal of attention.

Most results so far have been obtained mainly by using the fixed-point theorems in cones, such as the Guo-Krasnoselskii's fixed point theorem (1964), the Leggett-Williams theorem (1979), Avery and Henderson's theorem (2001), and so on. no work has been done for boundary value problem (1.1) - (1.2) by applying the Avery-Henderson fixed point theorem. Some authors have investigated boundary value problems with integral boundary conditions. In particular, we would like to mention some results of Karakostas and Tsamatos (2002) established existence of multiple positive solution of a boundary value problem of the form

$$x''(t) + q(t)f(x,t) = 0, \qquad 0 \leq t \leq 1,$$

$$x(0) = 0,\ x(1) = \int_{\alpha}^{\beta} x(s)dg(s)$$

Abdelkader, Boucherif(2009) considered the following problem,

$$y''(t) = f(t, y(t)), \qquad 0 \leq t \leq 1,$$

$$y(0) - ay'(0) = \int_0^1 g_0(s)y(s)ds,$$

$$y(1) + by'(1) = \int_0^1 g_1(s)y(s)ds$$

Motivated by the above mentioned results, in this thesis, we established the existence of at least 2n positive solutions for second order undamped integral boundary value problem of the form

$$-y''(t) + k^2y(t) = f(t, y(t)), \qquad 0 \leq t \leq 1, \tag{1.1}$$

$$\alpha y(0) - \beta y'(0) = \int_0^1 g_1(s)y(s)ds,$$

$$\gamma y(1) + \delta y'(1) = \int_0^1 g_2(s)y(s)ds, \tag{1.2}$$

where $k > 0$, α, β, γ and δ are positive constants such that

$g_1, g_2 : [0,1] \longrightarrow [0,\infty)$ are continuous, $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous function by applying Avery-Henderson fixed point theorem.

1.2 Statement of the problem

In this study we focused on establishing the existence of 2n positive solutions for second order undamped integral boundary value problems (1.1) - (1.2).

1.3 Objectives

1.3.1 General objective:

The main objective of this thesis was establishing the existence of 2n positive solutions for second order undamped integral boundary value problems by applying Avery-Henderson fixed point theorem.

1.3.2 Specific Objectives:

This study has the following specific objectives:

i) To construct Green's function by following its properties for corresponding homogeneous boundary value problems (1.1)-(1.2).

ii) To formulate the problem in the form of integral equation with considering conditions

iii) To prove the existence of 2n positive solution by using Avery-Henderson's fixed point theorem.

iv)To verify the applicability of the result by using specific example

1.4 Significance of the study:

The result of this thesis may have the following importance:

1) It may provide the techniques of constructing Green's function.

2) It may provide some background information for other researchers who want to conduct a research on related topics.

3) It may familiarize a researcher with scientific communication in applied mathematics.

1.5 Delimitation of the study

The study was delimited to finding the existence of 2n positive solution for second-order undamped integral boundary value problems by applying Avery-Henderson fixed point theorem.

The rest of this thesis organized as follows: We first present some definitions which are needed throughout this work and construct Green's function by using its properties for corresponding homogeneous boundary value problems and state fixed point result by applying the Avery-Henderson's fixed point theorem in a cone Banach space. Finally, we investigate the existence of at least 2n positive solution for second order undamped integral boundary value problems (1.1)-(1.2) and as an application, example were included to verify the applicability of our result.

2 Review of Related Literatures

2.1 Over view of the study

Positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important and significant result for the existence of positive solution was proved by Erbe and Wang in 1994. Due to the importance existence of positive solutions have been investigated heavily by many researchers:

Erbe, L.H and Haiyan Wang (1994), established positive solutions for the two-point boundary value problem,

$$u''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1,$$

$$\alpha u(0) - \beta u'(0) = 0,$$

$$\gamma u(1) + \delta u'(1) = 0.$$

Meiqiang, F et al(2008) investigated positive solutions for a class of boundary-value problem with integral boundary conditions in Banach spaces,

$$x''(t) + f(t,x) = 0, \qquad t \in (0,1)$$

$$x(0) = \int_0^1 g(t)x(t)dt, \quad x(1) = 0.$$

or

$$x(0) = \theta, \quad x(1) = \int_0^1 g(t)x(t)dt,$$

Moustafa and Tahari (2008) investigated existence and nonexistence of positive solutions for

$$u^4(t) + \lambda a(t)f(u(t)) = 0, \qquad 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u'''(0) = 0,$$

$$\alpha u'(1) + \beta u''(1) = 0$$

Abdelkader,Boucherif(2009) considered the following problem:

$$y''(t) = f(t, y(t)), \quad 0 < t < 1,$$

$$y(0) - ay'(0) = \int_0^1 g_0(s)y(s)ds$$

$$y(1) - by'(1) = \int_0^1 g_1(s)y(s)ds$$

Jinxiu,M et al (2010) investigated on existence and uniqueness of positive solutions for integral boundary value problems,

$$-u'' + k^2 u = f(t, u), \quad 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(t)dA(t).$$

