A FIXED POINT THEOREM FOR Z_Ω-CONTRACTION MAPPINGS IN P-METRIC SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

By: Dereje Abdissa

Advisor : Dr. Dula Tolera

CoAdvisor : Mr. Sisay Dibaba (M.Sc.)

August, 2020

Jimma, Ethiopia

Declaration

I, the undersigned declare that, this research paper entitled ''A Fixed Point Theorem for Z_{Ω}-Contraction Mappings in P-metric Spaces" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

Name: Dereje Abdissa

Signature: ....

Date:.............

The work has been done under the supervision of:

Name: Dula Tolera (Ph.D.)

Signature:............

Date:...........

Name: Sisay Dibaba (M.Sc.)

Signature: ....

Date: ....

Acknowledgment

First of all, I am indebted to God who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my Special heartfelt thanks go to my advisor, Dr. Dula Tolera and co-advisor Mr. Sisay Dibaba for their unreserved support, advice and guidance throughout the work of this research. Lastly, I express my deepest gratitude to all my friends whose strong stand with me in financial and moral support in preparing this thesis.

Abstract

In this thesis we established a Z_{Ω}-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P- metric Spaces. Our result extends the work of (Khojasteha et al., 2015) from metric space to P-metric spaces.

Contents

Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields. The contraction mapping principle introduced by (Banach, 1922) has wide range of applications in a fixed point theorem. In 1922, Banach proved the following famous fixed point theorem. (Banach, 1922) Let (X,d) be a complete metric space, $T : X \to X$ be a contraction, there exists a unique fixed point $x_0 \in X$ of T. This theorem, called the Banach contraction principle is a forceful tool in nonlinear analysis. Another category of contraction which is separate from Banach contraction, and does not imply continuity, was proposed by (Kannan, 1968) who also established in the same work that such mappings necessarily have unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type. In 1972, a new concept which is different from that of (Bannach, 1922) and (Kannan, 1968) for contraction type mapping was introduced by (Chatterjea, 1972) which gives a new direction to the study of fixed point theory. There are a class of contractive mappings which are different from Banach contraction and have unique fixed point in complete metric spaces. However,the prospect of fixed point theory charmed many researchers and so there is a vast literature available. Khojasteha et al. (2015) introduced the concept of Z-contractions by using simulation functions. This class of functions has received much recognition as these are convenient to exhibit a huge family of contractivity conditions that are renowned in fixed point theorem. Later on, Olgun et al.,(2016) provided a new class of Picard operators on complete metric spaces using the concept of generalized Z-contractions. In this exciting context,a lot of developments have been done in recent times. In (2019), Parvaneh and Ghoncheh introduced the notion of P-metric spaces as a generalization of metric spaces and b-metric spaces.

Inspired and motivated by the results of Khojasteha et al., (2015) and by the definition of Parvaneh and Ghoncheh (2019), we introduced a Z_Ω-contraction mappings with respect to simulation function $\zeta \in Z$ and we proved existence and uniqueness of fixed point for such mappings in complete P-metric spaces in our main result.

1.2 Statements of the problem

In this study we focused on establishing and proving a fixed point theorem for Z_{Ω}-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P-metric Spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to establish a fixed point theorem for Z_{Ω}-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P-metric Spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of fixed points for Z_{Ω}-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P- metric Spaces.
- To verify the uniqueness of the fixed point for Z_{Ω}-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P- metric Spaces .
- To provide an example in support of our main result.

1.4 Significance of the study

The study may have the following importance:

- The outcome of this study may contribute to research activities on study area.
- It may provide basic research skills to the researcher.
- It may help to show existence and uniqueness of solution of some problems involving integral and differential equations.

1.5 Delimitation of the Study

This study was delimited to finding the existence of a fixed point theorem for Z_{Ω}-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P-metric Spaces.