Mouffak,B et al (2010) established the existence of solutions for second order integral boundary value problem

$$-y''(t) = f(t, y(t)), \; a.e, \quad 0 < t < 1,$$

$$y(0) = 0, \quad y(1) = \int_0^1 g(s)y(s)ds.$$

Lingju,Kong (2010) established existence and uniqueness of positive solutions for second order singular boundary value problem

$$u''(t) + \lambda f(t, u) = 0, \quad t \in (0, 1)$$

$$u(0) = \int_0^1 u(s)d\zeta(s), \quad u(1) = \int_0^1 u(s)d\eta(s)$$

Lixin and Zuxing (2016) established the existence of three positive solutions for the boundary value problem with integral boundary conditions,

$$u''(t) + h(t)f(t, u(t), u'(t)) = 0, \quad 0 < t < 1,$$

$$u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds$$

$$u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds$$

Qiuyan and Xingqiu (2017) studied the existence of two positive solutions for the nonlinear second order differential equation involving integral boundary value problem,

$$u''(t) + a(t)u'(t) + b(t)u(t) + f(t, u(t) = 0, \qquad t \in (j_+)$$

$$u(0) = \int_0^1 g(s)u(s)ds, \quad u(1) = \int_0^1 h(s)u(s)ds.$$

2.2 Preliminaries

First we recall some known definitions and basic concepts on Green's function that we used in the proof our main results.

Definition 2.2.1[Ravi.P and Donal.O (2000)] We consider the second-order linear differential equation

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = r(t), \qquad t \in J = [0, 1]. \tag{2.1}$$

where the functions $p_0(t), p_1(t), p_2(t)$ and $r(t)$ are continuous in J and boundary conditions of the form

$$\begin{aligned} l_1[y] = a_0y(0) + a_1y'(0) + b_0y(1) + b_1y'(1) = A, \\ l_2[y] = c_0y(0) + c_1y'(0) + d_0y(1) + d_1y'(1) = B, \end{aligned} \tag{2.2}$$

where $a_i, b_i, c_i, d_i, i = 0, 1$ and A, B are given constants. The boundary value problems $(2.1), (2.2)$ is called a nonhomogeneous two-point linear boundary value problems, whereas the homogeneous differential equation

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = 0, \ t \in J = [0, 1]. \tag{2.3}$$

together with the homogeneous boundary conditions

$$\begin{aligned} l_1[y] &= 0, \\ l_2[y] &= 0, \end{aligned} \tag{2.4}$$

be called a homogeneous two-point linear boundary value problems.
The function called a Green's function $G(t,s)$ for the homogeneous boundary value problems (2.3), (2.4) and the solution of the nonhomogeneous boundary value problems (2.1), (2.2) can be explicitly expressed in terms of $G(t,s)$.

Obviously, for the homogeneous problems (2.3), (2.4) the trivial solution always exists. Green's function $G(t,s)$ for the boundary value problems (2.3), (2.4) is defined in the square $[0,1] \times [0,1]$ and possesses the following fundamental properties:

i) $G(t,s)$ is continuous in $[0,1] \times [0,1]$.

ii) $\frac{\partial G(t,s)}{\partial t}$ is continuous in each of the triangles $0 \le t \le s \le 1$ and $0 \le s \le t \le 1$ moreover,

$$\frac{\partial G(s^+,s)}{\partial t} - \frac{\partial G(s^-,s)}{\partial t} = \frac{1}{P_0(s)},$$

where $\frac{\partial G(s^+,s)}{\partial t} = \lim_{t \to s, t > s} \frac{\partial G(t,s)}{\partial t}$ and $\frac{\partial G(s^-,s)}{\partial t} = \lim_{t \to s, t < s} \frac{\partial G(t,s)}{\partial t}$.

iii) For every fixed $s \in [0,1]$, $z(t) = G(t,s)$ is a solution of the differential equation (2.3) in each of the intervals [0, s) and (s, 1],

iv) For every fixed $s \in [0,1]$, $z(t) = G(t,s)$ satisfies the boundary conditions (2.4).

These properties completely characterize Green's function $G(t,s)$.

Definition 2.2.2: A norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties

i) $\|x\| \ge 0$.

ii) $\|x\| = 0 \Longleftrightarrow x = 0$.

iii) $\|\alpha x\| = |\alpha| \|x\|$.

iv) $\|x + y\| \le \|x\| + \|y\|$.

Here, x and y are arbitrary vectors in X and α is any scalar.

Definition 2.2.3: Let X be a normed linear space with norm denoted by $\|.\|$. A sequence of elements x_n of X is a Cauchy sequence, if for every $\epsilon > 0$ there exists an integer N such that $\|x_n - x_m\| < \epsilon$, for all $m, n \geq N$.

Definition 2.2.4: A normed linear space X is said to be complete, if every Cauchy sequence in X converges to a point in X.

Definition 2.2.5: A complete normed space X is called a Banach space.

Definition 2.2.6: Let $-\infty < a < b < \infty$. A collection of real valued functions $A = \{f_i | f_i : [a, b] \to R\}$ is said to be Uniformly bounded, if there exists a constant $M > 0$ with $|f_i(t)| \leq M$, for all $t \in [a, b]$ and for all $f_i \in A$.

Definition 2.2.7: Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone, if it satisfies the following two conditions:

i) $y \in P,\ \alpha \geq 0$ implies $\alpha y \in P$, and

ii) $y \in P$ and $-y \in P$ implies $y = 0$.

Definition 2.2.8: Let X and Y be Banach spaces and $T : X \to Y$ an operator T is said to be completely continuous, if T is continuous and for each bounded sequence $x_n \subset X, (Tx_n)$ has a convergent subsequences.

Definition 2.2.9: Let $T : X \to X$ be self-map. A point $x \in X$ is called a fixed point of T if $Tx = x$.

Definition 2.2.10:[Jinxiu,M et al (2010)] The function $y(t) \in C[0, 1] \cap C^2[0, 1]$ is a positive solution of the boundary value problems (1.1)-(1.2). If y(t) is positive on the given interval and satisfies both the differential equation and the boundary conditions.

3 Methodology

This chapter contains study period and site, study design, source of information and mathematical procedures.

3.1 Study period and site

The study was conducted from September 2019 to August 2020 in Jimma University under the department of mathematics.

3.2 Study design

In order to achieve the objective of the study we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were different mathematics books, published articles, journals and related studies from internet.

3.4 Mathematical procedure

In this study we followed the following procedures:

i) Defining second order integral boundary value problems.

ii) Constructing the Green's function for the corresponding homogeneous equa tion .

iii) Formulating the equivalent operator equation for the boundary value prob lem (1.1)-(1.2).

iv) Determining the fixed point of the operator equation.

4 Main Result and Discussion

4.1 Construction of Green's Function

In this section, we construct Green's function for the homogeneous problem corresponding to $(1.1)-(1.2)$.
Let $G(t,s)$ be Green's function for the homogeneous problem,

$$-y''(t)+k^2y(t)=0, \qquad 0\le t\le 1,$$

with the boundary condition (1.2).
we can find the Green's function, from the homogeneous differential equation (1.1) ,thus two linearly independent solutions are $y_1(t)=-\sinh kt+\cosh kt$ and $y_2(t)=\sinh kt+\cosh kt$.
Hence, the problem $(1.1),(1.2)$ have only the trivial solution if and only if $\Delta\neq 0$

$$\Delta=\begin{bmatrix}\alpha y_1(0)-\beta y_1'(0) & \alpha y_2(0)-\beta y_2'(0)\\ \gamma y_1(1)+\delta y_1'(1) & \gamma y_2(1)+\delta y_2'(1)\end{bmatrix}$$

$$=\begin{bmatrix}\alpha+\beta k & \alpha-\beta k\\ \gamma(-sinhk+coshk)+\delta k(-coshk+sinhk) & \gamma(sinhk+coshk)+\delta k(\ coshk+sinhk)\end{bmatrix}$$

$$\Delta=\ 2sinhk(\alpha\gamma+\beta\delta k^2)+2kcoshk(\alpha\delta+\beta\gamma)\neq 0.$$

From the property (**iii**) there exist four functions, say, $\lambda_1(s),\lambda_2(s),\mu_1(s)$ and $\mu_2(s)$ such that

$$G(t,s)=\begin{cases}(-sinhkt+coshkt)\lambda_1(s)+\ (sinhkt+\cosh kt)\lambda_2(s), & 0\le t\le s\le 1,\\ (-sinhkt+coshkt)\mu_1(s)+\ (sinhkt+coshkt)\mu_2(s). & 0\le s\le t\le 1.\end{cases} \tag{4.1}$$

Now using properties (**i**) and (**ii**), we obtain the following two equations:

$$y_1(s)\lambda_1(s) + \; y_2(s)\lambda_2(s) = \; y_1(s)\mu_1(s) + \; y_2(s)\mu_2(s), \tag{4.2}$$

$$y_1'(s)\mu_1(s) + y_2'(s)\mu_2(s) - \; y_1'(s)\lambda_1(s) - \; y_2'(s)\lambda_2(s) = -1, \tag{4.3}$$

Let $v_1(s) = \mu_1(s) - \lambda_1(s)$ and $v_2(s) = \mu_2(s) - \lambda_2(s)$, so that (4.4), (4.5) can be written as

$$(-sinhks + coshks)v_1(s) + \; (sinhks + coshks)v_2(s) = 0, \tag{4.4}$$

$$(-coshks + sinhks)v_1(s) - \; (coshks + sinhks)v_2(s) = \frac{-1}{k}. \tag{4.5}$$