Chapter 2

Review of Related Literatures

Let X be a nonempty set and $T: X \to X$ be a self map. We say that x is a fixed point of T if $Tx = x$. Fixed point theory has been studied extensively, which can be seen from the works of many authors.
(Banach, 1922) Let (X,d) be a metric space and let $T : X \to X$. Then T is called a Banach contraction mapping if there exists $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point.
Kannan, (1968) Let (X,d) be a metric space and let $T : X \to X$. Then T is called a Kannan mapping if there exists $k \in [0,1/2)$ such that $d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)]$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point. (Chatterjea,1972), T is called Chatterjea mapping if there exists $k \in [0,1/2)$ such that $d(Tx,Ty) \leq k[d(x,Ty) + d(y,Tx)]$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point. Khajasteha et al.,(2015) Definition: Let $\zeta : R^+ \times R^+ \longrightarrow R$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:
$(\zeta_1) \zeta(0,0) = 0$
$(\zeta_2) \zeta(t,s) < s - t$ for all $t,s > 0$
(ζ_3) if t_n, s_n are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\lim_{n\to\infty} sup \zeta(t_n,s_n) < 0$. Khajasteha et al.(2015) Definition: Let X be a complete metric space with metric d, $T: X \to X$ a mapping and $\zeta \in Z$. Then T is called a Z-contraction with respect to ζ if the following condition is satisfied:
$\zeta(d(Tx,Ty),d(x,y)) \geq 0$ for all $x,y \in X$. This class of functions has received much recognition as these are convenient to exhibit a huge family of contractivity conditions that are renowned in fixed point theorem. Theorem (Khojasteha et al.,2015) Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a Z-contraction with respect to $\zeta \in Z$. Then the fixed point of $T \in X$ is unique, provided it exists. Theorem (Khonjasteha *et al.,* 2015) Let (X,d) be a complete metric space and $f: X \to X$ be a Z-contraction with respect to a certain simulation function ζ, then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim_{n\to\infty} f^n x_0 = u$ (say) in X and u is the unique fixed point of f in X. Theorem (Morgan *et al.*, 2016) Let

(X,d) be a complete metric space and $f\colon X \to X$ a selfmap on X. If there exists simulation function ζ such that $\zeta(d(fx,fy),M(x,y)) \geqslant 0$ for all $x,y \in X$, where $M(x,y) = max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\}$, then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim_{n\to\infty} f^n x_0 = u$ (say) in X and u is the unique fixed point of f in X. Later on,Olgun et al.,(2016) provided a new class of Picard operators on complete metric spaces using the concept of generalized Z-contractions. Ankush et al., (2017) introduced the notion of modified Z-contractions and explored the existence and uniqueness of fixed points of such functions on b-metric spaces. Babu and Dula (2017) Introduced a generalized $\alpha - \eta - Z$-contraction with respect to ζ in b-metric spaces .
Theorem (Babu and Dula ,2017) Let (X,d) be a complete b-metric space with coefficient $S \geq 1$.Let $T: X \longrightarrow X$ and $\alpha,\eta: X \times X \longrightarrow R^+$ be mappings.Suppose that the following conditions are satisfied: i) T is a generalized $\alpha - \eta - Z$-contraction with respect to ζ, ii) T is a triangular α-orbital admissible mapping with respect to η, iii) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$, and iv) T is an $\alpha - \eta$-continuous mapping. Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*. The basic idea of P-metric was initiated from the works of Parvaneh and Ghoncheh. In (2019), Parvaneh and Ghoncheh introduced the concept of P-metric space as a generalization of metric spaces, b-metric spaces. Theorem (Parvaneh and Ghoncheh) Let $(X,\preceq,d)$ be a partially ordered P-complete P-metric space. Let $f\colon X \longrightarrow X$ be an ordered non-decreasing continuous ordered $(\psi,\varphi)_\Omega$-contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point. Since then, some authors have been keeping their interest in finding the existence of fixed points of single valued selfmaps and set valued mappings in P-metric spaces.

Inspired and motivated by the results of Khojasteha et al., (2015) and by the definition of P-metric spaces introduced by (Parvaneh and Ghoncheh ,2019), we introduced a Z_Ω-contraction mappings with respect to simulation function $\zeta \in Z$ and we proved existence and uniqueness of fixed point for such mappings in complete P-metric spaces in our main result.

Chapter 3

Methodology

3.1 Study area and period

The study was conducted at Jimma University under the department of mathematics from September, 2019 G.C. to August, 2020 G.C.

3.2 Study Design

In this study we followed analytical method of design.

3.3 Source of Information

The relevant sources of information for this study were books and published articles related to the area of the study.

3.4 Mathematical Procedure of the Study

In this study we followed the procedures stated below:

- Establishing a fixed point theorem.
- Constructing a sequence.
- Showing the convergence of the sequence.
- Showing the constructed sequence is P-Cauchy .
- Proving the existence of a fixed point.
- Showing uniqueness of the fixed point.
- providing an example in support of our main result.

Chapter 4

Preliminaries and Main Result

4.1 Preliminaries

Notation:
1, Z is the set of all simulation function.
2, $R^{+} = [0,\infty)$.
3, N is the set of all natural number.
4, $R = (-\infty,\infty)$.