Since ($-sinhkt + coshkt$) and ($sinhkt + coshkt$) are linearly independent the Wronskian $\Delta \neq 0$ for all $t \in [0, 1]$.
Thus,

$$v_1(s) = \frac{1}{2k(\cosh ks - sinhks)}$$

and

$$v_2(s) = \frac{\sinh ks - coshks}{2k}.$$

Now using the relations

$$\mu_1(s) = \lambda_1(s) + \frac{1}{2k(\cosh ks - \sinh ks)}$$

and

$$\mu_2(s) = \lambda_2(s) + \frac{sinhks - coshks}{2k}$$

Green's function can be written as

$$G(t,s) = \begin{cases} (-sinhkt + coshkt)\lambda_1(s) + (\sinh kt + coshkt)\lambda_2(s), & 0 \le t \le s \le 1, \\ (-sinhkt + coshkt)(\lambda_1(s) + \frac{1}{2k(\; coshks - \sinh ks)}) + & \\ (\sinh kt + \cosh kt)(\lambda_2(s) + \frac{\sinh ks - \cosh ks}{2k}), & 0 \le s \le t \le 1. \end{cases} \tag{4.6}$$

Finally, using the property (**iv**) on the boundary condition (1.2) of Green's function with the given interval, we find $\lambda_1(s)$ and $\lambda_2(s)$,thus

$$\begin{cases} \alpha[y_1(0)\lambda_1(s) - y_2(0)\lambda_2(s)] - \beta[y_1'(0)\lambda_1(s) + y_2'(0)\lambda_2(s)] = 0, \\ \gamma[(y_1(1)(\lambda_1(s) + v_1(s)) + y_2(1)(\lambda_2(s) + v_2(s))] + \\ \delta[y_1'(1)(\lambda_1(s) + v_1(s)) + y_2'(1)(\lambda_2(s) + v_2(s))] = 0. \end{cases} \tag{4.7}$$

$$\begin{cases} (\alpha + \beta k)\lambda_1(s) + (\alpha - \beta k)\lambda_2(s) = 0, \\ (\gamma + \delta k)(-sinhk + coshk)\lambda_1(s) + (\gamma + \delta k)(sinhk + coshk)\lambda_2(s) = \\ \frac{\gamma \sinh k(1-s) + \delta k \cosh k(1-s)}{2k} \end{cases} \tag{4.8}$$

Hence,

$$\lambda_1(s) = \frac{1}{\Delta}\begin{bmatrix} 0 & (\alpha - \beta k) \\ \frac{\gamma sinhk(1-s) + \delta k \cosh k(1-s)}{2k} & (\gamma + \delta k)(\sinh k + \cosh k) \end{bmatrix}$$

$$\lambda_1(s) = \frac{-(\alpha - \beta k)(\gamma \sinh k(1-s) + \delta k \cosh k(1-s)}{\Delta 2k}$$

$$\lambda_2(s) = \frac{1}{\Delta}\begin{bmatrix} (\alpha + \beta k) & 0 \\ (\gamma + \delta k)(-\sinh k + \cosh k) & \frac{\gamma \sinh k(1-s) + \delta k \cosh k(1-s)}{2k} \end{bmatrix}$$

$$\lambda_2(s) = \frac{(\alpha + \beta k)(\gamma \sinh k(1-s) + \delta k \cosh k(1-s))}{\Delta 2k}.$$

Substituting the value of $\lambda_1(s)$ and $\lambda_2(s)$ in (4.1) this becomes

$$G(t,s) = \begin{cases} \frac{(\alpha \sinh kt + \beta k \cosh kt)(\gamma \sinh k(1-s) + \delta k \cosh k(1-s))}{(\alpha\gamma + \beta\delta k^2)k \sinh k + (\alpha\delta + \beta\gamma)k^2 \cosh k}, & 0 \le t \le s \le 1, \\ \frac{(\alpha \sinh ks + \beta k \cosh ks)(\gamma \sinh k(1-t) + \delta k \cosh k(1-t))}{(\alpha\gamma + \beta\delta k^2)k \sinh k + (\alpha\delta + \beta\gamma)k^2 \cosh k}, & 0 \le s \le t \le 1. \end{cases}$$

We make the following assumptions:

M_1) $g_1, g_2 \in C[0,1]$ and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continouos function.

M_2) There exists $k>0$, such that $\Delta> \int_0^1(\gamma \sinh k(1-t)+$ $\delta k \cosh k(1-t))y(t)dt$.

Lemma 4.1 *Suppose that $g \in C[0,1]$ and M_1 holds. then the linear boundary value problem $(1.1)-(1.2)$ has a unique solution*

$$y(t)=\int_0^1 H(t,s)g(s)ds$$

where

$$H(t,s)=G(t,s)+\frac{\alpha \sinh kt+\beta k \cosh kt}{\Delta}\int_0^1 g_1(s)y(s)ds$$

$$+\frac{\gamma \sinh k(1-t)+\delta k \cosh k(1-t)}{\Delta}\int_0^1 g_2(s)y(s)ds \tag{4.9}$$

.

$$G(t,s)=\frac{1}{\Delta k}\begin{cases}(\alpha \sinh kt+\beta k \cosh kt)(\gamma \sinh k(1-s)+\delta k \cosh k(1-s))\\(\alpha \sinh ks+\beta k \cosh ks)(\gamma \sinh k(1-t)+\delta k \cosh k(1-t))\end{cases} \tag{4.10}$$

Remark: *We call H(t,s) the Greens function of problem* $(1.1)-(1.2)$. *Suppose* (M_1) *and* (M_2) *hold. Then solutions of* $(1.1)-(1.2)$ *are equivalent to continouos solutions of the integral equation.*

$$y(t)=\int_0^1 H(t,s)f(s,y(s))ds$$

where H(t,s) is mentioned above.

Lemma 4.2 *Assume that the conditions (M_2) is satisfied. The Green's function $G(t,s)$ satisfies the following inequalities.*

i) $G(t,s)\le G(s,s)$, *for all* $t,s\in(0,1)$.

ii) $G(t,s) \geq NG(s,s)$, *for all* $t,s \in (0,1)$.

Proof: *i) The Green's function G(t, s) is positive for all* $t,s \in (0,1)$.

For $0 \leq s \leq t \leq 1$,*we have*

$$\frac{G(t,s)}{G(s,s)} = \frac{\alpha \sinh kt + \beta k \cosh kt}{\alpha \sinh ks + \beta k \cosh ks} \leq 1$$

$G(t,s) \leq G(s,s)$ *is bounded.*

similarly for $0 \leq t \leq s \leq 1$,*we have*

$$\frac{G(t,s)}{G(s,s)} = \frac{\gamma \sinh k(1-t) + \delta k \cosh k(1-t)}{\gamma \sinh k(1-s) + \delta k \cosh k(1-s)} \leq 1$$

Therefore,

$G(t,s) \leq G(s,s)$ *for all* $t,s \in (0,1)$

ii) *Let* $t \leq s$ *then*

$$\frac{G(t,s)}{G(s,s)} = \frac{\alpha \sinh kt + \beta k \cosh kt}{\alpha \sinh ks + \beta k \cosh ks} \geq \frac{\beta k}{\alpha \sinh ks + \beta k \cosh ks}$$

Let $s \leq t$, *then*

$$\frac{G(t,s)}{G(s,s)} = \frac{\gamma \sinh k(1-t) + \delta k \cosh k(1-t)}{\gamma \sinh k(1-s) + \delta k \cosh k(1-s)} \geqslant \frac{\delta k}{\gamma \sinh k + \delta k \cosh k}$$

Therefore, $G(t,s) \geq NG(s,s)$ *for all* $(t,s) \in (0,1)$

where

$$N = min\{\frac{\beta k}{\alpha \sinh k + \beta k \cosh k}, \frac{\delta k}{\gamma \sinh k + \delta k \cosh k}\} \tag{4.11}$$

This completes the proof.

Lemma 4.3 *For any* $t,s \in [0,1]$, *there exist constants* $a_1, a_2 > 0$ *such that*

$$a_2 e(t) e(s) \leq H(t,s) \leq a_1 e(s) \quad s,t \in [0,1] \tag{4.12}$$

where $e(s) = s(1-s)$.

Proof: *Suppose that*

$$I(t) = \alpha sinh(k)t + \beta k\cosh(k)t - \alpha\sinh(kt) + \beta k\cosh(kt),\ \ t \in [0,1]$$

then $I(0) = \beta k, I(1) = 2\beta k\cosh(k)$ *and* $I''(t) = -k^2(\alpha\sinh(kt) + \beta k\cosh(kt))$ $< 0,\ \ t \in [0,1].$ *so* $I(t) > 0,$ *i.e*

$$\alpha sinh(k)t + \beta k\cosh(k)t \leq \alpha\sinh(kt) + \beta k\cosh(kt),\ \ t \in [0,1] \tag{4.13}$$

similarly we have,

$$kt \leq \alpha\sinh(kt) + \beta k\cosh(kt), t \in [0,1] \tag{4.14}$$

From (4.10) *we know*

$$\frac{k}{\Delta}G(t,t)G(s,s) \leq G(t,s) \leq G(t,t) \tag{4.15}$$

By using (4.10),(4.12) *and* (4.13) *we obtain*

$$G(t,t) \geq \frac{(kt)(k(1-t))}{\Delta k} = \frac{ke(t)}{\Delta} \tag{4.16}$$

and

$$G(t,t) \leq \frac{(\alpha sinh(k)t + \beta k\cosh(k)t)(\alpha sinh(k)(1-t) + \beta k\cosh(k)(1-t)}{\Delta k} = \frac{\Delta e(t)}{k} \tag{4.17}$$