Definition 4.1.1 *Let X be a non empty set and $T: X \to X$ be a self map. We say that x is a fixed point of T if $Tx = x$.*

Definition 4.1.2 *(Browder and Petryshyn, 1966) Let (X,d) be a metric space. A mappings T of X into itself is said to be asymptotically regular at a point x in X, if* $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$.

Definition 4.1.3 *(Czerwik,1993) Let X be a nonempty set.A function $d: X \times X \longrightarrow R^{+}$ is said to be a b-metric if the following conditions are satisfied :*
(a) $0 \leqslant d(x,y)$ *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;
(b) $d(x,y) = d(y,x)$ *for all* $x,y \in X$*, and*
(c) There exists $s \geq 1$ *such that* $d(x,z) \leq s[d(x,y) + d(y,z)]$ *for all* $x,y \in X$.
In this case, the pair (X,d) is called a b-m etric space with coefficient s. Here, we observe that every metric space is a b-metric space, with $s = 1$.

Definition 4.1.4 *(Parvaneh and Ghoncheh) Let X be a nonempty set. A function $\tilde{d}: X \times X \longrightarrow R^{+}$ is p- metric, if there exists a strictly increasing continuous function $\Omega: R^{+} \to R^{+}$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for all $t \geq 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$. Such that for all $x;y;z \in X$, the following conditions hold:*

$$\begin{aligned}(P_1)\tilde{d}(x,y) &= 0 \text{ iff } x=y,\\ (p_2)\tilde{d}(x,y) &= \tilde{d}(y,x),\\ (P_3)\tilde{d}(x,z) &\leq \Omega[\tilde{d}(x,y)+\tilde{d}(y,z)].\end{aligned}$$

In this case, the pair $(X,\tilde{d})$ is called a p- metric space.

It should be noted that, the class of p- metric spaces is effectively larger than the class of b-metric spaces, Since a P- metric is a b-metric when $\Omega(t) = st$ for fixed $s \geq 1$, while a P-metric is metric when $\Omega(t) = t$. Here, we present an example to show that a P-metric need not a b-metric.

Example 1 (Parvaneh and Ghoncheh) Let $(X,\tilde{d})$ be a P-metric spaces and $\rho(x,y) = \sinh\tilde{d}(x,y)$. We show that ρ is a P-metric with $\Omega(t) = \sinh(t)$ for all $t \geq 0$.

Obviously,condition (i) and (ii) of Definition (4.1.4) are satisfied. For each $x,y,z \in X$,

$$\begin{aligned}\rho(x,y) = &\quad \sinh(\tilde{d}(x,y))\\ &\lneq \sinh(\tilde{d}(x,z)+\tilde{d}(z,y))\\ &\leq \sinh[\sinh(\tilde{d}(x,z)+\tilde{d}(z,y))]\\ &= \sinh(\rho(x,z)+\rho(z,y))\\ &= \Omega(\rho(x,z)+\rho(z,y)).\end{aligned}$$

So,condition (iii) of Definition 4.1.4 is also satisfied and ρ and ρ is a P-metric.
Note that,
$\sinh|x-y|$ is not a metric on R, as we know that
$\sinh 5 = 74.2032105778 \geq 3.62686040785 + 10.0178749274 = \sinh 2 + \sinh 3$
and $\sinh|x-y|$ is not also a b-metric for any $s \geq 1$.
In general, a P-metric is not necessarily continuous.

Definition 4.1.5 *(Khojasteha et al., 2015) Let X be a complete metric space with metric d, $T : X \to X$ a mapping and $\zeta \in Z$. Then T is called a Z-contraction with respect to ζ if the followig condition is satisfied*:
$\zeta(d(Tx,Ty),d(x,y)) \geq 0$*, for all* $x,y \in X$.

Theorem 4.1.1 *(Khojasteha et al., 2015) Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a Z-contraction with respect to $\zeta \in Z$. Then the fixed point of $T \in X$ is unique, provided it exists.*