From (4.9), (4.14), (4.15) *and* (4.16) *we have*

$$H(t,s) \geq G(t,s) \geq \frac{k}{\Delta}G(t,t)G(s,s) \geq (\frac{k}{\Delta})^3 e(t)e(s) \tag{4.18}$$

and

$$H(t,s) \geq G(t,s) + G(s,s)(\frac{\Delta}{\Delta - \int_0^1 \alpha sinh(k\tau) + \beta k\cosh(k\tau)g_1(s)y(s)ds})$$

$$(\frac{\Delta}{\Delta - \int_0^1 \alpha sinh(k\tau) + \beta k \cosh(k\tau) g_2(s) y(s) ds})$$

$$\leq \frac{\Delta}{k} e(s)[1 + (\frac{\Delta}{\Delta - \int_0^1 \alpha sinh(k\tau) + \beta k \cosh(k\tau) g_1(s) y(s) ds})$$

$$(\frac{\Delta}{\Delta - \int_0^1 \alpha sinh(k\tau) + \beta k \cosh(k\tau) g_2(s) y(s) ds})]$$

Letting

$$a_1 = \frac{\Delta}{k}[1 + (\frac{\Delta}{\Delta - \int_0^1 \alpha sinh(k\tau) + \beta k \cosh(k\tau) g_1(s) y(s) ds})$$

$$(\frac{\Delta}{\Delta - \int_0^1 \alpha sinh(k\tau) + \beta k \cosh(k\tau) g_2(s) y(s) ds}])$$

and

$$a_2 = (\frac{k}{\Delta})^3$$

we have,

$$a_2 e(t) e(s) \leq H(t,s) \leq a_1 e(s)$$

Thus (4.11) *holds.*

4.2 Existence of Positive Solution

In this section,we prove the existence of at least two positive solution for the boundary value problem (1.1) – (1.2) by using Avery-Henderson fixed point theorem .And then,we establish the existence of $2n$ positive solution to the boundary value problem (1.1) – (1.2) for an arbitrary positive integer n.

Let ϕ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for nonnegative real numbers c'_1 and c'_2. we define the sets

$$P(\phi, c'_1) = \{y \in P : \phi(y) < c'_1\}$$

and

$$Pc'_2 = \{y \in P : ||y|| < c'_2\}.$$

In obtaining multiple positive solutions of the boundary value problem $(1.1)-(1.2)$, the following theorem,also known as Avery-Henderson functional fixed point theorem will be the fundamental tool.

Theorem 4.2.1 Let P be a cone in a real Banach spce. If τ and ϕ are increasing,nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that for some positive constants r and ξ,

$$\phi(y) \leq \theta(y) \leq \tau(y) \; and \; ||y|| \leq \xi\phi(y)$$

for all $y \in \overline{P(\phi, r)}$. suppose that there exists positive numbers r' and s' with $r' < s' < t'$ such that $\theta(\lambda y) \leq \lambda\theta(y)$, for all $0 \leq \lambda \leq 1$ and $y \in \partial P(\theta, s')$. If $T : \overline{P(\phi, t')} \longrightarrow P$ compeletely continuous operator such that

i. $\phi(Ty) > t'$, for all $y \in \partial P(\theta, t')$,

ii. $\theta(Ty) < s'$, for all $y \in \partial P(\phi, s')$,

iii. $P(\tau, r') \neq \emptyset$ and $\tau(Ty) > r'$ for all $y \in \partial P(\tau, r')$

Then T has at least two fixed points $y_1, y_2 \in \overline{P(\phi, t')}$ such that $r' < \tau(y_1)$ with $\theta(y_1) < s'$ and $s' < \theta(y_2)$ with $\phi(y_2) < t'$.

Let the Banach space $B = \{y : y \in C[0, 1]\}$ with the norm
$\|y\| = \max_{t\in[0,1]}|y(t)|$.

Define the cone $P \subset B$ by $P = \{y \in B : y(t) \geq 0 \; and \; \min_{t\in[0,1]} y(t) \geq N\|y\|\}$, where N is defined as in (4.11).