Definition 4.1.6 *(Khojasteha et al., 2015) Let $\zeta : R^+ \times R^+ \longrightarrow R$ be a mapping, then $\zeta \in Z$ is called a simulation function if it satisfies the following conditions:*
$(\zeta_1)\zeta(0,0) = 0$
$(\zeta_2)\zeta(t,s) < s - t$ *for all* $t,s > 0$
(ζ_3) *if* t_n,s_n *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$
$\lim_{n\to\infty} sup\zeta(t_n,s_n) < 0$.
Remark *Let ζ be a simulation function.If $\{t_n\}$,$\{s_n\}$ are sequences in $(0,\infty)$ such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, *then* $\lim_{n\to\infty} sup\zeta(kt_n,s_n) < 0$ *for any* $k > 1$.
Example 2 *Let ζ_i: $R^+ \times R^+ \longrightarrow R$ defined by*
$(i)\zeta(t,s) = \lambda s - t$ *for all* $t,s \in [0,\infty)$ *,where* $\lambda \in [0,1)$,
$(ii)\zeta(t,s) = \frac{s}{1+s} - t$ *for all* $t,s \in [0,\infty)$,
$(iii)\zeta(t,s) = s - kt$ *for all* $t,s \in [0,\infty)$ *where* $k > 1$,
$(iv)\zeta(s,t) = \frac{s}{1+s} - te^t$ *for all* $t,s \in [0,\infty)$,
$(v)\zeta(t,s) = \frac{1}{s+1} - (t+1)$ *for all* $t,s \in [0,\infty)$.

Definition 4.1.7 *(Parvaneh and Ghoncheh) Let $(X,\tilde{d})$ be a p-metric space. Then a sequence $\{x_n\}$ in X is called :*
a. p-convergent if there exists $x \in X$ such that $\tilde{d}(x_n,x) \to 0$ as $n \to \infty$
b. p-Cauchy if $\tilde{d}(x_n,x_m) \to 0$ as $n,m \to \infty$
c. The p-metric space $(X,\tilde{d})$ is p-complete if every p-Cauchy sequence in X is p-convergent.

Lemma 4.1.1 (Parvaneh and Ghoncheh) Let $(X, \tilde{d})$ be a P- metric space with function Ω, and suppose that x_n and y_n P-converge to x, y, respectively. Then we have $\Omega^{-1}(\tilde{d}(x,y)) \leq \liminf_{n\to\infty} \tilde{d}(x_n, y_n) \leq \limsup_{n\to\infty} \tilde{d}(x_n, y_n) \leq \Omega^2(\tilde{d}(x,y))$
In particular, if $x = y$, then, $\lim_{n\to\infty} \tilde{d}(x_n, y_n) = 0$.
Moreover, for each $z \in X$ we have
$\Omega^{-1}(\tilde{d}(x,z)) \leq \liminf_{n\to\infty} \tilde{d}(x_n, z) \leq \limsup_{n\to\infty} \tilde{d}(x_n, z) \leq \Omega(\tilde{d}(x,y))$.

Inspired and motivated by the results of Khojasteha et al., (2015) and by the definition of P-metric spaces introduced by (Parvaneh and Ghoncheh ,2019), we introduced a Z_Ω-contraction mappings with respect to simulation function $\zeta \in Z$ and we proved existence and uniqueness of fixed point for such mappings in complete P-metric spaces in our main result.

4.2 Main Result

In this section,we introduced a fixed point theorem for Z_Ω-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P- metric Spaces and proved a fixed point result.

Definition 4.2.1 *Let $(X,\tilde{d})$ be a P- metric space, $T: X \longrightarrow X$ be a mapping and $\zeta \in Z$. Then T is said to be Z_Ω-contraction mapping with respect to $\zeta \in Z$, if the following condition is satisfied:*

$$\zeta(\Omega^4(\tilde{d}(Tx,Ty),\tilde{d}(x,y)) \geq 0, \textit{ for all } x,y \in X. \tag{4.1}$$

Remark A simple example of Z_Ω-contraction is the Banach contraction in P-metric space which can be obtained by taking $\lambda \in [0,1)$ and $\zeta(t,s) = \lambda s - t$ for all $s,t \in R^+$ and Ω is identity map in the above definition.

Lemma 4.2.1 Let $(X,\tilde{d})$ be a P-metric space and $T: X \longrightarrow X$ be a Z_Ω-contraction mapping with respect to $\zeta \in Z$. Then T is asymptotically regular at every $x \in X$.
Proof: Let $x \in X$ be arbitrary. If for some $p \in N$, we have $T^p x = T^{p-1}x$, that is, $Ty = y$, where $y = T^{p-1}x$, then $T^n y = T^{n-1}Ty = T^{n-1}y = ... = Ty = y$ for all $n \in N$. Now for sufficient large $n \in N$, we have

$$\begin{aligned}
\tilde{d}(T^n x, T^{n+1}x) \quad &= \tilde{d}(T^{n-p+1}T^{p-1}x, T^{n-p+2}T^{p-1}x) \\
&= \tilde{d}(T^{n-p+1}y, T^{n-p+2}y) \\
&= \tilde{d}(y,y) = 0
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \tilde{d}(T^n x, T^{n+1}x) = 0$.
Suppose $\tilde{d}(T^n x, T^{n+1}x) \geq \tilde{d}(T^n x, T^{n-1}x)$, for all $n \in N$, then it follows from (4.1) that

$$\begin{aligned}
0 \quad &\leq \zeta(\Omega^4(\tilde{d}(T^{n+1}x, T^n x), \tilde{d}(T^n x, T^{n-1}x)) \\
&= \zeta(\Omega^4(\tilde{d}(TT^n x, TT^{n-1}x), \tilde{d}(T^n x, T^{n-1}x)) \\
&\leq \tilde{d}(T^n x, T^{n-1}x) - \Omega^4(\tilde{d}(T^{n+1}x, T^n x)). \\
&< 0.
\end{aligned}$$

Which is a contradiction.So $\tilde{d}(T^n x, T^{n+1}x) < \tilde{d}(T^n x, T^{n-1}x)$, for all $n \in N$. This shows that $\{\tilde{d}(T^n x, T^{n+1}x)\}$ is a monotonically decreasing sequence of nonnegative reals and so it must be convergent. Let $\lim_{n\to\infty} \tilde{d}(T^n x, T^{n+1}x) = r \geq 0$. If $r > 0$ then since T is Z_Ω-contraction with respect to $\zeta \in Z$, therefore by ζ_3, we have

$$0 \leq \lim_{n\to\infty} sup\zeta(\Omega^4(\tilde{d}(T^{n+1}x, T^n x), \tilde{d}(T^n x, T^{n-1}x)) < 0,$$

a contradiction. This contradiction shows that $r = 0$, that is, $\lim_{n\to\infty} \tilde{d}(T^n x, T^{n+1}x) = 0$. Thus T is an asymptotically regular mapping at x.

The next Lemma shows that the Picard sequence $\{x_n\}$ generated by a Z_Ω is always bounded.

Lemma 4.2.2 Let $(X, \tilde{d})$ be a P-metric space and $T : X \longrightarrow X$ be a Z_Ω-contraction mapping with respect to ζ. Then the Picard sequence $\{x_n\}$ generated by T with initial value $x_0 \in X$ is a P-Cauchy sequence, where $x_n = Tx_{n-1}$ for all $n \in N$.

Proof: Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is, $x_n = Tx_{n-1}$ for all $n \in N$. We now prove that $\{x_n\}$ is a P-Cauchy sequence. Suppose $\{x_n\}$ is not P-Cauchy. Then there exist $\varepsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k > k$ such that

$$\tilde{d}(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{4.2}$$

and

$$\tilde{d}(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{4.3}$$

such $k > 0$,corresponding to m_k, we can choose n_k to be smallest positive integer such that $\tilde{d}(x_{m_k}, x_{n_k}) \geq \varepsilon$,$\tilde{d}(x_m, x_{n_k-1}) < \varepsilon$

Now we have the following two cases.

Case (i) when $\Omega(t)$ is identity map i.e $\Omega(t) = t$

Now by triangle inequality and using (4.2),(4.3), we have

$$\begin{aligned} \varepsilon \leq \tilde{d}(x_{m_k}, x_{n_k}) &\leq \tilde{d}(x_{m_k, n_k-1}) + \tilde{d}(x_{n_k-1}, x_{n_k}) \\ &\leq \varepsilon + \tilde{d}(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{4.4}$$

Taking limit as $k \longrightarrow \infty$ in (4.4) and using Lemma (4.2.1), we have

$$\varepsilon \leq \lim_{k\to\infty} \tilde{d}(x_{m_k}, x_{n_k}) \leq \varepsilon.$$

This implies,

$$\lim_{k\to\infty} \tilde{d}(x_{m_k}, x_{n_k}) = \varepsilon.$$

Since $\tilde{d}(x_{m_k+1}, x_{n_k+1} < \tilde{d}(x_{m_k}, x_{n_k})$ from Lemma (4.2.1), we have

$$\varepsilon \leq \tilde{d}(x_{m_k+1}, x_{n_k+1}) \leq \tilde{d}(x_{m_k}, x_{n_k}). \tag{4.5}$$

Taking the limit as $k \longrightarrow \infty$ in (4.5), we get

$$\lim_{k\to\infty} \tilde{d}(x_{m_k+1}, x_{n_k+1}) = \varepsilon.$$

By ζ_3, we have

$$0 \leq \lim_{k\to\infty} sup \zeta(\Omega^4(\tilde{d}(Tx_{m_k}, Tx_{n_k}), \tilde{d}(x_{m_k}, x_{n_k})) < 0$$

which is a contradiction. Hence $\{x_n\}$ is P-Cauchy.

case (ii) Suppose $\Omega(t)$ is not identity.