Define the nonnegative, increasing continuous functionals ϕ, θ and τ on the cone P by $\phi(y) = \min_{t\in[0,1]} y(t)$, $\theta(y) = \max_{t\in[0,1]} y(t)$, $\tau(y) = \max_{t\in[0,1]} y(t)$.

we observe that for any P,

$$\phi(y) \le \theta(y) = \tau(y) \tag{4.19}$$

and

$$\|y\| \le \frac{1}{N} \min_{t\in[0,1]} y(t) = \frac{1}{N}\phi(y) \le \frac{1}{N}\tau(y) \tag{4.20}$$

Define $E_j = \int_0^1 G(s,s)ds$

Theorem 4.2.2 Assume that the conditions $M_1 - M_2$ are satisfied. suppose that there exist real numbers r', s', t' with $0 < r' < s' < t'$, such that f satisfies the following conditions.

(A_1) $f(t,y) > \frac{t'}{NE_j}$,for $t \in [0,1]$ and $y \in [t', \frac{t'}{N}]$.
(A_2) $f(t,y) < \frac{s'}{NE_j}$,for $t \in [0,1]$ and $y \in [0, \frac{s'}{N}]$.
(A_3) $f(t,y) < \frac{r'}{NE_j}$,for $t \in [0,1]$ and $y \in [r', \frac{r'}{N}]$.
Then the boundary value problem $(1.1) - (1.2)$ has at least two positive solutions.

Proof: Define the operator $T : P \longrightarrow B$ by

$$Ty(t) = \int_0^1 H(t,s)f(s,y(s))ds \tag{4.21}$$

It is obvious that a fixed point of T is the solution of the boundary value problem $(1.1) - (1.2)$.we seek two fixed points $y_1, y_2 \in P$ of T.First, we show that $T : P \longrightarrow P$. Let $y \in P$, From above lemma (4.2) and (4.3),we obtain

$$\begin{aligned} Ty(t) &= \int_0^1 H(t,s)f(s,y(s))ds \\ &\ge N \int_0^1 G(s,s)f(s,y(s))ds \\ &\ge N||Ty|| \end{aligned}$$

Hence $Ty \in P$ and so $Tp \subset P$. More over, T is completely continuous. For each $y \in P$,we have

$$\phi(y) \le \theta(y) \le \tau(y) \ and \ \|y\| \le \frac{1}{N}\tau \tag{4.22}$$

Also for any $0 \leq \lambda \leq 1$ and $y \in P$, we have

$$\theta(\lambda y) = \max_{t\in[0,1]}(\lambda y)(t) = \lambda \max_{t\in[0,1]} y(t) = \lambda\theta(y)$$

It is clear that $\theta(0) = 0$. we show that the remaining condition of theorem 4.2.1 are satisfied.

Firstly,we shall verify that the condition (i) of theorem 4.2.1 is satisfied. since $y \in \partial p(\phi, t')$, from (4.22) we have that $t' = \min_{t\in[0,1]} y(t) \leq \|y\| \leq \frac{t'}{N}$,for $t \in [0, 1]$. then

$$\phi(Ty) = \min_{t\in[0,1]} \int_0^1 H(t, s)f(s, y(s))ds$$

$$\geq N \int_0^1 G(s, s)f(s, y(s))ds$$
$$> N \frac{t'}{NE_j} \int_0^1 G(s, s)ds = t'$$

using hypothesis (A_1).

Now ,we shall show that condition (ii) of theorem 4.2.1 is satisfied.since $y \in \partial p(\theta, s')$,from (4.22) we have that $0 \leq y(t) \leq ||y|| \leq \frac{s'}{N}$,for $t \in [0, 1]$. Thus

$$\theta(Ty) = \max_{t\in[0,1]} \int_0^1 H(t, s)f(s, y(s))ds$$
$$\leq \int_0^1 G(s, s)f(s, y(s))ds$$
$$\leq \frac{s'}{E_j} \int_0^1 G(s, s)ds = s'$$

by hypothesis (A_2).

Finally using hypothesis (A_3),we shall show that condition (iii) of theorem 4.2.1 is satisfied. since $0 \in P$ and $P > 0$, $P(\phi, p) \neq \varnothing$. we have $y \in \partial P(\tau, r'), Nr' \leq y(t) \leq \|y\| = r'$, for $t \in [0, 1]$.

Therefore,

$$\begin{aligned}\tau(Ty) &= \max_{t\in[0,1]} \int_0^1 H(t,s)f(s,y(s))ds \\ &\geq \int_0^1 H(t,s)f(s,y(s))ds \\ &\geq N\int_0^1 G(s,s)f(s,y(s))ds \\ &> \frac{r'}{NE_j}N\int_0^1 G(s,s)ds = r'.\end{aligned}$$

Thus ,all conditions of theorem (4.2.1) are satisfied .
So there exist at least two positive solutions $y_1, y_2 \in \overline{P(\phi, t')}$ for the boundary value problem $(1.1), (1.2)$.
This completes the proof.
Theorem 4.2.3 Let n be an arbitrary positive integer. suppose there exist positive constants $r'_a(a = 1, 2, ..., n+1)$ and $s'_b(b = 1, 2, ..., n)$ with $0 < r'_1 < s'_1 < r'_2 < s'_2 < ... < r'_n < s'_n < r'_{n+1}$ such that the function f satisfies the following conditions.