$t \leq \Omega(t)$, we have $\Omega(t) > t$.

By using triangle inequality and using (4.2),(4.3), we have

$$\varepsilon \leq \tilde{d}(x_{m_k}, x_{n_k}) \leq \Omega(\tilde{d}(x_{m_k}, x_{n_k-1}) + \tilde{d}(x_{n_k-1}, x_{n_k}))$$

$$\leq \Omega(\varepsilon + \tilde{d}(x_{n_k-1}, x_{n_k})). \tag{4.6}$$

Taking limit sup as $k \longrightarrow \infty$ in (4.6) and using Lemma (4.2.1), we have

$$\lim_{k\to\infty} sup \tilde{d}(x_{m_k}, x_{n_k}) \leq \Omega(\varepsilon). \tag{4.7}$$

Again by triangle inequality and using (4.2),(4.3), we have

$$\varepsilon \leq \tilde{d}(x_{m_k}, x_{n_k}) \leq \Omega(\tilde{d}(x_{m_k}, x_{n_k+1}) + \tilde{d}(x_{n_k+1}, x_{n_k}))$$

$$\leq \Omega(\Omega(x_{m_k}, x_{m_k+1}) + \tilde{d}(x_{m_k+1}, x_{n_k+1})) + \tilde{d}(x_{n_k+1}, x_{n_k})). \tag{4.8}$$

Taking the limit inf as $k \longrightarrow \infty$ in (4.8) and using Lemma (4.2.1), we get

$$\Omega^{-2}(\varepsilon) \leq \lim_{k\to\infty} inf \tilde{d}(x_{m_k+1}, x_{n_k+1}). \tag{4.9}$$

Now since T is a Z_Ω-contraction with respect to $\zeta \in Z$ and by (ζ_3), we have

$$\begin{aligned} 0 &\leq \zeta(\Omega^4(\tilde{d}(Tx_{m_k}, Tx_{n_k}), \tilde{d}(x_{m_k}, x_{n_k})) \\ &\leq \tilde{d}(x_{m_k}, x_{n_k})) - \Omega^4(\tilde{d}(Tx_{m_k}, Tx_{n_k})) \end{aligned}$$

That is $, 0 \leq \tilde{d}(x_{m_k}, x_{n_k})) - \Omega^4(\tilde{d}(Tx_{m_k}, Tx_{n_k}))$.

Taking the limit sup as $k \longrightarrow \infty$ in the above inequalities, we have

$$0 \leq \lim_{k\to\infty} sup \tilde{d}(x_{m_k}, x_{n_k}) - \lim_{k\to\infty} inf \Omega^4(\tilde{d}(Tx_{m_k}, Tx_{n_k})).$$

Then from (4.7) and (4.9), we have

$$0 \leq \Omega(\varepsilon) - \Omega^4(\Omega^{-2}(\varepsilon)) = \Omega(\varepsilon) - \Omega^2(\varepsilon) < 0$$

which is a contradiction. Hence from case (i) and case (ii) the sequence $\{x_n\}$ is P-Cauchy.

Theorem 4.2.1 *Let X be a P-complete P-metric space and $T : X \longrightarrow X$ be Z_Ω-contraction with respect to $\zeta \in Z$. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in N$ converges to the fixed point of T and this fixed point is unique.*

Proof: Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the Picard sequence, that is , $x_n = Tx_{n-1}$ for all $n \in N$. By above Lemma $\{x_n\}$ is a P-Cauchy.

Since X is P-complete there exists some $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now we show that u is a fixed point of T. That is, $Tu = u$.
Assume $\tilde{d}(u,Tu) > 0$.
As T is a Z_Ω-contraction with respect to $\zeta \in Z$, we drive that

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} sup\zeta(\Omega^4(\tilde{d}(Tx_{n-1},Tu),\tilde{d}(x_n,u) \\
&\leq \lim_{n\to\infty} sup\tilde{d}(x_n,u) - \lim_{n\to\infty} sup\Omega^4(\tilde{d}(Tx_{n-1},Tu)) \\
&= \lim_{n\to\infty} sup\tilde{d}(x_n,u) - \lim_{n\to\infty} sup\Omega^4(\tilde{d}(x_n,Tu)) \\
&\leq -\Omega^4(\tilde{d}(u,Tu)) < 0
\end{aligned}$$

which is a contradiction.
Hence $\Omega^4(\tilde{d}(u,Tu)) = 0$ implies $\tilde{d}(u,Tu) = 0$ gives $Tu = u$.
So we can conclude that u is a fixed point of T.