$$f(t,y) > \frac{r'_a}{NE_j}, for\ t\in[0,1]\ and\ y\in[r'_a, \frac{r'_a}{N}],\ \ a = 1, 2, ..., n+1 \tag{4.23}$$

$$f(t,y) < \frac{s'_b}{E_j}\ , for\ t\in[0,1]\ and\ y\in[s'_b, \frac{s'_b}{N}] \tag{4.24}$$

Then the boundary value problem $(1.1)-(1.2)$ has at least $2n$ positive solutions.
Proof : We use induction , for $n = 1, 0 < r'_1 < s'_1 < r'_2$, from (4.23) and (4.24), it is clear that $T : \overline{P}r'_2 \longrightarrow Pr'_2$,then it follows from Avery-Henderson fixed point theorem that the boundary value problem $(1.1)-(1.2)$ has at least two positive solutions in $\overline{P}r'_2$. Let us assume that this conclusion holds for $n = l$. In order to prove this conclusion holds for $n = l+1$,we suppose that there exist real numbers $r'_a(a = 1, 2, ..., l+2)$ and $s'_b(b = 1, 2, ..., l+1)$ with

$0 < r'_1 < s'_1 < r'_2 < s'_2 < ... < r'_{l+1} < s'_{l+2} < r'_{l+2}$ such that

$$f(t,y)>\frac{r'_a}{NE_j}, for\ t \in [0,1],\ and\ y \in [r'_a, \frac{r'_a}{N}]\ a = 1,2,...,l+2 \tag{4.25}$$

$$f(t,y)) < \frac{s'_b}{NE_j}, for\ t \in [0,1], [s'_b, \frac{s'_b}{N}]\ b = 1,2,...,l+1 \tag{4.26}$$

by assumption the $(1.1)-(1.2)$ has at least $2l$ positive solutions $y_i (i = 1,2,...,2l)$ in $\overline{P(\phi, r'_{l+1})}$. At the same time from theorem (4.2.2) and conditions on $f(t,y(t))$ on this theorem that the boundary value problem $(1.1) - (1.2)$ has at least two positive solutions y_1, y_2 in $\overline{P(\phi, r'_{l+2})}$ such that $r'_{l+1} < \tau(y_1)$ with $\theta(y_1) < s'_{l+1}$ and $s'_{l+1} < \theta(y_2)$ with $\tau(y_2) < r'_{l+2}$. obviously y_1 and y_2 are different $y_i, i = 1,2,...,l+2$.
Then the boundary value problem $(1.1) - (1.2)$ has at least $2l+2$ positive solution in $\overline{P(\phi, r'_{l+2})}$. which shows that conclusion holds for $n = l+1$.

4.3 Example

Let us consider example to illustrate our main result for second-order undamped integral boundary value problem.

$$-y''(t) + y(t) = f(t, y(t)) \tag{4.27}$$

satisfying the boundary condition

$$y(0) - y'(0) = \int_0^1 y(s)ds$$

$$2y(1) + y'(1) = \int_0^1 y(s)ds \tag{4.28}$$

where

$$f(t,y(t)) = \begin{cases} \frac{1}{100}t + (y+2)^2, y \le 2 \\ \frac{1}{100}t + (\frac{y+3}{3})^2, y > 2 \end{cases}$$

By direct calculation $E_j = \frac{1}{3}, N = 0.450$. Thus if we choose $r' = \frac{1}{2}, s' = 1, t' = 1.75$, then all the conditions of theorem 4.2.2 are satisfied and hence the boundary value problem $(4.27) - (4.28)$ has at least two positive solutions y_1 and y_2 such that

$\frac{1}{2} < \max_{t\in[0,1]} y_1(t)$ with $\max_{t\in[0,1]} y_1(t) < 1$

$1 < \max_{t\in[0,1]} y_2(t)$ with $\min_{t\in[0,1]} y_2(t) < 1.75$.

5 Conclusion and Future scope

5.1 Conclusion

Based on the obtained result the following conclusion can be derived:
In this study, we have considered second order undamped integral boundary value problems and used the properties of Green's function to construct corresponding homogeneous equation.

After these we formulated equivalent integral equation for the boundary value problem $(1.1) - (1.2)$ in the given interval and determined the existence of positive fixed point of the integral equation by applying Avery-Henderson's fixed point theorem.

We established the existence of positive solutions for second order undamped integral boundary value problem by applying Avery-Henderson's fixed point theorem. Finally,we established the existence of at least 2n positive solutions for second order undamped integral boundary value problem .

5.2 Future Scope

This study focused on existence of positive solutions for Second order undamped integral boundary value problems. Any interested researchers may conduct the research on existence of positive solutions by taking different coefficient and considering orders greater than two.

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