Now we show that T has a unique fixed point.

Proof: Suppose u and v in X be two distinct fixed points of T in X, that is $Tv = v$ and $Tu = u$. Suppose $u \neq v$, now from (4.1) we have:

$$0 \leq \zeta(\Omega^4(\tilde{d}(Tu,Tv),\tilde{d}(u,v)) = \zeta(\Omega^4(\tilde{d}(u,v),\tilde{d}(u,v)) = \tilde{d}(u,v) - \Omega^4(\tilde{d}(u,v)) < 0$$

which is a contradiction. Hence $u = v$.

Example 1: Let $X = [0, \frac{1}{5}] \cup \{\frac{1}{2}\} \cup [\frac{4}{5}, 1]$ and $\tilde{d} \colon X \times X \longrightarrow R^{+}$ be defined by $\tilde{d}(x,y) = \sinh|x-y|$ with $\Omega(t) = \sinh(t)$. Then $(X, \tilde{d})$ is a P-complete P-metric space. Define a mapping $T \colon X \longrightarrow X$ as:

$$Tx = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1]/\frac{1}{2}, \\ \frac{1}{16}, & \text{if } x = \frac{1}{2}. \end{cases}$$

We now define $\zeta \colon R^{+} \times R^{+} \longrightarrow R^{+}$ by $\zeta(t,s) = \lambda S - t$, for $\lambda = 0.9$.
We have the following possible cases.
Case (i) If $x \neq \frac{1}{2}, y = \frac{1}{2}$.
In this case, $Tx = 0$, $Ty = \frac{1}{16}$ and

$$\Omega^4(\tilde{d}(Tx,Ty)) = \sinh^4(\sinh(|Tx-Ty|)) = \sinh^4(\sinh(\frac{1}{16})) = 0.25278024.$$

Now we consider,

$$\begin{aligned} \zeta(\Omega^4(\tilde{d}(Tx,Ty), \tilde{d}(x,y)) &= \tilde{d}(x,y) - \Omega^4(\tilde{d}(Tx,Ty)) \\ &\geq \lambda \sinh|x-y| - \sinh^4(\sinh|Tx-Ty|) \\ &= \lambda \sinh|x-y| - sinh^4(\sinh(\frac{1}{16})) \\ &= \lambda \sinh|x-y| - 0.25278024 \\ &\geq \lambda \sinh|x - \frac{1}{2}| - 0.25278024 \\ &\geq 0. \end{aligned}$$

Case (ii): $x, y \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1]$.
In this case, $Tx = 0$, $Ty = 0$, $\Omega^4(\tilde{d}(Tx,Ty)) = \sinh^4(\tilde{d}(Tx,Ty)) = \sinh^4(\sinh(0)) = 0$.
Now we consider,

$$\begin{aligned} \zeta(\Omega^4(\tilde{d}(Tx,Ty), \tilde{d}(x,y)) &= \tilde{d}(x,y) - \Omega^4(\tilde{d}(Tx,Ty)) \\ &\geq \lambda \tilde{d}(x,y) - \Omega^4(\tilde{d}(Tx,Ty)) \\ &= \lambda \sinh|x-y| - 0 \\ &\geq 0. \end{aligned}$$

Hence applying Theorem (4.1), we get, T has a unique fixed point and $u = 0 \in X$.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

(Khojasteh et al., 2015) established a fixed point theorem for Z-contraction mappings with respect to simulation function $\zeta \in Z$ in complete metric Spaces and proved the existence and uniqueness of fixed point. In this thesis work, we established a fixed point theorem for Z_Ω-contraction mappings with respect to simulation function $\zeta \in Z$ in complete P- metric Spaces and proved the existence and uniqueness of fixed point. Our result extends the work of (Khojasteh et al., 2015) from metric space to P-metric space. We have also supported the main result of this research work by applicable example.

5.2 Future scope

There are some published results related to the existence of a fixed point theorem of mappings defined on P-metric space. The researcher believes the search for the existence and uniqueness of a fixed points of self-mappings satisfying Z_Ω-contraction mappings with respect to simulation function $\zeta \in Z$ in P-metric space is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this area.

References

Aghajani, A., Abbas, M.,& Roshan,J.(2014). Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, 64(4), 941-960.

Ansari,A. H., Chandok, S., Ionescu, C.(2014). Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions.Journal of inequalities and Applications 2014(1),429.

Babu, G.V.R.,& Sailaja, P.D.(2012). A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces.Thai Journal of Mathe matics,9(1), 1-10.

Bakhtin, I. A.(1989). The contraction mapping principle in almost metric spaces, Funct. Ana.Gos.Ped.Inst.Unianowsk,30, 26-37.

Banach, S. (1922). Sur les operations dans les ensembles abstraitet leur application aux equations integrals. *Fund Math, 3(1), 133 - 181.*

Boriceanu, M.(2009). Strict xed point theorems for multivalued operators in b-metric spaces. Int.J.Mod. Math,4(3), 285-301.

Boriceanu, M., Bota, M.,& Petrusel, A.(2010). Multivalued fractals in b-metric spaces.Centeral European Journal of Mathematics, 8(2), 367 - 377.

Boyd,D. W.,& Wong,J. S. W.(1969). On nonlinear contractions, Proc. Amer. Math. Soc. 20 , 458-464.

Bourbaki, N.(1974). Topologie Generale, Herman: Paris, France.

Browder, F.E., & Petryshyn, W.V.(1966). The solution by iteration of nonlinear functional equations in Banach spaces. Bulletin of american mathematical So ciety, 72(3), 571-575.

Chatterjea, S. K.(1972). Fixed point theorems.*Computes.Rend. Acad, Bulgaria*Sci., 25, 727-730.

Czerwik, S. (1993). Contraction mappings in b-metric spaces. Acta Mathemat ica et Informatica Universitatis Ostraviensis, 1(1) , 5 - 11.

Huang, H., Deng,G. & Radenovic, S.(2017). Fixed point theorems for c-class func tions in b-metric spaces and applications. J. Nonlinear Sci. Appl, 10, 5853-5868.

Hussain, N., Kadelburg, Z., Radenovic, S. & Al-Solamy, F. R.(2012).Comparison functions and xed point results in partial metric spaces, Abstract and Applied Analysis, vol., Article ID 605-781, 15 pp. Hussain, N., parvaneh, V.,
Roshan, J.R.,& Kadelburg, Z.(2013). Fixed points of cycle weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications. Fixed Point Theory and Applications, 2013(1),1- 18.

Hussain, N., Khaleghizadeh, S.,Salimi, P., & Abdou, A.A.(2014,Appril). A new approach to fixed point results in triangular intuionastic Fuzzy metric spaces. In Abstract and Applied Analysis (Vol.2014).Hindawi.

Jovanovic, M., Kadelburg, Z.,& Radenovic, S.(2010). Common xed point re sults in metric type spaces. Fixed Point Theory and Applications,2010(1), 978121.

Khojasteh,F.,& Rakocevic, V. (2012). Some new common xed point results for generalized contractive multi-valued non-self-mappings. Applied Mathematics Letter,25(3),287-293.

Khojasteh,F., Shukla,S.,& Radenovic,S.(2015). A new approach to the study of fixed point theory for simulation functions.Filomat, 29(6), 1189-1194.

Mizoguchi, N.,& Takahashi, W. (1989). Fixed point theorems for multivalued map pings on complete metric spaces. Journal of Mathematical Analysis and Appli cations, 141(1), 177-188.

Olgun, M.,ALYILDIZ, T.,& BiCER, O.(2016). A new aspect to Picard operators with simulation functions. Turkish Journal of Mathematics, 40(4), 832-837.

Ozturk, V., & Radenovic, S. (2016). Some remarks on b-(E.A)-property in b-metric spaces.Springerplus, 5(1), 544.

Parvaneh, V., & Hosseini Ghoncheh, S. J. Fixed points of $(\psi, \varphi)_\Omega$-contractive mappings in ordered P-metric spaces, submitted.

Rhoades,B. E.(2001). Some theorems on weakly contractive maps. Nonlinear Analysis:Theory Methods & Applications, 47(4) , 2683-2693.

Roshan, J. R., Parvaneh, V., & kadelburg, Z.(2014). Common xed point theo rems for weakly isotone increasing mappings in ordered b-metricspaces. J. Nonlinear Sci. Appl., 7(4), 229-245.

Shatanawi, W.(2016). Fixed and common xed point for mappings satisfying some nonlinear contractions in b-metric spaces, Journal Mathematica Analysis, 7, 1-12.